Double Smoothing Algorithm for a class of Optimal Control problems

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Outline

1. Studied problem class.
2. Dual Approach.
3. Double Regularization.
4. Solving the dual problem.
5. Reconstruction of a primal solution.
6. Conclusion and Further Research.

A class of Optimal Control problems.

$$P^* = \inf_{u \in L^2([0,T],\mathbb{R}^m)} \int_0^T G(t,u(t)) + \langle a(t), x(t) \rangle dt$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) = x_0$$

$$x(t_i) \in Q_i \quad i = 1, ..., N$$

$$u(t) \in P(t) \quad \text{a.e. in } [0, T]$$

where

- $Q_i \subset \mathbb{R}^n$ are convex, closed and bounded
- $P(t) \subset \mathbb{R}^m$ are convex, closed for all t such that $P = \cup_{t \in [0,T]} P(t)$ is bounded
- $G : [0, T] \times P \to \mathbb{R}$ is convex in u, bounded and continuously differentiable in (t, u)
- $A(t) \in \mathcal{C}([0, T], \mathbb{R}^{n \times n})$, $B(t) \in \mathcal{C}([0, T], \mathbb{R}^{n \times m})$.

State equation of the linear system

State generated by the linear system for a control $u(t)$ is linear in u:

$$x(t) = \Phi(t,0)x(0) + \int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau, \quad t \in [0, T]$$

where $\Phi(t,\tau)$ is the state transition matrix of the system i.e the unique solution of the matricial Cauchy problem:

$$\frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau)$$

$$\Phi(\tau,\tau) = I.$$

We can express all only in terms of the control

- Replacing $x(t)$ by this expression, we have a performance measure of the form:

$$\int_0^T F(t, u(t))dt.$$

- State constraint $x(t_i) \in Q_i$ becomes

$$\mathcal{L}_i u := \int_0^{t_i} \Phi(t_i, \tau)B(\tau)u(\tau)d\tau \in C_i := Q_i - \Phi(t_i, 0)x_0$$

where $\Phi(t_i, 0)x_0$ is the value at time t_i of the unique solution of the Cauchy problem:

$$\dot{x}(t) = A(t)x(t), \quad x(0) = x_0.$$

We can express all only in terms of the control

If we define $A_i : [0, T] \to \mathbb{R}^{n\times m}$ by:

$$A_i(\tau) = \begin{cases} \Phi(t_i, \tau)B(\tau), & \text{when } \tau \in [0, t_i], \\ 0, & \text{when } \tau \in]t_i, T]. \end{cases}$$

the optimal control problem becomes:

$$\inf_{u \in L^2([0,T],\mathbb{R}^m)} \int_0^T F(t, u(t))dt$$

$$\mathcal{L}_i(u) = \int_0^T A_i(t)u(t)dt \in C_i \quad \forall i = 1, ...N$$

$$u(t) \in P(t) \quad a.e \in [0, T].$$

Our problem is easy without the coupling constraint

$$P^* = \inf_{u \in L^2([0,T],\mathbb{R}^m)} \int_0^T F(t,u(t))dt$$

$$u(t) \in P(t) \quad \text{a.e. in } [0,T].$$

We can minimize in a pointwise way i.e. solve for each $t \in [0, T]$:

$$min_{u \in P(t)} F(t,u).$$

Basis idea: dualize the finite number of coupling constraints

$$\int_0^T A_i(t)u(t)dt \in C_i.$$

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Primal Problem

As C_i is convex, we have:

$$\mathcal{L}_i u \in C_i \Leftrightarrow \langle \mathcal{L}_i u, z^i \rangle \leq \sigma_{C_i}(z^i) \quad \forall z^i \in \mathbb{R}^n$$

where $\sigma_{C_i}(z^i) = \sup_{y \in C_i} \langle y, z^i \rangle$.
Dualizing the coupling constraints, we obtain the formulation:

$$P^* = \inf_{u \in U} \left[\int_0^T F(t, u(t))dt + \sup_{z \in \mathbb{R}^{N \times n}} \sum_{i=1}^{N} (\langle \mathcal{L}_i u, z^i \rangle - \sigma_{C_i}(z^i)) \right]$$

where $z = (z^1, ..., z^N) \in \mathbb{R}^{N \times n}$ and
$U = \{u \in L^2([0, T], \mathbb{R}^m) : u(t) \in P(t) \text{ a.e. in } [0, T]\}$.

