Double Smoothing Algorithm for a class of Infinite-dimensional Optimization Problems

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Outline

1. Studied problem class with application in optimal control.
2. Dual Approach.
3. Double Regularization.
4. Solving the dual problem.
5. Reconstruction of a primal solution.
6. Conclusion and Further Research.

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Linear optimal control problem with scalar control

Differential system:

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + B(t)u(t)$$

with $\mathbf{x}(t) = (x_1(t), ..., x_m(t))^T$, $A(t) \in \mathbb{R}^{m\times m}$ and $B(t) \in \mathbb{R}^{m\times 1}$.

Initial and final conditions for the state

$$\mathbf{x}(t_0) = \mathbf{x}_0 \quad \mathbf{x}(t_f) = \mathbf{x}_f.$$

Control constraints

$$0 \leq u(t) \leq 1 \quad \forall t \in [t_0, t_f].$$

Performance Measure

$$\min \int_{t_0}^{t_f} < \mathbf{c}(t), \mathbf{x}(t) > + < \mathbf{d}(t), \mathbf{x}'(t) > + e(t)u(t)dt.$$

State equation of the linear system

The state generated by the linear system for a control $u(t)$ is given by:

$$\mathbf{x}(t) = X(t, t_0)\mathbf{x}(t_0) + X(t, t_0)\left(\int_{t_0}^{t} X^{-1}(\tau, t_0)B(\tau)u(\tau)d\tau\right)$$

where $X(t, t_0)$ is the state transition matrix of the system i.e the unique solution of the matricial Cauchy problem:

$$\frac{d}{dt}X(t, t_0) = A(t)X(t, t_0)$$

$$X(t_0, t_0) = I.$$

Consequence: We can express all only in terms of the control

- Replacing $\mathbf{x}(t)$ by this expression, we have a performance measure of the form:

$$\int_{t_0}^{t_f} f(t)u(t)dt.$$

- Final conditions is satisfied by the state iff

$$\int_{t_0}^{t_f} X^{-1}(\tau, t_0)B(\tau)u(\tau)d\tau = X^{-1}(t_f, t_0)\mathbf{x}_f - \mathbf{x}_0.$$

  Using the explicit expression for $X(t, t_0)$ and $B(t)$, we have m linear equality constraints:

$$\int_{t_0}^{t_f} a_i(t)u(t)dt = b_i \quad i = 1, \ldots, m.$$

New formulation

Our initial optimal control problem can be expressed as an **infinite-dimensional linear programming** problem:

$$\inf_u \int_{t_0}^{t_f} f(t)u(t)dt$$

$$\int_{t_0}^{t_f} a_i(t)u(t)dt = b_i \quad \forall i = 1, ..., m$$

$$0 \leq u(t) \leq 1 \quad \forall t \in [t_0, t_f].$$

Studied problem class: Linear Programming in $L^\infty([\alpha, \beta])$.

$$P^* = \inf_{x \in L^\infty([\alpha,\beta])} \int_\alpha^\beta f(t)x(t)dt$$

$$\int_\alpha^\beta a_i(t)x(t)dt = b_i \quad \forall i = 1, ..., m$$

$$M_1 \leq x(t) \leq M_2 \quad \text{a.e. in } [\alpha, \beta]$$

where

- $b \in \mathbb{R}^m$
- $f \in L^1([\alpha, \beta])$
- $a_i \in L^1([\alpha, \beta])$ for all $i = 1...m$.

Outline

Primal Problem

$$P^* = \inf_{x \in L^\infty([\alpha,\beta])} \int_\alpha^\beta f(t)x(t)dt$$

$$\int_\alpha^\beta a_i(t)x(t)dt = b_i \quad \forall i = 1, \ldots, m$$

$$M_1 \leq x(t) \leq M_2 \quad \text{a.e. in } [\alpha, \beta].$$

If we dualize the linear equality constraints, we obtain the equivalent formulation:

$$P^* = \inf_{x \in \overline{X}} \sup_{y \in \mathbb{R}^m} \int_\alpha^\beta f(t)x(t)dt + \sum_{i=1}^m y_i \left(\int_\alpha^\beta a_i(t)x(t)dt - b_i \right)$$

where $\overline{X} = \{x \in L^\infty([\alpha, \beta]) : M_1 \leq x(t) \leq M_2 \text{ a.e. in } [\alpha, \beta]\}$.

