On the Wagner-Whitin Lot-Sizing Polyhedron*

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Abstract

We study a family of unbounded polyhedra arising in the study of uncapacitated lot-sizing problems with Wagner-Whitin costs. With n the number of periods, we completely characterize the bounded faces of maximal dimension, and derive an $\mathrm{O}(n^2)$ algorithm to express any point within the polyhedron as a convex combination of extreme points and extreme rays. We also study adjacency on the polyhedra, and give a simple $\mathrm{O}(n)$ test for adjacency. Finally we observe that if we optimize over these polyhedra, the face of optimal solutions can be found in $\mathrm{O}(n^2)$.

Keywords: Polyhedra, Adjacency, Maximal Faces, Dual Algorithm, Lot-Sizing, Wagner-Whitin costs.

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1 Introduction

The uncapacitated lot-sizing problem has been the subject of much research since the original paper of Wagner and Whitin [9]. Given n periods, demands $d_1, \ldots, d_n$ to be satisfied in each period, unit production, unit storage and set-up costs p'_t, h'_{t-1} and f_t respectively for $t = 1, \ldots, n$, the problem is to find a production schedule satisfying all the demands at minimum cost. Using variables x_t, s_t and $y_t \in \{0, 1\}$ to represent production, end-stock and set-up variables respectively, the problem can be formulated as the following mixed integer program:

$$\min \textstyle\sum_{t=1}^{n} p'_t x_t + \sum_{t=0}^{n-1} h'_t s_t + \sum_{t=1}^{n} f_t y_t \tag{1}$$

$$s_{t-1} + x_t = d_t + s_t \text{ for } t = 1, \ldots, n \tag{2}$$

$$x_t \leq M y_t \text{ for } t = 1, \ldots, n \tag{3}$$

$$s_{t-1}, x_t \geq 0, y_t \in \{0, 1\} \text{ for } t = 1, \ldots, n \tag{4}$$

where M is a large positive value. Note that we allow for a variable initial stock s_0, but there is no cost attached to the final stock s_n. Using the flow conservation equations (2), the objective function (1) can be rewritten as

$$\min \sum_{t=0}^{n-1} h_t s_t + \sum_{t=1}^{n} f_t y_t + K$$

where $h_0 = h'_0$, $h_t = h'_t + p'_t - p'_{t+1}$ for $t = 1, \ldots, n-1$ and $K = \sum_{t=1}^{n} p'_t d_t$.

When $h_t \geq 0$ for all t, the problem is said to have *Wagner-Whitin (WW) costs*. With WW costs, once the periods in which there is a set-up are known, it is optimal to produce as late as possible. This special case, that arises very often in practice, has been studied in a variety of papers. In [9] an $O(n^2)$ dynamic programming algorithm was presented for the general problem (1)–(4), and certain properties of optimal solutions were presented for WW costs. Almost thirty years later Wagelmans et al. [8], see also [1, 4], showed that the running time of the dynamic programming algorithm could be reduced to $O(n \log n)$ in the general case, and to $O(n)$ with WW costs. Barany et al. [3] presented a description of the convex hull of solutions in the general case, and Pochet and Wolsey [5] showed how an alternative formulation leads to a simpler polyhedral description of the convex hull in the presence of WW costs. Aghezzaf and Wolsey [2] showed that the number of set-ups in two neighboring vertices of the latter polyhedron differs by at most one.

Specifically Pochet and Wolsey considered the formulation

$$\min\{\sum_{t=0}^{n-1} h_t s_t + \sum_{t=1}^{n} f_t y_t : (s, y) \in P_n, y \in Z_+^n\}$$

where

$$P_n = \{(s, y) \in R^n \times [0, 1]^n : s_{k-1} \geq \sum_{j=k}^{t} d_j (1 - y_k - \ldots - y_j) \text{ for all } k, t \text{ with } 1 \leq k \leq t \leq n\}$$

$$= \{(s,y) \in R^n \times [0,1]^n : s_{k-1} \geq d_{kt} - \sum_{j=k}^{t} d_{jt} y_j \text{ for all } k,t \text{ with } 1 \leq k \leq t \leq n\},$$

where d_{kt} is used to denote $\sum_{j=k}^{t} d_j$ for all $1 \leq k \leq t \leq n$. They showed that P_n, the *Wagner-Whitin polyhedron*, is an integral polyhedron. This formulation clearly shows that the stock s_{k-1} at the end of period $k-1$ contains the demand d_j for period j only if there is no set up in the interval from k to j.

The goal of this paper is to obtain a better understanding of the family of polyhedra P_n. No knowledge of the lot-sizing literature is assumed. In Section 2 we study the facial structure of P_n, in particular the bounded faces where s is minimum, i.e. where

$$s_{k-1} = \max_{t=k,\ldots,n} [d_{kt} - \sum_{j=k}^{t} d_{jt} y_j] \text{ for } k = 1, \ldots, n.$$

Specifically we characterize the bounded faces of maximum dimension and show that their extreme points are integral, and thus provide an alternative proof of the integrality of P_n. We then use this characterization to show an $\mathrm{O}(n^2)$ procedure to represent an arbitrary point of P_n as a convex combination of extreme points and extreme rays of P_n.

