Level set methods for robustness measures

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What it is about

Fast and reliable computation of robustness measures for

- Stability
- Passivity
- Minimality

for (possibly structured) perturbations

$$\{\Delta_A, \Delta_B, \Delta_C, \Delta_D\}$$

of a (real or complex) plant

$$\{A, B, C, D\}$$

Link to structured singular values

In each of these problems we need to find a point $\lambda \in \mathbb{C}$ such that some singular value or eigenvalue of matrices derived from

$$\begin{bmatrix} A - \lambda I \end{bmatrix} \quad \begin{bmatrix} A - \lambda I & B \end{bmatrix}$$

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \quad \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$$

is minimal.

Level sets (parameterized by ξ) are sets where singular values or eigenvalues are larger than the parameter ξ.

This is clearly related to so-called pseudo-spectra

Setting

Given a minimal system

$$G(\lambda) := C(\lambda I - A)^{-1}B + D$$

and a perturbed system

$$G_\Delta(\lambda) := C_\Delta(\lambda I - A_\Delta)^{-1}B_\Delta + D_\Delta$$

we measure perturbations using

$$\Delta := \begin{bmatrix} \Delta_A & \Delta_B \\ \Delta_C & \Delta_D \end{bmatrix} := \begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and use the $\|\Delta\|_2$ norm

All matrices can be real or complex

Complex stability radius

We are looking for the radius $r_{\mathbb{C}} = \inf_{\Delta}\{\|\Delta\|_2 \mid A_{\Delta} \text{ is unstable}\}$ (Hinrichsen-Pritchard)

We need to find an eigenvalue λ of A_{Δ} in the unstable region Γ or on its boundary $\partial\Gamma$:

$$\det(A + \Delta_A - \lambda I) = 0, \quad \lambda \in \partial\Gamma$$

For the complex case, perturbation theory says

$$\|\Delta_A\|_2 = \sigma_{\min}(A - \lambda I)$$

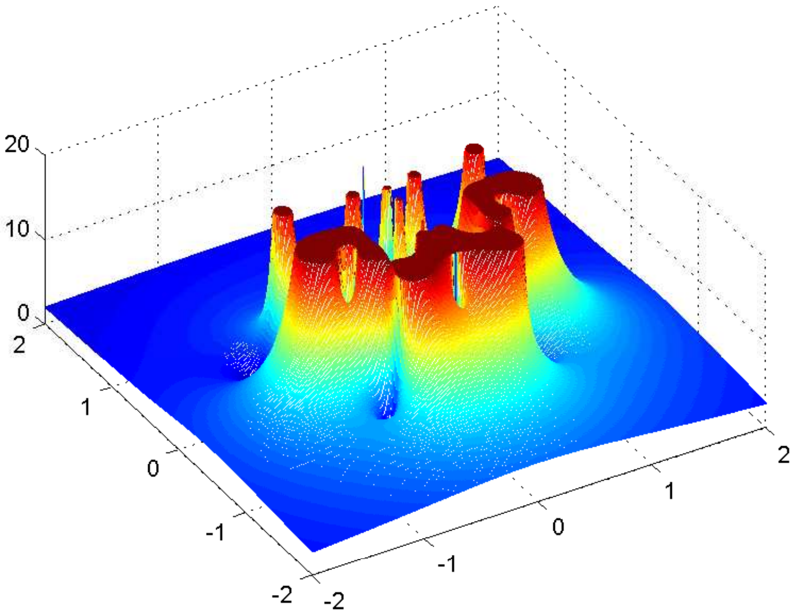
so we have (cfr pseudospectrum)

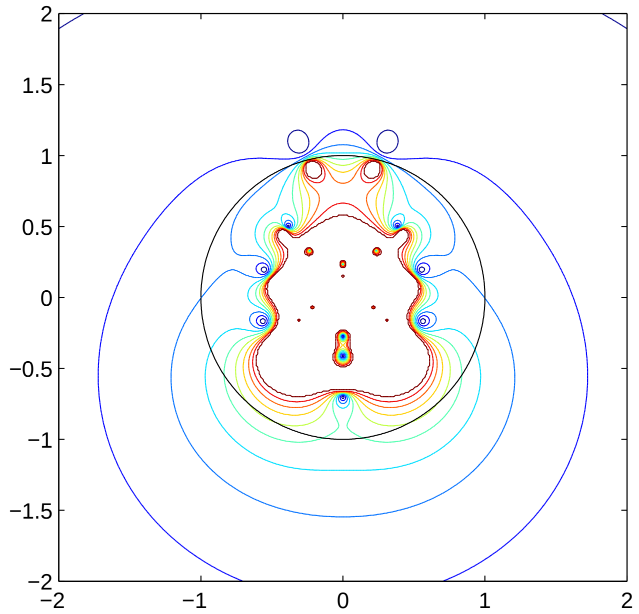
$$r_{\mathbb{C}} = \min_{\lambda \in \partial\Gamma} \sigma_{\min}(A - \lambda I) = 0, \quad \lambda \in \partial\Gamma$$