Dual Problem

$$
\begin{aligned}
D^* &= \sup_{y\in\mathbb{R}^m} \inf_{x\in\overline{X}} \int_\alpha^\beta f(t)x(t)dt + \sum_{i=1}^m y_i \left(\int_\alpha^\beta a_i(t)x(t)dt - b_i \right) \\
&= \sup_{y\in\mathbb{R}^m} -\sum_{i=1}^m y_i b_i - \phi(y)
\end{aligned}
$$

where

$$
\phi(y) = \sup_{x\in\overline{X}} \int_\alpha^\beta \left(-f(t) - \sum_{i=1}^m y_i a_i(t) \right) x(t)dt.
$$

Why a Dual Approach ?

Advantages of the dual problem:

- The dual is an unconstrained optimization problem in finite-dimension ($y \in \mathbb{R}^m$).
- The infinite-dimensional problem defining $\phi(y) = \sup_{M_1 \leq x(t) \leq M_2} \int_\alpha^\beta \left(-f(t) - \sum_{i=1}^m y_i a_i(t)\right) x(t) dt$ is easy:

$$x_y(t) = \begin{cases} M_1, & \text{when } f(t) + \sum_{i=1}^m y_i a_i(t) > 0, \\ M_2, & \text{when } f(t) + \sum_{i=1}^m y_i a_i(t) < 0 \\ \frac{M_1+M_2}{2}, & \text{when } f(t) + \sum_{i=1}^m y_i a_i(t) = 0 \end{cases}$$

  is an optimal solution.
- There is Strong Duality i.e : $P^* = D^*$ when the dual is solvable.

But θ can be non-differentiable...

If we rewrite the dual problem as a minimization problem:

$$-D^* = \min_{y \in \mathbb{R}^m} \theta(y) = \min_{y \in \mathbb{R}^m} b^T y + \phi(y)$$

where $\phi(y) = \sup_{x \in \overline{X}} \int_\alpha^\beta \left(-f(t) - \sum_{i=1}^m y_i a_i(t)\right) x(t) dt$,

θ can be **non-differentiable**:

$$\partial\theta(y) = \left\{ b - A\tilde{x} = \left(b_1 - \int_\alpha^\beta a_1(t)\tilde{x}(t)dt, ..., b_m - \int_\alpha^\beta a_m(t)\tilde{x}(t)dt \right)^T \right.$$

$$\left. \text{for any optimal solution } \tilde{x} \text{ of the problem defining } \phi(y) \right\}$$

and the optimization problem defining $\phi(y)$ can have multiple optimal solutions.

Conclusion:

We have to solve a non-smooth convex optimization problem.

How to solve a non-smooth convex problem ?

- **The classical approach: subgradient-type scheme.**
  Advantage : Can be applied directly on the dual objective function without any regularization
  Disadvantage: Slow Convergence
  $$\theta(y_k) \to \theta^* \text{ in } O\left(\frac{1}{\epsilon^2}\right).$$
- **The smoothing approach.**
  We modify the dual objective function in order to be able to apply more efficient scheme of smooth convex optimization.
  Advantage : Faster convergence, we will obtain a scheme such that
  $$\theta(y_k) \to \theta^* \text{ in } O\left(\frac{1}{\epsilon}\ln\left(\frac{1}{\epsilon}\right)\right).$$
  Disadvantage : We have to modify the dual objective function with two regularizations.

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Double Regularization of the dual objective function

In order to

- be able to solve efficiently the dual problem
- to be able to obtain a nearly optimal and feasible primal solution from a nearly optimal dual solution

we will modify the dual objective function with two regularizations:

1. A first regularization on the infinite-dimensional side of the problem
2. A second regularization on the finite-dimensional side of the problem.

First regularization

Why?
In order to obtain a smooth objective function for the dual problem with gradient Lipschitz-continuous i.e.:

$$\|\nabla g(y) - \nabla g(\bar{y})\| \leq L \|y - \bar{y}\| \quad \forall y, \bar{y} \text{ with } L < +\infty$$

and therefore be able to apply efficient schemes of smooth convex optimization.

Where?
On the infinite-dimensional side of the problem.

First regularization: How ?