In Section 3 we consider adjacency of the vertices of P_n. We give a simple characterization from which it follows that adjacency can be tested in linear time. In Section 4 we reinterpret earlier work of van Hoesel et al. [7] on a dual greedy algorithm as an $\mathrm{O}(n^2)$ algorithm to find all optimal solutions to the uncapacitated lot-sizing problem, or to find the face of optimal solutions when optimizing over P_n.

2 The Bounded Faces of P_n

Given a point $y \in [0,1]^n$, we use the notation $\sigma_{k-1,t}(y) \equiv d_{kt} - \sum_{j=k}^{t} d_{jt} y_j$, and $\sigma_{k-1}(y) \equiv \max_{t=k,\ldots,n} \sigma_{k-1,t}(y)$. So a point $(s,y) = (s_0, \ldots, s_{n-1}, y_1, \ldots, y_n) \in P_n$ if and only if $y \in [0,1]^n$ and $s_{k-1} \geq \sigma_{k-1}(y)$ for all k. We assume throughout that $d_t > 0$ for $t = 1, \ldots, n$.

Observation 1 *$dim(P_n) = 2n$.*

The bounded faces of P_n are obtained by setting one (or more) of each of the inequalities $s_{k-1} \geq \sigma_{k-1,t}(y)$ to equality for each k, so the bounded faces have dimension at most n. Given $\tau = (\tau(1), \ldots, \tau(n)) \in Z^n$ with $k \leq \tau(k) \leq n$, we consider the face

$$F(\tau) = \{(s,y) \in P_n, s_{k-1} = \sigma_{k-1,\tau(k)}(y) \text{ for } k = 1, \ldots, n\}.$$

Proposition 2.1 *For $y \in [0,1]^n$,*
i) $\sigma_{k-1,t}(y) > \sigma_{k-1,t+1}(y)$ if and only if $\sum_{j=k}^{t+1} y_j > 1$,
ii) $\sigma_{k-1,t}(y) > \sigma_{k-1,t+q}(y)$ for all $q \geq 1$ if $\sum_{j=k}^{t+1} y_j > 1$,

iii) $\sigma_{k-1,t}(y) > \sigma_{k-1,t-1}(y)$ *if and only if* $\sum_{j=k}^{t} y_j < 1$,
iv) $\sigma_{k-1,t}(y) > \sigma_{k-1,t-q}(y)$ *for all* $1 \le q \le t-k$ *if* $\sum_{j=k}^{t} y_j < 1$,
v) $\sigma_{k-1}(y) = \sigma_{k-1,t}(y) > \sigma_{k-1,l}(y)$ *for all* $l \ne t$ *if and only if* $\sum_{j=k}^{t} y_j < 1$ *and* $\sum_{j=k}^{t+1} y_j > 1$.

Proof.
i) $\sigma_{k-1,t}(y) - \sigma_{k-1,t+1}(y) = d_{kt} - \sum_{j=k}^{t} d_{jt}y_j - d_{k,t+1} + \sum_{j=k}^{t+1} d_{j,t+1}y_j = d_{t+1}(\sum_{j=k}^{t+1} y_j - 1)$.
ii) $\sigma_{k-1,t}(y) - \sigma_{k-1,t+q}(y) = d_{kt} - \sum_{j=k}^{t} d_{jt}y_j - d_{k,t+q} + \sum_{j=k}^{t+q} d_{j,t+q}y_j = d_{t+1,t+q}(\sum_{j=k}^{t+1} y_j - 1) + \sum_{j=t+2}^{t+q} d_{j,t+q}y_j > 0$ if $\sum_{j=k}^{t+1} y_j - 1 > 0$ as $y \ge 0$.
iii) and iv) are almost identical to i) and ii), and v) follows from i) – iv) and the definition of $\sigma_{k-1}(y)$. ∎

Theorem 2.2 *Every bounded face of* P_n *of dimension* n *is of the form* $F(\tau)$ *with* $\tau(k) \le \tau(k+1)$ *for* $k = 1, \dots, n-1$. *Such a face is described by the constraints:*

$$\begin{gathered} s_{k-1} = \sigma_{k-1,\tau(k)}(y) \text{ for } k = 1, \dots, n \\ \textstyle\sum_{j=k}^{\tau(k)} y_j \le 1 \text{ for } k = 1, \dots, n \\ \textstyle\sum_{j=k}^{\tau(k)+1} y_j \ge 1 \text{ for } k = 1, \dots, n \text{ with } \tau(k) < n \\ 0 \le y_j \le 1 \text{ for } j = 1, \dots, n. \end{gathered}$$

Proof. Suppose $\tau(k+1) < \tau(k)$. Then any point on the face $F(\tau)$ satisfies $\sum_{j=k+1}^{\tau(k+1)+1} y_j \ge 1$ and $\sum_{j=k}^{\tau(k)} y_j \le 1$ from Proposition (2.1). But $\sum_{j=k+1}^{\tau(k+1)+1} y_j \le \sum_{j=k}^{\tau(k)+1} y_j$ as $\tau(k+1) + 1 \le \tau(k)$, and thus $\sum_{j=k+1}^{\tau(k+1)+1} y_j = \sum_{j=k}^{\tau(k)+1} y_j = 1$. Thus $F(\tau)$ is of dimension less than n.

