Trajectory Clustering for Vibration Detection in Aircraft Engines

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Abstract. The automatic detection of the vibration signature of rotating parts of an aircraft engine is considered. This paper introduces an algorithm that takes into account the variation over time of the level of detection of orders, i.e. vibrations ate multiples of the rotating speed. The detection level over time at a specific order are gathered in a so-called trajectory. It is shown that clustering the trajectories to classify them into detected and non-detected orders improves the robustness to noise and other external conditions, compared to a traditional statistical signal detection by an hypothesis test. The algorithms are illustrated in real aircraft engine data.

1 Introduction

We tackle the issue of monitoring the behavior of an aircraft engine from the point of view of measured vibrations. Indeed, abnormal level or odd pattern of vibrations may be the consequence of mechanical or sensor malfunction, both of dramatic importance for engine manufacturers and airline operators.

More precisely this work focuses on the detection of the signature of specific parts of a turbofan engine (the fans), whose vibration levels are modelled as vibration *trajectories*, i.e. as the amount of vibration of a given fan with respect to time. The trajectories are then clustered thanks to standard clustering algorithms. The obtained clusters gather all trajectories that correspond to fans whose vibratory signature is present in the data.

Section 2 gives a more thorough introduction to the problem. In Section 4 the algorithms are presented. Results are discussed in Section 5, while conclusion and perspectives are sketched in Section 6.

2 Problem Description

A turbofan whose structure is presented by Fig. 1 is considered. Air from the outside enters an intake, then is successively compressed by the low-pressure (LP)
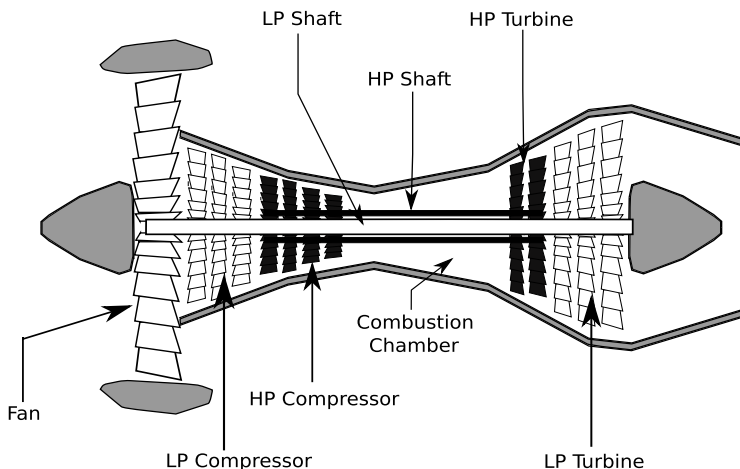
Fig. 1. Turbofan engine. Simplified diagram of fan, low-pressure and high-pressure compressors and turbines attached to their respective shafts.

and high-pressure (HP) compressors. Compressed air passes to a combustion chamber, where it is mixed with fuel and burnt. Both compressors are powered by turbines located at the rear of the engine, which transmit their energy to the compressors through two contra-rotating shafts, the low-pressure (LP) shaft and the high-pressure (HP) shaft.

Although turbofan condition monitoring can be achieved in various ways, we take the stand to focus on vibrations monitoring in this work. Two accelerometers provide vibration measurements at a constant 51 kHz frequency. Since compressors and turbines are fan-like components made of a varying number of blades mounted on the shafts, it is expected that their motion entails vibrations at frequencies which are multiples of shaft speeds. Although this is a strong simplification of the overall vibratory behavior of a bladed disk mounted on a rotor [1], it allows an efficient detection of malfunction and damages, given the low quantity of available information (two vibration sensors). Vibration patterns corresponding to multiples of shaft speed are known as "orders" in the engineering field.

