Robustness analysis tools for an uncertainty set obtained by prediction error identification ⋆

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Abstract

This paper presents a robust stability and performance analysis for an uncertainty set delivered by classical prediction error identification. This nonstandard uncertainty set, which is a set of parametrized transfer functions with a parameter vector in an ellipsoid, contains the true system at a certain probability level. Our robust stability result is a necessary and sufficient condition for the stabilization, by a given controller, of all systems in such uncertainty set. The main new technical contribution of this paper is our robust performance result: we show that the worst case performance achieved over all systems in such an uncertainty region is the solution of a convex optimization problem involving Linear Matrix Inequality (LMI) constraints. Note that we only consider Single Input Single Output (SISO) systems.

Key words: identification for control, robust stability, robust performance, LMI

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1 Introduction

This paper is part of our continuing effort to close the gap between time-domain prediction error identification and robust control theory (Bombois *et*

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al., 1999; Gevers *et al.*, 2000; Bombois, 2000). We have shown in (Bombois *et al.*, 1999; Gevers *et al.*, 2000; Bombois, 2000) that a prediction error identification step with a full order model structure can be used as a procedure for model set validation. The key advantage of using a full order model structure is that the estimated error is due to the noise only (no bias error): see (Ljung, 1999). This full order identification step can be performed either on the system itself (Bombois *et al.*, 1999; Bombois, 2000) or on the model error (Ljung, 1998; Gevers *et al.*, 2000). In both cases, it delivers an uncertainty region D containing the true system at a certain probability level, to be chosen by the designer. As shown in (Bombois, 2000), this uncertainty set D has a generic structure in which the elements are ratios of transfer functions parametrized by a real vector δ that belongs to an ellipsoid. Such uncertainty set is specific to prediction error identification (or validation), and is clearly nonstandard in mainstream robust control theory. It is described by the following Proposition (Bombois *et al.*, 1999; Gevers *et al.*, 2000; Bombois, 2000).

Proposition 1 *Consider a true linear time-invariant system $y = G_0 u + v$, where G_0 is a rational SISO transfer function and v is additive noise. Let $G(z,\delta)$ be a full order model structure such that $G_0 = G(z,\delta_0)$ for some δ_0. The uncertainty region D resulting from either open loop or closed loop prediction error identification with the unbiased model structure[2] $G(z,\delta)$, and which contains the true system G_0 at a prescribed probability level, can always be described in the following generic form:*

$$D = \left\{ G(z,\delta) \mid G(z,\delta) = \frac{e + Z_N\delta}{1 + Z_D\delta} \text{ and } \delta \in U = \{\delta \mid (\delta - \hat{\delta})^T R (\delta - \hat{\delta}) < 1\} \right\} \quad (1)$$

where $\delta \in \mathbf{R}^{k\times 1}$ is a real parameter vector, $\hat{\delta}$ is the parameter estimate resulting from the identification step, R is a symmetric positive definite matrix $\in \mathbf{R}^{k\times k}$ that is proportional to the inverse of the covariance matrix of $\hat{\delta}$, $Z_N(z)$ and $Z_D(z)$ are row vectors of size k of known transfer functions, and $e(z)$ is a known transfer function.

The uncertainty region D of Proposition 1, which we have baptized "generic prediction error (PE) uncertainty set" in (Gevers *et al.*, 2000), is a direct result of the use of a prediction error method with an unbiased model structure for the construction of an uncertainty set. The true system belongs to D with some probability level α which the user is free to select; that level is directly connected to the scaling of the matrix R.

[2] We have recently extended our analysis to the case of possibly biased models using a stochastic embedding approach (Goodwin *et al.*, 1992) (see (Bombois *et al.*, 2000)).

In this paper, we no longer discuss how this set arises from prediction error identification or validation. Rather, we take this generic set D as the starting point of a robust control analysis. We develop a necessary and sufficient condition for the stabilization of all plants in such a set D by a given controller C. More importantly, we show how to compute the worst case performance over all closed loop systems made up of the controller C and all plants in D. We show that this worst case performance can be computed exactly by an LMI-based optimization problem. Note that we only consider SISO systems.