When $\tau(k) \le \tau(k+1)$ for $k = 1, \dots, n$, a point y^* satisfying

$$\begin{gathered} \textstyle\sum_{j=k}^{\tau(k)} y_j < 1 \text{ for } k = 1, \dots, n \\ \textstyle\sum_{j=k}^{\tau(k)+1} y_j > 1 \text{ for } k = 1, \dots, n \\ 0 < y_j < 1 \text{ for } j = 1, \dots, n \end{gathered}$$

is easily constructed by selecting $y_n^*, y_{n-1}^*, \dots, y_1^*$ in that order. Therefore $F(\tau)$ is of dimension n. ∎

Corollary. *The number of bounded n-dimensional faces of* P_n *is* $(2n)!/(n!(n+1)!)$.

We now consider how to express a point $(s, y) \in P_n$ as a convex combination of its extreme points and extreme rays. Obviously it suffices to find which bounded face $F(\tau)$ contains $(\sigma(y), y)$, and then express $(\sigma(y), y)$ as a convex combination of extreme points of $F(\tau)$. Note that, once $F(\tau)$ is determined, it suffices to just consider y as $\sigma(y)$ is uniquely determined by y.

Given y, the corresponding τ is easily calculated. Then consider $F_m(\tau)$ described by

$$
\begin{gathered}
s_{k-1} = \sigma_{k-1,\tau(k)}(y) \text{ for } k = m, \dots, n \\
\textstyle\sum_{j=k}^{\tau(k)} y_j \leq 1 \text{ for } k = m, \dots, n \\
\textstyle\sum_{j=k}^{\tau(k)+1} y_j \geq 1 \text{ for } k = m, \dots, n \\
0 \leq y_j \leq 1 \text{ for } j = m, \dots, n.
\end{gathered}
$$

We use $\bar{y}$ to denote $(y_m, \dots, y_n)$ and $\tilde{y}$ to denote $(y_{m-1}, \dots, y_n)$. The extreme points of $F_m(\tau)$ are just referred to as $\{\bar{y}^i\}_{i\in T}$. In the next proposition, we suppose that $(s^*, y^*) \in F(\tau)$, and $(\bar{s}^*_m, \dots, \bar{s}^*_n, \bar{y}^*_m, \dots, \bar{y}^*_n)$ has already been expressed as a convex combination of extreme points of $F_m(\tau)$, i.e. $(\bar{s}^*, \bar{y}^*) = \sum_{i\in T} \lambda_i(\bar{s}^i, \bar{y}^i)$ with $\sum_{i\in T} \lambda_i = 1, \lambda_i \geq 0$ for $i \in T$.

Proposition 2.3 *Let $T_0 \subset T$ be the extreme points of $F_m(\tau)$ with $\sum_{j=m}^{\tau(m-1)} y_j = 1$, T_1 those with $\sum_{j=m}^{\tau(m-1)+1} y_j = 0$, and T_2 those with $\sum_{j=m}^{\tau(m-1)} y_j = 0$ and $y_{\tau(m-1)+1} = 1$.*
i) if $\bar{y}^i$ is an extreme point of $F_m(\tau)$ with $i \in T_0$, then $\tilde{y}^{i0} = (0, \bar{y}^i)$ is an extreme point of $F_{m-1}(\tau)$.
ii) if $\bar{y}^i$ is an extreme point of $F_m(\tau)$ with $i \in T_1$, then $\tilde{y}^{i1} = (1, \bar{y}^i)$ is an extreme point of $F_{m-1}(\tau)$.
iii) if $\bar{y}^i$ is an extreme point of $F_m(\tau)$ with $i \in T_2$, then both $\tilde{y}^{i0} = (0, \bar{y}^i)$ and $\tilde{y}^{i1} = (1, \bar{y}^i)$ are extreme points of $F_{m-1}(\tau)$.
iv) a) If $m-1 < \tau(m-1)$, $\sum_{i\in T_0} \lambda_i = \sum_{j=m}^{\tau(m-1)} y^*_j$.
b) If $m-1 = \tau(m-1)$, $\sum_{i\in T_1} \lambda_i = 1 - y^*_m$
c) If $m-1 < \tau(m-1) < \tau(m)$, $\sum_{i\in T_2} \lambda_i = y^*_{\tau(m-1)+1}$.
v) $y^*_{m-1} \geq \sum_{i\in T_1} \lambda_i$.

