Polynomial Approximation for Infinite-Dimensional Optimization Problems in Lebesgue Spaces

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Introduction : From the finite to the infinite...

$$\inf c^T x \qquad \inf \int_\alpha^\beta \gamma(t) f(t) dt$$

$$a_i^T x = b_i \quad \forall i = 1..., m \qquad \int_\alpha^\beta a_i(t) f(t) dt = b_i \quad \forall i = 1, ..., m$$

$$x_j \geq 0 \quad \forall j = 1, ..., n \qquad f(t) \geq 0 \quad \forall t \in [\alpha, \beta]$$

$$x \in \mathbb{R}^n \qquad f \in X$$

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Motivations

- Translation of the discrete case into the continuous case
- Continuous approximation of large-scale problems
- Optimal Control, Shape Optimization, PDE-constrained Optimization...

Outline

Functional Problem (F^q)

$$F^{q,*} = \inf_{f \in L^q([-1,1])} \int_{-1}^{1} f(t)\gamma(t)dt \tag{F^q}$$

$$\begin{aligned} \int_{-1}^{1} f(t)a_i(t)dt &= b_i \quad \forall i = 1, \ldots, L \\ \|f\|_q &\leq M \end{aligned}$$

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$$\int_{-1}^{1} f(t)a_i(t)dt = b_i \quad \forall i = 1, \ldots, L$$

$$\|f\|_q \leq M$$

Applications

- Supply problems in continuous time
- Large-scale portfolio allocation problems

Polynomial Approximation

Principle : Replace $f \in L^q([-1,1])$ by $f \in \pi^n([-1,1])$

where π^n is the space of the polynomials of degree at most n.

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We know :

- $\{$admissible sol. of $(P_n^q)\} \subset \{$admissible sol. of $(F^q)\}$
  $\Rightarrow F^{q,*} \leq P_n^{q,*}$
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We want :

- (P_n^q) soluble in practice (Part 2)
- Fast convergence of $P_n^{q,*}$ to $F^{q,*}$ when $n \to \infty$ (Part 3)

Outline

Resolving (P_n^q)

Aim : Be able to solve easily the problem (P_n^q) using finite-dimensional optimization methods.

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Results Obtained : Case $q = 2$ and $q = \infty \Rightarrow$: stuctured convex problems.

Translation of (P_n^q) in a basis

Let $\{X_0, ..., X_n\}$ be a basis of $\pi^n([-1,1])$: $p = \sum_{j=0}^n \mathbf{p}_j X_j$.

(P_n^q) becomes :

$$\inf_{\mathbf{p}\in\mathbb{R}^{n+1}} < c^{(n)}, \mathbf{p} >$$

$$A^{(n)}\mathbf{p} = b$$

$$\mathbf{p} \in Q = \left\{ \mathbf{p} \in \mathbb{R}^{n+1} : \left\| \sum_{j=0}^n \mathbf{p}_j X_j \right\|_q \leq M \right\}$$

with

$$c_j^{(n)} = \int_{-1}^{1} \gamma(t) X_j(t) dt$$

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Remaining : translate Q in finite dimensional

- If $q = 2$: $Q = \{\mathbf{p} \in \mathbb{R}^{n+1} :< B\mathbf{p}, \mathbf{p} >\leq M^2\}$ with $B_{ij} = \int_{-1}^{1} X_i(t)X_j(t)dt \quad \forall i, j = 0, ..., n.$
  $\rightarrow$ Convex quadratic problem

- If $q = \infty$: $\|p\|_\infty \leq M$ iff $\begin{cases} p(t) + M \geq 0 & \forall t \in [-1, 1] \\ M - p(t) \geq 0 & \forall t \in [-1, 1] \end{cases}$
  $\Rightarrow$ Finite representation of the cone of positive polynomials using the sum of squares approach.
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Conclusion : (P_n^2) and (P_n^∞) are soluble in practice.

Outline

Obtaining a convergence result

Aim : Prove, under some hypothesis, the convergence of $P_n^{q,*}$ to $F^{q,*}$ when $n \to +\infty$ with, if possible, a guarantee over the convergence speed.

