

Laguerre polynomials and transitional asymptotics of the modified Korteweg-de Vries equation for step-like initial data

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Abstract

We consider the compressive wave for the modified Korteweg–de Vries equation with background constants $c > 0$ for $x \rightarrow -\infty$ and 0 for $x \rightarrow +\infty$. We study the asymptotics of solutions in the transition zone $4c^2t - \varepsilon t < x < 4c^2t - \beta t^\sigma \ln t$ for $\varepsilon > 0$, $\sigma \in (0, 1)$, $\beta > 0$. In this region we have a bulk of nonvanishing oscillations, the number of which grows as $\frac{\varepsilon t}{\ln t}$. Also we show how to obtain Khruslov–Kotlyarov’s asymptotics in the domain $4c^2t - \rho \ln t < x < 4c^2t$ with the help of parametrices constructed out of Laguerre polynomials in the corresponding Riemann–Hilbert problem.

Contents

1	Introduction and results	1
2	Preliminaries	7
3	Asymptotics in the domain D_2	12
4	Logarithmic domain D_{2a}	19
4.1	Generalized Laguerre polynomials with index $\frac{1}{2}$	20
4.2	First approximation of $M^{(1)}$	21
4.2.1	Refined approximation of $M^{(1)}$	22
4.2.2	Second refined approximation of $M^{(1)}$	25
5	”Mesoscopic” regime	26
A	Asymptotics of modified Laguerre polynomials	32
A.1	Airy Parametrix	35

1 Introduction and results

We consider the Cauchy problem for the modified Korteweg–de Vries equation with step-like initial datum in the long-time limit, namely

$$q_t(x, t) + 6q^2(x, t)q_x(x, t) + q_{xxx}(x, t) = 0 \quad (1)$$

$$q(x, 0) = q_0(x) \rightarrow \begin{cases} 0 & \text{as } x \rightarrow +\infty, \\ c & \text{as } x \rightarrow -\infty, \end{cases} \quad c > 0, \quad (2)$$

The problem received a lot of attention in the span of more than four decades; the first asymptotic results were obtained for the Korteweg–de Vries equation. Physicists started to develop a qualitative description since the pioneer work of A. Gurevich and L. Pitaevsky [6] (1973). Later in this direction R. Bikbaev, V. Novokshenov and others actively worked on the topic (sf. [7]–[14]). In particular, the Cauchy problem (1) – (2) with more general type of initial data was considered by R. Bikbaev [12] in 1992.

The above research utilized the semi-heuristic Whitham method. The physical intuition suggested that the (x, t) -plane gets divided into three domains (see Figure 1), where the solution exhibits completely different asymp-

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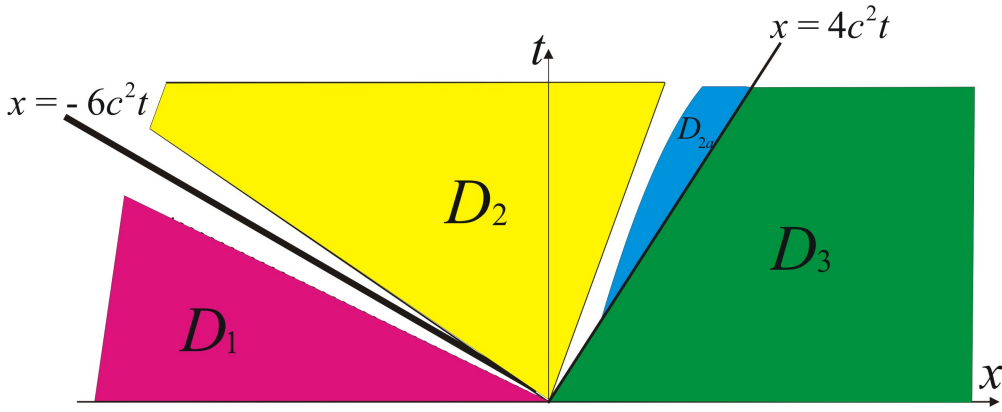


Figure 1: Domains in $x, t > 0$ -half-plane with qualitatively different asymptotic behaviour of $q(x, t)$: D_1 – asymptotics tends to the constant c , D_3 – possible solitons, otherwise the solution $q(x, t)$ of the Cauchy problem tends to 0, D_2 – modulated elliptic asymptotics, D_{2a} – asymptotic solitons.

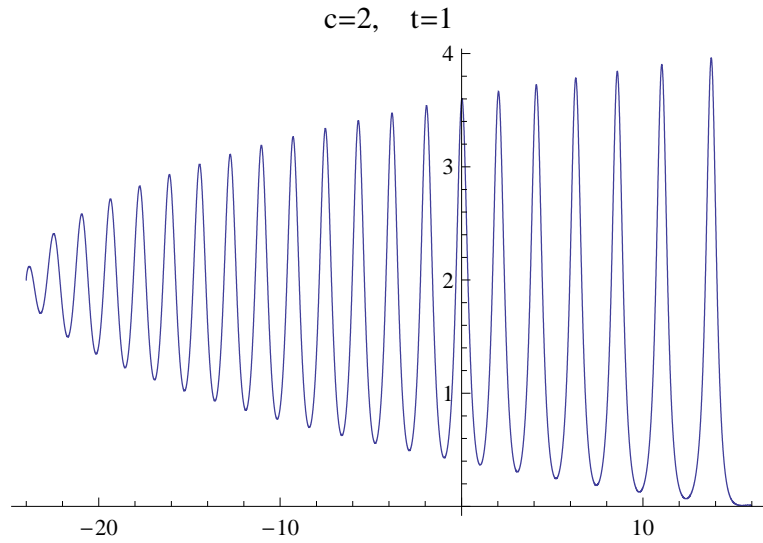


Figure 2: Graphics of q_{el} in the domain $-6c^2t < x < 4c^2t$ for initial data (3).

otic behaviours; in the left and in the right domains (D_1 and D_3) the solution tends to constants (one of them equals zero in our case) while in the middle domain it behaves as a modulated elliptic wave.

Rigorous mathematical papers were almost absent with the exception of papers concerning the region of asymptotic solitons. It was proved that near the wave front there exists a strip-like domain where the so-called *asymptotic solitons* arise. They are attributable to the presence of an endpoint of the simple continuous spectrum of the corresponding Lax operator, as opposed to usual solitons which are generated by points of the discrete spectrum. For the KdV equation this observation was pointed out in the pioneer work by E. Khruslov [25], [26] (1975, 1976), which appeared even before the first studies of the scattering and inverse scattering problem for Sturm-Liouville operators with step-like potential. This was done even before a full proof of existence of the time dynamics for spectral functions in the case of step-like data (see the book by Marchenko [36] 1977, Deift-Trubowitz [21] 1979, Koen-Kappeler [5] 1985). The work by Khruslov was inspired by prior results of Gurevich, Pitaevskii [6] and for the MKdV we refer to E. Khruslov and V. Kotlyarov [27] (1989). A comprehensive review of the results in this direction can be found in [28], [29] and references therein. Thus, besides the three specified regions, there is an additional transition region near the leading edge where a train of asymptotic solitons runs.

Similar phenomena occur in the semiclassical asymptotics of integrable equations and large size random matrices,

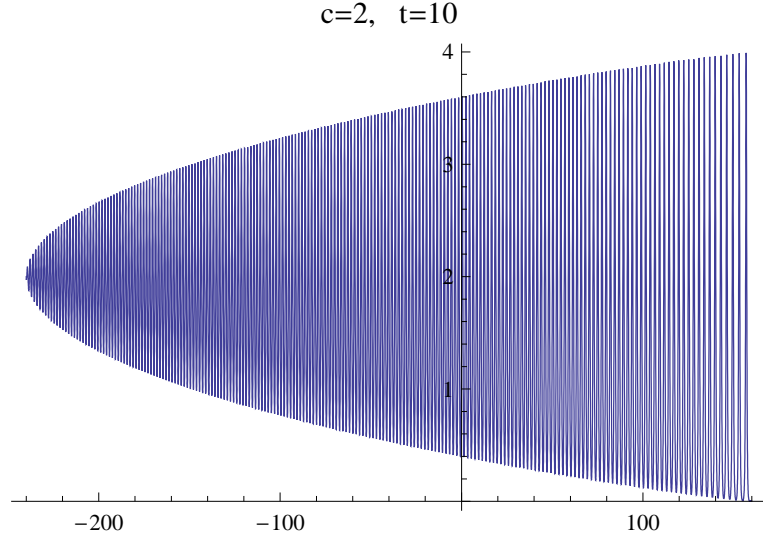


Figure 3: Graphics of q_{el} in the domain $-6c^2t < x < 4c^2t$ for initial data (3).

see [4], [2], [3].

The so-called *Riemann–Hilbert method* (a.k.a. *inverse scattering*) and the corresponding steepest descent method [22] have been actively developed now for more than 20 years. Recently these methods were successfully applied to the study of solutions of step type in other regions of the x, t half-plane, not only in the soliton domains ([15]–[19], [23], [30]–[41]).

Throughout the paper, we make some additional assumptions on the initial data, namely Assumptions 1, 2, 3 described in section 2. The Cauchy problem (1)–(2) with a pure step initial data

$$\tilde{q}_0(x) = \begin{cases} 0, & x > 0, \\ c > 0, & x < 0, \end{cases} \quad (3)$$

for the modified Korteweg – de Vries equation (1) was recently studied by V. Kotlyarov and A. Minakov [30] via the Riemann–Hilbert approach. (More general initial data with nonzero backgrounds, defined by two different and nonzero constants as $x \rightarrow \pm\infty$, were studied in [37], [38], [31], see also [34], [35]). In particular, in the domain $-6c^2t < x < 4c^2t$ the asymptotics of the Cauchy problem solution is described by a modulated elliptic wave.

Theorem 1. [30] *In the region $-6c^2t < x < 4c^2t$ the solution of the initial value problem (1), (2), (3) takes form of a modulated elliptic wave*

$$q(x, t) = q_{el}(x, t) + o(1), \quad t \rightarrow \infty,$$

where

$$q_{el}(x, t) = \sqrt{c^2 - d^2(\xi)} \frac{\Theta(\pi i + itB(d(\xi)) + i\Delta(d(\xi))|\tau(d(\xi)))}{\Theta(itB(d(\xi)) + i\Delta(d(\xi))|\tau(d(\xi)))}, \quad \xi = \frac{x}{12t}. \quad (4)$$

Here

$$\Theta(z, \tau) = \sum_{m \in \mathbb{Z}} e^{\frac{1}{2}\tau m^2 + zm}$$

is the theta function determined by its b -period $\tau = \tau(d)$, the functions $B(d)$, $\tau(d)$, $\Delta(d)$ are explicitly defined via (5)–(7)

$$B(d) = 24 \int_{id}^{ic} \frac{(k^2 + \mu^2(d))(k^2 + d^2)dk}{w_+(k, d)}, \quad \text{where } w(k, d) = \sqrt{(k^2 + c^2)(k^2 + d^2)}, \quad (5)$$

$$\tau(d) = -\pi i \int_{id}^{ic} \frac{dk}{w_+(k, d)} \left(\int_0^{id} \frac{dk}{w(k, d)} \right)^{-1}, \quad (6)$$

$$\Delta(d) = \int_{id}^{ic} \frac{\log(a_+(k)a_-(k)) dk}{w_+(k, d)} \left(i \int_0^{id} \frac{dk}{w(k, d)} \right)^{-1}, \quad (7)$$

and the functions $d(\xi)$, $\mu(d(\xi))$ are defined implicitly through formulas (8), (9)

$$\int_0^1 (\mu^2 - \lambda^2 d^2) \sqrt{\frac{1 - \lambda^2}{c^2 - \lambda^2 d^2}} d\lambda = 0, \quad (8)$$

$$\frac{c^2}{2} + \xi = \mu^2 + \frac{d^2}{2}. \quad (9)$$

The method used in [30] can be readily applied not only for the pure initial datum (3), but for more general initial datum of the form (2) satisfying the Assumptions 1, 2, 3 from section 2. Our first result improves on Thm. 1 by specifying the error term, namely we show that

Theorem 2. *Let $q(x, t)$ be the solution of initial value problem (1), (2), with initial datum satisfying Assumptions 1, 2, 3 of section 2. Then we have three typical behaviours in the following regions:*

1. Elliptic region (far from the leading edge and asymptotic solitons). Let $\varepsilon > 0$ be sufficiently small. Then

$$q(x, t) = q_{el}(x, t) + \mathcal{O}(t^{-1}) \quad \text{for} \quad \frac{-c^2}{2} + \varepsilon < \frac{x}{12t} < \frac{c^2}{3} - \varepsilon,$$

2. The beginning of transition. Let $\sigma \in (0, 1)$ and $\frac{1}{K} < \beta < K$, where $K > 0$. Then

$$q(x, t) = q_{el}(x, t) + \mathcal{O}(t^{-\sigma}) \quad \text{for} \quad \frac{c^2}{3} - \varepsilon < \frac{x}{12t} < \frac{c^2}{3} - \frac{\beta t^\sigma \ln t}{12t}.$$

3. In the middle of transition. Let $0 < \sigma_1 < \sigma_2 < 1$ and $\frac{1}{K} < \beta_1 < \beta_2 < K$, where $K > 0$. Then

$$q(x, t) = q_{el}(x, t) + \mathcal{O}(t^{-\sigma_1}) \quad \text{for} \quad \frac{c^2}{3} - \frac{\beta_2 t^{\sigma_2} \ln t}{12t} < \frac{x}{12t} < \frac{c^2}{3} - \frac{\beta_1 t^{\sigma_1} \ln t}{12t}.$$

Since we mainly consider the regime in which

$$\xi \equiv \frac{x}{12t} \rightarrow \frac{c^2}{3},$$

it is useful to introduce parameters associated with ξ, d , which are small in this regime. Namely, since in our regime we have

$$d \rightarrow c, \quad \xi \rightarrow \frac{c^2}{3},$$

we use the following notations

$$\eta = 1 - \frac{d}{c}, \quad v = 1 - \frac{3\xi}{c^2}. \quad (10)$$

It can be shown (see [32]) that the quantities B (5), v (10) have the following expansions:

$$\frac{B}{\pi} = 8c^3 \eta \left(1 + \sum_{j=1}^{M-1} \eta^j P_j(\eta) + \mathcal{O}(\eta^M \ln^M \eta) \right), \quad M = 2, 3, \dots \quad (11)$$

where P_j are polynomials of $\ln \eta$ of degree j , the first two of which are

$$P_1(\eta) = -\left(2 + \frac{1}{2} \ln \frac{\eta}{8}\right), \quad P_2(\eta) = \frac{1}{16} (13 - 42 \ln 2 + 36 \ln^2 2 + 2 \ln \eta (7 - 12 \ln 2 + 2 \ln \eta))$$

and

$$v = \eta \ln \frac{8e}{\eta} + \sum_{j=2}^M \eta^j Q_j(\eta) + \mathcal{O}(\eta^M \ln^M \eta), \quad M = 2, 3, \dots \quad (12)$$

where Q_j are polynomials of $\ln \eta$ of degree j . The first polynomials Q_j are

$$Q_2(\eta) = \frac{1}{2} \left(-2 - 9 \ln 2 + 9 \ln^2 2 + \left(3 + \ln \frac{\eta}{64} \right) \ln \eta \right),$$

$$Q_3(\eta) = \frac{1}{16} (9 - 6 \ln 2 + 18 \ln^2 2 (-5 + 6 \ln 2)) + \frac{\ln \eta}{8} (1 + 30 \ln 2 - 54 \ln^2 2) + \frac{\ln^2 \eta}{8} (-5 + 18 \ln 2) - \frac{\ln^3 \eta}{4}.$$

Further, as a consequence of Thm 2, we can decompose the elliptic wave into a train of asymptotic solitons (see the end of Section 3),

Corollary 1. *Let M be 1, 2, or 3. Suppose that (x, t) lies on a curve*

$$x = 4c^2 t - \beta t^\sigma \ln t, \quad \beta > 0, \quad \sigma \in (0, \frac{M}{M+1}). \quad (13)$$

Let $\tilde{\eta}$ be the unique solution of

$$v = \tilde{\eta} \ln \frac{8e}{\tilde{\eta}} + \sum_{j=2}^M \tilde{\eta}^j Q_j(\tilde{\eta})$$

that behaves as $\tilde{\eta} \asymp \frac{v}{\ln \frac{1}{v}}$ as $v \rightarrow 0$ (Q_j are as in (12)). Let

$$\tilde{z} = 8c^3 t \tilde{\eta} \left(1 + \sum_{j=1}^{M-1} \tilde{\eta}^j P_j(\tilde{\eta}) \right) - \frac{1}{2}, \quad (14)$$

where P_j are as in (11) and let n be the greatest integer not exceeding $\frac{\tilde{z}}{2}$, i.e. $n = \lfloor \tilde{z}/2 \rfloor$, $2n \leq \tilde{z} < 2n + 2$. Then, as $t \rightarrow \infty$

$$q(x, t) = \frac{2c}{\cosh \left[2c(x - 4c^2 t) + (2n + \frac{3}{2}) \ln t + \alpha_n^{(M)} \right]} + r^{(M)}(x, t), \quad (15)$$

where the errors $r^{(M)}$ are

$$r^{(M=1)} = \mathcal{O}(t^{2\sigma-1} \ln^2 t) + \mathcal{O}(t^{-\sigma}),$$

$$r^{(M=2)} = \mathcal{O}(t^{3\sigma-2} \ln^3 t) + \mathcal{O}(t^{\sigma-1} \ln t) + \mathcal{O}(t^{-\sigma}),$$

$$r^{(M=3)} = \mathcal{O}(t^{4\sigma-3} \ln^4 t) + \mathcal{O}(t^{\sigma-1} \ln t) + \mathcal{O}(t^{-\sigma}),$$

and the phases $\alpha_n^{(M)}$ are

$$\alpha_n^{(M=1)} = (2n + \frac{3}{2}) \ln \frac{32c^3}{n} + 2n - \ln \frac{4}{(h^*)^2},$$

$$\alpha_n^{(M=2)} = (2n + \frac{3}{2}) \ln \frac{32c^3}{n} + 2n - \ln \frac{4}{(h^*)^2} + \frac{n^2}{4c^3 t} \ln \frac{32c^3 t}{e^2 n},$$

$$\begin{aligned} \alpha_n^{(M=3)} &= (2n + \frac{3}{2}) \ln \frac{32c^3}{n} + 2n - \ln \frac{4}{(h^*)^2} + \frac{n^2}{4c^3 t} \ln \frac{32c^3 t}{e^2 n} + \\ &+ \frac{n^3}{64c^6 t^2} \left(-31 + \frac{25}{2} \ln \frac{32c^3 e t}{n} - \left(\ln \frac{32c^3 e t}{n} \right)^2 \right). \end{aligned}$$

Remark 1.1. *It is possible to find expressions for $\alpha_n^{(M)}$ and the error $r^{(M)}$ for any given integer M but the expressions become quickly unwieldy.*

Remark 1.2. We see that on the curve (13), we move through the bulk of solitons, and not constrained to a particular soliton. Hence the quantity n , which describes the number of soliton on which we are located, varies with time. We can give a rough approximation of how it changes. Namely, the quantity \tilde{z} (14) behaves like

$$\frac{\tilde{z}}{2} = \frac{4c^3tv}{\ln \frac{1}{v}} \left(1 + \mathcal{O} \left(\frac{1}{\ln v} \right) \right) - \frac{1}{4} + \mathcal{O} \left(\frac{1}{\ln v} \right),$$

hence on the curve

$$v = \frac{\beta t^\sigma \ln t}{4c^2t}$$

we have

$$n \sim \frac{\tilde{z}}{2} = \frac{\beta ct^\sigma}{1-\sigma} \left(1 + \mathcal{O} \left(\frac{\ln \ln t}{\ln t} \right) \right).$$

Remark 1.3. In section 5 we give an alternative proof of Corollary 1 (and more generally, for all integer M) by constructing parametrices in terms of Laguerre polynomials for the corresponding Riemann-Hilbert problem.

The Corollary 1 does not deal with the situation when $\sigma \rightarrow 0$ or $\sigma = 0$, that corresponds to the cases when the number of solitons grows but not very fast, or when it is bounded; for finite number of solitons Khruslov and Kotlyarov obtained the following formula:

Theorem 3 ([27]). Fix an integer $N \geq 1$. Then for

$$x > 4c^2t - \frac{(N + \frac{1}{2}) \ln t}{c}$$

the solution of the initial value problem (1)-(2) with initial datum satisfying

$$\int_{-\infty}^0 (1 + |x|) |q_0(x)| dx + \int_0^{+\infty} (1 + |x|) |q_0(x)| < \infty \quad (16)$$

admits the asymptotic representation:

$$q(x, t) = q_{as}(x, t) + \mathcal{O} \left(t^{-\frac{1}{2} + \varepsilon} \right), \quad (17)$$

$$q_{as}(x, t) = \sum_{n=0}^{N-1} \frac{2c}{\cosh \left(2c(x - 4c^2t) + \left(2n + \frac{3}{2} \right) \ln t + \tilde{\alpha}_n \right)}, \quad (18)$$

$$\tilde{\alpha}_n = -\ln \left[\frac{|h_0|}{4^{2n+1} (2c)^{6n+3}} \Gamma(n+1) \Gamma \left(n + \frac{3}{2} \right) \right] = -\ln \left[\frac{2\Gamma(n+1) \Gamma \left(n + \frac{3}{2} \right)}{(h^*)^2 \pi (32c^3)^{2n+\frac{3}{2}}} \right], \quad (19)$$

where $|h_0|$ is related to h^* (38) in the following way:

$$|h_0| = \frac{1}{h^{*2} \pi (2c)^{3/2}}. \quad (20)$$

We prove a refined version of this theorem with improved error estimates.