Dual Problem

$$
\begin{aligned}
D^* &= \sup_{z \in \mathbb{R}^{N\times n}} -\sum_{i=1}^{N} \sigma_{C_i}(z^i) + \inf_{u\in U} \left(\int_0^T F(t,u(t))dt + \sum_{i=1}^{N} \langle \mathcal{L}_i u, z^i \rangle \right) \\
&= \sup_{z \in \mathbb{R}^{N\times n}} -\sum_{i=1}^{N} \sigma_{C_i}(z^i) - \phi(z)
\end{aligned}
$$

where
$\phi(z) = \sup_{u\in U} \left(\int_0^T -F(t,u(t)) - \sum_{i=1}^{N} \langle u(t), A_i(t)^T z^i \rangle dt \right)$.
Advantages of the dual problem:

- Strong Duality holds i.e : $P^* = D^*$ when the dual is solvable
- The dual is an unconstrained optimization problem in finite-dimension ($z \in \mathbb{R}^{N\times n}$).

More advantages ?

- The infinite-dimensional problem defining $\phi(z) = \sup_{u\in U}\left(\int_0^T -F(t,u(t)) - \sum_{i=1}^N \langle u(t), A_i(t)^T z^i\rangle dt\right)$.
  can be solved pointwisely.
  The function u^* defined at each $t \in [0, T]$ by:

$$u^*(t) = \arg\max_{v\in P(t)} \left\{-F(t,v) - \sum_{i=1}^{N} \langle v, A_i(t)^T z^i\rangle\right\}$$

  is an optimal solution of this problem.

Remark:
$A_i(t)$ depends directly on the state transition matrix which is often not known but we can compute $A_i^T(t)z^i$ as $B(t)v(t)$ where:

$$\dot{v}(t) = -A(t)^T v(t), \quad v(t_i) = z^i, \quad t \in [0, t_i],$$

extended by zero for $t \in [t_i, T]$.

But the dual objective function can be non-differentiable...

If we rewrite the dual problem as a minimization problem:

$$-D^* = \theta^* = \min_{z \in \mathbb{R}^{N \times n}} \theta(z) = \min_{z \in \mathbb{R}^{N \times n}} \sum_{i=1}^{N} \sigma_{C_i}(z^i) + \phi(z)$$

σ_{C_i} and ϕ can be **non-differentiable**:

- $\partial \sigma_{C_i}(z^i) = \left\{ \tilde{y} \in C_i : \langle \tilde{y}, z^i \rangle = \sigma_{C_i}(z^i) \right\}$
- $\partial \phi(z) =$
  $\{-\mathcal{A}\tilde{u}$ for any optimal solution $\tilde{u}$ of the problem defining $\phi(z)\}$.

Conclusion: We have to solve a non-smooth convex problem.

How to solve a non-smooth convex problem ?

- **The classical approach: subgradient-type scheme.**
  Advantage : Can be applied directly on the dual objective function without any regularization
  Disadvantage: Slow Convergence
  $$\theta(z_k) \to \theta^* \text{ in } O\left(\frac{1}{\epsilon^2}\right).$$
- **The smoothing approach.**
  We modify the dual objective function in order to be able to apply more efficient scheme of smooth convex optimization.
  Advantage : Faster convergence, we will obtain a scheme such that
  $$\theta(z_k) \to \theta^* \text{ in } O\left(\frac{1}{\epsilon}\ln\left(\frac{1}{\epsilon}\right)\right).$$
  Disadvantage : We have to modify the dual objective function with some regularizations.

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Double Regularization of the dual objective function

In order to

- be able to solve efficiently the dual problem
- to be able to obtain a nearly optimal and feasible primal solution from a nearly optimal dual solution

we will modify the dual objective function with two regularizations:

1. A first regularization that make the **gradient** of the dual objective function **Lispchitz-continuous**
2. A second regularization that make the dual objective function **strongly convex.**

Why a double smoothing ?