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How : Introduction of a perturbed polynomial problem :

$$P_{n,\epsilon}^{q,*} = \inf_{p \in \pi^n([-1,1])} \int_{-1}^{1} p(t)\gamma(t)dt \qquad (\mathrm{P}_{n,\epsilon}^q)$$

$$\|Ap - b\|_2 \leq \epsilon$$

$$\|p\|_q \leq M$$

with $\epsilon > 0$ and $Ap = \begin{pmatrix} \int_{-1}^{1} p(t)a_1(t)dt \\ \vdots \\ \int_{-1}^{1} p(t)a_L(t)dt \end{pmatrix}$.

Sketch of the proof

We know :

- $P_{n,\epsilon}^{q,*} \leq P_n^{q,*} \quad \forall \epsilon \geq 0$
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We will show :

- $P_{n,\epsilon_q(n)}^{q,*} \leq F^{q,*} + \Delta_1^q(n)$ for a specific $\epsilon_q(n)$(STEP 1)
- $P_n^{q,*} \leq P_{n,\epsilon}^{q,*} + \epsilon \Delta_2^q(n)$ for all $\epsilon \geq 0$ (STEP 2)

Sketch of the proof

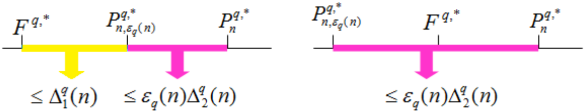
We know :

- $P^{q,*}_{n,\epsilon} \leq P^{q,*}_{n} \quad \forall \epsilon \geq 0$
- $F^{q,*} \leq P^{q,*}_{n}$

We will show :

- $P^{q,*}_{n,\epsilon_q(n)} \leq F^{q,*} + \Delta^q_1(n)$ for a specific $\epsilon_q(n)$(STEP 1)
- $P^{q,*}_{n} \leq P^{q,*}_{n,\epsilon} + \epsilon\Delta^q_2(n)$ for all $\epsilon \geq 0$ (STEP 2)

We will conclude :



In the worst case : $F^{q,*} \leq P^{q,*}_{n} \leq F^{q,*} + \Delta^q_1(n) + \epsilon_q(n)\Delta^q_2(n)$.

Outline

STEP 1 : Obtainment of $\Delta_1^q(n)$

Principle :

$\boxed{(F^q)}$ $\boxed{(P^q_{n,\varepsilon_q(n)})}$ $\boxed{(P^q_n)}$

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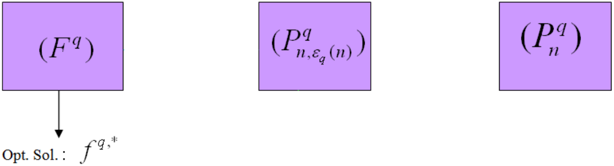

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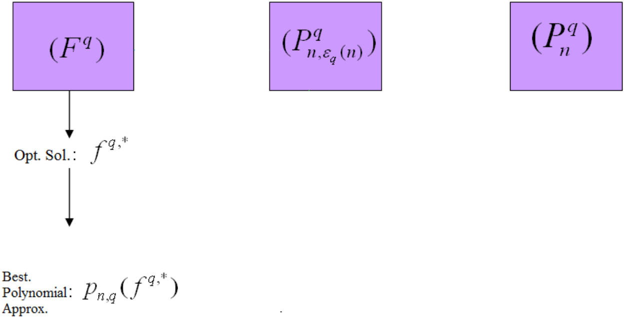

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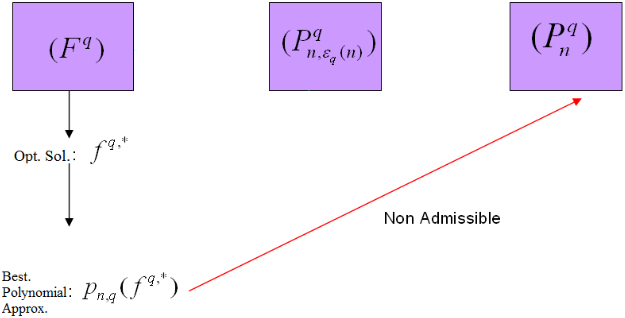