Resolvant

In systems theory we are more familiar with the resolvent and its norm :

$$G(\lambda) := (\lambda I - A)^{-1}, \quad r_{\mathbb{C}}^{-1} = \max_{\lambda \in \partial \Gamma} \sigma_{\max}[G(\lambda)]$$





Complex stability radius

We wanted the radius $r_{\mathbb{C}}(A) = \inf_{\Delta \in \mathbb{C}^{m \times p}} \{ \|\Delta\|_2 \mid A + \Delta_A \text{ is unstable} \}$

We needed to find an eigenvalue λ of $A + \Delta_A$ in the unstable region Γ or on its boundary $\partial\Gamma$:

$$\det(A + \Delta_A - \lambda I) = 0, \quad \lambda \in \partial\Gamma$$

For the complex case, this depended on the transfer function

$$G(\lambda) = (\lambda I - A)^{-1}$$

and yielded

$$r_{\mathbb{C}}^{-1} = \max_{\lambda \in \partial\Gamma} \sigma_{\max}[G(\lambda)]$$

Structured stability radius

If we impose a very simple structure, the problem becomes

$$r_{\mathbb{C}}(A, B, C) := \inf_{\Delta \in \mathbb{C}^{m \times p}} \{\|\Delta\|_2 : A + B\Delta C \text{ is stable}\}$$

Define $G(\lambda) := C(\lambda I - A)^{-1}B$ then

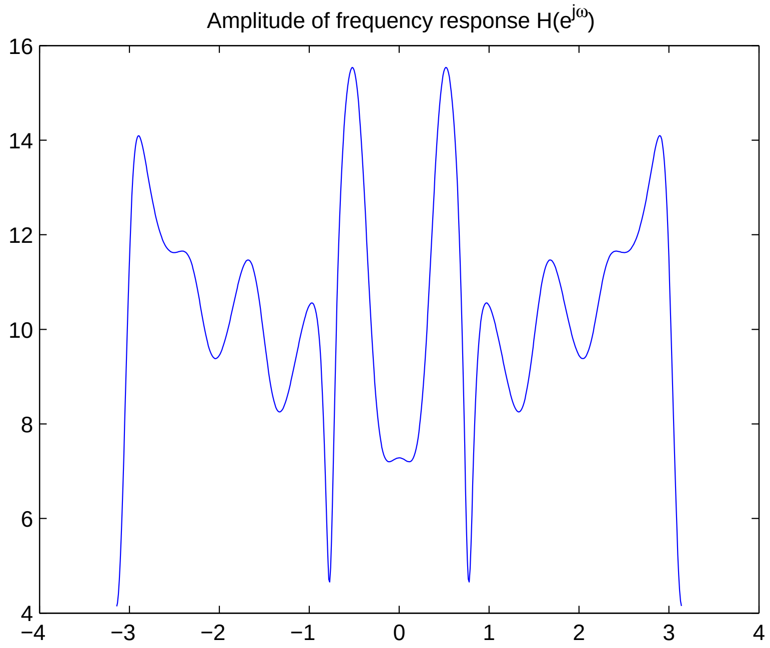
$$\boxed{r_{\mathbb{C}}^{-1}(A, B, C) = \max_{\lambda \in \partial\Gamma} \{\sigma_{\max}[G(\lambda)]\}}$$

See Hinrichsen Pritchard for more on this

Maximum frequency response

We thus need a reliable algorithm to find ($\lambda \in \partial\Gamma = j\omega$ or $e^{j\omega}$):

$$\max_{\omega} \sigma_{\max} C(e^{j\omega} I - A)^{-1} B$$



Alternative : gradient search but too many peaks may cost a lot and may be troublesome