Method	Dual function	Dual conv.	Primal conv.
Subgradient	Convex but Non-Smooth	$O\left(\frac{1}{\epsilon^2}\right)$	$O\left(\frac{1}{\epsilon^2}\right)$
Simple Smoothing	Convex ∇ Lipschitz-cont	$O\left(\frac{1}{\epsilon}\right)$	$O\left(\frac{1}{\epsilon^2}\right)$
Double Smoothing	Strongly convex ∇ Lipschitz-cont.	$O\left(\frac{1}{\epsilon}\ln\left(\frac{1}{\epsilon}\right)\right)$	$O\left(\frac{1}{\epsilon}\ln\left(\frac{1}{\epsilon}\right)\right)$

Modified Dual Objective Function

$$\theta_{\rho,\mu,\kappa}(z) = \sum_{i=1}^{N} \sigma_{\rho,C_i}(z^i) + \phi_\mu(z) + \frac{\kappa}{2}\|z\|_2^2$$

where

- $\sigma_{\rho,C_i}(z^i) = \sup_{y \in C_i}\{\langle y, z^i\rangle - \frac{\rho}{2}\|y\|_2^2\}$
- $\phi_\mu(z) =$ $\sup_{u \in U} \int_0^T \left(-F(t, u(t)) - \sum_{i=1}^{N}\langle u(t), A_i(t)^T z^i\rangle - \frac{\mu}{2}\|u(t)\|_2^2\right) dt$

Modified Dual Objective Function after the double smoothing

This function $\theta_{\rho,\mu,\kappa}(z)$ is:

- Gradient Lipschitz-continuous with constant

$$L_{\rho,\mu,\kappa} = \frac{1}{\rho} + \frac{\sum_{i=1}^{N} \|\mathcal{L}_i\|_2^2}{\mu} + \kappa$$

- Strongly convex with parameter κ
- A good approximation of the original dual objective function:

$$-\frac{\kappa}{2}\|z\|_2^2 \leq \theta(z) - \theta_{\rho,\mu,\kappa}(z) \leq \rho\widehat{D} + \mu D - \frac{\kappa}{2}\|z\|_2^2 \quad \forall z \in \mathbb{R}^{N\times n}$$

where $\widehat{D} = \sum_{i=1}^{N} \max_{y^i \in C_i} \frac{1}{2}\left\|y^i\right\|_2^2$ and $D = \max_{u\in U} \frac{1}{2}\|u\|_2^2$.

We need to know the Lipschitz constant $L_{\rho,\mu,\kappa}$

We have modified θ in $\theta_{\rho,\mu,\kappa} \in S^{1,1}_{\kappa,L_{\rho,\mu,\kappa}}(\mathbb{R}^{N\times n})$, the class of functions:

- Strongly convex with parameter κ
- Gradient Lipschitz-continuous with constant $L_{\rho,\mu,\kappa}$

That allows us to apply the very efficient optimal scheme for this class of function to our modified dual objective function.

However, in this scheme and in its complexity analysis, we need to know the Lipschitz constant

$$L_{\rho,\mu,\kappa} = \frac{1}{\rho} + \frac{\sum_{i=1}^{N} \|\mathcal{L}_i\|_2^2}{\mu} + \kappa$$

or an upper-bound for this quantity.

Computation of the Lipschitz constant $L_{\rho,\mu,\kappa}$

Exact expression for $\|\mathcal{L}_i\|_2$ If the system is reachable, define the reachability Gramian:

$$W_r(0,t_i) = \int_0^{t_i} \Phi(t_i,\tau)B(\tau)B(\tau)^T\Phi(t_i,\tau)^T d\tau = \mathcal{L}_i\mathcal{L}_i^*.$$

We have:

$$\|\mathcal{L}_i\|_2 = \lambda_{\max}^{1/2}(W_r(0,t_i)).$$

Computable bound for $\|\mathcal{L}_i\|_2$
Consider the time-invariant case i.e : $\dot{x}(t) = Ax(t) + Bu(t)$.
We can solve the following quasi-convex problem that gives a upper-bound for $\|\mathcal{L}_i\|_2^2$:

$$\min\left\{\frac{\eta_3^2}{\tau_1\eta_2} : A^TP + PA \preceq -\tau_1 I, \quad \eta_2 I \preceq P \preceq \eta_3 I, \quad \tau_1,\eta_2,\eta_3 \geq 0\right\}.$$

Outline

Optimal Scheme for $S^{1,1}_{\kappa,L}(\mathbb{R}^{N\times n})$

Let $g : \mathbb{R}^n \to \mathbb{R}$ be

- strongly convex with parameter $\kappa > 0$
- gradient Lipschitz-continuous with constant $L > 0$.