Comparison with other uncertainty regions deduced from measured data. Note that the uncertainty set D is directly deduced from measured data; no prior assumptions are required on the magnitude of the noise and of the impulse response. The only important restriction imposed in the Prediction Error procedure that leads to the uncertainty set D is that the model structure contains the true system; the same holds for the Model Error Model approach developed by Ljung (Ljung, 1998) for open-loop validation and extended to closed-loop validation in (Gevers *et al.*, 1999). This assumption has recently been relaxed in (Bombois *et al.*, 2000).

Other uncertainty regions, that do not have the structure of our set D, have been described in the literature, based on rather different assumptions and methods (Chen, 1997; Poolla *et al.*, 1994; Milanese, 1998; Hakvoort, 1994). In (Chen, 1997; Poolla *et al.*, 1994) and references therein, a method is proposed to decide whether a postulated region with bounded uncertainties is consistent with measured input-output data (the so-called model invalidation concept). In (Milanese, 1998) (and references therein) a hard bound assumption is made on the noise and on the impulse response of the true system in order to derive a parametric uncertainty set using *set membership identification*. In (Hakvoort, 1994) an additive uncertainty region is estimated on the basis of a stochastic noise assumption, but with a known prior bound again on the impulse response of the true system. Furthermore, the approach presented in (Hakvoort, 1994) is restricted to linearly parametrized models, such as FIR models, whereas our uncertainty set D is described by rational transfer functions with denominator uncertainty.

Robustness analysis of D. Our robust stability result for the uncertainty set D takes the form of a necessary and sufficient condition for the stabilization of all plants in D by a given controller C. This result is obtained by recasting the closed-loop connections of the controller C and the systems in D as a Linear Fractional Transformation (LFT) (see e.g. (Zhou *et al.*, 1995)) where the uncertainty part is a real vector. The necessary and sufficient condition is then deduced from the results in (Rantzer, 1992). A necessary and sufficient condition could also have been derived from the results in (Biernacki *et al.*, 1987). However, the advantage in our approach is that we recast the uncertainty region D as a rank-one LFT which allows us to possibly use the convex or quasi-convex optimization methods of (Rantzer and Megretski, 1994) for the solution of robust control design problems using such uncertainty set D.

The main technical contribution of this paper, however, lies in the procedure for robust performance analysis. We provide an exact computation, using an LMI-based optimization problem, of the worst case performance over all closed loops made up of the controller C and all systems $G(z, \delta)$ in the uncertainty region D. The performance of a particular loop $[C \; G(z, \delta)]$ is here defined as the largest singular value of a weighted version of the matrix containing the four closed-loop transfer functions of this loop. Our definition of the worst case performance is thus very general and, by an appropriate choice of the weights, allows one to derive most of the commonly used worst case performance measures, such as e.g. the largest modulus of the sensitivity function. The LMI formulation of the problem uses the fact that the uncertainty part (i.e. the real parameter vector) appears linearly in the expression of both the numerator and the denominator of the systems in the uncertainty region D and, as a consequence, also appears linearly in the expression of the different closed-loop transfer functions.

Some solutions have already been proposed for the computation of the worst case performance over model sets described by parametric uncertainties. However, they are not applicable to our model set D of Proposition 1. In (Fan and Tits, 1992; Ferreres and Fromion, 1997), the worst case performance in an uncertainty region described by an LFT is computed using an extension of the structured singular value μ. However, this is done only for a limited number of parametric uncertainties, which do not cover the case of a real vector as in our uncertainty region D. In (Bhattacharyya *et al.*, 1995, page 402), the authors give a procedure for the computation of the worst case performance in uncertainty regions defined by a real vector that is constrained to lie in a hypercube. This is achieved by replacing the original problem by a fixed number of simple optimization problems involving one parameter. This procedure can not be used for the computation of the worst case performance in D, where the real uncertainty vector is constrained to lie in an ellipsoid and not in a hypercube.

Paper outline. In Section 2, the robust stability analysis procedure for the uncertainty region D is developed. In Section 3, the concept of worst case performance level is introduced and the LMI-based optimization problem developed for its computation is given. Our procedures are illustrated by an example in Section 4. Finally, some conclusions are given in the last section.