Theorem 4. Let $N \geq 1$ be a fixed integer. Then in the domain $4c^2t \geq x \geq 4c^2t - \frac{N+\frac{1}{2}}{c} \ln t$ the solution of the initial value problem (1), (2) with initial datum satisfying Assumptions 1, 2, 3 of section 2 has the following asymptotics

$$\begin{aligned} q(x, t) &= \sum_{n=0}^{N-1} \frac{2c}{\cosh \left(2c(x - 4c^2t) + (2n + \frac{3}{2}) \ln t + \alpha_n(\frac{x}{t}) \right)} + \mathcal{O}(t^{-1}) = \\ &= \sum_{n=0}^{N-1} \frac{2c}{\cosh \left(2c(x - 4c^2t) + (2n + \frac{3}{2}) \ln t + \tilde{\alpha}_n \right)} + \mathcal{O}\left(\frac{\ln t}{t}\right), \end{aligned}$$

where $v = 1 - \frac{x}{4c^2t}$, and phase $\tilde{\alpha}_n$ is in (19),

$$\alpha_n = \alpha_n\left(\frac{x}{t}\right) = \ln \frac{\pi (h^*)^2}{2\Gamma(n+1)\Gamma(n+\frac{3}{2})} + \left(2n + \frac{3}{2}\right) \ln(16c^3(2+v)). \quad (21)$$

The method in [27] is based on the corresponding Gelfand-Levitan-Marchenko equation. In Sec. 4 we show how to obtain this asymptotics using the parametrices in terms of Laguerre polynomials in the corresponding Riemann-Hilbert problem.

Theorems 2, 3 and Corollary 1 do not cover the whole transition zone

$$4c^2t - \varepsilon t \leq x \leq 4c^2t,$$

where $\varepsilon > 0$. Indeed there remains an unexplored region

$$4c^2t - \beta t^\sigma \ln t \leq x \leq 4c^2t - \rho \ln t,$$

with fixed $\beta > 0$, $0 < \sigma < 1$, $\rho > 0$. We study the asymptotics in this region in section 5.

Comparing phases in the argument of cosh in (15) and (19), we see that those arguments are equal up to terms of order n^{-1} ,

$$\alpha_n^{(M=1)} - \tilde{\alpha}_n = \mathcal{O}\left(\frac{1}{n}\right).$$

This suggests that the formula (17) is valid in a wider region; to this end we have the following theorem

Theorem 5. *Let $\sigma \in [0, \frac{1}{2})$, $\beta \in (\frac{1}{K}, K)$, $K > 0$, and suppose that (x, t) lies on the curve $x = 4c^2t - \beta t^\sigma \ln t$. Let $\tilde{\eta}$, \tilde{z} and n be as in Corollary 1 with $M = 1$. Then, as $t \rightarrow \infty$*

$$q(x, t) = \frac{2c}{\cosh(2c(x - 4c^2t) + (2n + \frac{3}{2}) \ln t + \tilde{\alpha}_n)} + \mathcal{O}(t^{2\sigma-1} \ln^2 t), \quad (22)$$

where $\tilde{\alpha}_n$ is given in (19), and the \mathcal{O} estimate is uniform when σ, β change in compact domains of their domain, i.e. for $0 \leq \sigma \leq \sigma_0$, $\frac{1}{K_0} \leq \beta \leq K_0$.

Remark 1.4. *Theorem 5 covers the regime in which the point (x, t) travels along solitons with slowly growing number $\asymp t^\sigma$, $\sigma < 1$.*

Remark 1.5. *We leave open the question about direct verification of consistency of results of Theorem 2, Corollary 1 and results which can be obtained with the help of method of chapter 5, formula (110), for the regime*

$$x = 4c^2t - \beta t^\sigma \ln t$$

when σ is in the range

$$\sigma \in \left[\frac{M}{M+1}, \frac{M+1}{M+2} \right), \quad \text{for } M \geq 1.$$

While this might be checked for several first $M = 1, 2, 3$, to verify this for a general M requires deeper understanding of relation between z (45) and γ (107).

2 Preliminaries

In this section we collect some well-known results. The MKdV equation (1) admits a Lax pair representation in the form [49], [47]

$$\Phi_x(x, t; k) + ik\sigma_3\Phi(x, t; k) = Q(x, t)\Phi(x, t; k), \quad (23)$$

$$\Phi_t(x, t; k) + 4ik^3\sigma_3\Phi(x, t; k) = \hat{Q}(x, t; k)\Phi(x, t; k), \quad (24)$$

where

$$Q(x, t) = \begin{pmatrix} 0 & q(x, t) \\ -q(x, t) & 0 \end{pmatrix}, \quad \widehat{Q}(x, t; k) = 4k^2 Q - 2ik(Q^2 + Q_x)\sigma_3 + 2Q^3 - Q_{xx}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (25)$$

If we substitute $q(x, t) = c$ (constant) in (25), then the Lax pair equations (23), (24) admit explicit solutions [30]

$$E_0(x, t; k) = e^{-(ikx + 4ik^3 t)\sigma_3}, \quad \text{and} \quad E_c(x, t; k) = K(k)e^{-(ixX(k) + 2it(2k^2 - c^2)X(k))\sigma_3},$$

for $c = 0$ and $c \neq 0$, respectively. In the formula above we have set $X(k) = \sqrt{k^2 + c^2}$ and

$$K(k) = \begin{pmatrix} a_\gamma(k) & b_\gamma(k) \\ b_\gamma(k) & a_\gamma(k) \end{pmatrix}, \quad a_\gamma(k) = \frac{1}{2}(\gamma(k) + \gamma^{-1}(k)), \quad b_\gamma(k) = \frac{1}{2}(\gamma(k) - \gamma^{-1}(k)), \quad \gamma(k) = \sqrt[4]{\frac{k - ic}{k + ic}}.$$

Proposition 0.1. ([24]) *Let $q_0(x)$ satisfies (16) and*

$$\sup_{x \in \mathbb{R}} |q_0(x)| < \infty, \quad \lim_{x \rightarrow +\infty} q_0(x) = 0, \quad \lim_{x \rightarrow -\infty} q_0(x) = c > 0. \quad (26)$$

Denote

$$\sigma_l(x) = \int_{-\infty}^x |q_0(\tilde{x}) - c| d\tilde{x}, \quad \sigma_r(x) = \int_x^{+\infty} |q_0(\tilde{x})| d\tilde{x}, \quad \sigma_{l,1}(x) = \int_{-\infty}^x \sigma_l(\tilde{x}) d\tilde{x}, \quad \sigma_{r,1}(x) = \int_{-\infty}^x \sigma_r(\tilde{x}) d\tilde{x}$$

(these quantities exist under the above condition (16)), then there exist Jost solutions of the equation (23) that possess the following integral representations

$$\Psi_l(x, k) = E_c(x, k) + \int_{-\infty}^x L_l(x, y) E_c(y, k) dy, \quad \Psi_r(x, k) = E_0(x, k) + \int_x^{+\infty} L_r(x, y) E_0(y, k) dy \quad (27)$$

with kernels

$$L_l(x, y) = \begin{pmatrix} L_{l,1}(x, y) & -L_{l,2}(x, y) \\ L_{l,2}(x, y) & L_{l,1}(x, y) \end{pmatrix}, \quad L_r(x, y) = \begin{pmatrix} L_{r,1}(x, y) & -L_{r,2}(x, y) \\ L_{r,2}(x, y) & L_{r,1}(x, y) \end{pmatrix},$$

which satisfy the following estimates:

$$\begin{cases} |L_{l,1}(x, y)| \leq \frac{1}{2} \sigma_l\left(\frac{x+y}{2}\right) M_{2,l} M_{1,l}, \\ |L_{l,2}(x, y)| \leq \frac{|q_0(\frac{x+y}{2}) - c|}{2} + \sigma_l\left(\frac{x+y}{2}\right) \frac{(\sigma_l(x) - \sigma_l(\frac{x+y}{2}))}{2} M_{2,l} M_{1,l}, \\ |L_{r,1}(x, y)| \leq \frac{1}{2} \sigma_r\left(\frac{x+y}{2}\right) M_{2,r} M_{1,r}, \\ |L_{r,2}(x, y)| \leq \frac{|q_0(\frac{x+y}{2})|}{2} + \sigma_r\left(\frac{x+y}{2}\right) \frac{(\sigma_r(x) - \sigma_r(\frac{x+y}{2}))}{2} M_{2,r} M_{1,r}, \end{cases} \quad (28)$$

where we denoted for brevity

$$\begin{cases} M_{1,l} = \exp \left[\max_{z \in (-\infty, x)} |q_0(z) + c| \cdot (\sigma_{l,1}(x) - \sigma_{l,1}(\frac{x+y}{2})) \right], & M_{2,l} = \max_{z \in (-\infty, x)} |q_0(z) + c|, \\ M_{1,r} = \exp \left[\max_{z \in (x, +\infty)} |q_0(z)| \cdot (\sigma_{r,1}(x) - \sigma_{r,1}(\frac{x+y}{2})) \right], & M_{2,r} = \max_{z \in (x, +\infty)} |q_0(z)|. \end{cases} \quad (29)$$

Proof. For convenience of the reader we will sketch the proof for L_l . Substituting the first part of (27) into (23), we come to the system

$$\begin{cases} L_{l1,x}(x, y) + L_{l1,y}(x, y) = (q_0(x) + c)L_{l2}(x, y), & L_{l2,x}(x, y) - L_{l2,y}(x, y) = -(q_0(x) - c)L_{l1}(x, y), \quad y < x, \\ L_{l2}(x, x) = \frac{-(q_0(x) - c)}{2}, & \lim_{y \rightarrow -\infty} L_{l2}(x, y) = 0. \end{cases}$$

Making the change of variables

$$u = \frac{x+y}{2}, \quad v = \frac{x-y}{2}, \quad H_1(u, v) \equiv L_{l,1}(x, y), \quad H_2(u, v) \equiv L_{l,2}(x, y),$$

we come to the integral equations

$$H_1(u, v) = - \int_{-\infty}^u (q_0(\tilde{u} + v) + c) H_2(\tilde{u}, v) d\tilde{u}, \quad H_2(u, v) = \frac{-(q_0(u) - c)}{2} - \int_0^v (q_0(u + \tilde{v}) - c) H_1(u, \tilde{v}) d\tilde{v}, \quad (30)$$

or

$$H_1(u, v) = \frac{1}{2} \int_{-\infty}^u (q_0(\tilde{u} + v) + c) (q_0(\tilde{u}) - c) d\tilde{u} + \int_{-\infty}^u (q_0(\tilde{u} + v) + c) \int_0^v (q_0(\tilde{u} + \tilde{v}) - c) H_1(\tilde{u}, \tilde{v}) d\tilde{v} d\tilde{u}, \quad (31)$$

$$H_2(u, v) = \frac{-(q_0(u) - c)}{2} + \int_0^v (q_0(u + \tilde{v}) - c) \int_{-\infty}^u (q_0(\tilde{u} + \tilde{v}) + c) H_2(\tilde{u}, \tilde{v}) d\tilde{u} d\tilde{v}. \quad (32)$$

Applying the method of successive approximations (the integral operator becomes a contraction for u sufficiently large) to the above integral equations, we come to the statement of the proposition. \blacksquare

Proposition 0.2. ([24]) *Let initial function $q_0(x)$ satisfies (16) and also*

$$\sup_{x \in \mathbb{R}} |q'_0(x)| < \infty, \quad \lim_{x \rightarrow \pm\infty} q'_0(x) = 0, \quad \int_{-\infty}^{+\infty} |q'_0(x)| dx < \infty. \quad (33)$$

Denote

$$\sigma_{2,l}(x) = \int_{-\infty}^x |q'_0(\tilde{x})| d\tilde{x}, \quad \sigma_{2,r}(x) = \int_x^{+\infty} |q'_0(\tilde{x})| d\tilde{x}.$$

Then the partial derivatives of L_l , L_r satisfy the following estimates:

$$\begin{aligned} |L_{1l,y}| &\leq \frac{\sigma_l(\frac{x+y}{2})}{4} \left[M_{3,l} + M_{1,l} M_{2,l} M_{3,l} \left(\sigma_{l,1}(x) - \sigma_{l,1}\left(\frac{x+y}{2}\right) \right) + \left(2\sigma_l(x) - \sigma_l\left(\frac{x+y}{2}\right) \right) M_{2,l}^2 M_{1,l} \right] + \\ &\quad + \frac{|q_0(\frac{x+y}{2}) - c|}{4} |q_0(x) + c|, \\ |L_{2l,y}| &\leq \frac{|q'_0(\frac{x+y}{2})|}{4} + \frac{\sigma_l(\frac{x+y}{2}) (\sigma_{l,2}(x) - \sigma_{l,2}(\frac{x+y}{2}))}{4} M_{2,l} M_{1,l} \left(1 + \left(\sigma_l(x) - \sigma_l\left(\frac{x+y}{2}\right) \right) M_{2,l} \right) + \\ &\quad + \frac{|q_0(\frac{x+y}{2}) - c| (\sigma_l(x) - \sigma_l(\frac{x+y}{2}))}{4} M_{2,l} + \frac{\sigma_l(\frac{x+y}{2}) |q_0(x) - c|}{4} M_{2,l} M_{1,l}. \end{aligned}$$

Similar estimates are valid for L_r , we just need to replace index l with r , c with 0, and $-\infty$ with $+\infty$ in the above expressions. Here M_1, M_2 are defined in (29), and

$$M_{3,l} = \max_{z \leq x} |q'_0(z)|, \quad M_{3,r} = \max_{z \geq x} |q'_0(z)|$$

Proof. The statements of the lemma follows directly from analysis of integral equations (30), (31), (32) and Proposition 0.1. \blacksquare

Remark 2.1. Conditions (26) follow from (33).

Corollary 2. (a) *Provided that the conditions (16) are satisfied, the first column $\Psi_{l,1}$ is analytic in $\text{Im } \sqrt{k^2 + c^2} > 0$, the second column $\Psi_{l,2}$ is analytic in $\text{Im } \sqrt{k^2 + c^2} < 0$, and the first column of the right Jost solution $\Psi_{r,1}$ is analytic in $\text{Im } k < 0$, and the second column $\Psi_{r,2}$ is analytic in $\text{Im } k > 0$.*

(b) Suppose that in addition to (16) the following condition is satisfied:

$$\int_{-\infty}^0 |q_0(x) - c| e^{2|x|\sqrt{l_0^2 - c^2}} dx + \int_0^{+\infty} |q_0(x)| e^{2xl_0} dx < \infty, \quad \text{for some } l_0 > c > 0. \quad (34)$$

Then the Jost solutions Ψ_l, Ψ_r are analytic in a $l_0 - c$ neighborhood of the contour $k \in \Sigma = \mathbb{R} \cup [ic, -ic]$.

Proof. The first part of the corollary follows from the fact that under (16) the kernels in (27) are summable:

$$\int_{-\infty}^x |L_l(x, y)| dy < \infty, \quad \int_x^{+\infty} |L_r(x, y)| dy < \infty.$$

The second statement of the corollary follows directly from the estimates (28) and the fact that under (34)

$$\int_{-\infty}^0 e^{2|y|\sqrt{l_0^2 - c^2}} \sigma_l(y) dy < \infty, \quad \int_0^{+\infty} e^{2yl_0} \sigma_r(y) dy < \infty.$$

The first of the last estimates gives us that Ψ_l is analytic in $0 < |\operatorname{Im} \sqrt{k^2 + c^2}| < \sqrt{l_0^2 - c^2}$, which includes $l_0 - c$ neighborhood of Σ , and the second estimate gives us analyticity of Ψ_r in $|\operatorname{Im} k| < l_0$, which also includes $l_0 - c$ neighborhood of Σ . \blacksquare

Given the Jost solutions of (23) for $t = 0$, we define the transition matrix $T(k)$ by

$$\Psi_l(x, k) = \Psi_r(x, k) T(k).$$

The Jost solution have the (defining) property that Ψ_l is asymptotic to E_c as $x \rightarrow -\infty$ and Ψ_r is asymptotic to E_0 as $x \rightarrow +\infty$ (whence the subscripts, “l(ef)” and “r(ight)”). Due to symmetries of the x -equation (23) the transition matrix has the following structure:

$$T(k) = \begin{pmatrix} a(k) & -\overline{b(\overline{k})} \\ b(k) & \overline{a(\overline{k})} \end{pmatrix}, \quad \text{where } a(k) = \det(\Psi_{l1}, \Psi_{r2}), \quad b(k) = \det(\Psi_{r1}, \Psi_{l1}). \quad (35)$$

The functions $a^{-1}(k)$ and $r(k) := \frac{b(k)}{a(k)}$ are called the transmission and reflection coefficients, respectively. Under the condition (16), they have the following properties:

Lemma 2.1. 1. Under conditions (16), (33), the function $a(k)$ is analytic in $\{k : 0 < \operatorname{Im} k\} \setminus [0, ic]$, and it can be extended continuously up to the boundary with the exception of the point ic , where $a(k)$ may have at most a root singularity of the order $(k - ic)^{-1/4}$; further,

$$a(k) = 1 + \mathcal{O}(k^{-1}) \quad \text{as } k \rightarrow \infty;$$

function $r(k)$ is defined in $k \in \mathbb{R}$ except for points where $a(k) = 0$ and

$$r(k) = \mathcal{O}(k^{-1}).$$

2. $a(k), r(k)$ satisfy the symmetry condition

$$\overline{a(-\overline{k})} = a(k), \quad \overline{r(-\overline{k})} = r(k); \quad (36)$$

3. if we assume also (34), then $r(k)$ is meromorphic in a $l - c_0$ neighborhood of $\Sigma = \mathbb{R} \cup [ic, -ic]$ with poles at zeros of $a(k)$. In the specified domain $r(k)$ has the following asymptotics:

$$r(k) = \mathcal{O}(k^{-1}) \quad \text{as } k \rightarrow \infty.$$

Proof. The first two items have been proved in [24]. The third statement follows from Corollary 2, (b) and estimates in Proposition 0.2, which are used after integrating by parts in (27). ■

Assumption 1. For simplicity, we suppose the absence of usual solitons generated by the discrete spectrum:

$$\forall k \in \mathbb{C}_+ \setminus [ic, 0] \quad a(k) \neq 0.$$

We have the following

Lemma 2.2. The spectral coefficients $a(k), b(k)$ do not vanish on the segment $(-ic, ic)$.

Proof. Using the relations (they follow from the symmetry and determinantal properties of the corresponding Jost solutions, see [30], chapter II, p.4-5)

$$a(k)\overline{a(\bar{k})} + b(k)\overline{b(\bar{k})} = 1, \quad \overline{a(-\bar{k})} = a(k), \quad \overline{b(-\bar{k})} = b(k), \quad a_-(k) = -\overline{ib_+(\bar{k})}, \quad (37)$$

we get that $a_{\pm}(k) \neq 0, b_{\pm}(k) \neq 0$ on $(ic, -ic)$. ■

Assumption 2. From the definition of $a(k)$ (35) and the integral representations for Jost solutions (27) it follows that generically expansion of $a(k)$ in a vicinity of the point $k = ic$ starts from the term $(k - ic)^{-1/4}$ and then continues with $(k - ic)^{1/4}, (k - ic)^{3/4}, \dots$. However, it might happen that the coefficient in front of the term $(k - ic)^{-1/4}$ vanishes for some particular initial data. Here we suppose the generic situation, i.e.

$$a(k) = \frac{h^*}{2} \sqrt[4]{\frac{2ic}{k - ic}} \left(1 + O\left(\sqrt{\frac{k - ic}{i}}\right) \right), \quad k \rightarrow ic, \quad h^* \in \mathbb{R} \setminus \{0\}. \quad (38)$$

The fact that $h^* \in \mathbb{R}$ follows from the symmetry (36).

Assumption 3. We will also assume that the reflection coefficient admit an analytic continuation to a δ -neighborhood of $\mathbb{R} \cup [ic, 0]$ with some $\delta > 0$.

This assumption is satisfied for example in each of the following cases:

1. the initial function is smooth (33) and tends to its background limits exponentially fast (34). Lemma 2.1 then ensures analyticity of $r(k)$ in a neighbourhood of contour Σ under assumption 1 of absence of discrete spectrum. Here l_0 satisfies the inequality $l_0 > c + \delta > 0$.
2. the initial data is of the form (3). In this case the inverse of the transmission coefficient $a(k) = \tilde{a}(k)$ and the reflection coefficient $r(k) = \tilde{r}(k)$ can be computed explicitly and have the following form:

$$\tilde{a}(k) = \frac{1}{2} \left(\sqrt[4]{\frac{k - ic}{k + ic}} + \sqrt[4]{\frac{k + ic}{k - ic}} \right), \quad \tilde{r}(k) = \frac{\left(\sqrt[4]{\frac{k - ic}{k + ic}} - \sqrt[4]{\frac{k + ic}{k - ic}} \right)}{\left(\sqrt[4]{\frac{k - ic}{k + ic}} + \sqrt[4]{\frac{k + ic}{k - ic}} \right)},$$

where the cut is taken along the segment $[-ic, ic]$ and the branch of the root is taken such that $\tilde{a}(k)$ tends to 1 as $k \rightarrow \infty$. The constant h^* in (38) is $h^* = 1$.