Proof. i) - iii). To be extreme, the points must be integral, be extensions of the extreme points of $F_m(\tau)$, and in addition satisfy $\sum_{j=m-1}^{\tau(m-1)} y_j \leq 1$ and $\sum_{j=m-1}^{\tau(m-1)+1} y_j \geq 1$.
iv). Note that (T_0, T_1, T_2) form a partition of T. Also $\tau(m-1) = \tau(m)$ implies $T_1 = \emptyset$, and $\tau(m-1) = m-1$ implies $T_0 = \emptyset$.
a). $\sum_{j=m}^{\tau(m-1)} \bar{y}^i_j = 1$ for $i \in T_0$ and $\sum_{j=m}^{\tau(m-1)} \bar{y}^i_j = 0$ for $i \in T_1 \cap T_2$, so $\sum_{i\in T_0} \lambda_i = \sum_{i\in T_0} \lambda_i \sum_{j=m}^{\tau(m-1)} \bar{y}^i_j + \sum_{i\in T_1\cup T_2} \lambda_i \sum_{j=m}^{\tau(m-1)} \bar{y}^i_j = \sum_{j=m}^{\tau(m-1)} \sum_{i\in T} \lambda_i \bar{y}^i_j = \sum_{j=m}^{\tau(m-1)} y^*_j$.
b). $\sum_{j=m}^{\tau(m-1)+1} \bar{y}^i_j = \bar{y}^i_m = 0$ for $i \in T_1$, and $\bar{y}^i_m = 1$ for $i \in T_2$. Thus $\sum_{i\in T_2} \lambda_i = y^*_m$, and $\sum_{i\in T_1} \lambda_i = 1 - \sum_{i\in T_2} \lambda_i = 1 - y^*_m$ as $T_0 = \emptyset$.
c). $\sum_{j=m}^{\tau(m-1)} \bar{y}^i_j \leq \sum_{j=m}^{\tau(m)} \bar{y}^i_j \leq 1$. But for $i \in T_0$, $\sum_{j=m}^{\tau(m-1)} \bar{y}^i_j = 1$, and as $\tau(m-1) < \tau(m)$, $y^i_{\tau(m-1)+1} = 0$. So $\bar{y}^i_{\tau(m-1)+1} = 1$ if and only if $i \in T_2$, and thus $\sum_{i\in T_2} \lambda_i = y^*_{\tau(m-1)+1}$.
v) If $m-1 < \tau(m-1) < \tau(m)$, $\sum_{j=m-1}^{\tau(m-1)+1} y^*_j \geq 1$ and thus $y^*_{m-1} \geq 1 - \sum_{j=m}^{\tau(m-1)} y^*_j - y^*_{\tau(m-1)+1} = 1 - \sum_{i\in T_0} \lambda_i - \sum_{i\in T_2} \lambda_i = \sum_{i\in T_1} \lambda_i$.
If $\tau(m-1) = \tau(m)$, $T_1 = \emptyset$, as $y^*_{m-1} \geq 0$ always holds.
If $m-1 = \tau(m-1)$, $y^*_{m-1} + y^*_m \geq 1$ and so $y^*_{m-1} \geq 1 - y^*_m = \sum_{i\in T_1} \lambda_i$ as $T_0 = \emptyset$. ∎

Using this Proposition, it is easy to construct $(y^*_{m-1}, y^*_m, \ldots, y^*_n)$ as a convex combination of extreme points of $F_{m-1}(\tau)$ from the explicit representation $(y^*_m, \ldots, y^*_n) = \sum_{i \in T} \lambda_i \bar{y}^i$ with the extreme points of $F_m(\tau)$.

Specifically each extreme point $\bar{y}^i$ of $F_m(\tau)$ in $T_0 \cup T_1$ extends uniquely and keeps the same weight λ_i. These give a value $\sum_{i \in T_1} \lambda_i$ for y_{m-1}. As $y^*_{m-1} \geq \sum_{i \in T_1} \lambda_i$, it suffices to greedily extend extreme points of T_2 with $y^i_{m-1} = 1$ until the value of y^*_{m-1} is attained, and then extend all other extreme points of T_2 with $y^i_{m-1} = 0$. In this way at most one more extreme point of $F_{m-1}(\tau)$ receives a positive weight. Hence we obtain:

Theorem 2.4 *There is an $O(n^2)$ algorithm to express any point $(s, y) \in P_n$ as a convex combination of $n+1$ extreme points and n extreme rays.*

Example. Suppose $n = 3, d = (10, 4, 7)$ and $(s^*, y^*) = (10.0, 3.1, 4.9, 0.3, 0.5, 0.4)$.

Step 1. As $y^*_1 + y^*_2 \leq 1$ and $y^*_1 + y^*_2 + y^*_3 \geq 1$, $\tau(1) = 2$.
As $y^*_2 + y^*_3 \leq 1$, $\tau(2) = 3$. Thus $\tau = (2, 3, 3)$.

Step 2. $\sigma(y^*) = (\sigma_{0,2}(y^*), \sigma_{1,3}(y^*), \sigma_{2,3}(y^*)) = (7.8, 2.7, 4.2)$. As $s^* \geq \sigma(y^*)$, $(s^*, y^*) \in P_3$.