2 Robust stability analysis of D

Consider an uncertainty region D given by (1) and containing G_0 at some probability level. We now give a necessary and sufficient condition for the stabilization by a given controller C of all plants in D. This theorem is based on results presented in (Rantzer, 1992).

Theorem 2 *Consider an uncertainty set D of the form (1) and a controller $C(z) = X(z)/Y(z)$ [3] that stabilizes the center of that set, $G(z, \hat{\delta})$. Then all models in D are stabilized by $C(z)$ if and only if*

$$\max_{\Omega} \mu(M_D(e^{j\Omega})) \leq 1, \tag{2}$$

where

- *$M_D(z)$ is defined as*

$$M_D(z) = \frac{-(Z_D + \frac{X(Z_N - eZ_D)}{Y+eX})T^{-1}}{1 + (Z_D + \frac{X(Z_N - eZ_D)}{Y+eX})\hat{\delta}}, \tag{3}$$

- *T is a square root of the matrix R defining U : $R = T^T T$.*
- *$\phi = T(\delta - \hat{\delta})$, whereby $\delta \in U \Leftrightarrow |\phi|_2 < 1$*
- *$\mu(M_D(e^{j\Omega}))$ is called the (real) stability radius of the loop $[M_D(z) \quad \phi]$. For a real vector ϕ it is computed as follows:*

$$\begin{aligned} &\mu(M(e^{j\Omega})) = \sqrt{|Re(M)|_2^2 - \frac{(Re(M)Im(M)^T)^2}{|Im(M)|_2^2}} \quad \text{if } Im(M) \neq 0 \\ &\mu(M(e^{j\Omega})) = |M|_2 \quad \text{if } Im(M) = 0. \end{aligned} \tag{4}$$

Proof. The proof consists of showing that the set of feedback loops $[C\ G(z, \delta)]$ can be recast in a framework to which the results of (Rantzer, 1992) can be applied. It is easy to prove that the closed-loop connection of a plant $G(z, \delta)$ in D with the controller C can be restated in the general LFT framework of robust stability analysis by introducing the signals p_1 and q such that $p_1 = \delta q$

$$\begin{cases} y = \frac{e+Z_N\delta}{1+Z_D\delta} u = (e + \frac{(Z_N - eZ_D)\delta}{1+Z_D\delta})u \\ u = -Cy \end{cases} \Longleftrightarrow \begin{cases} p_1 = \delta q \\ q = \overbrace{(-Z_D - \frac{C(Z_N - eZ_D)}{1+eC})}^{M_1(z)} p_1 \end{cases} \tag{5}$$

[3] $X(z)$ and $Y(z)$ are the polynomials corresponding to the numerator and to the denominator of $C(z)$, respectively.

We now show the equivalence between the set of loops $[C\ G(z,\delta)]$ for all $\delta \in U$ and the set of loops $[M_D(z)\ \phi]$ for all ϕ such that $|\phi|_2 < 1$, by replacing δ and its uncertainty domain U by the real vector $\phi \triangleq T(\delta - \hat{\delta})$ and its uncertainty domain $|\phi|_2 < 1$. With $p \triangleq \phi q$ and $\delta = \hat{\delta} + T^{-1}\phi$, we have

$$\begin{cases} p_1 = \delta q \\ q = M_1(z) p_1 \end{cases} \Leftrightarrow \begin{cases} p = \phi q \\ q = \frac{M_1 T^{-1}}{1 - M_1 \hat{\delta}} p = M_D(z) p \end{cases} \tag{6}$$

The necessary and sufficient condition then follows from the fact that $M_D(z) \in H_\infty$ and from a result in (Rantzer, 1992). This result states that (4) is the stability radius of the set of loops $[M_D(z)\ \phi]$ whose uncertainty part is a real vector constrained to lie in a two-norm unit ball.

3 Robust performance analysis of D

In Section 2, we have presented a procedure to check whether a controller C stabilizes all plants in the uncertainty region D. However, stabilization does not imply good performance with all plants in D. In this section, we show that we can evaluate the worst case performance in the uncertainty region D, i.e. the worst level of performance of a closed loop made up of the connection of the considered controller and any plant in D. Modulo the probability that $G_0 \in D$, the worst case performance in D is of course a lower bound for the closed-loop performance achieved with the true system.