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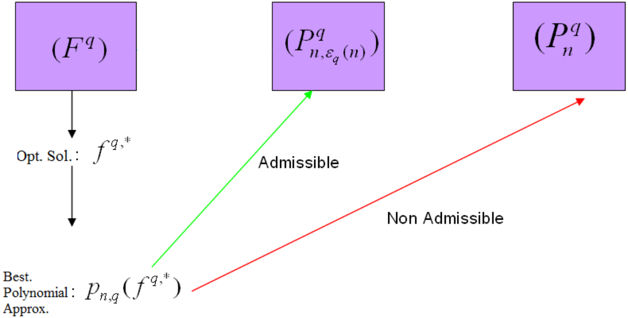

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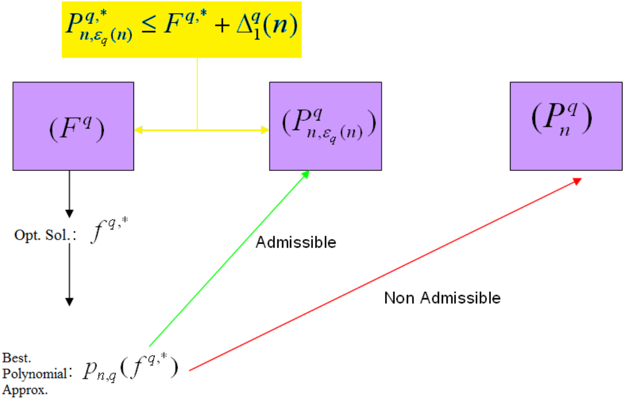

Obtained results

$$\text{Notation : } E_n(f)_q = \inf_{p \in \pi^n([-1,1])} \|f - p\|_q$$

- If $1 \leq q \leq +\infty$,

$$P^{q,*}_{n,\epsilon_q(n)} \leq F^{q,*} + \overbrace{2E_n(f^{q,*})_q \|\gamma\|_{q'}}^{\Delta_1^q(n)}$$

with $\epsilon_q(n) = 2E_n(f^{q,*})_q \sqrt{\sum_{i=1}^{L} \|a_i\|_{q'}^2}$

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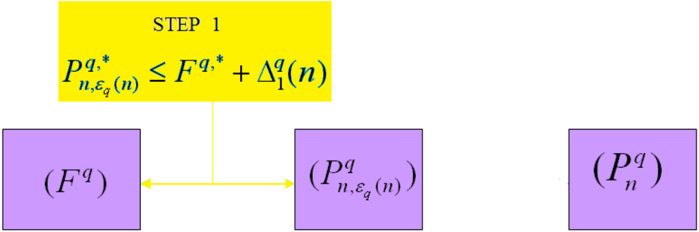
- Possible improvement if $q = 2$,

$$P^{2,*}_{n,\overline{\epsilon}(n)} \leq F^{2,*} + \overbrace{E_n(f^{2,*})_2 E_n(\gamma)_2}^{\Delta_1^2(n)}$$

with $\overline{\epsilon}(n) = E_n(f^{2,*})_2 \sqrt{\sum_{i=1}^{L} E_n(a_i)_2^2}$

Outline

STEP 2 : Obtainment of $\Delta_2^q(n)$



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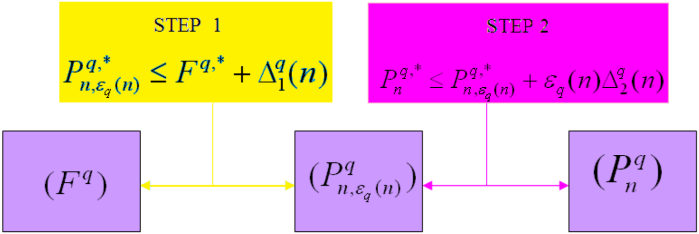

Tool : Regularity Theorem

Let

- X and Y be two normed vector spaces
- $A : X \to Y$ be a linear application
- $b \in Y$
- $Q \subset X$ be convex, closed, bounded with non empty interior

Let's consider the optimization problems :

$$g^* = \inf < x, c > \qquad g^*_\epsilon = \inf < x, c >$$

$$Ax = b \qquad \|Ax - b\|_Y \leq \epsilon$$

$$x \in Q \qquad x \in Q$$

Regularity Theorem (2)

- H.1 : $A : X \to Y$ is non degenerate which means :

$$\|Ax - b\|_Y \geq \sigma d(x, \mathcal{L}) \quad \forall x \in X$$

with $\mathcal{L} = \{x \in X : Ax = b\}$ and $\sigma > 0$

- H.2 : there is $\hat{x} \in Q$ such that $A\hat{x} = b$ and

$$B(\hat{x}, \rho) \subset Q \subset B(\hat{x}, R)$$

with $\rho > 0, R > 0$

$\Rightarrow$

$$g^* - \frac{\epsilon \|c\|_{X'}}{\sigma}\left(1 + \frac{R}{\rho}\right) \leq g^*_\epsilon \leq g^*.$$

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Purpose : Use to link $P_n^{q,*}$ and $P_{n,\epsilon}^{q,*}$.