Algorithm

1. Initialization
   Choose $w_0 = z_0 \in \mathbb{R}^m$.
2. Iteration $(k \geq 0)$
   Set

$$z_{k+1} = w_k - \frac{1}{L}\nabla g(w_k)$$

$$w_{k+1} = z_{k+1} + \frac{\sqrt{L} - \sqrt{\kappa}}{\sqrt{L} + \sqrt{\kappa}}(z_{k+1} - z_k).$$

Choice of the smoothing parameters

Let $\epsilon > 0$ and define:

- $\widehat{D} = \sum_{i=1}^{N} D_i$ where $D_i = \max_{y \in C_i} \frac{1}{2} \|y\|_2^2$
- $D = \max_{u \in U} \frac{1}{2} \|u\|_2^2$
- R is such that $\|z^*\|_2 \leq R$ where z^* is the optimal solution of the original dual problem.

We choose the parameters in the following way:

- $$\rho(\epsilon) = \frac{1}{4\widehat{D}}\epsilon = C_1\epsilon$$
- $$\mu(\epsilon) = \frac{1}{4D}\epsilon = C_2\epsilon$$
- $$\kappa(\epsilon) = \frac{1}{2R^2}\epsilon = C_3\epsilon.$$

Solving the dual problem using the optimal scheme for $S^{1,1}_{\kappa,L_{\rho,\mu,\kappa}}(\mathbb{R}^{N\times n})$

If we apply the optimal scheme for $S^{1,1}_{\kappa,L(\rho,\mu,\kappa)}(\mathbb{R}^{N\times n})$ to our modified dual function with the choosen value of the parameters, we have after a number of iteration

$$k(\epsilon) = O\left(\frac{1}{\epsilon}\ln\left(\frac{1}{\epsilon}\right)\right)$$

an iterate $z_{k(\epsilon)}$ such that

$$\theta(z_{k(\epsilon)}) - \theta^* \leq \epsilon.$$

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A nearly feasible and optimal primal solution

Using the dual iterate $z_{k(\epsilon)}$, we can obtain a primal solution with the same order of accuracy. Consider:

- $$u_{k(\epsilon)} = \arg \sup_{u \in U} \int_0^T \left(-F(t, u(t)) - \sum_{i=1}^{N} \langle u(t), A_i(t)^T z^i_{k(\epsilon)} \rangle - \frac{\mu(\epsilon)}{2} \|u(t)\|_2^2 \right) dt$$

  the unique optimal solution of the problem defining $\phi_{\mu(\epsilon)}(z_{k(\epsilon)})$.

- $$y^i_{k(\epsilon)} = \arg \sup_{y \in C_i} \{ \langle y, z^i_{k(\epsilon)} \rangle - \frac{\rho(\epsilon)}{2} \|y\|_2^2 \}$$

  the unique optimal solution of the problem defining $\sigma_{\rho(\epsilon), C_i}(z^i_{k(\epsilon)})$.

A nearly feasible and optimal primal solution

This function $u_{k(\epsilon)}$ is:

- In U by construction i.e $u_{k(\epsilon)}(t) \in P(t)$ a.e. in $[0, T]$
- Nearly optimal for the primal problem:

$$\left| \int_0^T F(t, u_{k(\epsilon)}(t))dt - P^* \right| \leq 2(1 + 2\sqrt{3})\epsilon$$

- Nearly feasible for the coupling constraints:

$$\mathrm{dist}(\mathcal{L}_i u_{k(\epsilon)}, C_i) \leq \left\| \mathcal{L}_i u_{k(\epsilon)} - y^i_{k(\epsilon)} \right\|_2 \leq \frac{2}{R}\epsilon.$$

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Double Smoothing Algorithm: Conclusion

After $k(\epsilon) = O\left(\frac{1}{\epsilon}\ln\left(\frac{1}{\epsilon}\right)\right)$ iterations, the double smoothing algorithm provide us with a nearly optimal:

$$\left|\int_0^T G(t, u_{k(\epsilon)}(t)) + \langle a(t), x(t)\rangle dt - P^*\right| \leq 2(1 + 2\sqrt{3})\epsilon$$

and nearly feasible:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) = x_0$$

$$\mathrm{dist}(x(t_i), Q_i) \leq \frac{2}{R}\epsilon \quad \forall i = 1, ..., N$$

$$u_{k(\epsilon)}(t) \in P(t) \quad \forall t \in [0, T]$$

solution of the original optimal control problem.

Further research

- **What is the efficiency of the double smoothing algorithm in practice ?**
  Numerical experimentation and comparison with methods based on preliminary discretization.
- How to compute $\phi_\mu(z)$ and $\nabla\phi_\mu(z)$?
  In order to obtain the exact value of these quantities, we need to compute an infinite number of pointwise minimization which is impossible in practice.
  What are the consequence on the optimal scheme, if we use inexact first-order informations ?

Thanks for your attention !

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