3.1 The general criterion measuring the worst case performance

There is no unique way of defining the performance of a closed-loop system. However, most commonly used performance criteria can be derived from some norm of a frequency weighted version of the stability matrix $H(G,C)$ of the closed-loop system $[C\ G]$ made up of G in feedback with the controller C.

Definition 3 *Given a plant $G(z)$ and a stabilizing controller $C(z)$, the performance of a closed-loop system $[C\ G]$ is defined as the following frequency function*

$$J(G, C, W_l, W_r, \Omega) = \sigma_1 \left(W_l H(G(e^{j\Omega}), C(e^{j\Omega})) W_r \right) \tag{7}$$

where $W_l(z) = diag(W_{l1}, W_{l2})$ and $W_r(z) = diag(W_{r1}, W_{r2})$ are diagonal weights, $\sigma_1(A)$ denotes the largest singular value of A, and $H(G,C)$ is de-

fined as follows:

$$H(G,C) = \begin{pmatrix} H_{11}(G,C) & H_{12}(G,C) \\ H_{21}(G,C) & H_{22}(G,C) \end{pmatrix} = \begin{pmatrix} \frac{GC}{1+GC} & \frac{G}{1+GC} \\ \frac{C}{1+GC} & \frac{1}{1+GC} \end{pmatrix}. \tag{8}$$

The worst case performance criterion over all plants in an uncertainty region D is then defined as follows.

Definition 4 *Consider an uncertainty region D of systems $G(z,\delta)$ with $\delta \in U$. Consider also a controller $C(z)$. The worst case performance achieved by this controller at a frequency Ω over all systems in D is defined as:*

$$J_{WC}(D,C,W_l,W_r,\Omega) = \max_{G(z,\delta)\in D} \sigma_1 \left(W_l H(G(e^{j\Omega},\delta), C(e^{j\Omega}))W_r \right). \tag{9}$$

Note that J_{WC} is a frequency function : it defines a template. Using appropriate weights W_r and W_l, more specific worst case performance measures (such as the largest modulus of one of the four closed loop transfer functions) can be defined. For instance, the largest modulus of the sensitivity function $H_{22}(G(z,\delta),C)$ for a system $G(z,\delta) \in D$ can be computed choosing $W_l = W_r = diag(0,1)$.

3.2 Computation of the worst case performance

We now present a procedure for the computation of $J_{WC}(D,C,W_l,W_r,\Omega)$ at a given frequency Ω. This procedure must be repeated at each frequency in order to obtain the shape of the frequency function J_{WC}.

Theorem 5 *Consider an uncertainty region D defined in (1) and a controller $C(z) = X(z)/Y(z)$ [4]. The worst case performance at Ω defined in (9) is equal to $\sqrt{\gamma_{opt}}$, where γ_{opt} is the optimal value of γ for the following standard convex optimization problem involving LMI constraints evaluated at the frequency Ω:*

$$\begin{aligned} &\textit{minimize } \gamma \\ &\textit{over} \qquad \gamma,\ \tau \\ &\textit{subject to } \tau \geq 0 \textit{ and} \end{aligned} \tag{10}$$

$$\begin{pmatrix} Re(a_{11}) & Re(a_{12}) \\ Re(a_{12}^*) & Re(a_{22}) \end{pmatrix} - \tau \begin{pmatrix} R & -R\hat{\delta} \\ (-R\hat{\delta})^T & \hat{\delta}^T R\hat{\delta} - 1 \end{pmatrix} < 0$$

where $a_{11} = (Z_N^* W_{l1}^* W_{l1} Z_N + Z_D^* W_{l2}^* W_{l2} Z_D) - \gamma(Q Z_1^* Z_1)$, $a_{12} = Z_N^* W_{l1}^* W_{l1} e + W_{l2}^* W_{l2} Z_D^* - \gamma(Q Z_1^*(Y+eX))$, $a_{22} = e^* W_{l1}^* W_{l1} e + W_{l2}^* W_{l2} - \gamma(Q(Y+eX)^*(Y+$

[4] $X(z)$ and $Y(z)$ are the polynomials corresponding to the numerator and to the denominator of $C(z)$, respectively.