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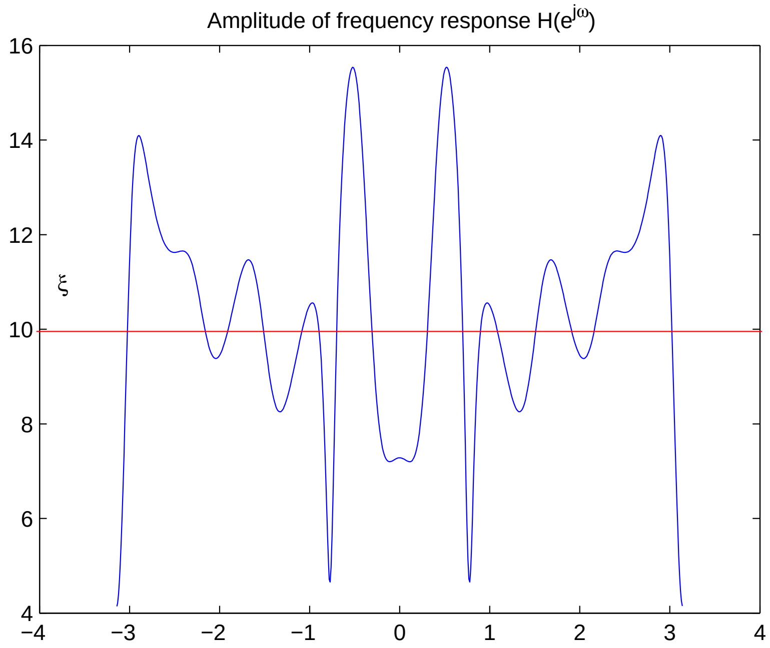


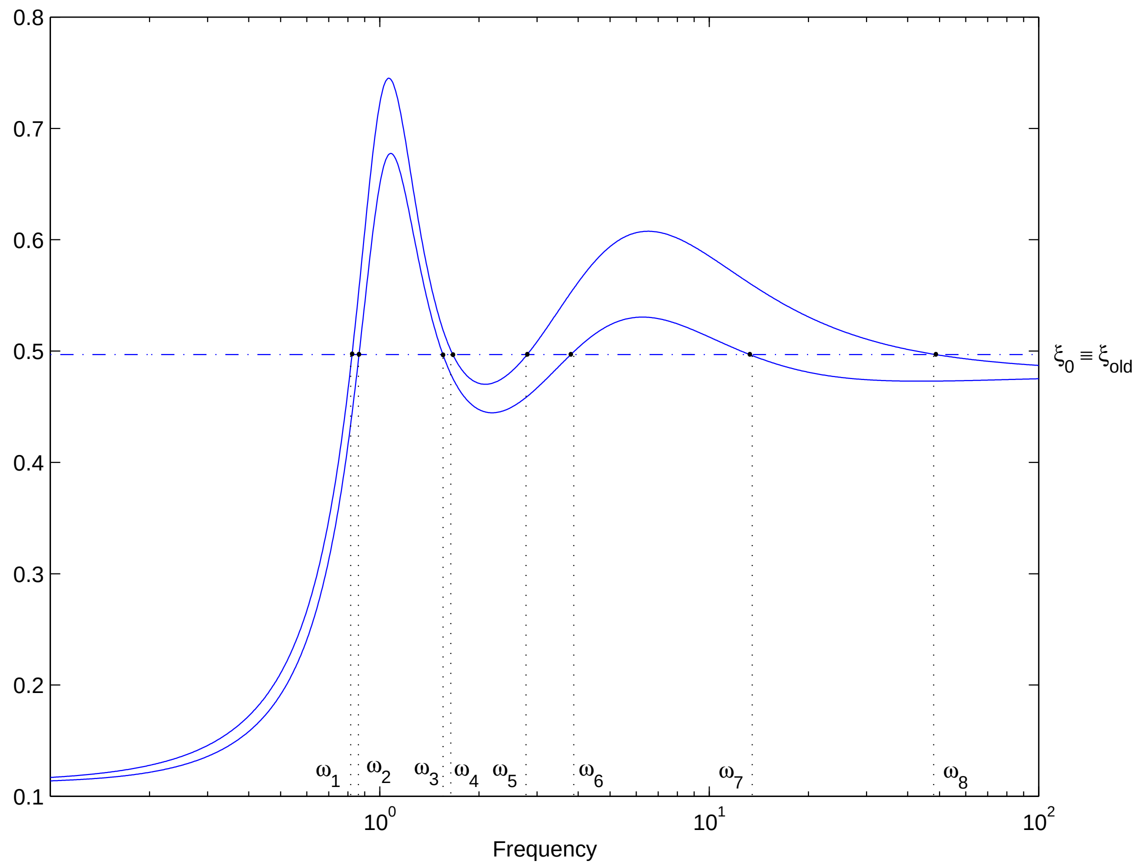
Alternative : gradient search but too many peaks may cost a lot and may be troublesome