How?
Modify the dual objective function:

$$\begin{aligned}\theta(y) &= b^T y + \phi(y)\\ &= b^T y + \sup_{x \in \overline{X}} \int_\alpha^\beta \left(-f(t) - \sum_{i=1}^m y_i a_i(t) \right) x(t) dt\end{aligned}$$

in

$$\begin{aligned}\theta_\mu(y) &= b^T y + \phi_\mu(y)\\ &= b^T y + \sup_{x \in \overline{X}} \int_\alpha^\beta \left(-f(t) - \sum_{i=1}^m y_i a_i(t) \right) x(t) - \frac{\mu}{2} x^2(t) dt\end{aligned}$$

with $\mu > 0$.

First regularization: θ_μ is differentiable

With the addition of the strongly concave function $-\frac{\mu}{2}\int_\alpha^\beta x^2(t)dt$, the optimization problem defining $\phi_\mu(y)$ has only one optimal solution given by:

$$x_{y,\mu}(t) = \begin{cases} M_1, & \text{if } f(t) + \sum_{i=1}^m y_i a_i(t) \geq -\mu M_1, \\ M_2, & \text{if } f(t) + \sum_{i=1}^m y_i a_i(t) \leq -\mu M_2 \\ \frac{-1}{\mu}(f(t) + \sum_{i=1}^m y_i a_i(t)), & \text{else.} \end{cases}$$

The function θ_μ is therefore differentiable.

First regularization: Further properties of θ_μ

θ_μ is gradient Lipschitz-continuous with constant

$$L_\mu = \frac{\|A\|^2}{\mu}$$

where

$$Ax = \left(\int_\alpha^\beta a_1(t)x(t)dt, \ldots, \int_\alpha^\beta a_m(t)x(t)dt \right)^T$$

and

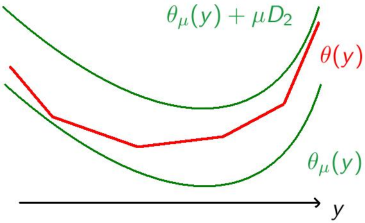
$$\|A\| = \sup_{x \in L^2([\alpha,\beta]), y \in \mathbb{R}^m} \{ < Ax, y >: \|x\|_2 = 1, \|y\|_2 = 1 \}.$$

First regularization: Further properties of θ_μ

θ_μ is a good approximation of θ with absolute accuracy bound depending on μ:

$$\theta_\mu(y) \leq \theta(y) \leq \theta_\mu(y) + \mu D_2 \quad \forall y \in \mathbb{R}^m$$

where $D_2 = \frac{1}{2}(\beta - \alpha) \max\{|M_1|, |M_2|\}^2$.



Second regularization

Why?

We want not only

- to solve the dual problem

but also

- to reconstruct from the obtained nearly optimal dual solution, a nearly optimal and feasible primal solution.

Reconstruction of a primal solution

For a given dual iterate y_k, if we consider the function

$$x_k(t) = x_{y_k,\mu}(t),$$

the unique optimal solution of the problem defining $\phi_\mu(y_k)$, we have:

- $$\left| \int_\alpha^\beta f(t)x_k(t)dt - P^* \right| \leq Cste \, |\theta(y_k) - \theta^*| + Cste \, \|\nabla\theta_\mu(y_k)\|$$
- $$\|Ax_k - b\| = \|\nabla\theta_\mu(y_k)\| \,.$$

The quality of this primal solution depends not only on the convergence rate of $\theta(y_k)$ to θ^* but also on the convergence rate of $\|\nabla\theta_\mu(y_k)\|$ to 0.

Convexity and Gradient Lipschitz-continuity are not enough

If the dual objective function is convex, gradient Lipschitz-continuous and if we apply the optimal scheme for $F_L^{1,1}(\mathbb{R}^m)$:

$$g(y_k) \to g^* \text{ in } O\left(\frac{1}{\sqrt{\epsilon}}\right)$$

but the convergence of the gradient is slower:

$$\|\nabla g(y_k)\| \to 0 \text{ in } O\left(\frac{1}{\epsilon}\right).$$

Convexity and Gradient Lipschitz-continuity are not enough

In our case, if we apply this scheme to our function $\theta_\mu \in F^{1,1}_{L_\mu}(\mathbb{R}^m)$ with a good choice for μ, we have

$$\theta(y_k) - \theta^* \to 0 \text{ in } O\left(\frac{1}{\epsilon}\right)$$

but

$$\|\nabla \theta_\mu(y_k)\| \to 0 \text{ in } O\left(\frac{1}{\epsilon^2}\right).$$

Therefore if the dual objective function is only convex and gradient Lipschitz-continuous, we have a convergence rate in $O\left(\frac{1}{\epsilon^2}\right)$ for the primal sequence.