$eX)),\ Z_1 = XZ_N + YZ_D$ *and* $Q = 1/(X^*W_{r1}^*W_{r1}X + Y^*W_{r2}^*W_{r2}Y)$.

Proof. It is important to note that $H_w(z,\delta) \triangleq W_l H(G(z,\delta), C(z))W_r$ is of rank one. Using the definition of the closed-loop transfer matrix H in (8) and the expression of $G(z,\delta)$ in (1), the weighted matrix $H_w(z,\delta)$ can therefore be rewritten as follows:

$$H_w(z,\delta) = \begin{pmatrix} \frac{W_{l1}(e+Z_N\delta)}{Y+eX+Z_1\delta} \\ \frac{W_{l2}(1+Z_D\delta)}{Y+eX+Z_1\delta} \end{pmatrix} \begin{pmatrix} XW_{r1} & YW_{r2} \end{pmatrix} \tag{11}$$

with $Z_1 = XZ_N + YZ_D$. Proving Theorem 5 is equivalent to proving that the solution γ_{opt} of the LMI problem (10), evaluated at Ω, is such that:

$$\sqrt{\gamma_{opt}} = \max_{\delta\in U} \sigma_1(H_w(e^{j\Omega},\delta)) \Longleftrightarrow \gamma_{opt} = \max_{\delta\in U} \lambda_1(H_w(e^{j\Omega},\delta)^* H_w(e^{j\Omega},\delta))$$

where $\sigma_1(A)$ and $\lambda_1(A)$ denote the largest singular value and the largest eigenvalue of A, respectively. An equivalent and convenient way of restating the problem of computing $\max_{\delta\in U} \lambda_1(H_w(e^{j\Omega},\delta)^* H_w(e^{j\Omega},\delta))$ is as follows:

$$\textit{minimize } \gamma \textit{ such that } \lambda_1(H_w(e^{j\Omega},\delta)^* H_w(e^{j\Omega},\delta)) - \gamma < 0 \ \forall \delta \in U.$$

Since $H_w(e^{j\Omega},\delta)$ is a rank one matrix, we have that $\lambda_1(H_w(e^{j\Omega},\delta)^* H_w(e^{j\Omega},\delta)) - \gamma < 0$ is equivalent with

$$\begin{pmatrix} \frac{W_{l1}(e+Z_N\delta)}{Y+eX+Z_1\delta} \\ \frac{W_{l2}(1+Z_D\delta)}{Y+eX+Z_1\delta} \end{pmatrix}^* \begin{pmatrix} \frac{W_{l1}(e+Z_N\delta)}{Y+eX+Z_1\delta} \\ \frac{W_{l2}(1+Z_D\delta)}{Y+eX+Z_1\delta} \end{pmatrix} (X^*W_{r1}^*W_{r1}X + Y^*W_{r2}^*W_{r2}Y) - \gamma < 0 \Longleftrightarrow$$

$$\begin{pmatrix} \frac{W_{l1}(e+Z_N\delta)}{Y+eX+Z_1\delta} \\ \frac{W_{l2}(1+Z_D\delta)}{Y+eX+Z_1\delta} \\ 1 \end{pmatrix}^* \begin{pmatrix} I_2 & 0 \\ 0 & -\gamma Q \end{pmatrix} \begin{pmatrix} \frac{W_{l1}(e+Z_N\delta)}{Y+eX+Z_1\delta} \\ \frac{W_{l2}(1+Z_D\delta)}{Y+eX+Z_1\delta} \\ 1 \end{pmatrix} < 0 \tag{12}$$

with Q as defined in (10). By pre-multiplying (12) by $(Y + eX + Z_1\delta)^*$ and post-multiplying the same expression by $(Y + eX + Z_1\delta)$, we obtain:

$$\begin{pmatrix} W_{l1}(e+Z_N\delta) \\ W_{l2}(1+Z_D\delta) \\ Y+eX+Z_1\delta \end{pmatrix}^* \begin{pmatrix} I_2 & 0 \\ 0 & -\gamma Q \end{pmatrix} \begin{pmatrix} W_{l1}(e+Z_N\delta) \\ W_{l2}(1+Z_D\delta) \\ Y+eX+Z_1\delta \end{pmatrix} < 0 \tag{13}$$