The solution is to look at the ξ level set

Bisection

The intersection points ω_i with the ξ level (of all singular values) are imaginary eigenvalues of a Hamiltonian (or symplectic) matrix

$$H(\xi) := \begin{bmatrix} A & -BB^*/\xi \\ C^*C/\xi & -A^* \end{bmatrix}$$



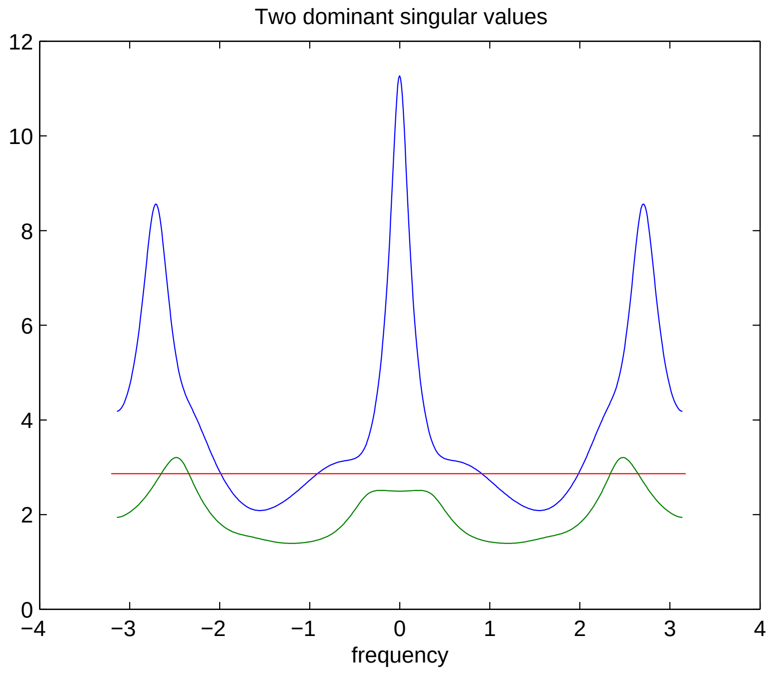
This can be used to do e.g. bisection

Linear convergence (Byers)

Interval midpoint rule is quadratic (Boyd et al, Bruinsma et al)

Correct intervals and interpolation

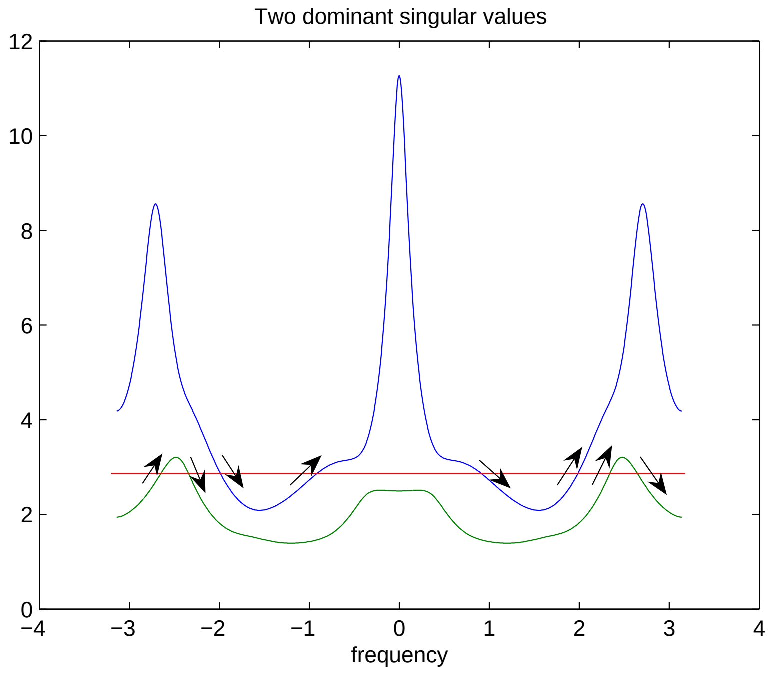
The eigenvalue problem also yields the derivatives of the singular value plots