Remark 2.1. We should notice that the "pure" step initial function (3) is not in the class

$$\int_{-\infty}^0 |x| |q(x, t) - c| dx + \int_0^{+\infty} x |q(x, t)| dx < \infty,$$

since $q(x, t) = c + \mathcal{O}(|x|^{-\frac{3}{4}})$ as $x \rightarrow -\infty$ for $t > 0$; nevertheless, due to the analyticity of $\tilde{a}(k), \tilde{r}(k)$, the asymptotic analysis in this case can be done in the same manner as in the case of smooth and fast decreasing initial functions.

The solution of the initial value problem (1), (2) can be reconstructed from the solution of the following Riemann-Hilbert problem ([30]):

Riemann-Hilbert problem 1. *find a 2×2 matrix-valued function $M(\xi, t; k)$ such that*

1. *analyticity: $M(\xi, t; k)$ is analytic in $\mathbb{C} \setminus \Sigma$, and continuous up to the boundary. Here the oriented contour Σ is $\Sigma = \mathbb{R} \cup [ic, -ic]$.*
2. *jump: $M_-(\xi, t; k) = M_+(\xi, t; k)J(\xi, t; k)$, where*

$$J(\xi, t; k) = \begin{pmatrix} 1 & -\overline{r(k)}e^{-2it\theta(k, \xi)} \\ -r(k)e^{2it\theta(k, \xi)} & 1 + |r(k)|^2 \end{pmatrix}, \quad k \in \mathbb{R} \setminus \{0\},$$

$$J(\xi, t; k) = \begin{pmatrix} 1 & 0 \\ f(k)e^{2it\theta(k, \xi)} & 1 \end{pmatrix}, \quad k \in (ic, 0),$$

$$J(\xi, t; k) = \begin{pmatrix} 1 & -\overline{f(k)}e^{-2it\theta(k, \xi)} \\ 0 & 1 \end{pmatrix}, \quad k \in (0, -ic),$$

where $r(k)$ is the reflection coefficient associated with spectral problem associated with MKdV, and

$$f(k) = r_-(k) - r_+(k) = \frac{i}{a_-(k)a_+(k)}, \quad k \in (ic, -ic) \quad (39)$$

where $a^{-1}(k)$ is the transmission coefficient.

3. *asymptotics: $M(x, t; k) \rightarrow I$ as $k \rightarrow \infty$.*

The jump matrices in the RH problem (1) satisfy the Schwartz symmetry $J^{-1}(k) = \left(\overline{J^T(\bar{k})}\right)$ for $k \in \Sigma \setminus \mathbb{R}$. Hence, it follows from the vanishing lemma [51] that the solution of the RH problem (1) exists. Further, from analyticity of the reflection coefficient it follows that we can deform the contour Σ into such a contour that the corresponding singular integral equation, which is equivalent to the RHP 1, admits x and t differentiation. Then, in the spirit of the well-known result of Zakharov – Shabat [50], one can prove that the solution of the initial value problem (1), (2) can be reconstructed by the following formula (see [42], chapter 2 for details):

$$q(x, t) = \lim_{k \rightarrow \infty} (2ikM(x, t; k))_{21} = \lim_{k \rightarrow \infty} (2ikM(x, t; k))_{12}.$$

3 Asymptotics in the domain D_2

The region D_2 corresponds to $\frac{-c^2}{2} + \varepsilon < \xi < \frac{c^2}{3} - \frac{\beta t^\sigma \ln t}{t}$, $\sigma \in (0, 1)$, $\varepsilon > 0$. The main result in this section is Theorem 2. It was shown in [30] that the RH problem 1 is equivalent to the following one:

Riemann-Hilbert problem 2. *Find a matrix-valued function $M^{(3)}(\xi, t, k)$ such that*

1. *analyticity: it is analytic in $k \in \mathbb{C} \setminus \Sigma^{(3)}$ (contour $\Sigma^{(3)}$ is shown in Figure 4),*
2. *asymptotics; we have $M^{(3)}(x, t; k) \rightarrow I$ as $k \rightarrow \infty$ and near the points $\pm ic$ we have that every entry is bounded by $\mathcal{O}(|z \mp ic|^{-\frac{1}{4}})$ (respectively).*
3. *jump: the boundary values satisfy the following jump conditions on the contour $\Sigma^{(3)}$*

$$M_-^{(3)}(\xi, t, k) = M_+^{(3)}(\xi, t, k)J^{(3)}(\xi, t, k), \quad M^{(3)}(\xi, t, k) \rightarrow I, \quad k \rightarrow \infty. \quad (40)$$

The jump matrix is given by the formula

$$\begin{aligned} J^{(3)}(\xi, t, k) &= \begin{pmatrix} 1 & 0 \\ -r(k)F^{-2}(k, \xi)e^{2itg(k, \xi)} & 1 \end{pmatrix}, & k \in L_1 \\ &= \begin{pmatrix} 1 & -\overline{r(k)}F^2(k, \xi)e^{-2itg(k, \xi)} \\ 0 & 1 \end{pmatrix}, & k \in L_2 \end{aligned}$$

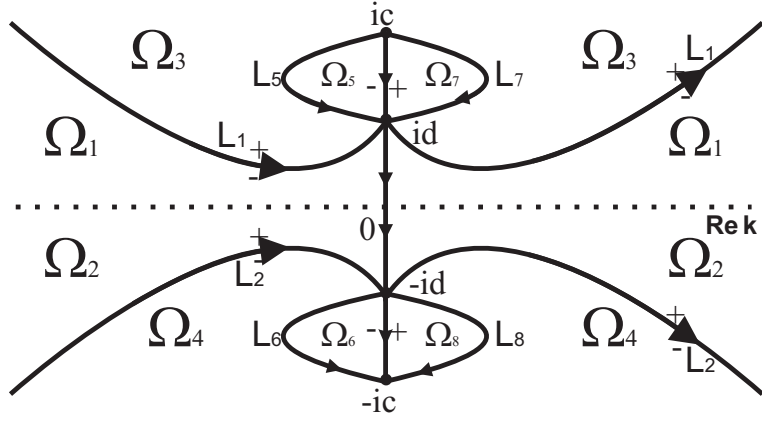


Figure 4: Contour $\Sigma^{(3)}$ for the RH problem for $M^{(3)}(\xi, t, k)$

$$\begin{aligned}
 J^{(3)}(\xi, t, k) &= \begin{pmatrix} 1 & \widehat{f}^{-1}(k)F^2(k, \xi)e^{-2itg(k, \xi)} \\ 0 & 1 \end{pmatrix}, & k \in L_7, \\
 &= \begin{pmatrix} 1 & -\widehat{f}^{-1}(k)F^2(k, \xi)e^{-2itg(k, \xi)} \\ 0 & 1 \end{pmatrix}, & k \in L_5
 \end{aligned}$$

$$\begin{aligned}
 J^{(3)}(\xi, t, k) &= \begin{pmatrix} 1 & 0 \\ -\widehat{f}^{-1}(\bar{k})F^{-2}(k, \xi)e^{2itg(k, \xi)} & 1 \end{pmatrix}, \quad k \in L_8, \\
 &= \begin{pmatrix} 1 & 0 \\ \widehat{f}^{-1}(\bar{k})F^{-2}(k, \xi)e^{2itg(k, \xi)} & 1 \end{pmatrix}, \quad k \in L_6 \\
 &= \begin{pmatrix} e^{itB_g(\xi)+i\Delta(\xi)} & 0 \\ 0 & e^{-itB_g(\xi)-i\Delta(\xi)} \end{pmatrix}, \quad k \in (-id, id) \\
 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k \in (ic, id) \cup (-id, -ic).
 \end{aligned} \tag{41}$$

The quantities B , Δ , τ in the formulation of the RHP 2 are defined by (5)-(7): the parameter d is determined by (8)-(9). The function F is determined by a scalar boundary value problem:

1. analyticity: F is analytic in $\mathbb{C} \setminus [ic, -ic]$;
2. jump conditions:

$$F_- F_+ = -if, \quad k \in [ic, id], \quad F_- F_+ = \frac{-i}{f(\bar{k})}, \quad k \in [-id, -ic], \quad \frac{F_+}{F_-} = e^{i\Delta}, \quad k \in [id, -id];$$

3. asymptotics: $F \rightarrow 1$ as $k \rightarrow \infty$.

The function \widehat{f} is the continuation of the function f (39) from the segment $[ic, -ic]$, using the last of the relations (37)

$$\widehat{f}(k) = \frac{-1}{a(k)b(\bar{k})}, \quad k \notin [ic, -ic] \quad \widehat{f}_+ = -\widehat{f}_- = f, \quad k \in [ic, -ic].$$

The function g appearing in (41) is analytic in $k \in \mathbb{C} \setminus [ic, -ic]$ and is given by

$$g(k, \xi) = 12 \int_{ic}^k \frac{(s^2 + \mu^2(\xi))\sqrt{s^2 + d^2(\xi)} \, ds}{\sqrt{s^2 + c^2}}.$$

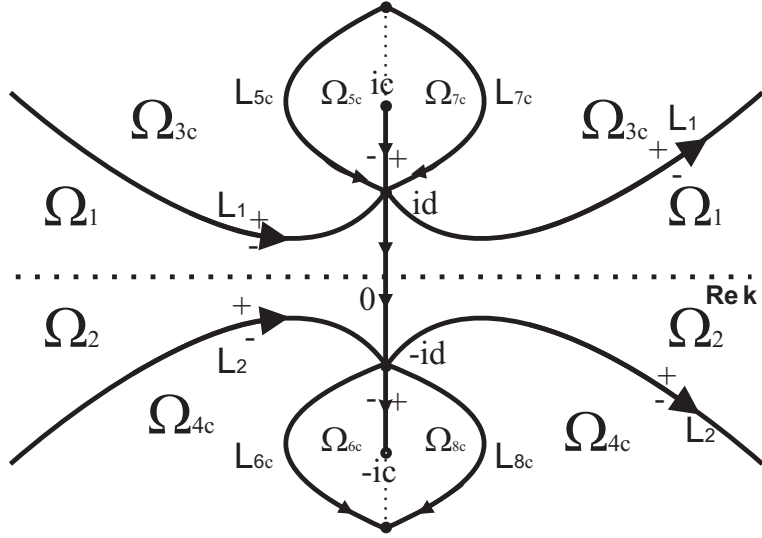


Figure 5: Contour $\Sigma^{(3)}$ for the RH problem for $M^{(3c)}(\xi, t, k)$

where the path of integration is arbitrarily chosen (notice that the integral on a loop surrounding the segment $[ic, -ic]$ is zero because the integrand is an even function, with the cuts of the integrand being $[id, ic] \cup [-id, -ic]$).

First we "move" the lenses L_7, L_5, L_8, L_6 in such a way, that they envelope the points $\pm ic$, respectively (see Figure 5). Further, we fold contours L_1, L_2 , to the intervals $(id, i(d - \tilde{r}))$, $(-i(d - \tilde{r}), -id)$, respectively. (Here $\tilde{r} = (c^2 - d^2)r$, and $r > 0$ is a fixed sufficiently small constant.)

This is achieved by the transformation

$$M^{(3c)}(\xi, t, k) = M^{(3)}(\xi, t, k)G^{(3c)}(\xi, t, k),$$

where

$$G^{(3c)}(\xi, t, k) = \begin{pmatrix} 1 & \hat{f}^{-1}(k)F^2(k, \xi)e^{-2itg(k, \xi)} \\ 0 & 1 \end{pmatrix}, \quad k \in (\Omega_{7c} \setminus \Omega_7) \cup (\Omega_{5c} \setminus \Omega_5),$$

$$G^{(3c)}(\xi, t, k) = \begin{pmatrix} 1 & 0 \\ -\overline{\hat{f}^{-1}(\bar{k})}F^{-2}(k, \xi)e^{2itg(k, \xi)} & 1 \end{pmatrix}, \quad k \in (\Omega_{8c} \setminus \Omega_8) \cup (\Omega_{6c} \setminus \Omega_6).$$

$$G^{(3c)} = \begin{pmatrix} 1 & 0 \\ r(k)F^{-2}(k, \xi)e^{2itg(k, \xi)} & 1 \end{pmatrix}, \quad k \in \Omega_1 \setminus \Omega_{1c},$$

$$G^{(3c)} = \begin{pmatrix} 1 & -\overline{r(\bar{k})}F^2(k, \xi)e^{-2itg(k, \xi)} \\ 0 & 1 \end{pmatrix}, \quad k \in \Omega_2 \setminus \Omega_{2c},$$

This transformation modifies the matrices in the jump relations on the intervals $(id, i(d - \tilde{r}))$, $(-i(d - \tilde{r}), -id)$, namely

$$J^{(3c)} = \begin{pmatrix} e^{itB+i\Delta} & 0 \\ \frac{f}{F_+F_-}e^{it(g_-+g_+)} & e^{-itB-i\Delta} \end{pmatrix}, \quad k \in (id, i(d - \tilde{r})),$$

$$J^{(3c)} = \begin{pmatrix} e^{itB+i\Delta} & -\overline{f(\bar{k})}F_+F_-e^{-it(g_-+g_+)} \\ 0 & e^{-itB-i\Delta} \end{pmatrix}, \quad k \in (-i(d - \tilde{r}), -id),$$

Under this transformation lines L_7, L_5 "move" up, so they do not intersect the segment $(ic, -ic)$, and the lines L_8, L_6 "moves" down. We have no additional jump across $(ic, i(c + \delta)) \cup (-i(c + \delta), -ic)$. The matrix $M^{(3c)}(\xi, t, k)$ still retains the same singularities near the points $\pm ic$ as $M^{(3)}$.

The phase function $g(k, \xi)$ has the following behaviour near the points $\text{id}(\xi)$, $-\text{id}(\xi)$:

$$g_{\pm}(k, \xi) = \mp \frac{B}{2} + \text{i}8(d^2 - \mu^2)\sqrt{2d}(c^2 - d^2) \left(\frac{k - \text{id}}{-\text{i}(c^2 - d^2)} \right)^{\frac{3}{2}} \times \\ \times \left(1 - \frac{3(9c^2d^2 - 5d^4 - 3d^2\mu^2 - c^2\mu^2)}{20d(d^2 - \mu^2)} \left(\frac{k - \text{id}}{-\text{i}(c^2 - d^2)} \right) + \mathcal{O} \left(\frac{k - \text{id}}{-\text{i}(c^2 - d^2)} \right)^2 \right), \\ k \rightarrow \text{id}(\xi),$$

$$g_{\pm}(k, \xi) = \mp \frac{B}{2} - \text{i}8(d^2 - \mu^2)\sqrt{2d}(c^2 - d^2) \left(\frac{k + \text{id}}{\text{i}(c^2 - d^2)} \right)^{\frac{3}{2}} \times \\ \times \left(1 - \frac{3(9c^2d^2 - 5d^4 - 3d^2\mu^2 - c^2\mu^2)}{20d(d^2 - \mu^2)} \left(\frac{k + \text{id}}{\text{i}(c^2 - d^2)} \right) + \mathcal{O} \left(\frac{k + \text{id}}{\text{i}(c^2 - d^2)} \right)^2 \right), \\ k \rightarrow -\text{id}(\xi).$$

The sign $+$ ($-$) is taken according to the boundary values as indicated in Fig. 4. In the region $\xi \rightarrow \frac{c^2}{3}$ we have $d \rightarrow c$, $\mu \rightarrow \frac{c}{\sqrt{3}}$. This suggests to introduce new local coordinates

$$g =: \mp \frac{B}{2} + \text{i}(c^2 - d^2)z^{3/2}, \quad tg = \mp \frac{tB}{2} + \text{i}\widehat{\tau}z^{3/2} =: \mp \frac{tB}{2} + \text{i}\frac{2}{3}\zeta^{3/2}, \quad |k - \text{id}| < r(c^2 - d^2), \\ g =: \mp \frac{B}{2} - \text{i}(c^2 - d^2)z_d^{3/2}, \quad tg = \mp \frac{tB}{2} - \text{i}\widehat{\tau}z_d^{3/2} =: \mp \frac{tB}{2} - \text{i}\frac{2}{3}\zeta_d^{3/2}, \quad |k + \text{id}| < r(c^2 - d^2),$$

where we have introduced the new fast variable

$$\widehat{\tau} := t(c^2 - d^2). \quad (42)$$

We also define the functions

$$\phi(k) := \begin{cases} \frac{F(k)e^{\frac{\text{i}tB}{2}}}{\sqrt{-\text{i}\widehat{f}(k)}}, & |k - \text{id}| < r(c^2 - d^2), \text{ Re } k > 0, \\ \frac{F(k)e^{-\frac{\text{i}tB}{2}}}{\sqrt{\text{i}\widehat{f}(k)}}, & |k - \text{id}| < r(c^2 - d^2), \text{ Re } k < 0, \end{cases} \\ \phi_d(k) := \begin{cases} F(k)e^{\frac{\text{i}tB}{2}}\sqrt{\text{i}\widehat{f}(k)}, & |k + \text{id}| < r(c^2 - d^2), \text{ Re } k > 0, \\ F(k)e^{-\frac{\text{i}tB}{2}}\sqrt{-\text{i}\widehat{f}(k)}, & |k + \text{id}| < r(c^2 - d^2), \text{ Re } k < 0, \end{cases}.$$

Inside the disks $|k \mp \text{id}| < r(c^2 - d^2)$ the jumps $J^{(3c)}$ can be written in the form

$$\phi^{\sigma_3} \begin{pmatrix} 1 & -\text{i}e^{\frac{2}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix} \phi^{-\sigma_3}, k \in L_7 \cup L_5, \quad \phi_+^{\sigma_3} \begin{pmatrix} 1 & 0 \\ \text{i}e^{-\frac{2}{3}\zeta^{3/2}} & 1 \end{pmatrix} \phi_-^{-\sigma_3}, k \in (\text{id}, \text{i}(d - \widetilde{r})), \\ \phi_+^{\sigma_3} \begin{pmatrix} 0 & \text{i} \\ \text{i} & 0 \end{pmatrix} \phi_-^{-\sigma_3}, k \in (\text{id}, \text{i}(d + \widetilde{r}))$$

and

$$\phi_d^{\sigma_3} \begin{pmatrix} 1 & 0 \\ -\text{i}e^{\frac{2}{3}\zeta_d^{3/2}} & 1 \end{pmatrix} \phi_d^{-\sigma_3}, k \in L_8 \cup L_6, \quad \phi_{d,+}^{\sigma_3} \begin{pmatrix} 1 & \text{i}e^{-\frac{2}{3}\zeta_d^{3/2}} \\ 0 & 1 \end{pmatrix} \phi_{d,-}^{-\sigma_3}, k \in (-\text{i}(d - \widetilde{r}), -\text{id}), \\ \phi_{d,+}^{\sigma_3} \begin{pmatrix} 0 & \text{i} \\ \text{i} & 0 \end{pmatrix} \phi_{d,-}^{-\sigma_3}, k \in (-\text{i}(d + \widetilde{r}), -\text{id}).$$

This leads to parametrices in terms of Airy functions; namely, we define

$$\Psi(\zeta) = \Psi_{Ai}(\zeta)e^{-\pi\text{i}\sigma_3/4}, \quad \Psi_d(\zeta_d) = \Psi(\zeta_d) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where Ψ_{Ai} is found in the Appendix ((113), (114)). Then

$$\begin{aligned}\Psi(\zeta) &= \zeta^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathcal{E}(\zeta) e^{\frac{2}{3}\zeta^{3/2}\sigma_3}, \quad \Psi_d(\zeta_d) = \zeta_d^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \mathcal{E}_d(\zeta_d) e^{-\frac{2}{3}\zeta_d^{3/2}\sigma_3}, \\ \mathcal{E}(\zeta) &= I + \begin{pmatrix} -\frac{1}{48} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{48} \end{pmatrix} \zeta^{-\frac{3}{2}} + \mathcal{O}(\zeta^{-3}), \quad \mathcal{E}_d(\zeta_d) = I + \begin{pmatrix} \frac{1}{48} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{48} \end{pmatrix} \zeta_d^{-\frac{3}{2}} + \mathcal{O}(\zeta_d^{-3}).\end{aligned}$$

Now we are ready to define the approximate solution to the RH problem for $M^{(3c)}$. Define

$$M_\infty = \begin{cases} M_{ell}(k), & |k \mp id| > r(c^2 - d^2), \\ B(k) \left(\frac{3\hat{\tau}}{2}\right)^{\sigma_3/6} \Psi(\zeta) \phi^{-\sigma_3}(k) e^{-\frac{2}{3}\zeta^{3/2}\sigma_3}, & |k - id| < r(c^2 - d^2), \\ B_d(k) \left(\frac{3\hat{\tau}}{2}\right)^{\sigma_3/6} \Psi_d(\zeta_d) \phi_d^{-\sigma_3}(k) e^{\frac{2}{3}\zeta_d^{3/2}\sigma_3}, & |k + id| < r(c^2 - d^2). \end{cases}$$

