Where are matrices coming from ?

ma·trix (mā'trĭks) n., pl., ma·tri·ces (mā'trĭ-sēz')

- ...
- *Anatomy.* The womb (uterus).
- ...
- *Geology*. The solid matter in which a crystal is embedded.
- ...
- *Metallurgy.* The principal metal in an alloy, as the iron in steel.
- ...
- *Printing*. A mold used in stereotyping. A metal plate used for casting typefaces.
- ...
- Computer Science. The network of intersections in a computer.
- .
- *Mathematics*. A rectangular table of numeric or algebraic quantities subject to mathematical operations, with a formation of elements into columns and rows.

• ...

[Middle English matrice, from Old French mātrīc-, from Latin, māter]

The early days ...

Solving a system of equations by is often credited to Carl Friedrich Gauss (Gaussian elimination)

3x+2y+1z=39 2x+3y+1z=34 1x+2y+3z=26 3x+2y+1z=39 5y+1z=24 4y+8z=39 3x+2y+1z=39 5y+1z=24 5y+1z=24 5y+1z=24 5y+1z=24 5y+1z=24 5y+1z=24 5y+1z=24 5y+1z=245y+1z=39

Gauss worked on many other things : least squares, statistics, heavenly bodies ...



Gauss 1777-1855



The early days ...

But this was also known in 202 BC in The Nine Chapters on the Mathematical Art

36z = 99 1z+5y = 24 1z+2y+3x=39 8z+4y = 39 1z+5y = 24 1z+2y+3x=39 3z+2y+1x=26 1z+3y+2x=341z+2y+3x=39



This book also talked about taxation, square roots, areas, volumes and the Pythagorean theorem ...

Positive quadratic forms

When is the form
$$f(x_1, ..., x_n) = \sum_i \sum_j s_{ij} x_i x_j > 0$$
 ?

J.J.Sylvester introduced matrices to rewrite this as

$$f(\mathbf{x}) = \mathbf{x}^T S \mathbf{x},$$
$$S = \begin{bmatrix} s_{11} & \dots & s_{1n} \\ \vdots & & \vdots \\ s_{n1} & \dots & s_{nn} \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

This also led to eigenvectors and eigenvalues

If
$$S\mathbf{x} = \lambda \mathbf{x}$$
 then $f(\mathbf{x}) = \lambda \mathbf{x}^T \mathbf{x} > 0 \Leftrightarrow \lambda > 0$



Sylvester 1814 - 1897

More eigenvectors and eigenvalues...

 $S\mathbf{x} = \lambda \mathbf{x}$ shows up in vibrations







Eigenvectors "avant-la-lettre"

Ernest Chladni studied acoustics (e.g. of guitars)





Chladni 1756 -1827

How did he (and Hooke) observe these ?



These are the nodal regions of the eigenvectors

Camille Jordan

 $\begin{bmatrix} \alpha & 1 \end{bmatrix}$

C. Jordan gives the complete algebraic theory of the standard eigenvalue problem (also Kronecker)

$$AX = X$$

$$\begin{array}{ccc}1\\\alpha\\&\beta\\&\beta\end{array}$$

1

This quickly led to matrix algebra (Cayley) and matrix equations (Sylvester) such as

Jordan 1838-1922

AX - XB = C

 $AX^2 + BX + C = 0$

and also to functions of matrices : $X(t) = exp^{At}$

All are very important for the study of dynamical systems

Where systems theory comes in

Rudolf Kalman and state-space models (1960)

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$



Kalman 1930 -

These models had a immense impact in several disciplines

- aeronautical, electrical and mechanical engineering
- signal processing, acoustics, statistics

A wealth of (old and new) matrix theoretic results and applications

- new canonical forms for 2, 4 and several matrices
- linear and quadratic matrix equations and inequalities, ...

Special matrix equations

In the context of linear control systems

 $AX + XA^T + BB^T = 0$ (stability)



Controllable system

 $\dot{x}(t) = Ax(t) + Bu(t)$

is stable iff $X \succ 0$

Aleksandr Lyapunov



1857-1918

Count Riccati



1676-1754

$AX + XA^T - XRX + Q = 0$

Optimal control of the above dynamical system

Structured matrix problems

Hankel matrices

$$H = \begin{bmatrix} a & b & c & d & e \\ b & c & d & e & f \\ c & d & e & f & g \\ d & e & f & g & h \\ e & f & g & h & i \end{bmatrix}$$



Hankel 1839-1873

Realization Model reduction (approximation) Input-output map (convolution)

Toeplitz matrices

$$T = \begin{bmatrix} a & b & c & d & e \\ f & a & b & c & d \\ g & f & a & b & c \\ h & g & f & a & b \\ i & h & g & f & a \end{bmatrix}$$



Toeplitz 1881-1940

Realization Correlation Spectral density and factorization

Strong links with function theory

Non-negative matrices

Perron-Frobenius theorem

Matrices with positive entries have an eigenvalue-eigenvector pair with positive entries as well



Oscar Perron 1880-1975

$$M > 0 \quad \Rightarrow \quad \exists x > 0, \rho > 0, Mx = \rho x$$

Applications

Markov processes, statistics (probabilities) Graphs and networks (links or edges) Datamining (occurrences of items) Google (reputation of a webpage)



F.G. Frobenius 1849-1917

Voting matrices



Voting matrices



$$f_{i} = d - \sum_{j=1}^{m} (X_{ij} - r_{j})^{2}$$
Belief divergence
= Variance

$$f_{i} = 4.6 \quad \textcircled{(1)}_{h} \quad \bigtriangledown$$

$$f_{i} = 4.6 \quad \textcircled{(2)}_{h} \quad \fbox{(2)}_{h} \quad \swarrow$$

$$f_{i} = 4.6 \quad \textcircled{(2)}_{h} \quad \fbox{(2)}_{h} \quad \cancel{(2)}_{h} \quad \cancel{(2)$$

r?