Step 3. Suppose that (y^*_2, y^*_3) has been expressed as a convex combination of extreme points of $F_2(\tau)$. Specifically $(y^*_2, y^*_3) = (0.5, 0.4) = 0.1 \times (0, 0) + 0.5 \times (1, 0) + 0.4 \times (0, 1)$. We now extend to (y^*_1, y^*_2, y^*_3).

$m = 2$. By Proposition 2.3, the extreme point $(1, 0) \in T_0$ extends to the unique extreme point $(0, 1, 0)$ the extreme point $(0, 0) \in T_1$ to the unique extreme point $(1, 0, 0)$, and the extreme point $(0, 1) \in T_2$ to the two extreme points $(0, 0, 1)$ and $(1, 0, 1)$.
The point $(0, 1, 0)$ inherits the weight 0.5 and the point $(1, 0, 0)$ the weight 0.1 giving together a weight of 0.1 to y_1. As $y^*_1 = 0.3$, $(1, 0, 1)$ receives the weight 0.3-0.1=0.2, and $(0, 0, 1)$ the remainder $0.4 - 0.2$.

Now $y^* = (0.3, 0.5, 0.4) = 0.5 \times (0, 1, 0) + 0.1 \times (1, 0, 0) + 0.2 \times (1, 0, 1) + 0.2 \times (0, 0, 1)$.

Finally $(s^*, y^*) = 0.5 \times (10, 0, 7, 0, 1, 0) + 0.1 \times (0, 11, 7, 1, 0, 0) + 0.2 \times (0, 4, 0, 1, 0, 1) + 0.2 \times (14, 4, 0, 0, 0, 1) + 2.2 \times (1, 0, 0, 0, 0, 0) + 0.4 \times (0, 1, 0, 0, 0, 0) + 0.7 \times (0, 0, 1, 0, 0, 0)$.

Theorem 2.2 also leads to a very simple proof of a result of Van Hoesel concerning multiple optimal solutions.

Proposition 2.5 *[6]. For the uncapacitated lot-sizing problem with Wagner-Whitin costs, if k_1 and k_2 are successive set-up periods in one optimal solution, then in every other optimal solution a set-up occurs at least once in the interval $[k_1, \ldots, k_2]$.*

Proof. All optimal solutions must lie in one of the faces $F(\tau)$. Suppose there is an optimal solution y^* with $\sum_{j=k_1}^{k_2} y^*_j = 0$ contradicting the claim. Then as $\sum_{j=k_1}^{\tau(k_1)+1} y_j \geq 1$ necessarily $k_2 < \tau(k_1) + 1$. But then $\sum_{j=k_1}^{k_2} y_j \leq \sum_{j=k_1}^{\tau(k_1)} y_j \leq 1$ and the first optimal point with $y_{k_1} = y_{k_2} = 1$ cannot lie on the face. ∎

3 Neighboring Vertices

The polyhedron P_n has 2^n vertices $(\sigma(y), y)$, one for each $y \in \{0,1\}^n$. Let $G_n = (V_n, E_n)$ be a graph with a node for each vertex of P_n, and an edge whenever two vertices of P_n are adjacent. We use the binary string $y_1 \dots y_n$ for the node corresponding to the vertex $(y_1, \dots, y_n) \in \{0,1\}^n$.

In Figure 1 below we show the adjacency matrix A_p of G_p for $p = 1, 2$ and of A_{n+1} in terms of the structure of A_n.

$$A_1 \quad \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 1 \end{array} \qquad\qquad A_2 \quad \begin{array}{c|cc|cc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 1 & 1 & 0 \\ 01 & 1 & 1 & 1 & 1 \\ \hline 10 & 1 & 1 & 1 & 1 \\ 11 & 0 & 1 & 1 & 1 \end{array}$$

$$A_n \quad \begin{array}{c|cc} & 0y & 1y \\ \hline 0y & B & C \\ 1y & C^T & B \end{array} \qquad\qquad A_{n+1} \quad \begin{array}{c|cc|cc} & 00y & 01y & 10y & 11y \\ \hline 00y & B & C & C & O \\ 01y & C^T & B & C^T & I \\ \hline 10y & C^T & C & B & C \\ 11y & O & I & C^T & B \end{array}$$

Figure 1: Adjacency Matrices

Theorem 3.1 *i). For $w, z \in V_n$, $(w, z) \in E_n$ if and only if $(0w, 0z) \in E_{n+1}$*
ii). For $w, z \in V_n$, $(w, z) \in E_n$ if and only if $(1w, 1z) \in E_{n+1}$
iii). For $w, z \in V_{n-1}$, $(0w, 1z) \in E_n$ if and only if $(00w, 10z) \in E_{n+1}$
iv). For $w, z \in V_{n-1}$, $(0w, 1z) \in E_n$ if and only if $(10w, 01z) \in E_{n+1}$
v). For $w, z \in V_{n-1}$, $(00w, 11z) \notin E_{n+1}$
vi). For $w, z \in V_{n-1}$, $(01w, 11z) \in E_{n+1}$ if and only if $w = z$.