Case q=2 : Link between $P_n^{2,*}$ and $P_{n,\epsilon}^{2,*}$

Satisfaction of the hypothesis :

- If the functions a_i are linearly independent $\Rightarrow$ H.1 OK
  with $\sigma = \sigma_n = \sqrt{\lambda_{\min}(A_n A_n^T)} > 0 \quad \forall n \geq N_1$ where
  $A_n = A|_{\pi^n([-1,1])}$.
- If there are N_2 and $\hat{p} \in \pi^{N_2}([-1,1])$ such that $A\hat{p} = b$ and
  $\|\hat{p}\|_2 < M \Rightarrow$ H.2 OK
  with $\frac{R}{\rho} = \frac{2M}{M - \mathcal{R}_n^2}$ and $\mathcal{R}_n^2 = \min_{p \in \pi^n([-1,1]), Ap=b} \|p\|_2$.

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Regularity Theorem :

$$P_n^{2,*} \leq P_{n,\epsilon}^{2,*} + \epsilon \overbrace{\frac{\|\gamma\|_2}{\sigma_n}\left(1 + \frac{2M}{M-\mathcal{R}_n^2}\right)}^{\Delta_2^2(n)} \forall n \geq N = \max\{N_1, N_2\}$$

Case $q \neq 2$: Link between $P_n^{q,*}$ and $P_{n,\epsilon}^{q,*}$

Satisfaction of the hypothesis :

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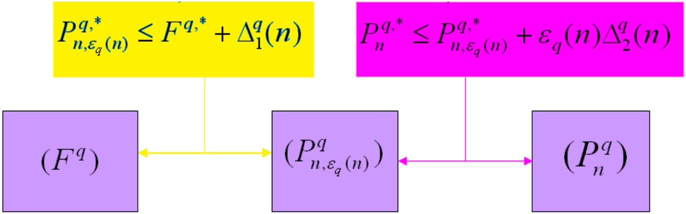
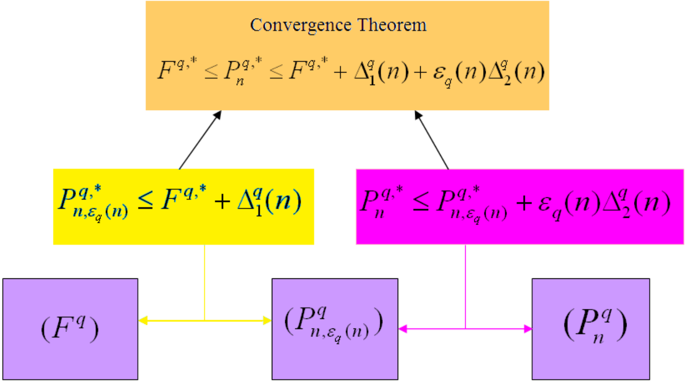
Outline

Final convergence results



Final convergence results



Case $q = 2$: Convergence of $P_n^{2,*}$ to $F^{2,*}$

Our results :

If

- the functions a_i are linearly independant
- the polynomial problems (P_n^2) are strictly admissible from a definite threshold

then for all $n \geq N = \max\{N_1, N_2\}$:

$$\left|F^{2,*} - P_n^{2,*}\right| \leq \Delta_1^2(n) + \epsilon_2(n)\Delta_2^2(n) =$$

$$E_n(f^{2,*})_2 \left(E_n(\gamma)_2 + \frac{\sqrt{\sum_{i=1}^{L} E_n(a_i)_2^2}}{\sigma_n} \|\gamma\|_2 \left(1 + \frac{2M}{M - \mathcal{R}_n^2}\right)\right)$$

Case $q = 2$: Convergence of $P_n^{2,*}$ to $F^{2,*}$ (2)

Convergence

We know : $E_n(f)_2 \to 0$ for all $f \in L^2([-1,1])$

We conclude : $\boxed{P_n^{2,*} \to F^{2,*}}$.