which is equivalent to the following constraint on δ with variable γ

$$\begin{pmatrix} \delta \\ 1 \end{pmatrix}^* \begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix} \begin{pmatrix} \delta \\ 1 \end{pmatrix} < 0 \Longleftrightarrow \overbrace{\begin{pmatrix} \delta \\ 1 \end{pmatrix}^T \begin{pmatrix} Re(a_{11}) & Re(a_{12}) \\ Re(a_{12}^*) & Re(a_{22}) \end{pmatrix} \begin{pmatrix} \delta \\ 1 \end{pmatrix}}^{\alpha(\delta)} < 0 \tag{14}$$

with a_{11}, a_{12} and a_{22} as defined in (10). The equivalence in (14) is due to the fact that δ is real.

Expression (14) is equivalent to stating that $\lambda_1(H_w(e^{j\Omega},\delta)^* H_w(e^{j\Omega},\delta)) - \gamma < 0$ for a particular δ in U. However, this must be true for all $\delta \in U$. Therefore the last expression must be true for all δ such that

$$\overbrace{\begin{pmatrix}\delta \\ 1\end{pmatrix}^T \begin{pmatrix} R & -R\hat{\delta} \\ (-R\hat{\delta})^T & \hat{\delta}^T R \hat{\delta} - 1 \end{pmatrix} \begin{pmatrix}\delta \\ 1\end{pmatrix}}^{\rho(\delta)} < 0$$

which is equivalent to the statement "$\delta \in U$".

Let us now recapitulate. Computing $\max_{\delta \in U} \lambda_1(H_w(e^{j\Omega},\delta)^* H_w(e^{j\Omega},\delta))$ is equivalent to finding the smallest γ such that $\alpha(\delta) < 0$ for all δ for which $\rho(\delta) < 0$. Since the domain of δ is only constrained by one LMI (i.e. $\rho(\delta) < 0$), by the **S** procedure (Boyd *et al.*, 1994, page 23), the last problem is equivalent to finding the smallest γ and a positive scalar τ such that $\alpha(\delta) - \tau\rho(\delta) < 0$, for all $\delta \in \mathbf{R}^{k\times 1}$, which is precisely (10).

4 Example

To illustrate our results, we present an example of controller validation for a model identified in closed-loop. Let us consider the following true system G_0 with an Output Error structure:

$$y = \overbrace{\frac{0.1047z^{-1} + 0.0872z^{-2}}{1 - 1.5578z^{-1} + 0.5769z^{-2}}}^{G_0} u + \tilde{e},$$

where $\tilde{e}$ is a unit-variance white noise. The sampling time is 0.05 second. We perform an indirect closed-loop identification of an unbiased closed-loop transfer function $T(\hat{\delta})$ by collecting 1000 reference and output data on the true system in closed loop with an output-feedback controller $K = 3 : u = 3(r - y)$ (see (Bombois *et al.*, 1999) for details). This controller stabilizes G_0. The open-loop model $G(\hat{\delta}) = T(\hat{\delta})/(K(1 - T(\hat{\delta})))$ corresponding to $T(\hat{\delta})$ is equal to

$$G(\hat{\delta}) = \frac{0.1060z^{-1} + 0.0928z^{-2}}{1 - 1.5308z^{-1} + 0.5467z^{-2}}$$

Control design. From the model $G(\hat{\delta})$, we have designed a controller with a phase advance : $C(z) = (1.8464 - 1.3647z^{-1})/(1 - 0.4545z^{-1})$. With this controller, the designed closed-loop $[G(\hat{\delta})\ C]$ has a stability margin of 57 degrees and a gain margin of 10dB. The cut-off frequency Ω_c is equal to 0.5 which corresponds to an *actual* frequency $\omega_c = 11\ rad/s$. Before applying this controller $C(z)$ to the true system, we verify whether it achieves satisfactory behaviour with all plants in the uncertainty region D_{CL}. This uncertainty region D_{CL} is constructed from the estimated covariance matrix P_δ of the parameters of the closed-loop model $T(\hat{\delta})$ (see (Bombois *et al.*, 1999) for details). It contains the true system G_0 at a probability level equal to 0.95, and is given by

$$D_{CL} = \{G(\delta) \mid G(\delta) = \frac{T(\delta)}{K(1-T(\delta))}\ and\ \delta \in U_{CL}\}$$

where $U_{CL} = \{\delta \mid (\delta - \hat{\delta})^T P_\delta^{-1}(\delta - \hat{\delta}) < 12.6\}$. As shown in (Bombois, 2000), it is easy to prove that D_{CL} has the general structure (1).