Here the analytic matrices $B(k), B_d(k)$ are to be determined in such a way that they minimize the jump on the boundaries of the disks for the error matrix $E = MM_{ell}^{-1}$. Hence we take

$$B(k) = M_{ell}(k) \phi^{\sigma_3}(k) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} z^{\sigma_3/4}, \quad B_d(k) = M_{ell}(k) \phi_d^{\sigma_3}(k) \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} z_d^{\sigma_3/4},$$

and it is straightforward to check that $B(k), B_d(k)$ are indeed have identity jumps inside the disks, and singularity at the points $k = \pm id$ (due to M_{ell}) has the order no more than $(k \mp id)^{-1/2}$, i.e. it is a removable singularity. Hence, these functions are indeed analytic inside the disks. The error matrix has a jump matrix J_E ($E_- = E_+ J_E$) as follows

$$\begin{aligned}J_E &= M_{ell} \phi^{\sigma_3} \mathcal{E} \phi^{-\sigma_3} M_{ell}^{-1} = B \left(\frac{3\hat{\tau}}{2}\right)^{\frac{\sigma_3}{6}} \zeta^{-\frac{\sigma_3}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathcal{E} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \zeta^{\frac{\sigma_3}{4}} \left(\frac{3\hat{\tau}}{2}\right)^{-\frac{\sigma_3}{6}} B^{-1}, \\ &|k - id| = \tilde{r},\end{aligned}\tag{43}$$

$$\begin{aligned}J_E &= M_{ell} \phi_d^{\sigma_3} \mathcal{E}_d \phi_d^{-\sigma_3} M_{ell}^{-1} = B_d \left(\frac{3\hat{\tau}}{2}\right)^{\frac{\sigma_3}{6}} \zeta_d^{-\frac{\sigma_3}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \mathcal{E}_d \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \zeta_d^{\frac{\sigma_3}{4}} \left(\frac{3\hat{\tau}}{2}\right)^{-\frac{\sigma_3}{6}} B_d^{-1}, \\ &|k + id| = \tilde{r}.\end{aligned}\tag{44}$$

The terms in the middle of the conjugations by B, B_d in (43), (44) admit the following estimate

$$\begin{aligned}\left(\frac{3\hat{\tau}}{2}\right)^{\frac{\sigma_3}{6}} \zeta^{-\frac{\sigma_3}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathcal{E} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \zeta^{\frac{\sigma_3}{4}} \left(\frac{3\hat{\tau}}{2}\right)^{-\frac{\sigma_3}{6}} &= \frac{2}{3\hat{\tau}} \begin{pmatrix} 0 & \frac{5}{48z^2} \\ \frac{-7}{48z} & 0 \end{pmatrix} + \mathcal{O}(\hat{\tau}^{-2}) \\ \left(\frac{3\hat{\tau}}{2}\right)^{\frac{\sigma_3}{6}} \zeta_d^{-\frac{\sigma_3}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \mathcal{E}_d \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \zeta_d^{\frac{\sigma_3}{4}} \left(\frac{3\hat{\tau}}{2}\right)^{-\frac{\sigma_3}{6}} &= \frac{2}{3\hat{\tau}} \begin{pmatrix} 0 & \frac{5}{48z_d^2} \\ \frac{-7}{48z_d} & 0 \end{pmatrix} + \mathcal{O}(\hat{\tau}^{-2}).\end{aligned}$$

Since the analytic matrices B, B_d are bounded for $\hat{\tau} \rightarrow \infty$ on the circles $|k \mp id| = r(c^2 - d^2)$, we get that uniformly for $k \in \Sigma^{(3c)} \cup \{k : |k \mp ic| = r(c^2 - d^2)\}$ we have

$$J_E(k) = I + \mathcal{O}(\hat{\tau}^{-1}), \text{ and hence } E(k) = I + \mathcal{O}\left(\frac{1}{\hat{\tau}k}\right) \text{ for } k \rightarrow \infty.$$

Hence, the solution to the MKdV equation $q(x, t)$ is estimated by the function $q_{el} = \lim_{k \rightarrow \infty} (2ik M_{el}(x, t; k))$ with the accuracy $\hat{\tau}^{-1}$:

$$q(x, t) = q_{el}(x, t) + \mathcal{O}(\hat{\tau}^{-1}).$$

To achieve the statement of Theorem 2 it remains to estimate $\hat{\tau}^{-1}$ in terms of t , which requires to unravel its definition from (42) and (8), (9). Denote $\eta = 1 - \frac{d(\xi)}{c}$, $v = 1 - \frac{3\xi}{c^2}$: then for $d, v \rightarrow 0$ we have $\eta = \frac{v}{\ln \frac{1}{v}}(1 + o(1))$

([32]). Hence, on the curve $\xi = \frac{c^2}{3} - \frac{\beta t^\sigma \ln t}{t}$, $\sigma \in (0, 1)$, $\beta > 0$, we have $\hat{\tau} = \frac{2c^2\beta}{1-\sigma} t^\sigma (1 + \mathcal{O}(1))$. Therefore, for fixed small $\varepsilon > 0$ we have that

$$\begin{cases} q(x, t) = q_{el}(x, t) + \mathcal{O}(t^{-1}) & \text{for } -\frac{c^2}{2} + \varepsilon < \frac{x}{12t} < \frac{c^2}{3} - \varepsilon, \\ q(x, t) = q_{el}(x, t) + \mathcal{O}\left(\frac{1-\sigma}{\beta t^\sigma}\right) & \text{for } \frac{c^2}{3} - \varepsilon < \frac{x}{12t} < \frac{c^2}{3} - \frac{\beta t^\sigma \ln t}{t}, \quad \sigma \in (0, 1). \end{cases}$$

This completes the proof of Theorem 2. ■

We now prove Corollary 1.

Proof. [Proof of Corollary 1.] Denote

$$z := \frac{tB + \Delta}{\pi}, \quad \tau^* = \frac{4\pi^2}{\tau}, \quad \eta := 1 - \frac{d}{c}, \quad v := 1 - \frac{3\xi}{c^2}. \quad (45)$$

Then (see [32], p.16-17)

$$q_{el}(x, t) = \frac{2c}{\cosh\left(\frac{\tau^*}{4}(z - 2n - 1)\right)} + \mathcal{O}\left(e^{\frac{\tau^*}{4}}\right) \quad \text{for } 2n \leq z \leq 2n + 2. \quad (46)$$

Furthermore, ([32] (B.43), (B.54)),

$$\eta = e^{\frac{\tau^*}{4}} + \mathcal{O}\left(e^{\frac{\tau^*}{2}}\right), \quad v = \eta \ln \frac{8e}{\eta} + \mathcal{O}(\eta^2 \ln^2 \eta).$$

Consequently, on a curve

$$\xi = \frac{c^2}{3} - \frac{\beta t^\sigma \ln t}{12t}$$

we have

$$v = \frac{\beta t^\sigma \ln t}{4c^2 t} \quad \text{and hence} \quad \eta \asymp t^{\sigma-1}, \quad e^{\frac{\tau^*}{4}} = \mathcal{O}(t^{\sigma-1}), \quad (47)$$

and we can substitute the \mathcal{O} estimate (47) in (46). We also have the expansions ([32])

$$\frac{\tau^*}{4} = \ln \frac{\eta}{8} + \frac{\eta}{2} + \frac{3\eta^2}{16} + \mathcal{O}(\eta^3), \quad \frac{4}{\tau^*} = \frac{1}{\ln \frac{\eta}{8}} + \mathcal{O}\left(\frac{\eta}{\ln^2 \eta}\right), \quad (48)$$

$$\frac{\Delta}{\pi} + \frac{1}{2} = \frac{\ln \frac{4}{(h^*)^2}}{\ln \frac{8}{\eta}} + \mathcal{O}(\eta), \quad (49)$$

$$\frac{B}{\pi} = 8c^3 \eta \left(1 + \sum_{j=1}^{M-1} \eta^j P_j(\eta) + \mathcal{O}(\eta^M \ln^M \eta) \right), \quad M \geq 2,$$

$$v = \eta \ln \frac{8e}{\eta} + \sum_{j=2}^M \eta^j Q_j(\eta) + \mathcal{O}(\eta^{M+1} \ln^{M+1} \eta), \quad M \geq 2.$$

The polynomials P_j, Q_j are described after (11), (12). Now, expressing directly the argument of cosh in (46) in terms of x, t (i.e. in terms of v, t) does not yet lead us to the statement of the corollary. An indication of this is that the n -dependence of the phases in (15) is much more involved than in (46). But the real issue is that, due to the inversion of the Lambert equation, η is invertible in terms of v as a series in $\frac{\ln \ln v}{\ln v}$, i.e. it has a very slow convergent rate. Hence, the leading term of τ^* (see 48), i.e. $\ln \eta$, being multiplied by the leading term from tB (see (11)), i.e. $t\eta \asymp t^\sigma$, gives us an expression of the type

$$t^\sigma \sum_{j=0}^{\infty} b_j \left(\frac{\ln \ln v}{\ln v} \right)^j,$$

and since $\ln v \asymp \ln t$, we can not truncate this series. To overcome this issue, let us notice that

$$\frac{2c}{\cosh\left(\frac{\tau^*}{4}(z-2n-1)\right)} =: \frac{2c}{\cosh a} \in [Kt^{-\delta}, 2c]$$

for some $K > 0, \delta > 0$ if and only if

$$a := \frac{\tau^*}{4}(z-2n-1) \in [-\delta \ln t - C, \delta \ln t + C]$$

for some $C > 0$. So, we treat the argument of \cosh only in a domain around the peak, where $\frac{2c}{\cosh a}$ is at least of the order $\asymp t^{-\delta}$, since in the remaining part it is $\mathcal{O}(t^{-\delta})$. We have

$$a = \frac{\tau^*}{4} \left(\frac{tB}{\pi} + \frac{\Delta}{\pi} + \frac{1}{2} - 2n - \frac{3}{2} \right), \quad \text{or} \quad 2n + \frac{3}{2} + \frac{4a}{\tau^*} = \frac{tB}{\pi} + \frac{\Delta}{\pi} + \frac{1}{2},$$

and substituting here (49), (11), (48), and expressing further the leading term of tB , i.e. $8c^3 t \eta$, we obtain

$$8c^3 t \eta = 2n + \frac{3}{2} - \frac{\ln \frac{4}{(h^*)^2} + a}{\ln \frac{8}{\eta}} - 8c^3 t \eta \sum_{j=1}^{M-1} \eta^j P_j(\eta) + \mathcal{O}(\eta) + \mathcal{O}\left(t \eta^{M+1} \ln^M \eta\right). \quad (50)$$

Now let us find an expression for $-2c(x-4c^2 t) = 8c^3 vt$. Substituting (12), and expressing then subsequently $8c^3 t \eta$, η and $\ln \frac{8e}{\eta}$ by the l.h.s. of (50), after some computation we obtain

$$\begin{aligned} 8c^3 vt &= \left(2n + \frac{3}{2}\right) \ln \frac{32c^3 e t}{n} - \ln \frac{4}{(h^*)^2} - a - \frac{\ln \frac{4}{(h^*)^2} + a}{\ln \frac{8}{\eta}} + \mathcal{O}(\eta \ln \eta) + \\ &+ 8c^3 t \left[\sum_{j=2}^M \eta^j Q_j(\eta) - \eta \ln \frac{8e}{\eta} \sum_{j=1}^{M-1} \eta^j P_j(\eta) + \mathcal{O}(\eta^{M+1} \ln^{M+1} \eta) \right] - \\ &- 2n \left(1 + \frac{3}{4n}\right) \ln \left[1 + \frac{3}{4n} - \frac{\ln \frac{4}{(h^*)^2} + a}{2n \ln \frac{8}{\eta}} - \frac{4c^3 t \eta}{n} \sum_{j=1}^{M-1} \eta^j P_j(\eta) + \mathcal{O}\left(\frac{t \eta^{M+1} \ln^M \eta}{n}\right) + \mathcal{O}(t^{-1}) \right]. \end{aligned}$$

Taking, in this formula $M = 1, 2, 3$, and expressing a as the third summand in the r.h.s., we obtain subsequently

$$a = -8c^3 vt + \left(2n + \frac{3}{2}\right) \ln \frac{32c^3 t}{n} + 2n - \ln \frac{4}{(h^*)^2} + \mathcal{O}(t^{2\sigma-1} \ln^2 t) + \mathcal{O}(t^{-\sigma}),$$

$$a = -8c^3 vt + \left(2n + \frac{3}{2}\right) \ln \frac{32c^3 t}{n} + 2n - \ln \frac{4}{(h^*)^2} + \frac{n^2}{4c^3 t} \left(-2 + \ln \frac{32c^3 t}{n}\right) + \mathcal{O}(t^{3\sigma-2} \ln^3 t) + \mathcal{O}(t^{\sigma-1} \ln^2 t) + \mathcal{O}(t^{-\sigma}).$$

$$\begin{aligned} a &= -8c^3 vt + \left(2n + \frac{3}{2}\right) \ln \frac{32c^3 t}{n} + 2n - \ln \frac{4}{(h^*)^2} + \frac{n^2 \ln \frac{8}{e^2 \eta}}{4c^3 t} + \frac{n^3}{8c^6 t^2} \left(Q_3 - \ln \frac{8}{\eta} P_2 - \ln \frac{8}{e^2 \eta} P_1 + \frac{1}{2} P_1^2\right) + \\ &+ \mathcal{O}(t^{4\sigma-3} \ln^4 t) + \mathcal{O}(t^{\sigma-1} \ln^2 t) + \mathcal{O}(t^{-\sigma}), \end{aligned}$$

and, after further computations and substitutions of η using (50),

$$\begin{aligned} a &= -8c^3 vt + \left(2n + \frac{3}{2}\right) \ln \frac{32c^3 t}{n} + 2n - \ln \frac{4}{(h^*)^2} + \\ &+ \frac{n^2 \ln \frac{32c^3 t}{e^2 n}}{4c^3 t} + \frac{n^3}{128c^6 t^2} \left(-39 + 63 \ln 2 - 18 \ln^2 2 - (21 - 12 \ln 2) \ln \frac{n}{4c^3 t} - 2 \ln^2 \frac{n}{4c^3 t}\right) + \\ &+ \mathcal{O}(t^{4\sigma-3} \ln^4 t) + \mathcal{O}(t^{\sigma-1} \ln^2 t) + \mathcal{O}(t^{-\sigma}). \end{aligned} \quad (51)$$

This gives us the statement of Corollary 1. ■

In Sec.5 we give an alternative proof of Corollary 1 which has the advantage that it is also valid up to $\sigma \in [0, \frac{M}{M+1})$. The number of the soliton to which the point (x, t) belongs at a particular time will be expressed by the solution (107) whereas in this section this number was determined by $\frac{z}{2}$ (formula 45). In order to compare z and γ we need the following lemma:

Lemma 3.1. *The quantity z in (45) satisfies the following asymptotic bound:*

$$8c^3vt + \left(z + \frac{1}{2}\right) \left(\ln \frac{z}{2t} - (\widehat{Q} + 1)\right) = \mathcal{O}(1), \quad t \rightarrow \infty$$

where \widehat{Q} satisfies

$$\widehat{Q} + 1 = \ln \frac{B}{2\pi} + \frac{8c^3v\pi}{B} + \mathcal{O}\left(\frac{1}{t\eta}\right).$$

Proof. The proof is straightforward using (12), (11), (49). We have

$$\begin{aligned} & 8c^3vt + \left(\frac{tB}{\pi} + \frac{\Delta}{\pi} + \frac{1}{2}\right) \left(\ln \frac{B}{2\pi} + \ln \left(1 + \frac{\Delta}{tB}\right) - (\widehat{Q} + 1)\right) = \\ & = 8c^3vt + \frac{tB}{\pi} \left(\ln \frac{B}{2\pi} - (\widehat{Q} + 1)\right) + \underbrace{\left(\frac{\Delta}{\pi} + \frac{1}{2}\right) \left(\ln \frac{B}{2\pi} + \ln \left(1 + \frac{\Delta}{tB}\right) - (\widehat{Q} + 1)\right)}_{\mathcal{O}(1)} + \underbrace{\frac{tB}{\pi} \ln \left(1 + \frac{\Delta}{tB}\right)}_{\mathcal{O}(1)}. \end{aligned}$$

The underbraced terms are of the order $\mathcal{O}(1)$. This proves lemma 3.1. For further use let us notice also that

$$\begin{aligned} \ln \frac{B}{2\pi} + \frac{8c^3v\pi}{B} &= (1 + \ln(32c^3)) + \left(-1 + \frac{1}{2} \ln \frac{8}{\eta}\right) \eta + \\ &+ \frac{1}{16} (9 - 21 \ln 2 + 18 \ln^2 2 + (7 - 12 \ln 2) \ln \eta + 2 \ln^2 \eta) \eta^2 + \mathcal{O}(\eta^3 \ln^4 \eta). \end{aligned}$$

Substituting here η in view of (50), we obtain

$$\ln \frac{B}{2\pi} + \frac{8c^3v\pi}{B} = (1 + \ln(32c^3)) + \left(-1 + \frac{3}{2} \ln 2 - \frac{1}{2} \ln \frac{n}{4c^3t}\right) \frac{n}{4c^3t} + \mathcal{O}(t^{2\sigma-2}).$$

■

4 Logarithmic domain D_{2a}

This is the region where $4c^2t - C \ln t < x < 4c^2t$, $C > 0$. In this domain we deal with a finite number, n , of asymptotic solitons and our goal is to prove Thm 4. We start from the original Riemann-Hilbert problem 1 (see also [30]). In this section we consider the regime $\xi = \frac{c^2}{3} - \frac{\rho}{12} \frac{\ln t}{t}$, for some fixed positive $\rho > 0$. We perform a first transformation of the RH problem which removes jump from the real line; fix a sufficiently small $r > 0$ and introduce the lines

$$L_1 = \mathbb{R} + i(c - r), \quad L_2 = \mathbb{R} - i(c - r),$$

with the natural orientation of \mathbb{R} . Denote by Ω_1 the domain bounded by \mathbb{R} and L_1 , the other domain bounded by L_1 is D_3 , the domain bounded by \mathbb{R} and L_2 denote by Ω_2 , and the other domain bounded by L_2 denote by Ω_4 . Introduce the following transformation of the RH problem:

$$M^{(1)} := M \begin{cases} \begin{pmatrix} 1 & 0 \\ -r(k)e^{2it\theta} & 1 \end{pmatrix}, & k \in \Omega_1, \\ \begin{pmatrix} 1 & r(\bar{k})e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & k \in \Omega_2, \\ I, & k \in \Omega_3 \cup \Omega_4. \end{cases} \quad (52)$$

Then the jump $M_-^{(1)} = M_+^{(1)} J^{(1)}$ is

$$J^{(1)} = \begin{cases} I, & k \in \mathbb{R} \cup (i(c - r), -i(c - r)), \\ J, & k \in (ic, i(c - r)) \cup (-i(c - r), -ic), \\ \begin{pmatrix} 1 & 0 \\ -r(k)e^{2it\theta} & 1 \end{pmatrix}, & k \in L_1, \\ \begin{pmatrix} 1 & -r(\bar{k})e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & k \in L_2. \end{cases} \quad (53)$$

We see that the jumps are exponentially close to the identity matrix everywhere except for the intervals $\pm(ic, i(c-r))$. We then define a local change of spectral variable k on the interval $\pm(ic, i(c-r))$ in terms of new variables y, y_d , respectively:

$$k = ic - iy, \quad k = -ic + iy_d. \quad (54)$$

Then we set

$$\begin{aligned} 2it\theta(k, \xi) &= 2c\rho \ln t - t \left[\left(16c^2 + \frac{2\rho \ln t}{t} \right) y - 24cy^2 + 8y^3 \right] =: 2c\rho \ln t - tz =: 2c\rho \ln t - \zeta, \\ -2it\theta(k, \xi) &= 2c\rho \ln t - t \left[\left(16c^2 + \frac{2\rho \ln t}{t} \right) y_d - 24cy_d^2 + 8y_d^3 \right] =: 2c\rho \ln t - tz_d =: 2c\rho \ln t - \zeta_d, \end{aligned}$$

where we denoted

$$\frac{\zeta}{t} := z := \left[\left(16c^2 + \frac{2\rho \ln t}{t} \right) y - 24cy^2 + 8y^3 \right]; \quad \frac{\zeta_d}{t} := z_d := \left[\left(16c^2 + \frac{2\rho \ln t}{t} \right) y_d - 24cy_d^2 + 8y_d^3 \right];$$

$$\begin{aligned} f(k) &= \frac{2\sqrt{2}i}{(h^*)^2\sqrt{c}} \sqrt{y}(1 + \mathcal{O}(\sqrt{y})) = \frac{2\sqrt{2}i}{(h^*)^2\sqrt{c}} \sqrt{\frac{y}{z}}(1 + \mathcal{O}(\sqrt{y})) \sqrt{z} =: i\phi(k, t)\sqrt{z}, \\ -\overline{f(\bar{k})} &= \frac{2\sqrt{2}i}{(h^*)^2\sqrt{c}} \sqrt{y_d}(1 + \mathcal{O}(\sqrt{y_d})) = \frac{2\sqrt{2}i}{(h^*)^2\sqrt{c}} \sqrt{\frac{y_d}{z_d}}(1 + \mathcal{O}(\sqrt{y_d})) \sqrt{z_d} =: i\phi_d(k, t)\sqrt{z_d}. \end{aligned}$$

Here $\phi(k, t)$, $\phi_d(k, t)$ are analytic in a neighborhood of $k = \pm ic$, respectively, where they are separated both from 0 and ∞ uniformly with respect to $t \geq 1$.

