Solving Infinite-Dimensional Optimization Problems using Polynomial Approximation

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What is infinite-dimensional optimization ?

Optimization problems where the decision variable belongs to a infinite-dimensional space i.e:

$$\inf_{x \in X} f(x)$$
$$x \in C$$

where X is a normed vector space of **infinite dimension**.

Furthermore, our interest is in **convex** optimization problems i.e. $f : X \to \mathbb{R}$ and $C \subset X$ must be convex.

Why infinite-dimensional optimization problems ?

- Domain of interest since the 17th century with the classical calculus of variations
- Natural generalization of many classical optimization problems to the continuous setting, for example network problems, supply problems, transportation problems, etc.
- Natural framework for Optimal Control problems using theory of PDE-constrained optimization
- Natural framework for Shape or Topology optimization problems

Studied problems class

Let X be a normed vector space and $X^{'}$ its topological dual.
We consider the following class of infinite-dim. problems:

$$P^* = \inf_{x \in X} < f, x > \qquad \text{(P)}$$
$$< a_i, x > = b_i \quad \forall i = 1, ..., L$$
$$< c_t, x > \leq d_t \quad \forall t \in T$$
$$\|x\|_X \leq M$$

where $f, a_i, c_t \in X'$, L is finite and T is a (possibly infinite) set.

All solvable convex optimization problems with closed feasible set can be included in this class (a closed convex set in a normed space is equal to the intersection of all closed half-spaces that contain it).

How to solve infinite-dimensional opt. prob.?

- Using infinite-dimensional algorithms whose *implementation* has been discretized
- Using a discretization of the *functions* in the original problem
- Our approach is different: using **polynomial approximation** by discretizing the description of elements of space X

Polynomial Approximation

Let $\{p_1, p_2, ..., p_n, ...\}$ be an infinite family of linearly independent elements of X, e.g. a basis of polynomials and $X_n = \mathrm{span}\{p_1, p_2, ..., p_n\}$.
We consider the sequence of finite-dimensional optimization problems

$$P_n^* = \inf_{x \in X_n} < f, x > \text{ s.t. } < a_i, x > = b_i, \ < c_t, x > \leq d_t, \ \|x\|_X \leq M \quad (\mathrm{P}_n)$$

which are restrictions of problem (P) to the finite-dimensional subspaces X_n (e.g. spaces of polynomials of degree at most $n-1$).

Why consider Polynomial Approximation ?

Polynomial approximation is useful if we can prove the following two properties:

1. The problem (P_n) can be solved in practice using existing (finite-dimensional) optimization methods.
2. The sequence of optimal values P_n^* converges to the optimal value of the original problem P^* when $n \to \infty$.

Resolution of the Polynomial Approximation

- When X is an Hilbert space: P_n is a convex quadratic problem.
- When $X = L^\infty([a,b])$ or $X = W^{k,\infty}([a,b])$ with $X_n = \mathrm{span}\{1, t, .., t^{n-1}\}$: P_n is a semidefinite problem.
- $X = L^q([a,b])$ or $X = W^{k,q}([a,b])$ with q even and $X_n = \mathrm{span}\{1, t, .., t^{n-1}\}$: P_n is a structured convex problem.

Conclusion: for these cases, we are **able to solve** (P_n) **in polynomial time** using interior-point methods.

Convergence of the Polynomial Approximation

Let $A : X \to R^L$ defined by $(Ax)_i = < a_i, x >$. If

1. $\cup X_n$ is dense in X
2. there exists N_1 and $x \in X_{N_1}$ such that $Ax = b$, $\|x\|_X < M$ and $\inf_{t \in T} \left(\frac{d_t - < c_t, x >}{\|c_t\|_{X'}} \right) > 0$
3. there exist N_2 and $\sigma_{N_2} > 0$ such that $\|Ax - b\| \geq \sigma_{N_2} \inf_{\tilde{x} \in X_{N_2}, A\tilde{x} = b} \|x - \tilde{x}\|_X \quad \forall x \in X_{N_2}$

Then P_n^* **converges to** P^* **when** $n \to \infty$, with a rate of convergence that we can characterize quantitatively (i.e. give an upper bound).
In particular, if there are no inequality constraints, $P_n^* \to P^*$ in $O(E_n(x_{opt}))$ where x_{opt} is an optimal solution of (P) and $E_n(x_{opt})$ is its best polynomial approximation error in X_n.