Note that i) and ii) show that A_n reappears in the two main diagonal blocks of A_{n+1}, iii) shows that C appears in the $(00y, 10y)$ block of A_{n+1}, iv) that C appears in the $(10y, 01y)$ block of A_{n+1}, and hence by symmetry that C^T appears in the $(01y, 10y)$ block, v) that the $(00y, 11y)$ block is the null matrix, and vi) that the $(01y, 11y)$ block is an identity matrix.

Proof. v) immediately follows from Proposition 2.5. i) and iii) follow from the observation that the face $y_i = 0$ of P_{n+1} is isomorphic to P_n with demands $d_1, \dots, d_{i-2}, d_{i-1,i}$, $d_{i+1}, \dots, d_{n+1}$, and $s_{i-1} \geq d_i + \sigma_i(y)$. i)-iv) can also be proved using the property that $(w, z) \in E_n$ if and only if there exists an objective function (h, f) such that $(\sigma(w), w)$ and $(\sigma(z), z)$ are the only two optimal extreme point solutions of $\min\{hs + fy : (s, y) \in P_n\}$. We now prove iv) in this way.

First suppose that $(0w, 1z) \in E_n$. As we need to add a new coordinate in the first position, we consider a generic vector in V_n to be $(y_2, y_3, \ldots, y_{n+1})$, which we shorten to $(y_2, \bar{y})$ where $\bar{y} = (y_3, \ldots, y_{n+1}) \in \{0,1\}^{n-1}$.

Now suppose that there exists an objective function $\sum_{t=1}^{n} h_t s_t + \sum_{t=2}^{n+1} f_t y_t$ with corresponding demands $d_2, \ldots, d_{n+1}$ such that, if $\phi_n(y_2, y)$ denotes $\sum_{t=1}^{n} h_t \sigma_t(y_2, y) + \sum_{t=2}^{n+1} f_t y_t$, then

$\phi_n(0w) = \phi_n(1z)$,

$\phi_n(0w) < \phi_n(0y)$ for $y \neq w$ and

$\phi_n(1z) < \phi_n(1y)$ for $y \neq z$.

Now define a new objective function $\sum_{t=0}^{n} h'_t s_t + \sum_{t=1}^{n+1} f'_t y_t$ with $h'_0 = 1, h'_t = h_t$ for $t = 1, \ldots, n$, $f'_1 = d_1 + d_2/2$, $f'_2 = f_2 + d_2/2$ and $f'_t = f_t$ for $t = 3, \ldots, n+1$. We use $\phi_{n+1}(y_1, y_2, y)$ to denote $\sum_{t=0}^{n} h'_t \sigma_t(y_1, y_2, y) + \sum_{t=1}^{n+1} f'_t y_t$, and $\psi(y)$ to denote $\sum_{t=2}^{n} h_t \sigma_t(y) + \sum_{t=3}^{n+1} f_t y_t$ where we use the fact that $\sigma_t(y_1, y_2, y)$ depends only on $y_t, \ldots, y_{n+1}$.

Now by definition

$\phi_n(0w) = h_1\sigma_1(0w) + \psi(w)$, so $h_1\sigma_1(0w) + \psi(w) < h_1\sigma_1(0y) + \psi(y)$ for $y \neq w$, and

$\phi_n(1z) = f_2 + \psi(z)$, so $\psi(z) < \psi(y)$ for $y \neq z$.

Consider now the new objective function. We will show that $10w$ and $01z \in V_{n+1}$ are the unique 0-1 optimal solutions which suffices to show that $(10w, 01z) \in E_{n+1}$. We distinguish five cases.

$\phi_{n+1}(10w) = f'_1 + h'_1\sigma_1(0w) + \psi(w)$
$= d_1 + d_2/2 + h_1\sigma_1(0w) + \psi(w)$
$= d_1 + d_2/2 + \phi_n(0w)$.

Also $\phi_{n+1}(01z) = h'_0\sigma_0(01z) + f'_2 + \psi(z)$
$= d_1 + f_2 + d_2/2 + \psi(z)$
$= d_1 + d_2/2 + \phi_n(1z) = d_1 + d_2/2 + \phi_n(0w) = \phi_{n+1}(10w)$.

$\phi_{n+1}(00y) = h'_0\sigma_0(00y) + h'_1\sigma_1(0y) + \psi(y)$
$= d_1 + h'_0\sigma_1(0y) + h'_1\sigma_1(0y) + \psi(y)$
$\geq d_1 + d_2 + h_1\sigma_1(0y) + \psi(y)$ as $\sigma_1(0y) \geq d_2$
$= d_1 + d_2 + \phi_n(0y)$
$\geq d_1 + d_2 + \phi_n(0w)$ for all y
$> f'_1 + \phi_n(0w) = \phi_{n+1}(10w)$.

$\phi_{n+1}(10y) = f'_1 + h'_1\sigma_1(0y) + \psi(y)$
$= f'_1 + h_1\sigma_1(0y) + \psi(y)$
$> f'_1 + h_1\sigma_1(0w) + \psi(w)$ for $y \neq w$
$= f'_1 + h'_1\sigma_1(0w) + \psi(w) = \phi_{n+1}(10w)$.