Correct intervals and interpolation

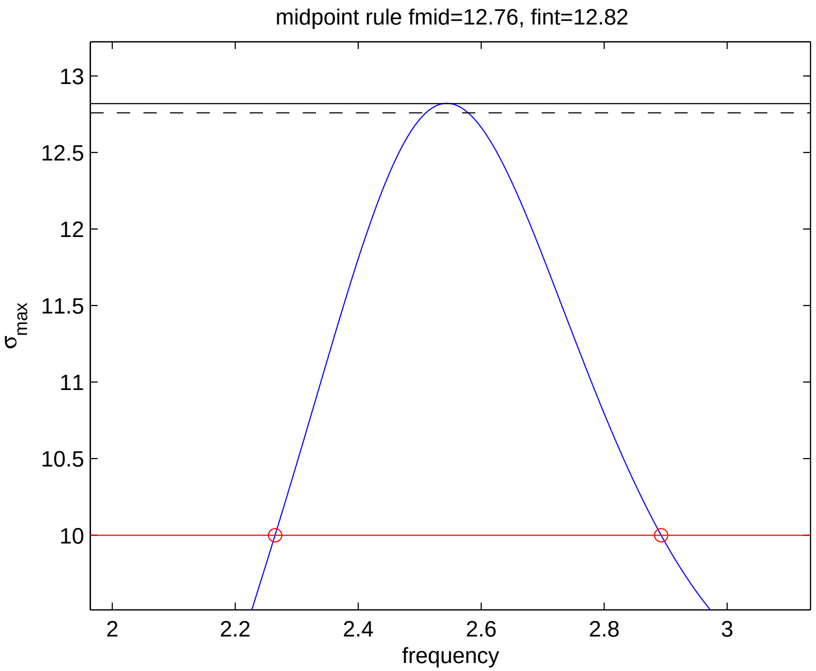
The eigenvalue problem also yields the derivatives of the singular value plots

They yield the info to find each individual singular value



Midpoint vs interpolation

Acceleration techniques yielding superlinear convergence



Midpoint rule (Boyd et al)

Choose ω_+ as midpoint of interval

Choose ξ_+ as function at that ω_+

$$|\xi_+ - \xi^*| = O|\xi - \xi^*|^2$$

Interpolation rule (Genin-V)

Choose ξ_{++} as maximum of

cubic interpolating polynomial

$$|\xi_{++} - \xi^*| = O|\xi - \xi^*|^4$$

Quartic convergence

Iteration	ξ (midp.)	Intervals (midp.)	ξ (cubic)	Intervals (cubic)
1	0.5224	[0,1.1991]	0.5224	[0,1.1991]
2	0.7980	[0.1867,0.5995] [0.7097,1.0153]	6.5148	[0.7804,0.7994]
3	1.7669	[0.7472,0.8625]	8.4043	[0.78942,0.78943]
4	5.3027	[0.7762,0.8048]	8.4043	Convergence
5	8.3691	[0.7884,0.7905]		
6	8.4043	[0.78942,0.78943]		

Complexity results (per iteration):

ω_i : $a(2n)^3$, $a \approx 50$ (exploit Hamiltonian structure)

$\partial\sigma_j/\partial\omega$: bn^2, $b < n$ (exploit Hamiltonian structure)

ξ_k : $cn^2(m+p)$, $c < n$ (use condensed forms)

Overall complexity is $O(n)^3$

Real stability radius

$$r_{\mathbb{R}}(A,B,C) := \inf_{\Delta\in\mathbb{R}^{m\times p}} \{\|\Delta\|_2 : A + B\Delta C \text{ is stable}\}$$

Define $G(\lambda) := C(\lambda I - A)^{-1}B$ and $G := G_r + iG_i$ then (Qiu et al)

$$\boxed{r_{\mathbb{R}}^{-1}(A,B,C) = \sup_{\lambda\in\partial\Gamma}\{\mu_{\mathbb{R}}[G(\lambda)]\}}$$

where

$$\mu_{\mathbb{R}}(G) := \inf_{\gamma\in(0,1]} \sigma_2\left[P_\gamma\right] := \inf_{\gamma\in(0,1]} \sigma_2 \begin{bmatrix} G_r & -\gamma G_i \\ G_i/\gamma & G_r \end{bmatrix}$$