Remark 4.1. In the case of pure step initial function (3) we have

$$\tilde{f}(k) = \frac{2i}{c} \sqrt{k^2 + c^2} = \frac{2i}{c} \sqrt{y(2c - y)}, \quad h^* = 1$$

In terms of the z, z_d variables, the jump matrix $J^{(1)}$ on the intervals $\pm(ic, i(c-r))$ is expressed as

$$J^{(1)} = \begin{pmatrix} 1 & 0 \\ f(k)e^{2it\theta(k, \xi)} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ i\phi t^{2c\rho - \frac{1}{2}} \sqrt{zt} e^{-tz} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ i\phi t^{2\gamma} \sqrt{\zeta} e^{-\zeta} & 1 \end{pmatrix}, k \in (ic, i(c-r)), \quad (55)$$

$$J^{(1)} = \begin{pmatrix} 1 & -\overline{f(\bar{k})}e^{-2it\theta(k, \xi)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & i\phi_d t^{2c\rho - \frac{1}{2}} \sqrt{z_d t} e^{-tz_d} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & i\phi_d t^{2\gamma} \sqrt{\zeta_d} e^{-\zeta_d} \\ 0 & 1 \end{pmatrix}, k \in (-i(c-r), -ic), \quad (56)$$

$$\gamma := c\rho - \frac{1}{4}. \quad (57)$$

4.1 Generalized Laguerre polynomials with index $\frac{1}{2}$.

Denote $p_n(\zeta) = L_n^{(1/2)}(\zeta) = \frac{(-1)^n}{n!} \zeta^n + \dots$, $\pi_n(\zeta) = (-1)^n n! p_n(\zeta) = \zeta^n + \dots$

$$\int_0^{+\infty} \zeta^{1/2} e^{-\zeta} p_n(\zeta) p_m(\zeta) d\zeta = \frac{\Gamma(n + \frac{3}{2})}{n!} \delta_{m,n}, \quad \int_0^{+\infty} \zeta^{1/2} e^{-\zeta} \pi_n(\zeta) \pi_m(\zeta) d\zeta = \Gamma(n + \frac{3}{2}) n! \delta_{m,n}.$$

The generalized Laguerre polynomials with index $\frac{1}{2}$ and degree n solve a RHP of the form

$$\begin{aligned} L_-(\zeta) &= L_+(\zeta) J_L(\zeta), \quad \zeta \in \mathbb{R}_+, \\ J_L(\zeta) &= \begin{pmatrix} 1 & 0 \\ -\sqrt{\zeta} e^{-\zeta} & 1 \end{pmatrix}, \\ L(\zeta) &= (\mathbf{1} + \mathcal{O}(\zeta^{-1})) \zeta^{-n\sigma_3}, \quad \zeta \rightarrow \infty. \end{aligned} \quad (58)$$

and the solution is written as follows: for $n \geq 1$

$$L(\zeta) = \begin{pmatrix} \frac{-2\pi i}{\Gamma(n+\frac{1}{2})\Gamma(n)} \frac{1}{2\pi i} \int_0^{+\infty} \frac{\sqrt{s} e^{-s} \pi_{n-1}(s) ds}{s - \zeta} & \frac{-2\pi i}{\Gamma(n+\frac{1}{2})\Gamma(n)} \pi_{n-1}(\zeta) \\ \frac{1}{2\pi i} \int_0^{+\infty} \frac{\sqrt{s} e^{-s} \pi_n(s) ds}{s - \zeta} & \pi_n(\zeta) \end{pmatrix}, \quad (59)$$

and for $n = 0$

$$L(\zeta) = \begin{pmatrix} 1 & 0 \\ \frac{1}{2\pi i} \int_0^{+\infty} \frac{\sqrt{s} e^{-s} ds}{s - \zeta} & 1 \end{pmatrix}.$$

Furthermore, the matrix function

$$L_d(\zeta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L(\zeta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

solves a RHP of the form

$$\begin{aligned} L_{d,-}(\zeta) &= L_{d,+}(\zeta) J_{L_d}(\zeta), \quad \zeta \in (+\infty, 0) \text{ (the orientation is from } +\infty \text{ to } 0), \\ J_{L_d}(\zeta) &= \begin{pmatrix} 1 & \sqrt{\zeta_d} e^{-\zeta_d} \\ 0 & 1 \end{pmatrix}, \\ L_d(\zeta_d) &= (\mathbf{1} + \mathcal{O}(\zeta_d^{-1})) \zeta_d^{n\sigma_3}, \quad \zeta_n \rightarrow \infty. \end{aligned} \quad (60)$$

To show the relation with our RH problem for $M^{(1)}$, consider the functions

$$L^{(1)} = (-i\phi t^{2\gamma})^{-\sigma_3/2} L(-i\phi t^{2\gamma})^{\sigma_3/2}, \quad L_d^{(1)} = (i\phi_d t^{2\gamma})^{\sigma_3/2} L_d(i\phi_d t^{2\gamma})^{-\sigma_3/2};$$

they have the following jumps on $\zeta \in (0, +\infty)$, $\zeta_d \in (+\infty, 0)$, respectively (compare with (55), (56)):

$$(L_+^{(1)})^{-1} L_-^{(1)} = \begin{pmatrix} 1 & 0 \\ i\phi t^{2\gamma} \sqrt{\zeta} e^{-\zeta} & 1 \end{pmatrix}, \quad (L_{d,+}^{(1)})^{-1} L_{d,-}^{(1)} = \begin{pmatrix} 1 & i\phi_d t^{2\gamma} \sqrt{\zeta_d} e^{-\zeta_d} \\ 0 & 1 \end{pmatrix}.$$

Further, developing up to ζ^{-1} , ζ_d^{-1} term in the asymptotics (58), (60) of L , L_d as $\zeta, \zeta_d \rightarrow \infty$, we obtain

$$L(\zeta) = \left[\begin{pmatrix} 1 + \frac{n^2 + \frac{n}{2}}{\zeta} & \frac{-2\pi i}{\Gamma(n+\frac{1}{2})} \frac{n!}{\zeta} \\ \frac{-n!}{2\pi i} \frac{\Gamma(n+\frac{3}{2})}{\zeta} & 1 - \frac{n^2 + \frac{n}{2}}{\zeta} \end{pmatrix} + \mathcal{O}(\zeta^{-2}) \right] \zeta^{-n\sigma_3}, \quad \zeta \rightarrow \infty, \quad (61)$$

$$L_d(\zeta) = \left[\begin{pmatrix} 1 - \frac{n^2 + \frac{n}{2}}{\zeta_d} & \frac{-n!}{2\pi i} \frac{\Gamma(n+\frac{3}{2})}{\zeta_d} \\ \frac{-2\pi i}{\Gamma(n+\frac{1}{2})} \frac{n!}{\zeta_d} & 1 + \frac{n^2 + \frac{n}{2}}{\zeta_d} \end{pmatrix} + \mathcal{O}(\zeta_d^{-2}) \right] \zeta_d^{n\sigma_3}, \quad \zeta_d \rightarrow \infty. \quad (62)$$

The formulas (61), (62) include also the case $n = 0$.

4.2 First approximation of $M^{(1)}$.

Let $\alpha \in \mathbb{N}$ be an integer to be determined later. We look for an approximation of $M^{(1)}$ of the form

$$M_\infty = \begin{cases} \left(\frac{k-ic}{k+ic} \right)^{\alpha\sigma_3}, & |k \mp ic| > r, \\ BL(-i\phi)^{\sigma_3/2} t^{\gamma\sigma_3}, & |k - ic| < r, \\ B_d L_d(i\phi_d)^{-\sigma_3/2} t^{-\gamma\sigma_3}, & |k + ic| < r, \end{cases} \quad (63)$$

In the process of the construction we will also determine the matrix-valued functions B , B_d analytic inside the disks $|k \mp ic| < r$, respectively. The driving logic is that of minimizing the distance of the error matrix $E = MM_\infty^{-1}$ from the identity matrix. To this end we inspect its jump $J_E = (E_+)^{-1} E_- = M_{\infty,+} J M_{\infty,-}^{-1}$.

On the interval $(ic, i(c-r))$ the jump is

$$J_E = BL_+(-i\phi)^{\sigma_3/2} t^{\gamma\sigma_3} \begin{pmatrix} 1 & 0 \\ i\phi t^{2\gamma} \sqrt{\zeta} e^{-\zeta} & 1 \end{pmatrix} t^{-\gamma\sigma_3} (-i\phi)^{-\sigma_3/2} L_-^{-1} B^{-1} = I.$$

Similarly, on $(-i(c-r), -ic)$ the jump $J_E = I$. The jump J_E on the disks

$$\partial C = \{k : |k - ic| = r\}, \quad \partial C_d = \{k : |k + ic| = r\}, \quad (\text{we take counterclockwise orientation})$$

is

$$J_E = BL(-i\phi)^{\sigma_3/2} t^{\gamma\sigma_3} \left(\frac{k - ic}{k + ic} \right)^{-\alpha\sigma_3}, \quad k \in \partial C, \quad J_E = B_d L_d (i\phi_d)^{-\sigma_3/2} t^{-\gamma\sigma_3} \left(\frac{k - ic}{k + ic} \right)^{-\alpha\sigma_3}, \quad k \in \partial C_d.$$

To have J_E close to I , taking into account the asymptotics (58), (60) of L , L_d on the circles (we have $\zeta = zt \rightarrow \infty$ as $t \rightarrow \infty$ when $k \in \partial C$), we take

$$B = \left(\frac{k - ic}{k + ic} \right)^{\alpha\sigma_3} z^{n\sigma_3} (-i\phi)^{-\sigma_3/2} t^{(n-\gamma)\sigma_3}, \quad B_d = \left(\frac{k - ic}{k + ic} \right)^{\alpha\sigma_3} z_d^{-n\sigma_3} (i\phi_d)^{\sigma_3/2} t^{(\gamma-n)\sigma_3}. \quad (64)$$

In order not to have poles at $z = 0$, $z_d = 0$, we henceforth choose $\alpha = -n$. Then

$$J_E = \left(\frac{k + ic}{k - ic} \cdot z \right)^{n\sigma_3} (-i\phi)^{-\frac{\sigma_3}{2}} t^{(n-\gamma)\sigma_3} \begin{pmatrix} 1 + \mathcal{O}(\frac{1}{zt}) & \mathcal{O}(\frac{1}{zt}) \\ \mathcal{O}(\frac{1}{zt}) & 1 + \mathcal{O}(\frac{1}{zt}) \end{pmatrix} (-i\phi)^{\frac{\sigma_3}{2}} t^{(\gamma-n)\sigma_3} \left(\frac{k - ic}{k + ic} \cdot \frac{1}{z} \right)^{n\sigma_3}, \quad k \in \partial C, \quad (65)$$

$$J_E = \left(\frac{k + ic}{k - ic} \cdot \frac{1}{z_d} \right)^{n\sigma_3} (i\phi_d)^{\frac{\sigma_3}{2}} t^{(\gamma-n)\sigma_3} \begin{pmatrix} 1 + \mathcal{O}(\frac{1}{z_d t}) & \mathcal{O}(\frac{1}{z_d t}) \\ \mathcal{O}(\frac{1}{z_d t}) & 1 + \mathcal{O}(\frac{1}{z_d t}) \end{pmatrix} (i\phi_d)^{-\frac{\sigma_3}{2}} t^{(n-\gamma)\sigma_3} \left(\frac{k - ic}{k + ic} \cdot z_d \right)^{n\sigma_3}, \quad k \in \partial C_d, \quad (66)$$

and hence

$$J_E = \begin{pmatrix} 1 + \mathcal{O}(t^{-1}) & \mathcal{O}(t^{-2\gamma+2n-1}) \\ \mathcal{O}(t^{2\gamma-2n-1}) & 1 + \mathcal{O}(t^{-1}) \end{pmatrix}, \quad k \in \partial C, \quad J_E = \begin{pmatrix} 1 + \mathcal{O}(t^{-1}) & \mathcal{O}(t^{2\gamma-2n-1}) \\ \mathcal{O}(t^{-2\gamma+2n-1}) & 1 + \mathcal{O}(t^{-1}) \end{pmatrix}, \quad k \in \partial C_d. \quad (67)$$

For every value of γ which is not half-integer we can choose n such that $\gamma - \frac{1}{2} < n < \gamma + \frac{1}{2}$, and both off-diagonal terms in the r.h.s. of the first expression in (67) will be vanishing. However, for a half-integer $\gamma = m + \frac{1}{2}$ we cannot make both off-diagonal terms in (67) small: indeed, if we choose $n = m$, then the $(1, 2)$ entry is of the order $\mathcal{O}(t^{-2})$, but the $(2, 1)$ entry is just $\mathcal{O}(1)$; vice versa, for $n = m + 1$, then $(2, 1)$ entry is $\mathcal{O}(t^{-2})$, but the $(1, 2)$ entry is just $\mathcal{O}(1)$. As we shall see below, this indicates the presence of asymptotic solitons, which correspond to half-integer γ . Away from the asymptotic solitons, the solution of MKdV equation is asymptotically vanishing.

To capture the asymptotic solitons, we are lead to make a further correction in the approximate solution; this is accomplished in the next section.

4.2.1 Refined approximation of $M^{(1)}$

In order to have more freedom in choosing n in (67), the idea is that of “removing” the $\frac{1}{\zeta}$ term in $(2, 1)$ entry in asymptotics (61) of L , and in $(1, 2)$ entry in asymptotics (62) of L_d at ∞ , so that they will start from ζ^{-2} , ζ_d^{-2} ,

$$\begin{pmatrix} 1 & 0 \\ \frac{R_1}{\zeta} & 1 \end{pmatrix} L = \begin{pmatrix} 1 + \mathcal{O}(\frac{1}{\zeta}) & \mathcal{O}(\frac{1}{\zeta}) \\ \mathcal{O}(\frac{1}{\zeta^2}) & 1 + \mathcal{O}(\frac{1}{\zeta}) \end{pmatrix}, \quad \begin{pmatrix} 1 & \frac{R_1}{\zeta_d} \\ 0 & 1 \end{pmatrix} L_d = \begin{pmatrix} 1 + \mathcal{O}(\frac{1}{\zeta_d}) & \mathcal{O}(\frac{1}{\zeta_d^2}) \\ \mathcal{O}(\frac{1}{\zeta_d}) & 1 + \mathcal{O}(\frac{1}{\zeta_d}) \end{pmatrix},$$

where

$$R_1 = \frac{n! \Gamma(n + \frac{3}{2})}{2\pi i}.$$

However, this will bring poles at $k = \pm ic$ of the approximate solution and to compensate for this issue we multiply all M_∞ by an appropriate meromorphic matrix function from the left, which will remove these poles. In concrete, the above idea requires to define

$$G = I + \frac{A}{k - ic} + \frac{\tilde{A}}{k + ic} \quad (68)$$

$$M_{\infty}^{(1)} = \begin{cases} G \left(\frac{k-ic}{k+ic} \right)^{-n\sigma_3}, & |k \mp ic| > r, \\ GB \begin{pmatrix} 1 & 0 \\ \frac{R_1}{\zeta} & 1 \end{pmatrix} L(-i\phi)^{\sigma_3/2} t^{\gamma\sigma_3}, & |k-ic| < r, \\ GB_d \begin{pmatrix} 1 & \frac{R_1}{\zeta_d} \\ 0 & 1 \end{pmatrix} L_d(i\phi_d)^{-\sigma_3/2} t^{-\gamma\sigma_3}, & |k+ic| < r, \end{cases} \quad (69)$$

where B, B_d are as in (64) with $\alpha = -n$. The new error matrix

$$E^{(1)} := M^{(1)} \left(M_{\infty}^{(1)} \right)^{-1}$$

has the jump

$$J_{E^{(1)}} = (E_+^{(1)})^{-1} E_-^{(1)} = M_{\infty,+}^{(1)} J^{(1)} \left(M_{\infty,-}^{(1)} \right)^{-1};$$

in the intervals $\pm(ic, i(c-r))$ the jump is $J_{E^{(1)}} = I$, and on the circles $k \in \partial C, k \in \partial C_d$ it is of the form

$$J_{E^{(1)}} = G \begin{pmatrix} 1 + O(t^{-1}) & O(t^{-2\gamma+2n-1}) \\ O(t^{2\gamma-2n-2}) & 1 + O(t^{-1}) \end{pmatrix} G^{-1}, k \in \partial C, \quad J_{E^{(1)}} = G \begin{pmatrix} 1 + O(t^{-1}) & O(t^{2\gamma-2n-2}) \\ O(t^{-2\gamma+2n-1}) & 1 + O(t^{-1}) \end{pmatrix} G^{-1}, k \in \partial C_d. \quad (70)$$

We see that (70) provides us with better estimate than (67) provided that G, G^{-1} are uniformly bounded, as $t \rightarrow \infty$, on the circles $|k \mp ic| = r$. Now, for such γ that

$$\{\gamma\} \in [0, \frac{1}{2}]$$

we choose

$$n = \lfloor \gamma \rfloor,$$

where $\{\gamma\}, \lfloor \gamma \rfloor$ denote the fractional part of γ and the greatest integer not exceeding γ , respectively. Then $J_{E^{(1)}}$ in (70) admits the estimate

$$J_{E^{(1)}} = I + \mathcal{O}(t^{-1}).$$

Other values of γ , such that

$$\{\gamma\} \in (\frac{1}{2}, 1)$$

are considered in the next section 4.2.2.

Remark 4.2. We could assign n for wider range of γ (actually for all γ), namely take $n - \frac{1}{4} \leq \gamma < n + \frac{3}{4}$. Then $J_{E^{(1)}}$ in (70) would be of the order

$$J_{E^{(1)}} = I + \mathcal{O}(t^{-\frac{1}{2}}).$$

This would give us a worse estimate $\mathcal{O}(t^{-\frac{1}{2}})$ instead of $\mathcal{O}(t^{-1})$.

We determine now the matrix G (68) in such a way that $M_{\infty}^{(1)}$ is bounded (has no poles) at $k = \pm ic$. Expanding the product, the terms responsible for poles at $k = \pm ic$ in $M^{(1)}$ are

$$GB \begin{pmatrix} 1 & 0 \\ \frac{R_1}{\zeta} & 1 \end{pmatrix} = \left(I + \frac{A}{k-ic} + \frac{\tilde{A}}{k+ic} \right) \begin{pmatrix} 1 & 0 \\ \frac{R_1}{z} z^{-2n} \left(\frac{k-ic}{k+ic} \right)^{2n} t^{2\gamma-2n-1} (-i\phi) & 1 \end{pmatrix} B,$$

$$GB_d \begin{pmatrix} 1 & \frac{R_1}{\zeta_d} \\ 0 & 1 \end{pmatrix} = \left(I + \frac{A}{k-ic} + \frac{\tilde{A}}{k+ic} \right) \begin{pmatrix} 1 & \frac{R_1}{z_d} z_d^{-2n} \left(\frac{k-ic}{k+ic} \right)^{-2n} (i\phi_d) t^{2\gamma-2n-1} \\ 1 & 1 \end{pmatrix} B_d.$$

We see that at most we can have the poles of the second order at $z = 0, z_d = 0$. The requirement that the singular part vanishes yields a linear system for the matrices A, \tilde{A} : from the vanishing of the double-pole coefficient it is seen that they must be of the form

$$A = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0 & \tilde{b} \\ 0 & \tilde{a} \end{pmatrix}.$$

Writing down the conditions of vanishing of the residue we get the system of equations

$$\begin{cases} \frac{1}{-2ic} \frac{a}{16c^2 + \frac{2\rho \ln t}{t}} - \tilde{b} + \frac{\widehat{R}_{1,d}}{16c^2 + \frac{2\rho \ln t}{t}} = 0, & \begin{cases} \frac{1}{2ic} \frac{\tilde{a}}{16c^2 + \frac{2\rho \ln t}{t}} + ib + \frac{\widehat{R}_1}{16c^2 + \frac{2\rho \ln t}{t}} = 0, \\ \frac{1}{2ic} \frac{b}{16c^2 + \frac{2\rho \ln t}{t}} + i\tilde{a} = 0, \end{cases} \end{cases} \quad (71)$$

which decomposes into 2 linear systems: one for a, \tilde{b} , another for \tilde{a}, b . Here

$$\begin{aligned} \widehat{R}_1 &= R_1 t^{2\gamma-2n-1} \lim_{k \rightarrow ic} z^{-2n} \left(\frac{k-ic}{k+ic} \right)^{2n} (-i\phi) = \frac{-2 n! \Gamma(n + \frac{3}{2}) t^{2\gamma-2n-1}}{\pi(h^*)^2 \left(2c(16c^2 + \frac{2\rho \ln t}{t}) \right)^{2n+\frac{1}{2}}}, \\ \widehat{R}_{1,d} &= R_1 t^{2\gamma-2n-1} \lim_{k \rightarrow -ic} z_d^{-2n} \left(\frac{k-ic}{k+ic} \right)^{-2n} (i\phi_d) = \frac{2 n! \Gamma(n + \frac{3}{2}) t^{2\gamma-2n-1}}{\pi(h^*)^2 \left(2c(16c^2 + \frac{2\rho \ln t}{t}) \right)^{2n+\frac{1}{2}}}, \\ -\widehat{R}_1 &= \widehat{R}_{1,d} > 0. \end{aligned}$$