Moreover, vibrations signals are usually processed not in the time-domain, but in the frequency [2] or in the time-frequency domain [3, 15.6]. When the rotation speed of the engine is constant, vibration signals can be considered as stationary, and the classical Fourier transform is sufficient. However when the rotation speed is varying, signals are non-stationary and the spectrogram or more advanced time-frequency distributions might be of some help. Examples of a Fourier transform and a spectrogram on a vibration signal are given in Fig. 2(a,b).

In addition to accelerometers, the sensors give an approximation of both shaft velocities. Since many mechanical parts of the engine rotate at a speed multiple of these shafts speeds (compressors and turbines for example), we can anticipate
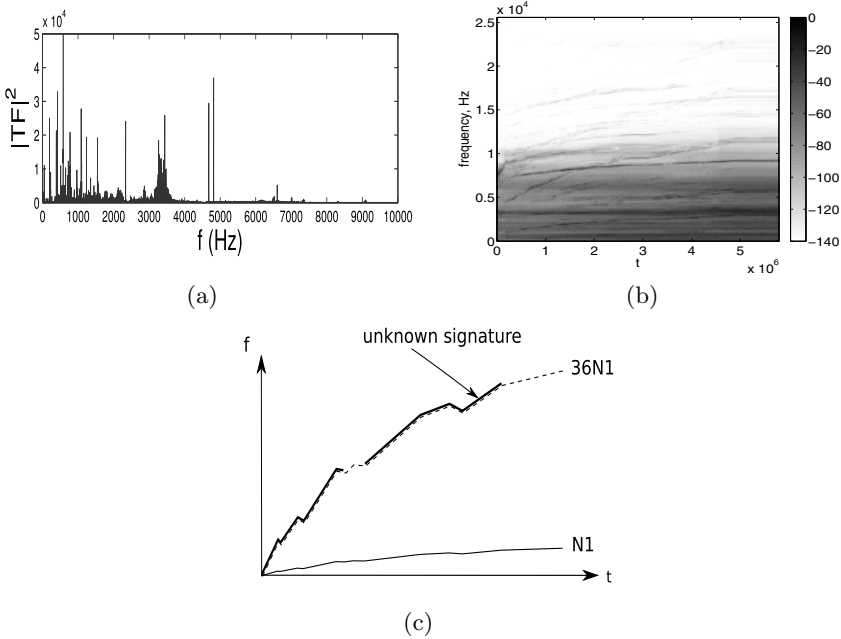
(a)

(b)

(c)

Fig. 2. Vibration analysis of accelerometric data: (a) in the frequency domain with the Discrete Fourier Transform; (b) in the time-frequency domain with the spectrogram; (c) idealized signature of fan-like mechanical part in accelerometer data. An unknown signature in the spectogram and an expected signature at 36 times the frequency of LP shaft superpose.

the position of their signature in the frequency or time-frequency domain, as illustrated in Fig. 2(c) where an unknown signature in the spectrogram and an expected signature superpose at a particular order.

As we can notice from Fig. 2(b), the signature is embedded in noise. For specific orders, it remains clearly visible whereas for others it vanishes. Furthermore, there is a potentially large number of orders. For both reasons, order detection is most of the time the task of an expert. Automatic detection would hence accelerate the process, and relieve the expert of this daunting task.

Several difficulties arise when considering the issue of automatic detection of orders in vibration data. First, as already mentioned, the signal is noisy. Second, the magnitude of vibration of the orders is a function of the excitation. The latter is a direct function of the shafts speeds. Indeed, the fans behave as mechanical parts that are excited and resonate for specific frequencies only. Consequently, if we examine a sufficiently long recording, the orders will appear with varying magnitude along time. The frequency response of rotating machines is classically summarized in Campbell diagrams where the experimentally[1] measured

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[1] Such diagrams can be drawn from analytical models as well, if the eigenfrequencies can be computed.
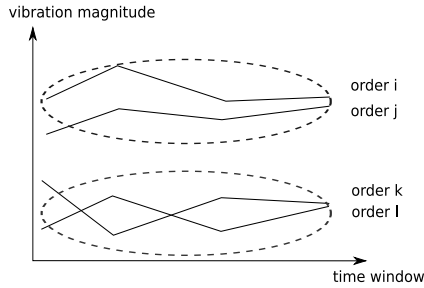
Fig. 3. Idealized diagram of trajectories clustering

vibration response spectrum appear as a function of the rotation speed of the shaft. If we suppose that the underlying shaft rotation speed follows a regular enough time pattern, then there should be some degree of continuity in the vibration magnitude of the orders.