Associated Hamiltonian

We use

$$T_\gamma := \frac{1}{\sqrt{2}} \begin{bmatrix} I & \gamma I \\ I/\gamma & -I \end{bmatrix}, \quad \begin{bmatrix} G_r & -\gamma G_i \\ G_i/\gamma & G_r \end{bmatrix} = T_\gamma \begin{bmatrix} G & 0 \\ 0 & \overline{G} \end{bmatrix} T_\gamma$$

to derive an associated real Hamiltonian matrix

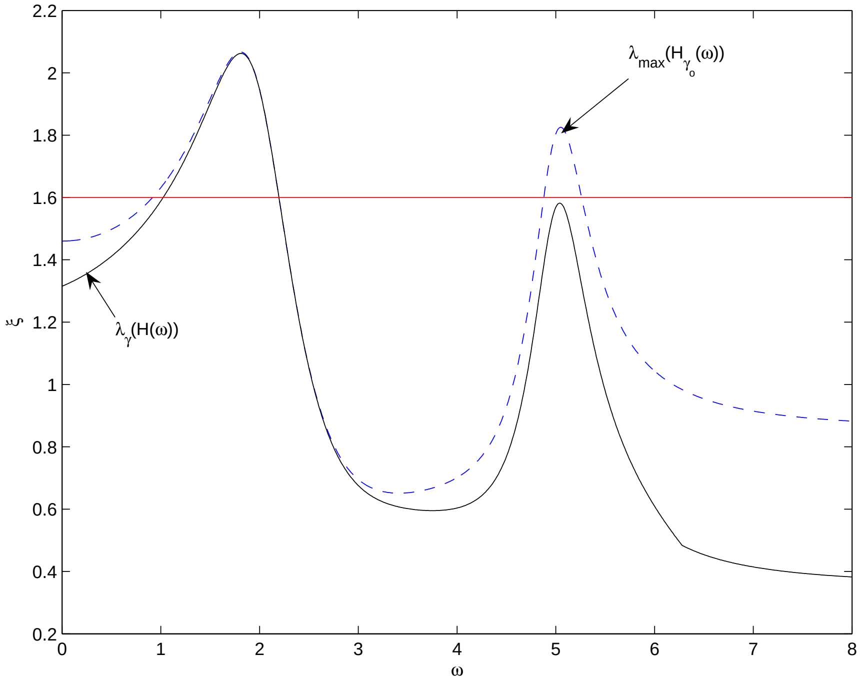
$$H_\gamma(\xi) := \left[\begin{array}{cc|cc} A & 0 & \alpha BB^T & -\beta BB^T \\ 0 & -A & -\beta BB^T & \alpha BB^T \\ \hline -\alpha C^T C & \beta C^T C & -A^T & 0 \\ \beta C^T C & -\alpha C^T C & 0 & A^T \end{array}\right]$$

where $\alpha := (1+\gamma^2)/(2\gamma\xi), \quad \beta := (1-\gamma^2)/(2\gamma\xi)$

We can find the $j\omega$ eigenvalue of $H_\gamma(\xi)$ that correspond to $\sigma_2[P_\gamma(\omega)]$

Lower envelope

For each γ_o value there is a σ_2 plot whose levels we can check with $H_{\gamma_o}(\xi_o)$



The (solid) curve $\mu_{\mathbb{R}}(\omega)$ we need to maximize is the lower envelope of these (dotted) curves

Each one is tangent to $\mu_{\mathbb{R}}(\omega)$ in one frequency ω_o

Convergence is quadratic or cubic (Sreedhar-Tits-V)

Passivity radius

Let $G(\lambda) := C(\lambda I_n - A)^{-1}B + D$ be strictly passive i.e. stable and positive real

$$Re\ \lambda_i(A) < 0, \quad G(j\omega) + [G(j\omega)]^* \succ 0,\ \forall \omega \in \mathbb{R}$$

Consider the perturbed system $G_\Delta(\lambda) := C_\Delta(\lambda I_n - A_\Delta)^{-1}B_\Delta + D_\Delta$

We wish to find the passivity radius of the system $G(\lambda)$

$$pr_{\mathbb{C}}(G) := \inf_{\Delta}\{\|\Delta\|_2 \mid G_\Delta(\lambda) \text{ is not passive}\}.$$