Solving system (71) for $a, b, \tilde{a}, \tilde{b}$, we obtain

$$a = \frac{-2ic \widehat{R}_1 \widehat{R}_{1,d}}{4c^2 \left(16c^2 + \frac{2\rho \ln t}{t} \right)^2 - \widehat{R}_1 \widehat{R}_{1,d}}, \quad b = \frac{4ic^2 \left(16c^2 + \frac{2\rho \ln t}{t} \right) \widehat{R}_1}{4c^2 \left(16c^2 + \frac{2\rho \ln t}{t} \right)^2 - \widehat{R}_1 \widehat{R}_{1,d}}, \quad (72)$$

$$\tilde{a} = \frac{2ic \widehat{R}_1 \widehat{R}_{1,d}}{4c^2 \left(16c^2 + \frac{2\rho \ln t}{t} \right)^2 - \widehat{R}_1 \widehat{R}_{1,d}}, \quad \tilde{b} = \frac{-4ic^2 \left(16c^2 + \frac{2\rho \ln t}{t} \right) \widehat{R}_{1,d}}{4c^2 \left(16c^2 + \frac{2\rho \ln t}{t} \right)^2 - \widehat{R}_1 \widehat{R}_{1,d}}. \quad (73)$$

We see, that $a, b, \tilde{a}, \tilde{b}$ are all bounded for $t \rightarrow \infty$, hence, G does not contribute to the error estimate (70) of J_E . Hence,

$$\begin{aligned} q_\infty^{(1)}(x, t) &:= \lim_{k \rightarrow \infty} (M_\infty)_{21} = \lim_{k \rightarrow \infty} (M_\infty)_{12} = 2ib = 2\tilde{b} = \frac{2c}{\cosh \left[\ln \left(\frac{\widehat{R}_{1,d}}{2c(16c^2 + \frac{2\rho \ln t}{t})} \right) \right]} = \\ &= \frac{2c}{\cosh \left[\ln \left(\frac{2 n! \Gamma(n + \frac{3}{2}) t^{2\gamma-2n-1}}{\pi(h^*)^2 \left(2c(16c^2 + \frac{2\rho \ln t}{t}) \right)^{2n+\frac{3}{2}}} \right) \right]} = \frac{2c}{\cosh \left[(2\gamma - 2n - 1) \ln t + \ln \left(\frac{2 n! \Gamma(n + \frac{3}{2})}{\pi(h^*)^2 \left(2c(16c^2 + \frac{2\rho \ln t}{t}) \right)^{2n+\frac{3}{2}}} \right) \right]}, \end{aligned} \quad (74)$$

and the solution of the initial value problem $q(x, t)$ is approximated by $q_\infty^{(1)}$ with the accuracy $\mathcal{O}(t^{-1})$:

$$q(x, t) = q_\infty^{(1)}(x, t) + \mathcal{O}(t^{-1/2}).$$

On the curve $\xi = \frac{c^2}{3} - \frac{\rho \ln t}{12t}$ we have $v = 1 - \frac{3\xi}{c^2} = 1 - \frac{x}{4c^2t} = \frac{\rho \ln t}{4c^2t}$, and

$$-2c(x - 4c^2t) = 8c^3vt = 2c\rho \ln t = (2\gamma + \frac{1}{2}) \ln t,$$

and substituting this into (74), we obtain

$$\begin{aligned} q_\infty^{(1)}(x, t) &= \frac{2c}{\cosh \left(2c(x - 4c^2t) + (2n + \frac{3}{2}) \ln t - \ln \frac{4}{(h^*)^2} - \ln \frac{n! \Gamma(n + \frac{3}{2})}{2\pi} + (2n + \frac{3}{2}) \ln[16c^3(2 + v)] \right)} = \\ &= \frac{2c}{\cosh \left(2c(x - 4c^2t) + (2n + \frac{3}{2}) \ln t - \ln \frac{2n! \Gamma(n + \frac{3}{2})}{\pi(h^*)^2} + (2n + \frac{3}{2}) \ln(32c^3) \right)} + \mathcal{O}\left(\frac{\ln t}{t}\right), \end{aligned} \quad (75)$$

This coincides with the formula by Khruslov and Kotlyarov [27], Thm. 3 on p. 6.

4.2.2 Second refined approximation of $M^{(1)}$

To deal with those γ such that $\{\gamma\} \in (\frac{1}{2}, 1)$, we introduce another refined approximation of $M^{(1)}$. Namely, by following a similar strategy as in the previous section we now "remove" the $\frac{1}{\zeta}$ term in the $(1, 2)$ entry in the asymptotics (61) of L , and in the $(2, 1)$ entry in asymptotics (62) of L_d at ∞ ,

$$\begin{pmatrix} 1 & \frac{R_2}{\zeta} \\ 0 & 1 \end{pmatrix} L = \begin{pmatrix} 1 + \mathcal{O}(\frac{1}{\zeta}) & \mathcal{O}(\frac{1}{\zeta^2}) \\ \mathcal{O}(\frac{1}{\zeta}) & 1 + \mathcal{O}(\frac{1}{\zeta}) \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ \frac{R_2}{\zeta_d} & 1 \end{pmatrix} L_d = \begin{pmatrix} 1 + \mathcal{O}(\frac{1}{\zeta_d}) & \mathcal{O}(\frac{1}{\zeta_d}) \\ \mathcal{O}(\frac{1}{\zeta_d}) & 1 + \mathcal{O}(\frac{1}{\zeta_d}) \end{pmatrix},$$

where

$$R_2 = \frac{2\pi i n}{n! \Gamma(n + \frac{1}{2})}, \quad n \geq 0, \quad \text{and } R_2 = 0 \quad \text{for } n = 0.$$

In keeping with the previous strategy, we introduce

$$G_2 = I + \frac{A_2}{k - ic} + \frac{\tilde{A}_2}{k + ic} \quad (76)$$

and define the refined approximation of $M^{(1)}$ by

$$M_\infty^{(2)} = \begin{cases} G_2 \left(\frac{k - ic}{k + ic} \right)^{-n\sigma_3}, & |k \mp ic| > r, \\ G_2 B \begin{pmatrix} 1 & \frac{R_2}{\zeta} \\ 0 & 1 \end{pmatrix} L(-i\phi)^{\sigma_3/2} t^{\gamma\sigma_3}, & |k - ic| < r, \\ G_2 B_d \begin{pmatrix} 1 & 0 \\ \frac{R_2}{\zeta_d} & 1 \end{pmatrix} L_d(i\phi_d)^{-\sigma_3/2} t^{-\gamma\sigma_3}, & |k + ic| < r, \end{cases} \quad (77)$$

with B, B_d as in (64) with $\alpha = -n$. The error matrix

$$E^{(2)} := M^{(1)} \left(M_\infty^{(2)} \right)^{-1}$$

has no discontinuity in the disks $|k \mp ic| < r$, and on the circles the jump is

$$J_{E^{(2)}} = G \begin{pmatrix} 1 + \mathcal{O}(t^{-1}) & \mathcal{O}(t^{-2\gamma+2n-2}) \\ \mathcal{O}(t^{2\gamma-2n-1}) & 1 + \mathcal{O}(t^{-1}) \end{pmatrix} G^{-1}, k \in \partial C, \quad J_{E^{(2)}} = G \begin{pmatrix} 1 + \mathcal{O}(t^{-1}) & \mathcal{O}(t^{2\gamma-2n-1}) \\ \mathcal{O}(t^{-2\gamma+2n-2}) & 1 + \mathcal{O}(t^{-1}) \end{pmatrix} G^{-1}, k \in \partial C_d. \quad (78)$$

We see that estimates in (78) are shifted with respect to estimates in (70), which allows to use both of them for different ranges of γ . For $\{\gamma\} \in (\frac{1}{2}, 1)$ we take

$$n = \lfloor \gamma \rfloor + 1,$$

then $J_{E^{(2)}}$ in (78) is of the order

$$J_{E^{(2)}} = I + \mathcal{O}(t^{-1}).$$

The minimal possible value of γ according to (57), is $\frac{-1}{4}$, and in this case we take $n = 0$. Hence, we are able to handle all the cases $\gamma \geq \frac{-1}{4}$ with constructions in terms of Laguerre polynomials with nonnegative index n . The matrix G_2 (76) is determined by the requirement that $M^{(2)}$ is regular at the points $k = \pm ic$: similar arguments lead to

$$A_2 = \begin{pmatrix} 0 & b_2 \\ 0 & a_2 \end{pmatrix}, \quad \tilde{A}_2 = \begin{pmatrix} \tilde{a}_2 & 0 \\ \tilde{b}_2 & 0 \end{pmatrix},$$

where $a_2, b_2, \tilde{a}_2, \tilde{b}_2$ satisfy the system (71) with $\hat{R}_1, \hat{R}_{1,d}$ replaced by $\hat{R}_2, \hat{R}_{2,d}$, respectively, and $a, b, \tilde{a}, \tilde{b}$ replaced with $a_2, b_2, \tilde{a}_2, \tilde{b}_2$, respectively. Hence, $a_2, b_2, \tilde{a}_2, \tilde{b}_2$ are defined by (72), (73), where we replace $\hat{R}_1, \hat{R}_{1,d}$ with $\hat{R}_2, \hat{R}_{2,d}$. Here

$$\hat{R}_2 = R_2 t^{2n-2\gamma-1} \lim_{k \rightarrow ic} \left(\frac{k - ic}{k + ic} \right)^{-2n} z^{2n} (-i\phi)^{-1} = \frac{-2\pi}{\Gamma(n)\Gamma(n + \frac{1}{2})} \frac{(h^*)^2}{4} \left[2c(16c^2 + \frac{2\rho \ln t}{t}) \right]^{2n+\frac{1}{2}} t^{2n-2\gamma-1},$$

$$\widehat{R}_{2,d} = R_2 t^{2n-2\gamma-1} \lim_{k \rightarrow -ic} \left(\frac{k-ic}{k+ic} \right)^{2n} z_d^{2n} (i\phi_d)^{-1} = -\widehat{R}_{2geq0}.$$

Hence, again G_2 does not contribute into asymptotics (78), and

$$q_\infty^{(2)} = 2i \lim_{k \rightarrow \infty} k M_{12}^{(2)} = 2i \lim_{k \rightarrow \infty} k M_{21}^{(2)} = 2ib_2 = 2i\tilde{b}_2 = \frac{2c}{\cosh \left[\ln \left(\frac{2\Gamma(n)\Gamma(n+\frac{1}{2})t^{2\gamma-2n+1}}{\pi(h^*)^2 (2c(16c^2 + \frac{2\rho \ln t}{t}))^{2n-\frac{1}{2}}} \right) \right]} \quad (79)$$

for $n \geq 1$, and $q_\infty^{(2)} = 0$ for $n = 0$. Further, $q(x, t) = q_\infty^{(2)}(x, t) + \mathcal{O}(t^{-1})$. We see that (79) coincides with (74), if we replace in the latest n with $n-1$.

To finish with the proof of Theorem 4 it is enough to notice that all the estimates in subsections 4.2.1, 4.2.2 are uniform with respect to finite shifts of the parameter ρ .

5 "Mesoscopic" regime

We extend the result of the previous section to the case when the number of the soliton n grows unboundedly but at a fixed rate in t . Let (x, t) lie on the curve

$$\frac{x}{12t} \equiv \xi = \frac{c^2}{3} - \frac{\beta t^\sigma \ln t}{12t}, \quad \sigma \in [0, 1), \quad \beta > 0. \quad (80)$$

The results of Section 4 suggests that (x, t) is constrained to an asymptotic soliton with number n of the order βt^σ (indeed we will see further that it is of the order $n \sim \frac{\beta t^\sigma}{1-\sigma}$), so that (x, t) will not be constrained to a one particular asymptotic soliton, but will move through the bulk of them.

It is not quite trivial to offer uniform estimates for the behaviour with respect to n in the constructions of Sec. 4. For example, consider the construction of Sec. 4 using the matrix-valued function $L(\zeta)$ (59). The monic Laguerre polynomial $\pi_n(\zeta)$ there has the following representation:

$$\begin{aligned} \pi_n(\zeta) &= \zeta^n \left(1 - \frac{(n+\frac{1}{2})n}{\zeta} + \frac{(n+\frac{1}{2})n(n-\frac{1}{2})(n-1)}{2!\zeta^2} + \dots \right) = \zeta^n \left(1 - \frac{(n)_{(1)}}{\zeta} + \frac{(n)_{(2)}}{2!\zeta^2} + \dots \right) = \\ &= \zeta^n \sum_{j=0}^n \frac{(-1)^j (n)_{(j)}}{j! \zeta^j}, \end{aligned} \quad (81)$$

where we denoted for brevity

$$(a)_{(b)} = (a + \frac{1}{2})a(a - \frac{1}{2})(a-1)\dots(a + \frac{3}{2} - b)(a - b + 1), \quad \text{for } b \geq 1, b \in \mathbb{N}, \quad (a)_{(0)} = 1. \quad (82)$$

Its Cauchy transform

$$C_{n-1}(\zeta) := \frac{1}{2\pi i} \int_0^{+\infty} \frac{\pi_{n-1}(s) \sqrt{s} e^{-s} ds}{s - \zeta} \quad (83)$$

admits an asymptotic expansion

$$\begin{aligned} \frac{-\zeta^n}{\Gamma(n)\Gamma(n+\frac{1}{2})} \int_0^{+\infty} \frac{\pi_{n-1}(s) \sqrt{s} e^{-s} ds}{s - \zeta} &= 1 + \frac{n(n+\frac{1}{2})}{\zeta} + \frac{n(n+\frac{1}{2})(n+1)(n+\frac{3}{2})}{2!\zeta} + \dots = \\ &= 1 + \frac{(n)_{(1)}}{\zeta} + \frac{(n)_{(2)}}{2!\zeta^2} + \dots = \sum_{j=0}^{\infty} \frac{(n)_{(j)}}{j! \zeta^j}, \end{aligned} \quad (84)$$

where we denoted for brevity

$$(a)^{(b)} = a(a + \frac{1}{2})(a+1)(a + \frac{3}{2})\dots(a+b-1)(a+b-\frac{1}{2}), \quad \text{for } b \geq 1, b \in \mathbb{N}, \quad (a)^{(0)} = 1. \quad (85)$$

While the n -dependence in $\pi_n(\zeta)$ can be controlled by making a change of variable $\zeta \mapsto n^2 \zeta$, this does not help in the asymptotic series for the Cauchy transform. Instead, the next lemma provides us with a uniform control in n of the asymptotic expansion of L .

Lemma 5.1. Define the functions $h(\zeta), \delta(\zeta)$, analytic in $\zeta \in \mathbb{C} \setminus [0, 1]$

$$h(\zeta) = 2 \int_1^\zeta \sqrt{\frac{s-1}{s}} ds - 2\zeta + \ln \zeta + \ln 4e \quad (86)$$

$$= 2\sqrt{\zeta(\zeta-1)} - 2\zeta + \ln \left(4e\zeta(-1+2\zeta-2\sqrt{\zeta(\zeta-1)}) \right) \quad (87)$$

$$\delta(\zeta) = \left(\zeta(-1+2\zeta-2\sqrt{\zeta(\zeta-1)}) \right)^{\frac{1}{4}}. \quad (88)$$

Given the Laurent expansions of $h(\zeta)$

$$h(\zeta) = \sum_{j=1}^{\infty} \frac{2(2j-1)!}{(j+1)(j!)^2(4\zeta)^j} =: \sum_{j=1}^{\infty} \frac{b_j}{\zeta^j}; \quad (89)$$

we denote by

$$h_K(\zeta) = \sum_{j=1}^{K-1} \frac{b_j}{\zeta^j}, \quad r_K(\zeta) = \sum_{j=K}^{\infty} \frac{b_j}{\zeta^j}.$$

the partial sums and tails. Define also the matrix-valued function $M_{mod}(\zeta)$ as follows:

$$M_{mod}(\zeta) = \begin{pmatrix} \frac{1}{2}(\gamma(\zeta) + \gamma^{-1}(\zeta)) & \frac{i}{2}(\gamma(\zeta) - \gamma^{-1}(\zeta)) \\ \frac{-i}{2}(\gamma(\zeta) - \gamma^{-1}(\zeta)) & \frac{1}{2}(\gamma(\zeta) + \gamma^{-1}(\zeta)) \end{pmatrix}, \quad \gamma(\zeta) = \sqrt[4]{\frac{\zeta-1}{\zeta}}.$$

Finally, let L be as in (59) and consider the matrix-valued functions

$$\mathcal{Q}_n(\zeta) = \left(\frac{n^{n+\frac{1}{4}}}{e^n} \right)^{\sigma_3} L(4n\zeta)(4n\zeta)^{n\sigma_3} e^{-nh(\zeta)\sigma_3} \left(\frac{e^n}{n^{n+\frac{1}{4}}} \right)^{\sigma_3}, \quad (90)$$

$$\mathcal{E}_n(\zeta) = \left(\frac{n^{n+\frac{1}{4}}}{e^n} \right)^{\sigma_3} L(4n\zeta)(4n\zeta)^{n\sigma_3} e^{-nh(\zeta)\sigma_3} \left(\frac{e^n}{n^{n+\frac{1}{4}}} \right)^{\sigma_3} \left(\sqrt{2}\delta(\zeta) \right)^{-\sigma_3} (M_{mod}(\zeta))^{-1},$$

Let U be any open set $U \supset [0, 1]$ and fix $\mathbb{N} \ni M \geq 0$. The functions $\mathcal{Q}_n(\zeta)$ and $\mathcal{E}_n(\zeta)$ admit the expansion into series

$$\begin{aligned} \mathcal{Q}_n(\zeta) &= I + \sum_{j=1}^{M-1} \frac{q_j(n)}{\zeta^j} + r_{\mathcal{Q},M}(\zeta; n), \\ \mathcal{E}_n(\zeta) &= I + \sum_{j=1}^{M-1} \frac{e_j(n)}{\zeta^j} + r_{\mathcal{E},M}(\zeta; n). \end{aligned}$$

The expansions are uniform in $\zeta \in \mathbb{C} \setminus U$ in the sense that there is a constant $C_M < \infty$ such that

$$\sup_{n \geq 1} |q_j(n)| < \infty, \quad \sup_{n \geq 1} |r_{\mathcal{Q},M}(\zeta; n)| \leq \frac{C_M}{|\zeta|^M}, \quad \sup_{n \geq 1} |r_{\mathcal{E},M}(\zeta; n)| \leq \frac{C_M}{|\zeta|^M}.$$

Moreover,

$$\lim_{n \rightarrow \infty} e_j(n) = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} r_{\mathcal{E},M}(\zeta; n) = 0 \text{ for any fixed } \zeta, M.$$

We prove Lemma 5.1 in Appendix A.

Remark 5.1. As a byproduct of Lemma 5.1 we can control the n -behaviour of the asymptotic series (84) for the Cauchy transform $C_{n-1}(\zeta)$ (83) of the Laguerre polynomials, namely (we use the notation (85) below):

$$e^{-nh(\zeta)} \frac{-(4n\zeta)^n C_{n-1}(4n\zeta)}{\Gamma(n)\Gamma(n+\frac{1}{2})} = e^{-nh(\zeta)} \sum_{j=0}^{\infty} \frac{(n)^{(j)}}{j!(4n\zeta)^n} = e^{-nh(\zeta)} \sum_{j=0}^{\infty} \frac{\Gamma(n+j)\Gamma(n+j+\frac{1}{2})}{j!\Gamma(n)\Gamma(n+\frac{1}{2})(4n\zeta)^n} =$$

$$= 1 + \sum_{j=1}^{M-1} \frac{s_j(n)}{\zeta^j} + r_M(\zeta; n), \quad \text{for any integer } M \geq 1$$

and

$$\sup_{n \geq 1} |s_j(n)| \leq C_j, \quad \sup_{n \geq 1} |r_M(s; n)| \leq \frac{C_M}{|\zeta|^M}, \quad \text{for any } \zeta \in \mathbb{C} \setminus U,$$

where $M \geq 1$ and C_j are independent of n , which we believe is also of independent interest.