Admitting that signals are split into a finite number of windows for computational reasons, we propose to consider vibration magnitude *trajectories*, where to each time window corresponds a robust measure of the vibration magnitude at a given order, for a selected shaft. This needs to be done systematically since we ignore whether a given order k of LP or HP shaft is visible for a specific regime. At the end, a large number of trajectories will be available, which we propose to cluster thanks to dedicated algorithms in order to segregate those who correspond to orders that are present in the data from those who do not. This is summarized by Fig.3. The expected outcome of this algorithm is thus a cluster of trajectories that correspond to what the expert would have singled-out as orders that are actually present in the recordings.

Related works are given a quick review in Section 3, while a detailed presentation of available data is given in Section 4.1. The proposed method is discussed in Section 4.2. Canonical signal processing detection procedures in the case of partially unknown sine wave are evoked in Section 4.3 for benchmarking purposes.

3   Related Work

This work addresses the problem of feature detection, where the features are the vibration signatures of bladed disks that compose compressors and turbines of an aircraft engine.

It is intended to be part of a Condition Based Monitoring (CBM) framework. CBM for industrial machines has been attracting increasing attention over the years in both academic and industrial areas. According to [4] it consists in four main steps: data acquisition, feature extraction, feature selection, and decision-making. The first two steps rely on mechanical modeling or rotor dynamics [5], noise and vibration phenomena in rotating machines [6,4] and data analysis [7]. The latter builds on the general tools and methods developed in signal

processing [8,3], statistical signal estimation and detection [9,10,11], learning theory, change detection [12], fault detection and isolation [13].

Aircraft CBM deals with many problems such as structural health monitoring. It treats engine health monitoring (EHM) as a special case [14,15]. As a subtopic of EHM, vibrations monitoring in engines addresses the following issues: rotor/stator contact [16], rotor unbalance, blade defects [17], bearing [18] and gearings defects [19]. Another important topic is *order tracking*, i.e. the precise estimation of the frequency and amplitude of a periodic signal whose leading frequency is varying in time. This may involve data resampling in the case of rotating machines [20,21], non-parametric time frequency methods such as the Gabor transform [22] or parametric methods such as the Vold-Kalman filter [23,24], that models the vibrating machine and use estimation tools to track the frequencies and amplitudes of orders.

When features have been extracted, decisions can be made, such as change (or novelty) detection. The decision concerning the health of the engine is either taken from a statistical viewpoint [25,12], or can derive from learning techniques such as neural networks [26,27].

In this work the aim is not to detect an anomaly, but a normal bladed disk signal. Up to our knowledge, there is no example of such order detection in the litterature. The case of novelty or abrupt change detection in engine have been covered, as well as feature extraction for many particular mechanical parts, but not for bladed disks in turbines and compressors. The works dedicated to these parts aim at estimating their rotation speed, amplitude and phase, but not to detect their presence. In the following we show that classical tools such as sine wave detection and unsupervised clustering enable us to tackle this issue.

4 Methods and Data

4.1 Data

The recordings under study were provided by the Health Monitoring Department of SNECMA[2] and correspond to a dual-shaft turbofan mounted on a testbench, that undergoes a continuous acceleration during several minutes. They include raw vibration outputs of an accelerometer sampled at 51kHz, as well as LP and HP shaft angular velocity computed from raw keyphasor data and sampled at 6.25 Hz. Sample time series are plotted in Fig. 4.