This is not better than with the subgradient scheme!

Is the smoothing approach useless ?

No!!!

If the dual objective function is also strongly convex, we can apply the optimal scheme for $S_{\sigma,L}^{1,1}(\mathbb{R}^m)$ for wich we have the same rate of convergence for $g(y_k) - g^*$ and $\|\nabla g(y_k)\|$ in

$$O\left(\exp(\left(-k\sqrt{\frac{\sigma}{L}}\right)\right).$$

Conclusion: We want a strongly convex objective function on the dual.

Second Regularization

Where?
On the finite-dimensional side of the problem.
How?
Let $\sigma > 0$ and $y_0 \in \mathbb{R}^m$, adding to the function θ_μ, the strongly convex function $\frac{\sigma}{2}\|y - y_0\|^2$, we obtain the function:

$$\theta_{\mu,\sigma}(y) = \theta_\mu(y) + \frac{\sigma}{2}\|y - y_0\|^2$$

wich is strongly convex with parameter σ .

Modified Dual Objective Function

$$\theta_{\mu,\sigma}(y) = b^T y + \phi_\mu(y) + \frac{\sigma}{2} \|y - y_0\|^2$$

where

- $\phi_\mu(y) = \sup_{x \in \overline{X}} \int_\alpha^\beta \left(-f(t) - \sum_{i=1}^m y_i a_i(t)\right) x(t) - \frac{\mu}{2} x^2(t) dt$
- y_0 is any element in $\mathbb{R}^m$
- $\sigma > 0$
- $\mu > 0$.

This function is:

- Strongly convex with parameter σ
- still Gradient Lipschitz-continuous now with constant $L_\mu + \sigma$.

Outline

Optimal Scheme for $S^{1,1}_{\sigma,L}(\mathbb{R}^m)$

Let $g : \mathbb{R}^m \to \mathbb{R}$ be

- strongly convex with parameter $\sigma > 0$
- gradient Lipschitz-continuous with constant $L > 0$.

Algorithm

1. Initialization
   Choose $w_0 = y_0 \in \mathbb{R}^m$.
2. Iteration $(k \geq 0)$
   Set

$$y_{k+1} = w_k - \frac{1}{L}\nabla g(w_k)$$

$$w_{k+1} = y_{k+1} + \frac{\sqrt{L} - \sqrt{\sigma}}{\sqrt{L} + \sqrt{\sigma}}(y_{k+1} - y_k).$$

Optimal Scheme for $S_{\sigma,L}^{1,1}(\mathbb{R}^m)$: Convergence Results

1. **Convergence of the objective values:**

$$g(y_k) - g^* \le \left(g(y_0) - g^* + \frac{\sigma}{2}\|y_0 - y^*\|^2\right) \exp\left(-k\sqrt{\frac{\sigma}{L}}\right)$$

2. **Convergence of the gradient:**

$$\|\nabla g(y_k)\| \le \sqrt{2L}\sqrt{g(y_0) - g^* + \frac{\sigma}{2}\|y_0 - y^*\|^2} \exp\left(-\frac{k}{2}\sqrt{\frac{\sigma}{L}}\right)$$

3. **Convergence of the iterates:**

$$\|y_k - y^*\| \le \sqrt{\frac{2}{\sigma}}\sqrt{g(y_0) - g^* + \frac{\sigma}{2}\|y_0 - y^*\|^2} \exp\left(-\frac{k}{2}\sqrt{\frac{\sigma}{L}}\right).$$

Solving the dual problem using the optimal scheme for $S^{1,1}_{\sigma,L_\mu+\sigma}(\mathbb{R}^m)$