$\phi_{n+1}(01y) = h'_0\sigma_0(01y) + f'_2 + \psi(y)$

$= d_1 + f_2 + d_2/2 + \psi(y)$
$> d_1 + f_2 + d_2/2 + \psi(z) = \phi_{n+1}(01z)$.

Finally $\phi_{n+1}(11y) = f'_1 + f'_2 + \psi(y)$
$= d_1 + d_2 + f_2 + \psi(y)$
$> d_1 + d_2/2 + f_2 + \psi(z) = \phi_{n+1}(01z)$.

Conversely suppose that $(10w, 01z) \in E_{n+1}$. Then there exists an objective function $\sum_{t=0}^{n} h'_t s_t + \sum_{t=1}^{n+1} f'_t y_t$ such that $\phi_{n+1}(10w) = \phi_{n+1}(01z) < \phi_{n+1}(y_1 y_2 y)$ for $y_1 y_2 y \neq 10w, 01z$.

Now let $\psi(y) = \sum_{t=2}^{n} h'_t \sigma_t(y) + \sum_{t=3}^{n+1} f'_t y_t$, so that
$\phi_{n+1}(10w) = f'_1 + h'_1\sigma_1(0w) + \psi(w) < f'_1 + h'_1\sigma_1(0y) + \psi(y)$ for $y \neq w$, and
$\phi_{n+1}(01z) = h'_0\sigma_0(01z) + f'_2 + \psi(z) = h'_0 d_1 + f'_2 + \psi(z) < h'_0 d_1 + f'_2 + \psi(y)$ for $y \neq z$.

Now consider the objective function $\sum_{t=1}^{n} h_t s_t + \sum_{t=2}^{n+1} f_t y_t$ with $h_t = h'_t$ for $t = 1, \ldots, n$, $f_2 = f'_2 - f'_1 + h'_0 d_1$, $f_t = f'_t$ for $t = 3, \ldots, n+1$.

One can check that
$\phi_n(0w) = h_1\sigma_1(0w) + \psi(w)$
$\phi_n(1z) = f_2 + \psi(z) = f'_2 - f'_1 + h'_0 d_1 + \psi(z) = \phi_{n+1}(10w) - f'_1 = h'_1\sigma_1(0w) + \psi(w) = h'_1\sigma_1(0w) + \psi(w) = \phi_n(0w)$.
$\phi_n(0y) = h_1\sigma_1(0y) + \psi(y) = \phi_{n+1}(10y) - f_1 > \phi_{n+1}(10w) - f_1 = \phi_n(0w)$ for $y \neq w$.
$\phi_n(1y) = f_2 + \psi(y) > f_2 + \psi(z)$ for $y \neq z$
$= \phi_n(1z)$.

Hence we have shown that $0w$ and $1z \in V_n$ are the unique integer optimal solutions, and so $(0w, 1z) \in E_n$.

The proofs of the other cases are similar. ∎

Knowing the structure of the adjacency matrix A_n, it is easy to obtain a more direct way to characterize adjacency. This immediately leads to a linear time algorithm.

Theorem 3.2 *Two distinct nodes y and z are adjacent if and only if after removal of those entries where $y_i = z_i = 0$, the remaining vectors of length $r \geq 1$ have the form $y_i = z_i = 1$ for $i = 1, \ldots, p$ and for $i = q, \ldots, r$, and $y_i + z_i = 1$ for $i = p+1, \ldots, q-1$ and $y_i + y_{i+1} = 1$ for $i = p+1, \ldots q-2$ with $0 \leq p < q \leq r+1$.*

Corollary [2]. If $(w, z) \in E_n$, then $|\sum_{t=1}^{n} w_t - \sum_{t=1}^{n} z_t| \leq 1$.

4 Finding All Optimal Solutions

In [7], van Hoesel et al. present a dual greedy algorithm for uncapacitated lot-sizing that runs in $O(n^2)$. Here we analyze their algorithm and draw the conclusion that it finds all optimal solutions, or alternatively the optimal face of P_n.

Consider the problem $\min\{cx : x \in X\}$. For simplicity we suppose that $X \subset R^n$ is full-dimensional. For a given c, let $M(c)$ be the set of all optimal solutions.

Dual Face Algorithm. For all $c \neq 0$, there is an algorithm specifying a valid inequality $a^{i(c)}x \geq b_{i(c)}$ and a bound $\bar{u} > 0$ such that $M(c - ua^{i(c)}) \subseteq \{x : a^{i(c)}x = b_{i(c)}\}$ for all $0 \leq u < \bar{u}$, but $M(c - \bar{u}a^{i(c)})$ is not a subset of $\{x : a^{i(c)}x = b_{i(c)}\}$.

Proposition 4.1 *If $a^{i(c)}x \geq b_{i(c)}$ and $\bar{u} > 0$ are specified as in the dual face algorithm,*
i) $M(c) = M(c - ua^{i(c)})$ for all $0 \leq u < \bar{u}$,
ii) $M(c) = M(c - \bar{u}a^{i(c)}) \cap \{x : a^{i(c)}x = b_{i(c)}\}$.