4.2 Clustering Contrast Trajectories for Order Detection

First we give a precise meaning to what was termed "vibration magnitude trajectory" in Section 2. Conforming to current practice in the field of vibration monitoring for rotating machine, we choose a classical time-frequency representation, the Discrete Gabor Transform (DGT) [22], which is a special kind of Discrete Short-Time Fourier Transform [8,28]. The representation of a given vibratory signal is a matrix, indexed in both time and frequency.
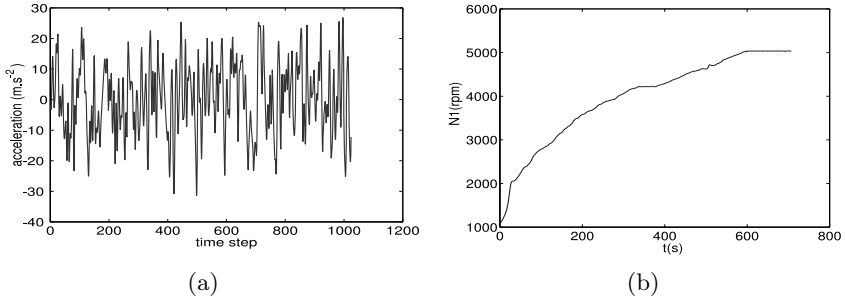
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[2] http://www.snecma.fr

Fig. 4. (a) Raw accelerometric data; (b) LP shaft angular velocity in rpm

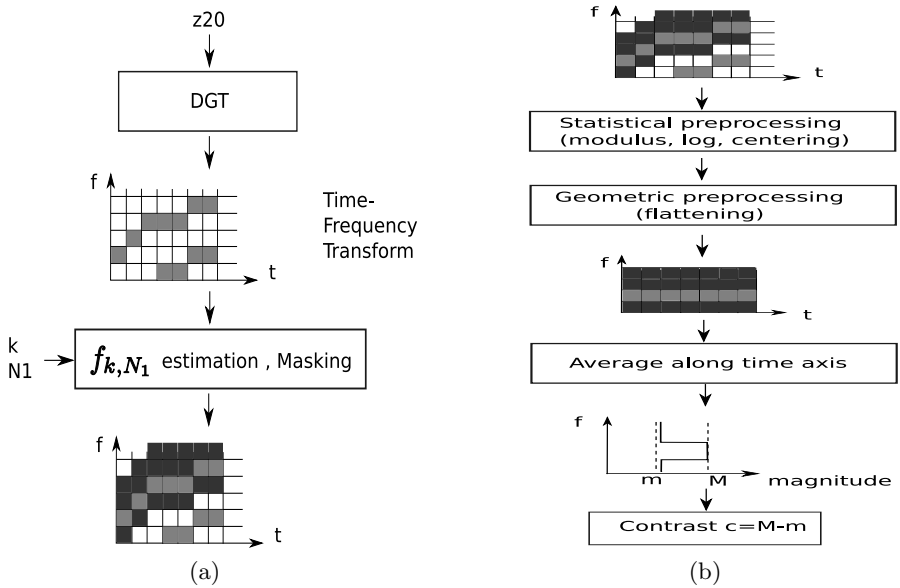**Fig. 5.** (a) Order masking takes as its inputs the vibratory signal, a specific order integer k and the shaft rotation speed N_1 or N_2. The expected main frequency is f_{k,N_1}, and the region that surrounds ths central frequency has a fixed user-defined width ; (b) contrast.

As stated earlier, we focus on the signature of "orders", produced by the bladed disks that compose the compressors and turbines. Their simplified signature in the time-frequency domain has been illustrated by Fig. 2(c). We write $G(t, \nu)$ the complex-valued time-frequency transform at discrete time t and discrete frequency ν. Since we know approximately the frequency of an order from the choice of an integer number (the order) and the estimation of the shaft speed, we can build a region in the time-frequency plane in which we expect the order

signature to appear, if it is present in vibratory data. This operation is called *masking* and is illustrated in Fig. 5(a).