If we apply the optimal scheme for $S^{1,1}_{\sigma,L_\mu+\sigma}(\mathbb{R}^m)$ to our modified dual function, we have:

$$\theta(y_k) - \theta^* \leq \mu D_2 + \frac{\sigma}{2}\|y^{**} - y_0\|^2$$

$$+2(\theta(y_0) - \theta^* + \mu D_2)\exp\left(-k\sqrt{\frac{\sigma}{L_\mu + \sigma}}\right)$$

$$+\sqrt{2\sigma}\sqrt{\|y^{**} - y_0\|^2 + \frac{2\mu}{\sigma}D_2}\sqrt{\theta(y_0) - \theta^* + \mu D_2}\exp\left(-\frac{k}{2}\sqrt{\frac{\sigma}{L_\mu + \sigma}}\right)$$

where y^{**} is an optimal solution of the original dual problem $\min_{y \in \mathbb{R}^m} \theta(y)$ and $\theta^* = -P^*$ is the optimal value of this problem. If we want an accuracy $\theta(y_k) - \theta^* \leq \epsilon$, we can choose μ, σ and k such that each of the four terms are $\leq \epsilon/4$.

Choice of μ and σ

- If we want

$$\mu D_2 \leq \frac{\epsilon}{4}$$

we choose

$$\mu(\epsilon) = \frac{1}{4D_2}\epsilon = C_1\epsilon.$$

- If we want

$$\frac{\sigma}{2}\|y^{**} - y_0\|^2 \leq \frac{\epsilon}{4}$$

we choose

$$\sigma(\epsilon) = \frac{\epsilon}{2\|y^{**} - y_0\|^2} = C_2\epsilon.$$

Number of iterations needed

- If we want $2(\theta(y_0) - \theta^* + \mu(\epsilon) D_2) \exp\left(-k\sqrt{\frac{\sigma(\epsilon)}{L_\mu(\epsilon)+\sigma(\epsilon)}}\right) \leq \frac{\epsilon}{4}$ we have to choose

$$k(\epsilon) \geq g_1(\epsilon) = O\left(\frac{1}{\epsilon}\ln\left(\frac{1}{\epsilon}\right)\right).$$

- If we want $\sqrt{2\sigma(\epsilon)}\sqrt{\|y^{**} - y_0\|^2 + \frac{2\mu(\epsilon)}{\sigma(\epsilon)} D_2}\sqrt{\theta(y_0) - \theta^* + \mu(\epsilon) D_2}$ $\exp\left(-\frac{k}{2}\sqrt{\frac{\sigma(\epsilon)}{L_\mu(\epsilon)+\sigma(\epsilon)}}\right) \leq \frac{\epsilon}{4}$ we have to choose

$$k(\epsilon) \geq g_2(\epsilon) = O\left(\frac{1}{\epsilon}\ln\left(\frac{1}{\epsilon}\right)\right).$$

Convergence of the gradient

In order to reconstruct from the nearly optimal dual solution, a nearly feasible and optimal primal solution with a given accuracy, we need the rate of convergence of $\|\nabla\theta_\mu(y_k)\|$.

We have:

$$\|\nabla\theta_\mu(y_k)\| \leq \sigma\sqrt{\|y^{**} - y_0\|^2 + \frac{2\mu}{\sigma}D_2}$$

$$+\left(\sqrt{2(L_\mu+\sigma)} + \sqrt{2\sigma}\right)\sqrt{\theta(y_0) - \theta^* + \mu D_2}\,\exp\left(-\frac{k}{2}\sqrt{\frac{\sigma}{L_\mu+\sigma}}\right).$$

Convergence of the gradient: Number of iterations

With $\mu = C_1\epsilon$ and $\sigma = C_2\epsilon$, we have:

$$\sigma\sqrt{\|y^{**}-y_0\|^2+\frac{2\mu}{\sigma}D_2} = \left(C_2\sqrt{\|y^{**}-y_0\|^2+\frac{2C_1}{C_2}D_2}\right)\epsilon = C_3\epsilon.$$

Furthermore, if we want

$$\left(\sqrt{2(L_\mu+\sigma)}+\sqrt{2\sigma}\right)\sqrt{\theta(y_0)-\theta^*+\mu D_2}\,\exp\left(-\frac{k}{2}\sqrt{\frac{\sigma}{L_\mu+\sigma}}\right) \leq \epsilon$$

we have to take

$$k(\epsilon) \geq g_3(\epsilon) = O\left(\frac{1}{\epsilon}\ln\left(\frac{1}{\epsilon}\right)\right).$$