Case $q = 2$: Convergence of $P_n^{2,*}$ to $F^{2,*}$ (2)

Convergence

We know : $E_n(f)_2 \to 0$ for all $f \in L^2([-1,1])$
We conclude : $P_n^{2,*} \to F^{2,*}$.

Convergence speed

We know :

- if $f \in \mathcal{C}^{r-1,r-1}([-1,1])$: $E_n(f)_2 = O(\frac{1}{n^r})$ where $\mathcal{C}^{k,r}([-1,1]) = \{f \in \mathcal{C}^k([-1,1]) | f^{(r)}$ is Lipschitz-continuous$\}$.
- $f^{2,*}$ is unique and expresses itself as a linear combination of the a_i and γ

We conclude :

If the a_i and $\gamma \in \mathcal{C}^{r-1,r-1}$ then $P_n^{2,*} \to F^{2,*}$ in $O(\frac{1}{n^{2r}})$

Case $q \neq 2$: Convergence of $P_n^{q,*}$ to $F^{q,*}$

Our results :

If

- the functions a_i are linearly independant, in $L^2([-1,1])$ such that $\sup_n \max_i \left\| \sum_{j=0}^n < a_i, L_j > L_j \right\|_q < \infty$
- the polynomial problems (P_n^q) are strictly admissible from a definite threshold

then for all $n \geq N = \max\{N_1, N_2\}$:

$$\left| F^{q,*} - P_n^{q,*} \right| \leq \Delta_1^q(n) + \epsilon_q(n)\Delta_2^q(n) =$$
$$2E_n(f^{q,*})_q \left\|\gamma\right\|_{q'} \left(1 + \frac{\sqrt{\sum_{i=1}^L \|a_i\|_{q'}^2}\sqrt{\sum_{i=1}^L \left\| \sum_{j=0}^n < a_i, L_j > L_j \right\|_q^2}}{\sigma_n^2} \left(1 + \frac{2M}{M - \mathcal{R}_n^q} \right) \right)$$

Case $q \neq 2$: Convergence of $P_n^{q,*}$ to $F^{q,*}$ (2)

Convergence

- If $q < +\infty$
  We know : $E_n(f)_q \to 0$ for all $f \in L^q([-1,1])$
  We conclude : $\boxed{P_n^{q,*} \to F^{q,*}}$
- If $q = +\infty$
  We know : $E_n(f)_\infty \to 0$ iff $f \in \mathcal{C}([-1,1])$
  We conclude : the convergence is not guaranteed.

Case $q \neq 2$: Convergence of $P_n^{q,*}$ of $F^{q,*}$ (3)

Convergence speed

We know : If $f \in \mathcal{C}^{r-1,r-1}([-1,1])$: $E_n(f)_q = O(\frac{1}{n^r})$

We conclude :

If $f^{q,*} \in \mathcal{C}^{r-1,r-1}$ then $P_n^{q,*} \to F^{q,*}$ in $O(\frac{1}{n^r})$.

Outline

1. Problem description
2. Resolving the polynomial approximation
3. Convergence of the optimal values
4. Conclusion

From the infinite to the finite...

New approach of resolution for the (F^q) class : Polynomial approximations scheme.

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Otained results

- Polynomial problems = structured convex problems in finite dimension (for $q = 2$ and $q = \infty$)
- Convergence of $P_n^{q,*}$ to $F^{q,*}$ under some hypothesis

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New approach of resolution for the (F^q) class : Polynomial approximations scheme.

Otained results

- Polynomial problems = structured convex problems in finite dimension (for $q = 2$ and $q = \infty$)
- Convergence of $P_n^{q,*}$ to $F^{q,*}$ under some hypothesis

$\Rightarrow$ Obtention of an effective resolution method for the problems (F^q).

Further researches

- Improve, if possible, the convergence bound obtained in the case $q = \infty$
- Obtain a translation of the problems (P_n^q) for $q \neq 2$ and $q \neq \infty$ in a convex structured problem
- Generalize the polynomial approximation scheme for other classes of infinite problems (derivatives, punctual constraints,...)

Thanks for your attention !

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