Then, having focused on a specific order, we still need to compute a measure of magnitude for that order. Several statistical preprocessing steps (computation of the modulus, of the logarithm, then centering and scaling of the data) are followed by a geometric flattening of the signature as shown in Fig. 5(b). We call $G'(t, \nu)$ the resulting quantity. Finally the flattened signature is time-average, then the contrast function is computed.

$$\forall \nu, \ \tilde{G}(\nu) = \frac{1}{T} \sum_{t=t_1}^{t_2} G'(t, \nu) \tag{1}$$

$$c_k(n) = \max_{\nu} \ \tilde{G}(\nu) - \min_{\nu} \ \tilde{G}(\nu) \tag{2}$$

where k and n are respectively the order and the window indices, T is the length of the time window, and t_1 and t_2 are respectively the start and end of that time window.

Secondly, we describe the trajectory clustering algorithm announced in 2. The previous masking step returns a contrast value for each order k in a user-defined range $[k_1, k_2]$, and for each window $n \leq N$ that divides the full vibration recording. Hence for each order k we get an ordered set of scalar real-valued contrasts $T_k = \{c_k(1), \ldots, c_k(N)\}$ indexed by the order k. This set T is what we call a trajectory. We then propose to cluster it into a fixed number $C = 3$ of clusters, with the k-means algorithm. The whole procedure is illustrated in Fig. 6.

Clustering trajectories is a way to classify them into classes, corresponding respectively to orders that are detected, not detected, and a possible intermediate, ambiguous classes. One could argue that if an order k is effectively present in the recordings, any contrast $c_k(n)$ belonging to T_k should be high, allowing a detection on each time window n. However, one has to take into account the fact that the vibratory signals are extremely noisy. Splitting into windows is thus a way to expect an improved detection robustness in some windows. Averaging the detection hits over the windows could then improve the detection results, compared to a single detection (over one window or the whole signal).

Still, this average would completely lose the temporal correlation between orders. Indeed it is naturally expected that at some engine speeds (corresponding to some of the windows), all or most of the orders will be hardly detected (either because a large part of the engine will be less subject to vibrations, or because the latter will be polluted by higher noise). It is therefore of primary importance to take into account the longitudinal aspect of the detections at various orders, i.e. the temporal correlation. This is the justification for the original use of trajectories in this work.

In Section 4.3 we summarize classical results relevant to our problem in the field of statistical signal detection, in order to assess the performance of the proposed method.
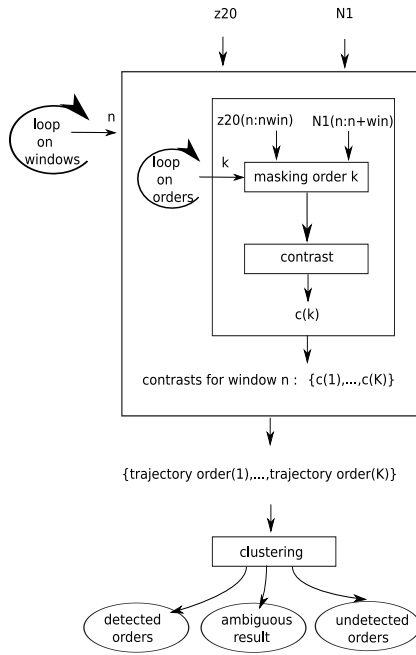
Fig. 6. Clustering algorithm