Proof. i) Suppose $x, y \in X$ with $x \in M(c - ua^{i(c)})$. Then $(c - ua^{i(c)})x \leq (c - ua^{i(c)})y$. Thus $cx \leq cy - u(a^{i(c)}y - b_{i(c)}) + u(a^{i(c)}x - b_{i(c)}) \leq cy$ as $u \geq 0, a^{i(c)}y \geq b_{i(c)}$ as $y \in X$, and $a^{i(c)}x = b_{i(c)}$. Thus $M(c - ua^{i(c)}) \subseteq M(c)$.

Now suppose $x^1 \in M(c) \setminus M(c - ua^{i(c)})$, and $x^2 \in M(c - ua^{i(c)})$. As $x^1, x^2 \in M(c), cx^1 = cx^2$. But as $a^{i(c)}x^1 = a^{i(c)}x^2 = b_{i(c)}$, $(c - ua^{i(c)})x^1 = (c - ua^{i(c)})x^2$, a contradiction as $x^1 \notin M(c - ua^{i(c)}), x^2 \in M(c - ua^{i(c)})$.
ii) Let $z(u) = \min\{(c - ua^{i(c)})x : x \in X\}$. By i), if $x^* \in M(c)$ and $0 \leq u < \bar{u}$, $z(u) = (c - ua^{i(c)})x^*$. As $z(u)$ is a continuous function of u, $z(\bar{u}) = (c - \bar{u}a^{i(c)})x^*$, and hence $M(c) \subseteq M(c - \bar{u}a^{i(c)})$. The proof of the converse, that $M(c - \bar{u}a^{i(c)}) \cap \{x : a^{i(c)}x = b_{i(c)}\} \subseteq M(c)$, is identical to the first part of the proof of i). ∎

Consider now a dual algorithm based on the dual face algorithm.

The Dual Greedy Algorithm.
Initialization. Let $c^1 = c$.
Iteration t. Find a valid inequality $a^{i(t)}x \geq b_{i(t)}$ for X and a value u_t using the dual face algorithm, and set $c^{t+1} = c^t - u_t a^{i(t)}$. If $c^{t+1} \neq 0$, increase t, and repeat.
Termination $c^{T+1} = 0$.

Theorem 4.2 *The face of all optimal solutions of* $\min\{cx : x \in conv(X)\}$ *is* $\{x : a^{i(t)}x = b_{i(t)}$ for $t = 1, \ldots, T\}$. *$M(c)$ is the set of integral points of this face (extreme points for 0-1 problems).*

Proof. Using ii) of the previous Proposition, $M(c) = M(c^1) = M(c^2) \cap \{x : a^{i(1)}x = b_{i(1)}\} = \ldots = M(c^{T+1}) \cap_{t=1}^{T} \{x : a^{i(t)}x = b_{i(t)}\}$. As $M(c^{T+1}) = M(0) = R^n$, the claim follows. ∎

The following algorithm is a specialization of the algorithm from [7].
A Dual Face Algorithm for Lot-Sizing with Wagner-Whitin costs.
$M(h, f)$ is the set of optimal solutions of $z = \min\{hs + fy : (s, y) \in P_n, y \in Z^n\}$.
If $h_j < 0$, $z = -\infty$.
If $f_j < 0$, $M(h, f) \subseteq \{(s, y) : -y_j = -1\}$ and $\bar{u} = -f_j$.

For $h, f \geq 0$, let $k-1 = \max\{j : h_j > 0\}$.
If $f_j = 0$ for all $j \geq k$, $M(h,f) \subseteq \{(s,y) : s_{k-1} = 0\}$ and $\bar{u} = h_{k-1}$.
Otherwise let $l = \min\{j : f_j > 0, j \geq k\}$.
If $k < l$, $M(h,f) \subseteq \{(s,y) : y_l = 0\}$ and $\bar{u} = f_l$.
If $k = l$, let $t = \max\{j : f_i > 0 \text{ for } k \leq i \leq j\}$, then $M(h,f) \subseteq \{(s,y) : s_{k-1} + \sum_{j=k}^{t} d_{jt} y_j = d_{kt}\}$ and $\bar{u} = \min\{h_{k-1}, f_k/d_{kt}, \ldots, f_j/d_{jt}, \ldots, f_t/d_t\}$.

Theorem 4.3 *The optimal face of* $\min\{hs + fy : (s,y) \in P_n\}$ *can be found by the dual greedy algorithm in* $O(n^2)$.

Proof. In the dual face algorithm, $\bar{u}$ is chosen so that one of the objective coefficients becomes zero and then never changes. Thus the algorithm terminates after at most $2n$ iterations, and the work per iteration is $\mathrm{O}(n)$. ∎

5 Concluding Remarks

Here we have studied the simplest lot-sizing polyhedron over which it is possible to optimize in polynomial time. There are many more complicated polyhedra that are much less well understood, see [5]. It is to be hoped that the geometric viewpoint taken here can be generalized to some of these more complicated polyhedra. It would also be interesting to examine whether dual greedy algorithms terminate with the set of all optimal solutions in other contexts.

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