Remark 5.2. From the representation of Laguerre polynomials (81) and asymptotic series for its Cauchy transform (84) we find an asymptotic series for $\mathcal{Q}_n(\zeta)$ (using the notations (82), (85))

$$\left(\begin{array}{c} e^{-nh(\zeta)} \sum_{j=0}^{\infty} \frac{(n)^{(j)}}{j!(4n\zeta)^j} \left(\frac{2\pi n^{2n-\frac{1}{2}}}{\Gamma(n+\frac{1}{2})\Gamma(n)e^{2n}} \right) \frac{(-i)}{4\zeta} \sum_{j=0}^{n-1} \frac{(-1)^j (n-1)^{(j)}}{j!(4n\zeta)^j} \\ \left(\frac{\Gamma(n+1)\Gamma(n+\frac{3}{2})e^{2n}}{2\pi n^{2n+\frac{3}{2}}} \right) \frac{i}{4\zeta} \sum_{j=0}^{\infty} \frac{(n+1)^{(j)}}{j!(4n\zeta)^j} e^{nh(\zeta)} \sum_{j=0}^n \frac{(-1)^j (n)^{(j)}}{j!(4n\zeta)^j} \end{array} \right).$$

In particular,

$$\mathcal{Q}_n(\zeta) = \left(\begin{array}{cc} 1 + \frac{1}{8\zeta} & \frac{2\pi n^{2n-\frac{1}{2}}}{\Gamma(n+\frac{1}{2})\Gamma(n)e^{2n}} \frac{-i}{4\zeta} \\ \frac{\Gamma(n+1)\Gamma(n+\frac{3}{2})e^{2n}}{2\pi n^{2n+\frac{3}{2}}} \frac{i}{4\zeta} & 1 - \frac{1}{8\zeta} \end{array} \right) + \mathcal{O}(\zeta^{-2})$$

and

$$\left(\begin{array}{cc} 1 & 0 \\ \frac{R_1}{\zeta} & 1 \end{array} \right) \mathcal{Q}_n(\zeta) = \left(\begin{array}{cc} 1 + \mathcal{O}(\frac{1}{\zeta}) & \mathcal{O}(\frac{1}{\zeta}) \\ \mathcal{O}(\frac{1}{\zeta^2}) & 1 + \mathcal{O}(\frac{1}{\zeta}) \end{array} \right), \quad (91)$$

$$\left(\begin{array}{cc} 1 & \frac{R_2}{\zeta} \\ 0 & 1 \end{array} \right) \mathcal{Q}_n(\zeta) = \left(\begin{array}{cc} 1 + \mathcal{O}(\frac{1}{\zeta^2}) & \mathcal{O}(\frac{1}{\zeta}) \\ \mathcal{O}(\frac{1}{\zeta}) & 1 + \mathcal{O}(\frac{1}{\zeta}) \end{array} \right), \quad (92)$$

with

$$R_1 = \frac{-i}{4} \frac{\Gamma(n+1)\Gamma(n+\frac{3}{2})e^{2n}}{2\pi n^{2n+\frac{3}{2}}}, \quad R_2 = \frac{i}{4} \frac{2\pi n^{2n-\frac{1}{2}}}{\Gamma(n+\frac{1}{2})\Gamma(n)e^{2n}}, \quad (93)$$

and all \mathcal{O} estimates are uniform with respect to $n \geq 1$.

Corollary 3. Consider the matrix-valued function

$$Y(\zeta) := \Lambda^{(n+\frac{1}{4})\sigma_3} L(\zeta\Lambda) \Lambda^{-\sigma_3/4} := \left(\begin{array}{cc} \frac{-\Lambda^{n+1/4}}{2\pi i \Gamma(n+\frac{1}{2})\Gamma(n)} \int_0^{+\infty} \frac{\pi_{n-1}(s\Lambda)\sqrt{s} e^{-\Lambda s} ds}{s-\zeta} & \frac{-2\pi i \Lambda^{n+\frac{1}{2}}}{\Gamma(n+\frac{1}{2})\Gamma(n)} \pi_{n-1}(\zeta\Lambda) \\ \frac{1}{2\pi i \Lambda^n} \int_0^{+\infty} \frac{\pi_n(s\Lambda)\sqrt{s} e^{-\Lambda s} ds}{s-\zeta} & \frac{\pi_n(\zeta\Lambda)}{\Lambda^n} \end{array} \right).$$

The matrix $Y(\zeta)$ solves a RHP of the form

$$Y_- = Y_+ \left(\begin{array}{cc} 1 & 0 \\ -\sqrt{\zeta} e^{-\Lambda\zeta} & 1 \end{array} \right), \quad \zeta \in (0, +\infty),$$

$$Y(\zeta) = \left(I + \mathcal{O}\left(\frac{1}{\zeta}\right) \right) \zeta^{-n\sigma_3},$$

and it can be written as follows

$$Y(\zeta) = \left(\frac{\Lambda^{n+\frac{1}{4}} e^n}{n^{n+\frac{1}{4}}} \right)^{\sigma_3} \mathcal{Q}_n \left(\frac{\zeta\Lambda}{4n} \right) \zeta^{-n\sigma_3} e^{nh(\frac{\zeta\Lambda}{4n})\sigma_3} \left(\frac{n^{n+\frac{1}{4}}}{\Lambda^{n+\frac{1}{4}} e^n} \right)^{\sigma_3}, \quad (94)$$

where \mathcal{Q} is defined in (90).

Lemma 5.1 allows us to control the large n -behaviour of L in the approximate solutions M_∞ (63), $M_\infty^{(1)}$ (69), $M_\infty^{(2)}$ (77), regardless of whether $n \geq 1$ is bounded or growing together with t .

But there is also another issue for growing n , namely providing bounds for the terms

$$\left(\frac{k - ic}{k + ic}\right)^{-n} z^n, \quad \left(\frac{k + ic}{k - ic}\right)^n z_d^n$$

on the circles $|k \mp ic| = r$ in the estimates (65), (66). While, for bounded n , these quantities are separated both from 0 and ∞ , for growing n this is no longer the case. We also need to control the term $e^{nh(\zeta)}$ in the asymptotics (90), since $nh(\zeta)$ will be of the order $\frac{n^2}{t}$ (as we will see shortly), and for growing fast enough n this term is not negligible. To overcome these difficulties, we need to make a local conformal change of variables z, z_d , as described in Lemma 5.2 below.

Mimicking the approach of Sec. 4, we first transform our original RH problem 1 to a RH problem for a new function $M^{(1)}$ (52) with jump $J^{(1)}$ (53). The following estimates hold in the intervals $k \in \pm(ic, i(c - r))$ (small enough parameter $r > 0$ will still be fixed throughout this section)

$$\begin{aligned} f(k)e^{2it\theta} &= \frac{2\sqrt{2}i}{(h^*)^2\sqrt{c}}\sqrt{y}(1 + \mathcal{O}(\sqrt{y}))t^{2c\beta t^\sigma}e^{-tz}, \\ -\overline{f(\bar{k})}e^{-2it\theta} &= \frac{2\sqrt{2}i}{(h^*)^2\sqrt{c}}\sqrt{y_d}(1 + \mathcal{O}(\sqrt{y_d}))t^{2c\beta t^\sigma}e^{-tz_d}, \end{aligned}$$

and

$$z = z(y; t) = (16c^2 + \frac{2\beta t^\sigma \ln t}{t})y - 24cy^2 + 8y^3, \quad z_d = z_d(y; t) = (16c^2 + \frac{2\beta t^\sigma \ln t}{t})y_d - 24cy_d^2 + 8y_d^3, \quad (95)$$

and y, y_d are as in (54).

Let \tilde{z}, \tilde{z}_d be some local conformal transformations of variables z, z_d , to be specified later, and denote

$$\zeta := \frac{\tilde{z}t}{\Lambda}, \quad \zeta_d := \frac{\tilde{z}_dt}{\Lambda}, \quad (96)$$

where Λ is a large parameter growing with t as $\Lambda \asymp t^\sigma$. Then

$$\begin{aligned} f(k)e^{2it\theta} &= \frac{2\sqrt{2}i}{(h^*)^2\sqrt{c}}\sqrt{\frac{y}{\tilde{z}}}(1 + \mathcal{O}(\sqrt{y}))t^{2c\beta t^\sigma}\sqrt{\tilde{z}}e^{-tz} =: i\phi(k, t)t^{2c\beta t^\sigma}\sqrt{\tilde{z}}e^{-tz}, \\ -\overline{f(\bar{k})}e^{-2it\theta} &= \frac{2\sqrt{2}i}{(h^*)^2\sqrt{c}}\sqrt{\frac{y_d}{\tilde{z}_d}}(1 + \mathcal{O}(\sqrt{y_d}))t^{2c\beta t^\sigma}\sqrt{\tilde{z}_d}e^{-tz_d} =: i\phi_d(k, t)t^{2c\beta t^\sigma}\sqrt{\tilde{z}_d}e^{-tz_d}, \end{aligned}$$

The functions $\phi(k, t), \phi_d(k, t)$, analytic near $k = \pm ic$, are uniformly separated from 0 and ∞ with respect to $t \geq 1$. Let us choose an integer $K \geq 0$ such that

$$K \geq \frac{2\sigma - 1}{1 - \sigma}, \quad \sigma \leq \frac{K + 1}{K + 2}. \quad (97)$$

For $\sigma \in [0, \frac{1}{2}]$ we take $K = 0$, for $\sigma \in (\frac{1}{2}, \frac{2}{3}]$ we take $K = 1$, etc. The next lemma specifies the conformal maps $z \mapsto \tilde{z}, z_d \mapsto \tilde{z}_d$.

Lemma 5.2. *Denote*

$$v := 1 - \frac{3\xi}{c^2} = \frac{\beta t^\sigma \ln t}{4c^2 t}, \quad \delta = \frac{2n}{c^3 t}.$$

For sufficiently small v, δ there exists a local conformal map $\tilde{z}(y, \delta, v)$ such that

$$z(y) - \frac{2n}{t} \ln \frac{y}{2c - y} + \frac{2n}{t} \sum_{j=1}^K \frac{A_j(ic - iy)}{(y(2c - y))^j} = \tilde{z} - \frac{2n}{t} \ln \tilde{z} + \frac{2n}{t} Q + \frac{2n}{t} \sum_{j=1}^K \frac{b_j(4n)^j}{t^j \tilde{z}^j}. \quad (98)$$

($z(y) = z(y; t)$ as in (95)). Furthermore, \tilde{z} and Q admit expansions of the form

$$\begin{aligned} \tilde{z} &= yc^2 \left[16 + 8v - \frac{8+v}{2+v} \frac{n}{c^3 t} + \mathcal{O}\left(\frac{n^2}{t^2}\right) \right] y + cy^2 \left(-24 - \frac{24 - 4v + v^2}{4(2+v)^2} \frac{n}{c^3 t} + \mathcal{O}\left(\frac{n^2}{t^2}\right) \right) + \\ &+ y^3 \left(8 + \mathcal{O}\left(\frac{n}{t}\right) \right) + \mathcal{O}(y^4 nt^{-1}), \end{aligned} \quad (99)$$

and

$$\begin{aligned}
Q &= \ln(16c^3(2+v)) + \mathcal{O}\left(\frac{n}{t}\right) \quad \text{for } K=0, \\
Q &= \ln(16c^3(2+v)) - \frac{3(8+v)}{16(2+v)^2} \frac{n}{c^3t} + \mathcal{O}\left(\frac{n^2}{t^2}\right) \quad \text{for } K=1, \\
Q &= \ln(16c^3(2+v)) - \frac{3(8+v)}{16(2+v)^2} \frac{n}{c^3t} - \frac{400+48v+5v^2}{128(2+v)^4} \left(\frac{n}{c^3t}\right)^2 + \mathcal{O}\left(\frac{n^3}{t^3}\right) \quad \text{for } K=2. \quad (100)
\end{aligned}$$

Statements of the type of Lemma 5.2 fall within the general class of “normal form” of singularities. A completely similar lemma can be found in [2] (Prop. 2.2) with the only difference that in our case $z(y; t) = \mathcal{O}(y)$ uniformly as $t \rightarrow \infty$ while in the case of [2] we have $\mathcal{O}(y^2)$ and hence the normal form starts with \tilde{z}^2 in the right side of (98). We omit the detailed proof because it is not substantially different from the one referred to above.

Let \tilde{z}_d be the conformal map of z_d determined by (98), where we substituted y with y_d , z with z_d . Then we have

$$e^{-tz} = e^{-t\tilde{z}} e^{-2nQ} \cdot \frac{\tilde{z}^{2n}(k+ic)^{2n}}{(k-ic)^{2n}} \cdot e^{-2n \sum_{j=1}^K \frac{b_j(4n)^j}{t^j \tilde{z}^j}} \cdot e^{2n \sum_{j=1}^K \frac{A_j k}{(k^2+c^2)^j}}, \quad (101)$$

and

$$e^{-tz_d} = e^{-t\tilde{z}_d} e^{-2nQ} \cdot \frac{\tilde{z}_d^{2n}(k-ic)^{2n}}{(k+ic)^{2n}} \cdot e^{-2n \sum_{j=1}^K \frac{b_j(4n)^j}{t^j \tilde{z}_d^j}} \cdot e^{-2n \sum_{j=1}^K \frac{A_j k}{(k^2+c^2)^j}}. \quad (102)$$

Now we are ready to construct two approximations $M_\infty^{(j)}$, $j=1,2$, of the function $M^{(1)}$ (analogues of (69), (77) in section 4):

$$M_\infty^{(j)} = \begin{cases} G_j \left(\frac{k-ic}{k+ic}\right)^{-n\sigma_3} \exp\left(n \sum_{j=1}^K \frac{A_j k}{(k^2+c^2)^j} \sigma_3\right), & |k \mp ic| > r, \\ G_j B \Delta_{u,j} Y(\zeta) (-i\phi)^{\frac{\sigma_3}{2}} \left(t^{c\beta t^\sigma} \sqrt[4]{\frac{\Lambda}{t}} e^{-nQ}\right)^{\sigma_3} \left(\frac{\tilde{z}(k+ic)}{k-ic}\right)^{n\sigma_3} \times \\ \times \exp\left(-n \sum_{j=1}^K \frac{b_j(4n)^j}{\Lambda^j \zeta^j} \sigma_3\right) \exp\left(n \sum_{j=1}^K \frac{A_j k}{(k^2+c^2)^j} \sigma_3\right), & |k-ic| < r. \\ G_j B_d \Delta_{d,j} Y_d(\zeta_d) (i\phi_d)^{-\frac{\sigma_3}{2}} \left(t^{c\beta t^\sigma} \sqrt[4]{\frac{\Lambda}{t}} e^{-nQ}\right)^{-\sigma_3} \left(\frac{k+ic}{\tilde{z}_d(k-ic)}\right)^{n\sigma_3} \times \\ \times \exp\left(n \sum_{j=1}^K \frac{b_j(4n)^j}{\Lambda^j \zeta_d^j} \sigma_3\right) \exp\left(n \sum_{j=1}^K \frac{A_j k}{(k^2+c^2)^j} \sigma_3\right), & |k+ic| < r. \end{cases} \quad (103)$$

Here b_j are as in (89), function $Y(\zeta)$ as in (94), and

$$Y_d(\zeta_d) := \sigma_1 Y(\zeta_d) \sigma_1, \quad \mathcal{Q}_{d,n}(\zeta_d) = \sigma_1 \mathcal{Q}_n(\zeta_d) \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Formulas (101), (102) ensure that $M_\infty^{(j)}$, $j=1,2$, have the same jump on the intervals $\pm(ic, i(c-r))$ as $M^{(1)}$ has. For the same reasons as in section 4, i.e. in order to minimize jump on the circles ∂C , ∂C_d of the error matrix $E^{(j)} := M^{(1)} (M_\infty^{(j)})^{-1}$, $j=1,2$, we take

$$B = \left(\frac{t^{n+\frac{1}{4}} e^{nQ}}{\Lambda^{n+\frac{1}{4}} t^{c\beta t^\sigma}}\right)^{\sigma_3} (-i\phi)^{-\sigma_3/2}, \quad B_d = \left(\frac{\Lambda^{n+\frac{1}{4}} t^{c\beta t^\sigma}}{t^{n+\frac{1}{4}} e^{nQ}}\right)^{\sigma_3} (i\phi_d)^{\sigma_3/2}.$$

Then the jump matrix for $E_-^{(j)} = E_+^{(j)} J_{E^{(j)}}$, $j=1,2$, on $k \in \partial C$, $k \in \partial C_d$, respectively, is (see (90))

$$J_{E^{(j)}} = G_j \left(\frac{e^{n(Q+1)} t^{n+\frac{1}{4}}}{\Lambda^{n+\frac{1}{4}} t^{c\beta t^\sigma}}\right)^{\sigma_3} \Delta_{u,j} \mathcal{Q}_n\left(\frac{\zeta \Lambda}{4n}\right) (-i\phi)^{\frac{\sigma_3}{2}} e^{n(h(\frac{\zeta \Lambda}{4n}) - \sum_{j=1}^K \frac{b_j 4^j n^j}{\Lambda^j \zeta^j}) \sigma_3} \left(\frac{n^{n+\frac{1}{4}} t^{c\beta t^\sigma}}{e^{n(Q+1)} t^{n+\frac{1}{4}}}\right)^{\sigma_3} G_j^{-1}. \quad (104)$$

$$J_{E^{(j)}} = G_j \left(\frac{n^{n+\frac{1}{4}} t^{c\beta t^\sigma}}{e^{n(Q+1)} t^{n+\frac{1}{4}}} \right)^{\sigma_3} \Delta_{d,j} \mathcal{Q}_{d,n} \left(\frac{\zeta_d \Lambda}{4n} \right) (\text{id}_d)^{-\frac{\sigma_3}{2}} e^{-n(h(\frac{\zeta_d \Lambda}{4n}) - \sum_{j=1}^K \frac{b_j 4^j n^j}{\Lambda^j \zeta^j}) \sigma_3} \left(\frac{e^{n(Q+1)} t^{n+\frac{1}{4}}}{n^{n+\frac{1}{4}} t^{c\beta t^\sigma}} \right)^{\sigma_3} G_j^{-1}. \quad (105)$$

Under the assumption that $n \asymp t^\sigma$, we have that the middle exponential term in (104), ((105)) is of the order

$$e^{n \left(h(\frac{\zeta_d \Lambda}{4n}) - \sum_{j=1}^K \frac{b_j 4^j n^j}{\Lambda^j \zeta^j} \right) \sigma_3} = I + \mathcal{O}(t^{K(\sigma-1)+2\sigma-1}),$$

due to the choice of K (97), and the definition of ζ (96). Hence,

$$\begin{aligned} J_{E^{(1)}} &= G_1 \begin{pmatrix} 1 + \mathcal{O}\left(\frac{\Lambda}{\zeta t}\right) & \mathcal{O}\left(\frac{\Lambda}{\zeta t}\right) \frac{1}{\omega} \\ \mathcal{O}\left(\frac{\Lambda^2}{\zeta^2 t^2}\right) \omega & 1 + \mathcal{O}\left(\frac{\Lambda}{\zeta t}\right) \end{pmatrix} \Omega G_1^{-1}, \partial C, & J_{E^{(1)}} &= G_1 \begin{pmatrix} 1 + \mathcal{O}\left(\frac{\Lambda}{\zeta t}\right) & \mathcal{O}\left(\frac{\Lambda^2}{\zeta^2 t^2}\right) \omega \\ \mathcal{O}\left(\frac{\Lambda}{\zeta t}\right) \frac{1}{\omega} & 1 + \mathcal{O}\left(\frac{\Lambda}{\zeta t}\right) \end{pmatrix} \Omega G_1^{-1}, \partial C_d, \\ J_{E^{(2)}} &= G_1 \begin{pmatrix} 1 + \mathcal{O}\left(\frac{\Lambda}{t}\right) & \mathcal{O}\left(\frac{\Lambda^2}{t^2}\right) \frac{1}{\omega} \\ \mathcal{O}\left(\frac{\Lambda}{t}\right) \omega & 1 + \mathcal{O}\left(\frac{\Lambda}{t}\right) \end{pmatrix} \Omega G_1^{-1}, \partial C, & J_{E^{(2)}} &= G_1 \begin{pmatrix} 1 + \mathcal{O}\left(\frac{\Lambda}{t}\right) & \mathcal{O}\left(\frac{\Lambda}{t}\right) \omega \\ \mathcal{O}\left(\frac{\Lambda^2}{t^2}\right) \frac{1}{\omega} & 1 + \mathcal{O}\left(\frac{\Lambda}{t}\right) \end{pmatrix} \Omega G_1^{-1}, \partial C_d, \end{aligned}$$

where we denoted for brevity

$$\omega = \frac{t^{2c\beta t^\sigma} n^{2n+\frac{1}{2}}}{e^{2n(Q+1)} t^{2n+\frac{1}{2}}}, \quad \text{and} \quad \Omega = I + \mathcal{O}\left(t^{K(\sigma-1)+2\sigma-1}\right). \quad (106)$$

We need to choose such an n that makes ω as close to 1 as possible. Denote by γ exact solution of equation

$$\omega|_{n \mapsto \gamma} = 1, \quad \text{i.e.} \quad \frac{t^{2c\beta t^\sigma} \gamma^{2\gamma+\frac{1}{2}}}{e^{2\gamma(Q(\gamma)+1)} t^{2\gamma+\frac{1}{2}}} = 1. \quad (107)$$

Then

$$\gamma \sim \frac{\beta t^\sigma}{1-\sigma} \quad \text{and} \quad \omega \asymp t^{2(1-\sigma)(\gamma-n)}.$$

For the same reasons as the triangular factors in (69), (77) in the previous section 4 we introduce the triangular matrices $\Delta_{u,j}$, $\Delta_{d,j}$, $j = 1, 2$:

$$\begin{aligned} \Delta_{u,1} &= \begin{pmatrix} 1 & 0 \\ \frac{R_1 4n}{\Lambda \zeta} \frac{n^{2n+\frac{1}{2}}}{\Lambda^{2n+\frac{1}{2}} e^{2n}} & 1 \end{pmatrix}, & \Delta_{d,1} &= \begin{pmatrix} 1 & \frac{R_1 4n}{\Lambda \zeta_d} \frac{n^{2n+\frac{1}{2}}}{\Lambda^{2n+\frac{1}{2}} e^{2n}} \\ 0 & 1 \end{pmatrix}, \\ \Delta_{u,2} &= \begin{pmatrix} 1 & \frac{R_2 4n}{\zeta \Lambda} \frac{\Lambda^{2n+\frac{1}{2}} e^{2n}}{n^{2n+\frac{1}{2}}} \\ 0 & 1 \end{pmatrix}, & \Delta_{d,2} &= \begin{pmatrix} 1 & 0 \\ \frac{R_2 4n}{\zeta_d \Lambda} \frac{\Lambda^{2n+\frac{1}{2}} e^{2n}}{n^{2n+\frac{1}{2}}} & 1 \end{pmatrix}. \end{aligned}$$

where, due to (91), (92), we have set R_1, R_2 as in 93.