4.3   Statistical Signal Detection Method

From the point of view of classical statistical signal detection theory [9,10,11], the problem formulated in Section 2 is an instance of the detection of a deterministic sine wave in noise, with unknown parameters [9, 7.6.2]. Indeed, with the simplifying hypotheses stated in Section 2, the compressor and turbine disk blades whose signature is under study are seen as curves in the time-frequency plane, i.e. as sine waves with slowly varying leading frequency. On time windows where the frequency can be considered as constant, this sine wave can be parametrized as follows:

$$s(i) = A\cos(2\pi f i + \phi) \tag{3}$$

where A is the amplitude, i the time instant, f the frequency and ϕ the phase difference. Apart from this wave the signal contains a vast amount of other deterministic or random contributions which we call "noise" since they are irrelevant to the purpose of detecting a given order of a specific shaft. The detection of the sine wave in the noise is formalized as the following hypothesis test:

$$\begin{aligned}
\mathcal{H}_0 &: z20(i) = w(i) & \forall i \in [0, I-1] \\
\mathcal{H}_1 &: z20(i) = w(i) + A\cos(2\pi f i + \phi) & \forall i \in [0, I-1]
\end{aligned}$$

where $z20$ is the accelerometric signal, $w(i)$ is the noise term. It should be noted that, while the frequency f is approximately known, the amplitude A and phase

difference ϕ are unknown to the observer. Under noise normality and independence hypotheses that diserve further discussion, this hypothesis test is amenable to analytic treatment under the Generalized Likelihood Ratio Test framework. Since this material is standard, we merely state the corresponding results, which are given proper development in classical textbooks (see for example [9, 7.6.2]):

- the decision rule is written $I(f_0) \gtrless_{\mathcal{H}_0}^{\mathcal{H}_1} \gamma'$, where the decision statistics is $I(f_0)$, the value of the periodogram evaluated at target frequency f_0 which can be approximated by Discrete Fourier Transform. γ' is the detection threshold.
- the false alarm probability is $\alpha = \exp\left(-\frac{\gamma'}{\sigma^2}\right)$, where σ is the standard deviation of the noise, supposedly known. Usually α takes a fixed user-defined value such as $\alpha = 1\%$ so that the value of the detection threshold γ' can be easily derived.
- the performance of this test is quantified by the power π of the test, whose analytic expression can be computed.

With these results, we can build a sine wave detector. Given a vibratory time series and a few contextual parameters such as the noise variance, the detector will accept or reject hypothesis $\mathcal{H}_0$ depending if the signal is composed of noise or if an order is present in the signal. Note that several improvements over this standard procedure should be made, because the estimated noise does not respect exactly the white noise hypothesis. For example, the energy is not homogenous in the frequency spectrum. Consequently we divide the frequency spectrum into bins and perform the above detection separately in each bin. In Section 5 we compare the results with those elicited by the clustering algorithm.

5   Results

We first illustrate the computation of the contrast function defined in Section 4.2. The first steps depicted by Fig. 5 are the statistical and geometric preprocessing steps. Fig.7(a) shows the resulting preprocessed time-frequency transforms for six chosen orders of the HP shaft. Three orders (38,53,68) correspond to the number of blades of three bladed disks belonging to the compressor, therefore we expect the corresponding signatures to be present in vibration signals. Three ordrers (20,90,100) were selected because no significant vibratory activity is expected. In Fig.7(a) thick lines appear clearly in the first three cases, whereas no specific pattern except the background noise can be noticed in the last three cases. Time averaged values of the time-frequency transforms are plotted in Fig. 7(b), as well as their peak value in Fig. 7(c). The latter clearly shows that peak values are higher for orders 38, 53, 68 that correspond to actual mechanical rotating parts.

Secondly we comment on the contrast trajectory clusters that are found out by the clustering algorithm mentioned in Fig. 6. The number of clusters was set to $C = 3$. Results are shown in Fig. 8(a). The first cluster gathers trajectories with high mean value over the window range, that increases at the beginning,
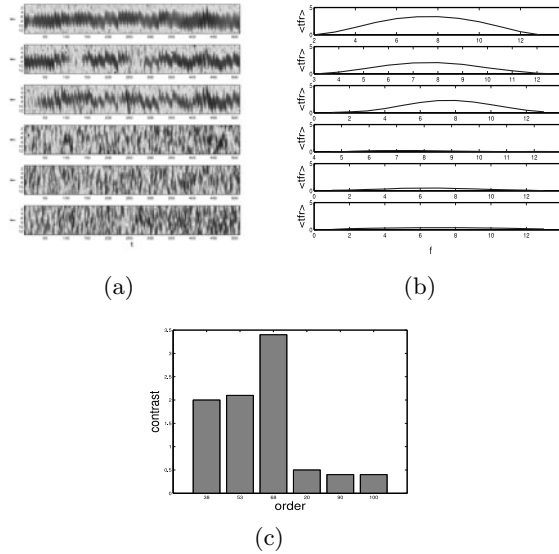
(a) (b)