Complexity

Let $\epsilon > 0$ and choose $\mu(\epsilon) = C_1\epsilon$, $\sigma(\epsilon) = C_2\epsilon$, after

$$k(\epsilon) = \max\{g_1(\epsilon), g_2(\epsilon), g_3(\epsilon)\} = O\left(\frac{1}{\epsilon}\ln\left(\frac{1}{\epsilon}\right)\right)$$

iterations, we have:

- $$\theta(y_{k(\epsilon)}) - \theta^* \leq \epsilon$$
- $$\left\|\nabla\theta_\mu(y_{k(\epsilon)})\right\| \leq (C_3 + 1)\epsilon.$$

Outline

A nearly feasible and optimal primal solution

Using the dual iterate $y_{k(\epsilon)}$, we can obtain a primal solution with the same order of accuracy.
Consider

$$x_{k(\epsilon)}(t) = x_{y_{k(\epsilon)},\mu(\epsilon)}(t)$$

the unique optimal solution of the problem defining $\phi_{\mu(\epsilon)}(y_{k(\epsilon)})$ that we can compute analytically.

A nearly feasible and optimal primal solution (2)

This function $x_{k(\epsilon)}$ is:

- In $\overline{X}$ by construction.
- Nearly optimal for the primal problem:

$$\left| \int_\alpha^\beta f(t) x_{k(\epsilon)}(t) dt - P^* \right| \leq ((C_3+1)C_4 + C_1 D_2 + \max\{1, C_1 D_2\})\epsilon.$$

where $C_4 = \sqrt{\|y^{**} - y_0\|^2 + \frac{2C_1}{C_2} D_2} + \|y_0\|$.

- Nearly feasible for the linear equality constraints:

$$\|\nabla \theta_\mu(y_k(\epsilon))\| = \left\| A x_{k(\epsilon)} - b \right\| \leq (C_3 + 1)\epsilon.$$

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Double Smoothing Algorithm: Conclusion

After $k(\epsilon) = \max\{g_1(\epsilon), g_2(\epsilon), g_3(\epsilon)\} = O\left(\frac{1}{\epsilon} \ln\left(\frac{1}{\epsilon}\right)\right)$ iterations, we have:

- A nearly optimal dual solution:

$$\theta(y_k(\epsilon)) - \theta^* \leq \epsilon$$

- A nearly optimal and feasible primal solution:

$$\left| \int_\alpha^\beta f(t) x_{k(\epsilon)}(t) dt - P^* \right| \leq ((C_3+1)C_4 + C_1 D_2 + \max\{1, C_1 D_2\})\epsilon$$

$$\left\| A x_{k(\epsilon)} - b \right\| \leq (C_3 + 1)\epsilon.$$

Come back to the Optimal Control Problem

Applying the double smoothing algorithm to the optimal control problem, we generate after $k(\epsilon) = O\left(\frac{1}{\epsilon}\ln\left(\frac{1}{\epsilon}\right)\right)$ iterations, a control $u_{k(\epsilon)}(t)$:

- Nearly optimal for the performance measure:

$$\left|\int_{t_0}^{t_f} < \mathbf{c}(t), \mathbf{x}(t) > + < \mathbf{d}(t), \mathbf{x}'(t) > + e(t)u_{k(\epsilon)}(t)dt - P^*\right|$$

$$\leq ((C_3 + 1)C_4 + C_1 D_2 + \max\{1, C_1 D_2\})\epsilon$$

- Satisfying nearly the initial and final conditions for the states:

$$\mathbf{x}(t_0) = \mathbf{x}_0 \quad \mathbf{x}(t_f) = \mathbf{x}_f + \Delta$$

  where $\|\Delta\| \leq \|X(t_f, t_0)\| (1 + C_3)\epsilon$

- Satisfying the control constraints:

$$0 \leq u_{k(\epsilon)}(t) \leq 1 \quad \forall t \in [t_0, t_f].$$

Further Research

- Introduction of integral inequality constraints

$$\int_{\alpha}^{\beta} a_j(t)x(t) \leq b_j$$

- Generalization to vector-valued function $\mathbf{x}(t)$ in order to consider optimal control problem with multiple controls

$$\mathbf{u}(t) = (u_1(t), ..., u_n(t))^T$$

- Consider more general pointwise constraints:

$$\mathbf{x}(t) \in Q(t) \quad \forall t \in [\alpha, \beta]$$

  where $Q(t)$ is a bounded convex set for all $t \in [\alpha, \beta]$

- ...

Thanks for your attention !