Similarly to Sec. 4, the matrices $M_\infty^{(j)}$, $j = 1, 2$, are regular at $k = \pm ic$ provided that G_j , $j = 1, 2$, are of the form

$$G_1 = \begin{pmatrix} 1 + \frac{a_1}{k-ic} & \frac{\tilde{b}_1}{k+ic} \\ \frac{b_1}{k-ic} & 1 + \frac{\tilde{a}_1}{k+ic} \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 + \frac{\tilde{a}_2}{k+ic} & \frac{b_2}{k-ic} \\ \frac{b_2}{k+ic} & 1 + \frac{a_2}{k-ic} \end{pmatrix},$$

and $a_j, b_j, \tilde{a}_j, \tilde{b}_j$, $j = 1, 2$, are determined by the regularity condition at $k = \pm ic$ of the expressions

$$G_1 \begin{pmatrix} 1 & 0 \\ -\frac{\hat{R}_1}{\tilde{z}} & 1 \end{pmatrix}, \quad G_1 \begin{pmatrix} 1 & \frac{\hat{R}_1}{\tilde{z}_d} \\ 0 & 1 \end{pmatrix}, \quad G_2 \begin{pmatrix} 1 & -\frac{\hat{R}_2}{\tilde{z}} \\ 0 & 1 \end{pmatrix}, \quad G_2 \begin{pmatrix} 1 & 0 \\ \frac{\hat{R}_2}{\tilde{z}_d} & 1 \end{pmatrix},$$

where

$$\hat{R}_1 := \frac{\Gamma(n+1)\Gamma(n+\frac{3}{2})t^{2c\beta t^\sigma} \lim_{k \rightarrow ic} \phi(k)}{2\pi t^{2n+\frac{3}{2}} e^{2nQ}}, \quad \hat{R}_2 := \frac{2\pi t^{2n-\frac{1}{2}} e^{2nQ}}{\Gamma(n+\frac{1}{2})\Gamma(n) t^{2c\beta t^\sigma} \lim_{k \rightarrow ic} \phi(k)}.$$

Solving the analogues of (71), we obtain analogs of (72), (73) (we denote by \tilde{z}_y the derivative of \tilde{z} (99) with respect to y at the point $y = 0$):

$$a_1 = \frac{2ic \hat{R}_1^2}{4c^2 (\tilde{z}_y)^2 + \hat{R}_1^2}, \quad b_1 = \frac{-4ic^2 \tilde{z}_y \hat{R}_1}{4c^2 (\tilde{z}_y)^2 + \hat{R}_1^2}, \quad (108)$$

$$\tilde{a}_1 = \frac{-2ic \hat{R}_1^2}{4c^2 (\tilde{z}_y)^2 + \hat{R}_1^2}, \quad \tilde{b}_1 = \frac{-4ic^2 \tilde{z}_y \hat{R}_1}{4c^2 (\tilde{z}_y)^2 + \hat{R}_1^2}. \quad (109)$$

Finally, for $\{\gamma\} \in [0, \frac{1}{2}]$ we take the first approximation $M_\infty^{(1)}$ with index $n = \lfloor \gamma \rfloor$, and for $\{\gamma\} \in (\frac{1}{2}, 1)$ the second approximation $M_\infty^{(2)}$ with index $n = \lfloor \gamma \rfloor + 1$. We then obtain the following asymptotics for $q(x, t)$: let (x, t) be on curve (80), and let γ be solution of equation (107). Take $n = \lfloor \gamma \rfloor$. Then

$$q(x, t) = \frac{2c}{\cosh \ln \left[2c(x - 4c^2t) + (2n + \frac{3}{2}) \ln t + 2nQ + \frac{3}{2} \ln(2c\tilde{z}_y) - \ln \frac{\Gamma(n+1)\Gamma(n+\frac{3}{2})}{2\pi} - \ln \frac{4}{(h^*)^2} \right]} + \mathcal{O}(t^{K(\sigma-1)+2\sigma-1}). \quad (110)$$

Remark 5.3. Multiplication by $e^{\sum_{j=1}^K \frac{A_j k}{(k^2+c^2)^j} \sigma_3}$ in (103) does not affect the error estimates for the jumps outside of the disks $|k \mp ic| < r$. Indeed, $A_j \asymp \frac{n^j}{t^j} \asymp t^{j(\sigma-1)}$, hence the j^{th} term gives the contribution of the order $\exp(t^{\sigma(j+1)-j})$, and since the error matrix $J^{(1)}$ outside of the disks is of the order $J^{(1)} = I + \mathcal{O}(e^{-bt})$ for some positive $b > 0$, the jump for the error matrix

$$E^{(j)} = M^{(1)} \left(M_\infty^{(j)} \right)^{-1}, j = 1, 2$$

is of the order

$$J_{E^{(j)}} - I = \mathcal{O} \left(\exp \left(-bt + t^{\sigma(j+1)-j} \right) \right) = \mathcal{O} \left(\exp \left[-bt(1 + \mathcal{O}(t^{-(1-\sigma)(j+1)})) \right] \right),$$

which is still of the order $\mathcal{O}(e^{-\tilde{b}t})$ with some other constant $\tilde{b} > 0$.

Let us now focus on the case $\sigma \in [0, \frac{1}{2})$. Lemma 3.1 allows us to conclude that the quantity ω in (106) will be of the order $1 + \mathcal{O}(t^{2\sigma-1} \ln t)$ provided that we substitute γ with $\frac{\gamma}{2}$ in (107). This gives us the statement of Theorem 5.

Beyond the regime $\sigma \in [0, \frac{1}{2})$, we obtain

Theorem 6. Let (x, t) be on a curve

$$\frac{x}{12t} \equiv \xi = \frac{c^2}{3} - \frac{\beta t^\sigma \ln t}{4c^2 t}.$$

Let $K \geq 0$ be an integer satisfying (97), i.e., $K \geq \frac{2\sigma-1}{1-\sigma}$. Let Q, \tilde{z}_y be as in (100), (99) and γ be solution of (107). Let n be an integer such that $n \leq \gamma \leq n+1$.

Then the solution of the initial value problem (1), (2) is given by formula (110). In particular, for $\sigma \leq \frac{1}{2}$, $K = 0$, $n \asymp t^\sigma$ we have

$$q = \frac{2c}{\cosh \left(2c(x - 4c^2t) + (2n + \frac{3}{2}) \ln t + (2n + \frac{3}{2}) \ln(32c^3) - \ln \frac{\Gamma(n+1)\Gamma(n+\frac{3}{2})}{2\pi} - \ln \frac{4}{(h^*)^2} \right)} + \mathcal{O}(t^{2\sigma-1} \ln t).$$

Remark 5.4. To check the consistency of Theorem 6 with Theorem 2 and Corollary 1, one need to match the phases and the center lines of the peaks, in the spirit of the end of Sec. 3. We verified this for $K = 0$. While this can be done for several first $K = 1, 2, \dots$ it is unclear how to deal with a general integer K . The reason is that γ and $\frac{\gamma}{2}$, which determine the number of soliton to which point (x, t) is constrained, are defined in quite a different implicit manner. In any case, this would not bring a new result since theorems 2, 4, 5 and Corollary 1 together give a complete description of asymptotics of $q(x, t)$ in the transition zone $4c^2t - \varepsilon t \leq x \leq 4c^2t$.

A Asymptotics of modified Laguerre polynomials

In this section we prove Lemma 5.1 and hence Corollary 3. Large $n \rightarrow \infty$ asymptotics of the RHP as in Corollary 3 with $\Lambda = n$ was studied in [48], but that treatment is not suitable for our present purposes.

In order to study large $n \rightarrow \infty$ asymptotics, first we introduce a new matrix-valued function

$$W(\zeta) = e^{n\ell\sigma_3} Y(\zeta) e^{n(g(\zeta)-\ell)\sigma_3},$$

where $g(\zeta)$ is the function analytic in $\mathbb{C} \setminus (-\infty, a]$ defined by:

$$g(\zeta) = \frac{2}{\pi a} \int_0^a \ln(\zeta - s) \sqrt{\frac{a-s}{s}} ds = \frac{-2}{a} \sqrt{\zeta(\zeta-a)} - \ln a + \ln \left(2\zeta - a + 2\sqrt{\zeta(\zeta-a)} \right) + \frac{2\zeta}{a} + \ell,$$

A simple computation shows that $g(\zeta)$ has the asymptotic expansion

$$g(\zeta) = \ln \zeta - \frac{a}{4\zeta} - \frac{a^2}{16\zeta^2} - \frac{5a^3}{192\zeta^3} + \cdots = \ln \zeta - \sum_{j=1}^{\infty} \frac{2(2n-1)!}{(n+1)n!^2} \frac{a^n}{4^n \zeta^n}. \quad (111)$$

The constant a, ℓ are given by

$$a = \frac{4n}{\Lambda}, \quad \ell := \ln \frac{a}{4e}, \quad .$$

At $\zeta = \infty$ we have $W(\zeta) = (I + \mathcal{O}(\zeta^{-1}))$, and W satisfies the boundary value condition

$$W_+ = W_- \begin{pmatrix} e^{n(g_+-g_-)} & 0 \\ \sqrt{\zeta} e^{n(g_-+g_+-2\ell-\frac{\Lambda\zeta}{n})} & e^{-n(g_+-g_-)} \end{pmatrix}, \zeta \in (0, +\infty), \quad W_+ = W_- e^{n(g_+-g_-)\sigma_3}, \zeta \in (-\infty, 0).$$

Further, we introduce the "effective" potential

$$\varphi(\zeta) := -\frac{\Lambda}{2n} \zeta + g(\zeta) - \ell,$$

so that the jump condition can be written in the following form:

$$W_+ = W_- \begin{pmatrix} e^{n(\varphi_+-\varphi_-)} & 0 \\ \sqrt{\zeta} e^{n(\varphi_-+\varphi_+)} & e^{-n(\varphi_+-\varphi_-)} \end{pmatrix}.$$

It is convenient now to scale the interval $(0, a)$ into $(0, 1)$, i.e.

$$W^{(1)}(\lambda) := W(\zeta), \quad \lambda := \frac{\zeta}{a} = \frac{\zeta \Lambda}{4n}.$$

Denote

$$\begin{aligned} \psi(\lambda) &:= \varphi(\zeta) = -2 \int_1^\lambda \sqrt{\frac{s-1}{s}} ds = -2\sqrt{\lambda(\lambda-1)} + \ln \left(-1 + 2\lambda + 2\sqrt{\lambda(\lambda-1)} \right) = \\ &= -2\lambda + \ln \lambda + \ln 4e - h(\lambda) = -2\lambda + \ln \lambda + \ln 4e + \mathcal{O}(\lambda^{-1}), \end{aligned}$$

where $h(\cdot)$ is as in (89). The scaled effective potential $\psi(\lambda)$, which is analytic in $\lambda \in \mathbb{C} \setminus (-\infty, 1]$, has the following properties on the interval $(-\infty, 1]$:

$$\psi_+ - \psi_- = 2\pi i, \quad \lambda \in (-\infty, 0), \quad \psi_- + \psi_+ = 0, \quad \lambda \in (0, 1). \quad (112)$$

Due to the above properties (112), function

$$W^{(2)}(\lambda) = a^{\frac{\sigma_3}{4}} W^{(1)}(\lambda) a^{-\frac{\sigma_3}{4}}$$

does not have jump along $\lambda \in (-\infty, 0)$, and solves a RHP of the form

$$\begin{aligned} W_+^{(2)} &= W_-^{(2)} \begin{pmatrix} e^{n(\psi_+-\psi_-)} & 0 \\ \sqrt{\lambda} & e^{-n(\psi_+-\psi_-)} \end{pmatrix}, \lambda \in (0, 1), \quad W_+^{(2)} = W_-^{(2)} \begin{pmatrix} 1 & 0 \\ \sqrt{\lambda} e^{2n\psi} & 1 \end{pmatrix}, \lambda \in (1, \infty), \\ W^{(2)}(\lambda) &= I + \mathcal{O}\left(\frac{1}{\lambda}\right). \end{aligned}$$

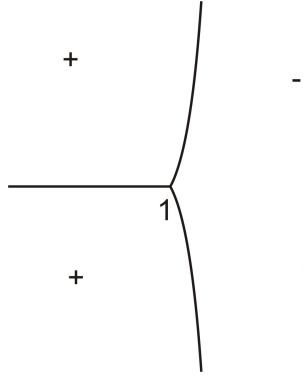


Figure 6: Distribution table of signs for $\text{Re } \psi(\lambda)$.

Tracking back the relation between $W^{(2)}(\lambda)$ and $W^{(1)}(\lambda)$, $W(\zeta)$, $Y(\zeta)$, $L(\zeta)$, we see that

$$W^{(2)}(\lambda) = \left(\frac{\sqrt{2} n^{n+\frac{1}{4}}}{e^n} \right)^{\sigma_3} L(4n\lambda) (4n\lambda)^{n\sigma_3} e^{-nh(\lambda)\sigma_3} \left(\frac{e^n}{\sqrt{2} n^{n+\frac{1}{4}}} \right)^{\sigma_3},$$

Important role now is played by the signature table of $\text{Re } \psi$. It is shown in the figure 6.

Now we introduce the δ - transformation, which removes the term $\sqrt{\lambda}$ from the jump matrix. It must solve the following scalar conjugation problem:

$$\delta_+ \delta_- = \sqrt{\lambda}, \quad \lambda \in (0, 1).$$

The solution is given by the formula

$$\delta(\lambda) = \exp \left\{ \frac{R(\lambda)}{2\pi i} \int_0^1 \frac{\ln \sqrt{s} ds}{(s-\zeta) R_+(s)} \right\}, \quad R(\lambda) = \sqrt{\lambda(\lambda-1)},$$

$$\delta(\lambda) = \left(\frac{\lambda}{2\lambda-1+2\sqrt{\lambda(\lambda-1)}} \right)^{1/4}, \quad \delta(\infty) = \frac{1}{\sqrt{2}}.$$

With a series of transformations, we now show how to remove the oscillating jump from the interval $(0, 1)$. First of them is

$$W^{(3)}(\lambda) := \delta^{\sigma_3}(\infty) W^{(2)}(\lambda) \delta^{-\sigma_3}(\lambda).$$

The jump matrix $W_+^{(3)} = W_+^{(3)} J^{(3)}$ is

$$J^{(3)} = \begin{pmatrix} \frac{\delta_-}{\delta_+} e^{n(\psi_+ - \psi_-)} & 0 \\ e^{n(\psi_- + \psi_+)} & \frac{\delta_+}{\delta_-} e^{-n(\psi_+ - \psi_-)} \end{pmatrix}, \quad \lambda \in (0, 1), \quad J^{(3)} = \begin{pmatrix} 1 & 0 \\ \frac{\sqrt{\lambda}}{\delta^2} e^{2n\psi} & 1 \end{pmatrix}, \quad \lambda \in (1, +\infty).$$

Now we need to factorize the jump matrix on the interval $[0, 1]$ according to the signature table, namely

$$\begin{pmatrix} \frac{\delta_-}{\delta_+} e^{n(\psi_+ - \psi_-)} & 0 \\ e^{n(\psi_- + \psi_+)} & \frac{\delta_+}{\delta_-} e^{-n(\psi_+ - \psi_-)} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\delta_-^2 e^{-2n\psi_-}}{\sqrt{\lambda}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -e^{-n(\psi_- + \psi_+)} \\ e^{n(\psi_- + \psi_+)} & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{\delta_+^2 e^{-2n\psi_+}}{\sqrt{\lambda}} \\ 0 & 1 \end{pmatrix}.$$

The next step is to "open lenses", we transform the RH problem, "moving" the left and the right factor in the above formula to the lower and upper half-planes, respectively. For a new RH problem $W^{(4)}$ the jumps $W_+^{(4)} = W_-^{(4)} J^{(4)}$ are

$$J^{(4)} = \begin{cases} \begin{pmatrix} 1 & \frac{\delta_-^2 e^{-2n\psi}}{\sqrt{\lambda}} \\ 0 & 1 \end{pmatrix}, \lambda \in l_u(\text{ the upper lense }), & \begin{pmatrix} 1 & \frac{\delta_+^2 e^{-2n\psi}}{\sqrt{\lambda}} \\ 0 & 1 \end{pmatrix}, \lambda \in l_d(\text{ the lower lense }), \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \lambda \in (0, 1), & \begin{pmatrix} 1 & 0 \\ \frac{\sqrt{\lambda}}{\delta^2} e^{2n\psi} & 1 \end{pmatrix}, \lambda \in (1, +\infty). \end{cases}$$

The RHP for $W^{(4)}$ gives us the first statement of Lemma 5.1 concerning function

$$\mathcal{Q}_n(\lambda) := W^{(4)}(\lambda) (\delta(\lambda)\delta^{-1}(\infty))^{\sigma_3}.$$

Indeed, the fact that $\mathcal{Q}_n(\lambda)$ decomposes into an asymptotic series follows, for example, from an explicit representation (59) of function $L(\cdot)$. Then RHP for $W^{(4)}$ gives us uniformity with respect to $n \geq 1$ of all the ingredients in this asymptotic series. Further, since for large $n \rightarrow \infty$ the jump matrix is concentrated mainly on the interval $(0, 1)$, the solution to the RH problem for $W^{(4)}$ is close everywhere except for a neighborhood of the points 0, 1 to the solution

$$M_{mod} = \begin{pmatrix} a(\lambda) & ib(\lambda) \\ -ib(\lambda) & a(\lambda) \end{pmatrix}, \quad a(\lambda) = \frac{1}{2}(\gamma(\lambda) + \gamma^{-1}(\lambda)), \quad b(\lambda) = \frac{1}{2}(\gamma(\lambda) - \gamma^{-1}(\lambda)), \quad \gamma(\lambda) = \sqrt[4]{\frac{\lambda-1}{\lambda}}$$

to the model problem with jumps only on $(0, 1)$. This gives us the second statement of Lemma 5.1 concerning function

$$\mathcal{E}_n(\lambda) := W^{(4)}(\lambda) M_{mod}^{-1}(\lambda).$$

However, rigorous proof of the fact that ingredients in asymptotic series for \mathcal{E}_n tend to 0 as $n \rightarrow \infty$ involves construction of local parametrices around the points 0, 1, since jump matrix for $W^{(4)}$ is not uniformly close to I in vicinities of these points. The details can be found for example in [48].

A.1 Airy Parametrix

For the benefit of the reader we recall the form of the so-called ‘‘Airy’’ parametrix, which is used to provide local solutions of a large class of Riemann Hilbert problems [1]. This is defined by

$$\Psi_{Ai}(\zeta) = \left\{ \begin{array}{ll} \begin{pmatrix} v_1 & v_0 \\ v'_1 & v'_0 \end{pmatrix}, \arg \zeta \in (0, 2\pi/3), & \begin{pmatrix} v_{-1} & v_0 \\ v'_{-1} & v'_0 \end{pmatrix}, \arg \zeta \in (0, -2\pi/3), \\ \begin{pmatrix} v_1 & -iv_{-1} \\ v'_1 & -iv'_{-1} \end{pmatrix}, \arg \zeta \in (2\pi/3, \pi), & \begin{pmatrix} v_{-1} & iv_1 \\ v'_{-1} & iv'_1 \end{pmatrix}, \arg \zeta \in (-\pi, -2\pi/3), \end{array} \right\} \times e^{\pi i \sigma_3 / 4}. \quad (113)$$

where we have used the notation

$$v_0(\zeta) = \sqrt{2\pi} Ai(\zeta), \quad v_1(\zeta) = \sqrt{2\pi} e^{-\pi i/6} Ai(\zeta e^{-2\pi i/3}), \quad v_{-1}(\zeta) = \sqrt{2\pi} e^{\pi i/6} Ai(\zeta e^{2\pi i/3}). \quad (114)$$

We have

$$v_0 - iv_1 + iv_{-1} \equiv 0.$$

This function $\Psi_{Ai}(\zeta)$ has the following jumps $\Psi_{Ai,+} = \Psi_{Ai,-} J_{\Psi_{Ai}}$:

$$J_{\Psi_{Ai}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \arg \zeta = 0, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \arg \zeta = \frac{2\pi}{3}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \arg \zeta = \frac{-2\pi}{3}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \arg \zeta = \pi.$$

and the following behaviour at infinity:

$$\Psi_{Ai}(\zeta) = \zeta^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(I + \begin{pmatrix} \frac{-1}{48} & \frac{-1}{8} \\ \frac{1}{8} & \frac{1}{48} \end{pmatrix} \zeta^{-3/2} + \mathcal{O}(\zeta^{-3}) \right) e^{\frac{2}{3}\zeta^{3/2}\sigma_3} e^{\pi i \sigma_3 / 4}.$$

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