(c)

Fig. 7. Contrasts of orders 38,53,68,20,90,100 of HP-shaft N_2: (a) DGT after preprocessing; (b) mean DGT along time axis; (c) contrasts

decreases at the end, and experiences a sudden decrease at window index 6. In the second one, trajectories are flat and have low mean value. The third cluster mixes different types of trajectories that have an average intermediate value. In first approximation, only the trajectories in cluster 1 are meaningful to the detection task.

In order to assess the significance of results, we use prior mechanical information. Indeed, the number of blades that compose the compressor and turbine mounted on HP shaft is known. Because of many mechanical factors it is not certain whether or not each bladed disk will have a noticeable vibratory activity, but this information can be helpful for comparison purposes. In Fig. 8(b), we plot the composition of the clusters obtained above (indices 1 and 3), and compare it with the cluster built with prior information which is given cluster label 0. Lastly, clusters found by the statistical detection algorithm from Section 4.3 are labelled as cluster D. We see that between orders 35 to 80 many orders are shared by clusters 0 and 1. This is true also for clusters 0 and D. However, clusters 2 and 3 show little similarity with cluster 0. This is coherent with our initial expectation, stated at the end of Section 2. In addition we remark that many low orders are detected in cluster D. This could be explained by the fact that low frequencies bear more energy that higher ones, as evidenced in Fig. 2(b). The clustering method is less prone to overweigthing such orders because of the higher energy content in the low orders area. Lastly the fact that orders not expected from mechanical knowledge appear both in clusters 1 and D suggest that interesting information not provided by naive mechanical data was actually discovered.
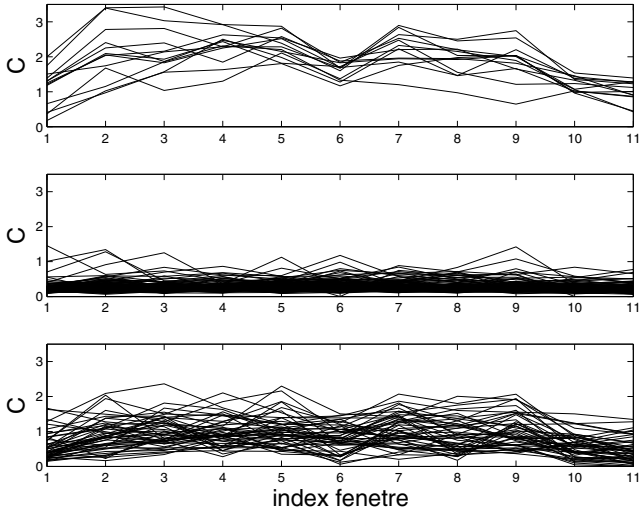
(a)
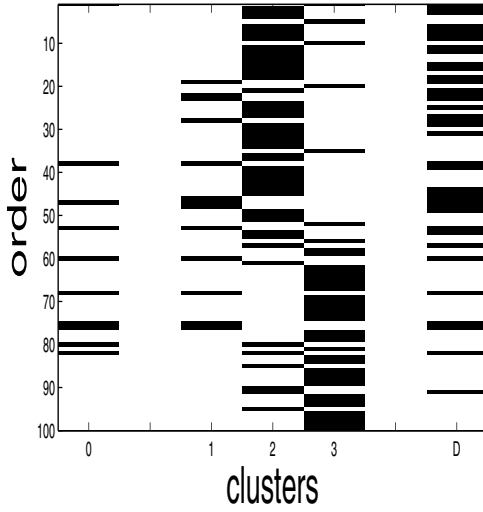
(b)