r

Social networks

World with n citizens each with k "friends" (say n=20, k=4)



In a regular world most of my friends know each other

In a random world the "diameter" is small

Adjacency matrix A

Random connections with probability *p* quickly decrease diameter

Adjacency matrix A

 $(i,j)=1 \leftrightarrow i \text{ and } j \text{ are "friends"}$





The ${\bf k}\mbox{-th}$ power of the adjacency matrix give the paths of length ${\bf k}$

Adjacency matrix A²



The ${\bf k}\mbox{-th}$ power of the adjacency matrix give the paths of length ${\bf k}$

Adjacency matrix A³

(i,j)=1 ↔ *i* and *j* are "friends of friends of my friends"



The k-th power of the adjacency matrix give the paths of length k

Adjacency matrix A⁴



The ${\bf k}\mbox{-th}$ power of the adjacency matrix give the paths of length ${\bf k}$

Adjacency matrix A⁵



The k-th power of the adjacency matrix give the paths of length k

Adjacency matrix A⁶



The ${\bf k}\mbox{-th}$ power of the adjacency matrix give the paths of length ${\bf k}$

Adjacency matrix A⁷



"Six degrees" of separation of a small word

Movie world {co-actors} Kevin Bacon distance

Scientific publications {coauthors} Erdös number



of people
1
991
56479
82792
12183
1337
308
42
8



Google matrix

The web carries tens of billions of pages organized by tens of thousands of keywords

Each query selects relevant pages from a low rank matrix approximation

Selected pages are ordered according to PageRank (computed for all pages)



PageRank x is an eigenvector of an NxN stochastic matrix (N~10¹⁰) Largest eigenvector ever computed ? Has to be adapted every month or so ...

Is the steady state probability of a random walker being on a webpage if he hops around from one webpage to another with equal probability

Perron-Frobenius vector of an appropriately defined Google matrix

Google search on MATRIX

Go	ogle matrix		Search	Advanced Search
前前前前前 前前前前前 Google		web O pages from Belgium		
Web	Show options	Results 1 - 10 of about 101,000	0,000 for matrix [de	efinition]. (0.19 seconds

1. The Matrix - Wikipedia, the free encyclopedia

The Matrix is a 1999 American science fiction-action film directed by Larry and Andy Wachowski; starring Keanu Reeves, Laurence Fishburne, Carrie-Anne Moss, ... <u>Matrix Revolutions - Matrix Reloaded</u> - <u>Matrix (franchise)</u> en.wikipedia.org/wiki/The_Matrix - 8 hours ago - <u>Cached</u> - <u>Similar</u>

2. Matrix (mathematics) - Wikipedia, the free encyclopedia

In mathematics, a **matrix** (plural matrices, or less commonly matrixes) is a rectangular array of numbers, such as. \begin{bmatrix} 1 & 9 & 13 \\ 20 & 55 & ... Definition - Basic operations - Matrix multiplication, linear ... en.wikipedia.org/wiki/Matrix_(mathematics) - <u>Cached</u> - <u>Similar</u>

Mathematics comes only in second place !

3. The Matrix (1999)

Directed by Andy Wachowski, Lana Wachowski. With Keanu Reeves, Laurence Fishburne, Carrie-Anne Moss. A computer hacker learns from mysterious rebels about ... www.imdb.com/title/tt0133093/ - 8 hours ago - <u>Cached</u> - <u>Similar</u>

4. The Matrix Reloaded (2003)

Directed by Andy Wachowski, Lana Wachowski. With Ray Anthony, Christine Anu, Andy Arness. Neo and the rebel leaders estimate that they have 72 hours until ... www.imdb.com/title/tt0234215/ - 3 hours ago - Cached - Similar

1/845914	9000 9000 9000 9000 9000 9000 9000 90000 90000 90000 90000 90000 90000 9000000	axeoneoo	1911-9787 1911-9787		422#X29 404062 98065 747099 1747099 2747099 174707 174709 174709 174709 174709 174709 174709 174709 1747000 1747000 1747000 1747000 17470000000000		987 - 48 - 4 8-87 - 48 - 4 06 / W + 40 - 4 06 / W + 40 - 4	+12000000000000000000000000000000000000
00000000000000000000000000000000000000	ext fou	r lectu	res wi	l do m	natrix r	eloadir	ng on state	
avera avera avera	Distan		blems		tra an		dospect	7日の時7日の時2 1100時7日の時2 6日年以前7日の時2 6日年以前7日の時4日日 7日の時1200時時6日 100時1200時時6日 100時1205月7日 6日子7日から 6日子7日から 6日子7日から 6日子7日から 6日子7日から 6日子7日の時2 6日子7日の 6日子7日の 6日子7日の 7 6日子7日の 7 6日子7日の 7 7 6日子710 6日子710 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7
** knek4#	Model		tion of	linear	r syste		日 1 1 1 1 1 1 1 1 1 1 1 1 1	18041914 18041914 18041914 18041914 18041914 18094191 18094191 180
1411 1411 1411 1411 1411 1411 1411 141	Domin	ant fe	aturee	xtract	ion an	dstruc	tured m	atrices
10000000000000000000000000000000000000	Netwo	rks ar	dgrap		448465 448440 5×8×240 5×8×240 5×8×25 5×24 0 5	10000112 10000112 10000112 10000112 10000112 10000112 10000112	11、11、11、11、11、11、11、11、11、11、11、11、11、	1000X
1 A A	25465451	1 2242	* + a	462	8.0 9.0 9.0 9.0 9.0 9.0	170 V 470	A CTX 80	2 C C C C C