Robust stability analysis. Using the procedure presented in Section 2, we check whether C stabilizes all plants in D_{CL}. For this purpose, we construct the row vector $M_{D_{CL}}(z)$ defined in Theorem 2 and we compute the corresponding stability radius $\mu(M_{D_{CL}}(e^{j\Omega}))$ at all frequencies in $[0\ \pi]$. The maximum over these frequencies is 0.1313. Since this maximum is smaller than 1, we conclude that $C(z)$ stabilizes all plants in D_{CL} and therefore also the true system G_0, at least with probability 0.95.

Robust performance analysis. In order to verify that C gives satisfactory performance with all plants in D_{CL}, we compute at each frequency the worst case modulus $t_{D_{CL}}(\Omega, H_{22})$ of the sensitivity function "H_{22}" achieved by C over all plants in D_{CL}. This can be done by computing $J_{WC}(D_{CL}, C, W_l, W_r, \Omega)$ using Theorem 5 with the particular weights $W_l = W_r = diag(0, 1)$. The worst case modulus of the sensitivity function over all models in D_{CL} is represented in Figure 1. In this figure, the worst case performance level $t_{D_{CL}}(\Omega, H_{22})$ is compared with the sensitivity functions of the designed closed loop $[C\ G(\hat{\delta})]$ and of the achieved closed loop $[C\ G_0]$. From $t_{D_{CL}}(\Omega, H_{22})$, we can find that the worst case static error ($=t_{D_{CL}}(0, H_{22})$) resulting from a constant disturbance of unit amplitude is equal to 0.1692, whereas this static error is 0.0834 in the designed closed-loop. The achieved static error is 0.1017. Using $t_{D_{CL}}(\Omega, H_{22})$, we can also see that the bandwidth of $\Omega_c = 0.5$ in the designed closed-loop is preserved for all closed loops with a plant in D_{CL}, since $t_{D_{CL}}(\Omega, H_{22})$ is equal to 1 at $\Omega_c \simeq 0.5$. The difference between the resonance peak of the designed sensitivity function (i.e. $\max_\Omega \parallel H_{22}(G(\hat{\delta}), C) \parallel = 1.6184$) and the worst case reasonance peak achieved by a plant in D_{CL} (i.e. $\max_\Omega t_{D_{CL}}(\Omega, H_{22}) = 1.7075$) also remains small. Note that the actually achieved resonance peak (i.e. $\max_\Omega \parallel H_{22}(G_0, C) \parallel$) is equal to 1.6229.

We may therefore conclude that the controller C achieves sufficient performance with all plants in D_{CL} since the difference between the nominal and worst case performance level remains very small at every frequency. With

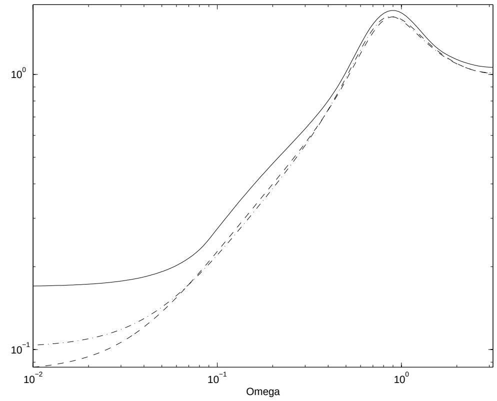

Fig. 1. $t_{D_{CL}}(\Omega, H_{22})$ (solid), $\parallel H_{22}(G(\hat{\delta}), C) \parallel$ (dashed) and $\parallel H_{22}(G_0, C) \parallel$ (dashdot)

such stability and performance analysis results, one would confidently apply the controller to the real system, assuming that the nominal performance is judged to be satisfactory.