Fig. 8. Comparison of clustering and classical detection results for $z20$ vibratory data: (a) contrast vectors clustered by k-means with $C = 3$ clusters; (b) composition matrix for each cluster. Dark line with matrix coordinate i, j means that order i belongs to cluster j. Column 0 represents orders for which fan signature is expected from mechanical knowledge. Columns 1 to 3 are produced by the clustering algorithm. Column D is output by the statistical signal detection algorithm.

Lastly we confirm quantitatively these observations. Mutual information [29] is used here to measure the similarity between clusters. Setting cluster 0 as the default one, we compute the mutual information between the corresponding column of the composition matrix displayed in Fig. 8(b) and the other columns. What we expect is that self-information (i.e. the information between column 0 and itself) is high, while information between unrelated clusters such as 0 and 2 should be low. Moreover, mutual information between cluster pairs 0/1 and 0/D should be between the self-information value, and the unrelated clusters value. Indeed this is what we observe from Fig. 9, where the highest value corresponds to cluster pairs 0/0, while cluster pair 0/1 and 0/D have high mutual information. The negative value, which is theoretically meaningless, is a known limitation of the estimation algorithm. Lastly, the mutual information for cluster pair 0/D is lower than for cluster pair 0/1, which seems to indicate a higher performance of the proposed algoritm.
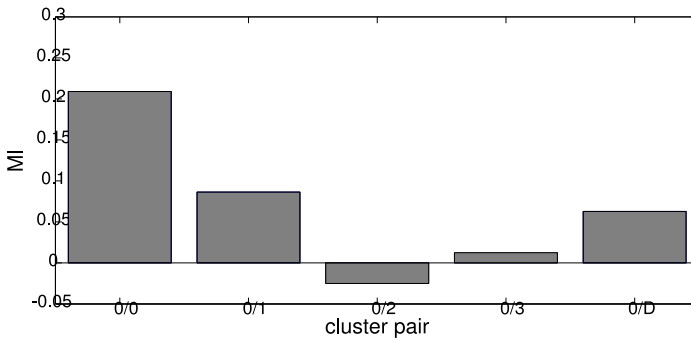

Fig. 9. Mutual information between pairs of clusters

The good results obtained by unsupervised clustering over signal-processing algorithms are suprising at first mainly because additional knowledge is embedded in the latter. Indeed, one needs to model the signal and the noise components before deriving a decision algorithm. We suggest that the results of the signal detection algorithm could be explained first by poor agreement between real data and noise hypotheses. This should be examined in further experiments.

Nevertheless, it remains that in the clustering approach we take into account the continuity of contrast values in time, which is a consequence of the continuity of shafts rotation speeds as a function of time. This continuity remains unexploited by the classical signal-detection algorithm, which iterates the decision process over successive windows without relating them.

6   Conclusion and Perspectives

In this work we have tackled the issue of order detection, i.e. the discovery of vibration patterns in noisy aircraft engine vibration signals. We proposed a

contrast measure whose aim is to single out significant vibration patterns that correspond to compressor and turbines mounted on the engine shafts. Then a clustering algorithm is built, and compared to a statistical signal detection procedure. We show with real data that the clustering method performs well, and give quantitative measure of this performance. Future works will aim at:

- increasing the statistical significance by enlarging the database to several engines, in both acceleration and deceleration situations. Theoretical properties of the estimators of contrast, and of mutual information could be studied as well.
- improving the signal detection algorithm, for example by considering extension to colored noise situation. In addition the continuity from one window to the following could be used.
- merging the decisions from both methods.
- refining the clustering, mainly the interpretation of the intermediate cluster.
- assessing the continuity hypothesis and using it as prior for clustering, from the knowledge of theoretical Campbell diagrams.

Acknowledgements

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