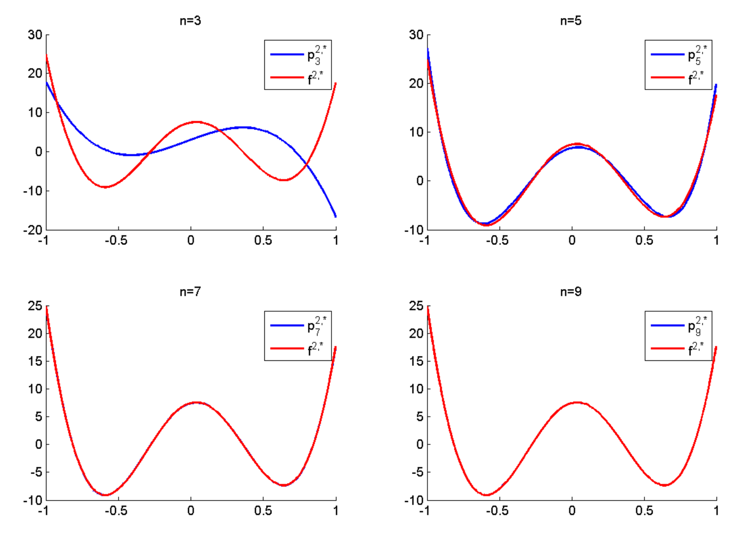
Example of Numerical results

We take $X = L^2([-1,1])$ and no inequality constraints, define the (relative) approximation error as $e_n = \frac{P_n^* - P^*}{P^*}$ and illustrate two cases:

Problem data s.t. $x_{opt} \in \mathcal{C}^\infty$
Convergence of the opt. values:

n	e_n
3	4,2119
5	0.0020408
7	4.5711 $\times 10^{-5}$
9	2.27 $\times 10^{-7}$

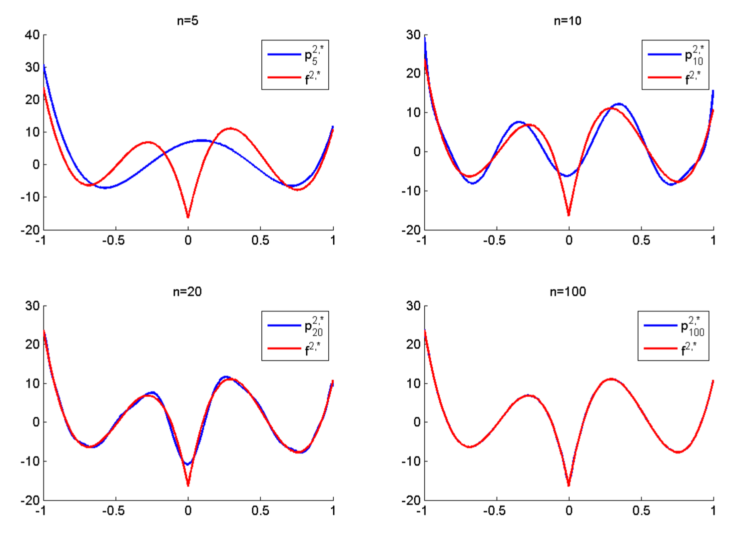
Convergence of the opt. sol.:



Problem data s.t. $x_{opt} \in \mathcal{C} \backslash \mathcal{C}^1$
Convergence of the opt. values:

n	e_n
5	0.50585
10	0.05980
20	0.0089009
100	8.4327$\times 10^{-5}$

Convergence of the opt. sol.:



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