5 Conclusions

This paper matches prediction error identification and robustness theory which have often been seen as incompatible theories. Indeed, we have shown in (Bombois *et al.*, 1999; Gevers *et al.*, 2000; Bombois, 2000) that prediction error identification, in open or closed loop, yields a parametric uncertainty region D that takes the generic form described in Proposition 1. We have shown in this paper that this generic uncertainty region D is amenable to robust stability and robust performance analysis. Our solution to the robust stability problem is obtained by recasting the set of closed loops in a standard form using a sequence of transformations. Our solution to the robust performance problem, which is the main technical contribution of this paper, is the exact LMI-based computation of the worst case performance achieved by a given controller in closed loop with all models in the uncertainty set D.

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References

Bhattacharyya, S.P., H. Chapellat and L.H. Keel (1995). *Robust Control - The Parametric Approach.* Prentice Hall, Upper Saddle River, New Jersey.

Biernacki, R.M., H. Hwang and S.P. Bhattacharyya (1987). Robust stability with structured real parameter perturbations. *IEEE Transactions on Automatic Control* **32**(6), 495–506.

Bombois, X. (2000). Connecting Prediction Error Identification and Robust Control Analysis: a new framework. PhD thesis. Université Catholique de Louvain, Belgium.

Bombois, X., M. Gevers and G. Scorletti (1999). Controller validation based on an identified model. In: *Proc. 38th IEEE Conference on Decision and Control*, Phoenix, Arizona. pp. 2816–2821.

Bombois, X., M. Gevers and G. Scorletti (2000). Controller validation for stability and performance based on a frequency domain uncertainty region obtained by stochastic embedding. In: *Proc. 39th IEEE Conference on Decision and Control*, Sydney, Australia. Paper TuM06-5.

Boyd, S., L. El Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequalities in Systems and Control Theory.* Vol. 15 of *Studies in Appl. Math.*. SIAM. Philadelphia.

Chen, J. (1997). Frequency-domain tests for validation of linear fractional uncertain models. *IEEE Transactions on Automatic Control* **42**(6), 748–760.

Fan, M. K. H. and A. L. Tits (1992). A measure of worst-case H_∞ performance and of largest acceptable uncertainty. *Syst. Control Letters* **18**, 409–421.

Ferreres, G. and V. Fromion (1997). Computation of the robustness margin with the skewed μ-tool. *Syst. Control Letters* **32**, 193–202.

Gevers, M., B. Codrons and F. De Bruyne (1999). Model validation in closed-loop. In: *Proc. American Control Conference,* San Diego, California. pp. 326–330.

Gevers, M., X. Bombois, B. Codrons, G. Scorletti and B.D.O. Anderson (2000). Model validation for control and controller validation: a prediction error identification approach. Submitted to Automatica.

Goodwin, G.C., M. Gevers and B. Ninness (1992). Quantifying the error in estimated transfer functions with application to model order selection. *IEEE Trans. Automatic Control* **37**, 913–928.

Hakvoort, R.G. (1994). System Identification for Robust Process Control. PhD thesis. Delft University of Technology, The Netherlands.

Ljung, L. (1998). Identification for control - what is there to learn. *Workshop on Learning, Control and Hybrid Systems, Bangalore.*

Ljung, L. (1999). *System Identification: Theory for the User, 2nd Edition*. Prentice-Hall. Englewood Cliffs, NJ.

Milanese, M. (1998). Learning models from data: the set membership approach. In: *Proc. American Control Conference*. Philadelphia, USA. pp. 178–182.

Poolla, K., P. Khargonekar, A. Tikku, J. Krause and K. Nagpal (1994). A time-domain approach to model validation. *IEEE Transactions on Automatic Control* **39**(5), 951–959.

Rantzer, A. (1992). Convex robustness specifications for real parametric uncertainty in linear systems. In: *Proc. American Control Conference*. pp. 583–585.

Rantzer, A. and A. Megretski (1994). A convex parametrization of robustly stabilizing controllers. *IEEE Trans. Aut. Control* **39**(9), 1802–1808.

Zhou, K., J.C. Doyle and K. Glover (1995). *Robust and Optimal Control*. Prentice Hall, New Jersey.