KYP lemma

Passivity (stability and positive realness)

$$Re\ \lambda_i(A) < 0, \quad G_\Delta(j\omega) + [G_\Delta(j\omega)]^* \succ 0, \quad \forall \omega \in \mathbb{R}$$

iff there exists a Hermitian matrix P such that

$$\begin{bmatrix} -A_\Delta P - PA_\Delta^* & B_\Delta - PC_\Delta^* \\ B_\Delta^* - C_\Delta P & D_\Delta + D_\Delta^* \end{bmatrix} \succ 0, \quad P \succ 0$$

Stability $Re\ \lambda_i(A) < 0$ iff there exists a Hermitian matrix P such that

$$-A_\Delta P - PA_\Delta^* \succ 0, \quad P \succ 0$$

Therefore stability can not be lost "before" positive realness is lost

(It can at one frequency if minimality is also lost)

Positive realness $G_\Delta(j\omega) + [G_\Delta(j\omega)]^* \succ 0$ gets lost as soon as

$$\det \left[\begin{array}{cc|c} 0 & A_\Delta - j\omega I_n & B_\Delta \\ A_\Delta^* + j\omega I_n & 0 & C_\Delta^* \\ \hline B_\Delta^* & C_\Delta & D_\Delta + D_\Delta^* \end{array}\right] = 0, \ \text{for some } \omega \in \mathbb{R}$$

or as soon as

$$\det\left(\mathcal{H}_\omega + E \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} E^T\right) = 0$$

where

$$\mathcal{H}_\omega := \begin{bmatrix} 0 & A - j\omega I_n & B \\ A^* + j\omega I_n & 0 & C^* \\ B^* & C & D + D^* \end{bmatrix}, \ E := \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_m & 0 & I_m \end{bmatrix}$$

We need a closed expression for

$$\min \|\Delta\|_2 : \det \left(\mathcal{H}_\omega + E \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} E^T \right) = 0$$

The corresponding transfer function is $H(\omega) := E^T \mathcal{H}(\omega)^{-1} E$

and one shows (Hu-Qiu, Overton-V) that the passivity radius is then

$$\boxed{pr_{\mathbb{C}}^{-1}(A, B, C, D) = \sup_\omega \{\nu_{\mathbb{C}}[H(\omega)]\}}$$

where

$$\nu_{\mathbb{C}} := \max\{\inf_\gamma \lambda_{\max}(H_\gamma), \inf_\gamma \lambda_{\max}(-H_\gamma)\}$$

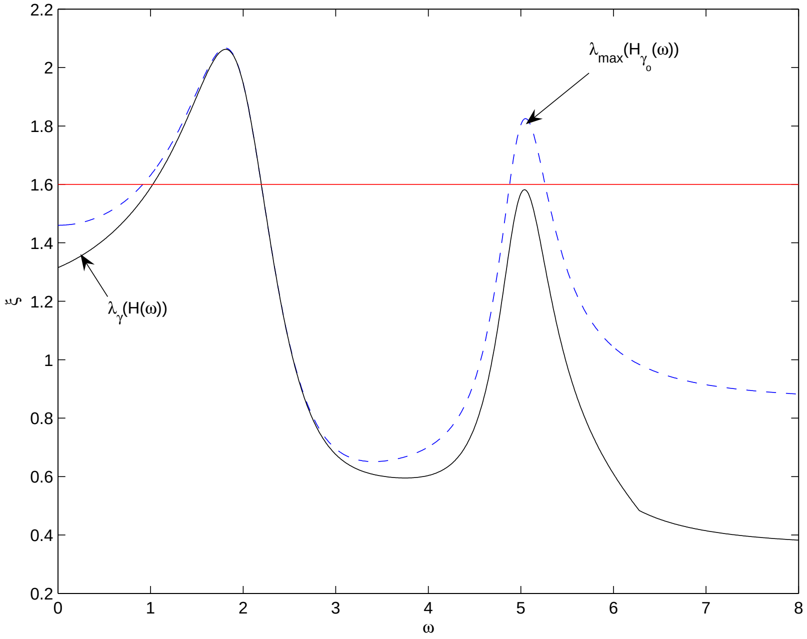
and

$$H_\gamma := T_\gamma H T_\gamma, \quad T_\gamma := \begin{bmatrix} \gamma I_{n+m} & 0 \\ 0 & I_{n+m}/\gamma \end{bmatrix}$$

Associated Hamiltonian

For each γ_o value there is a $\lambda_{\max}$ plot whose levels we can check with

$$\begin{bmatrix} -\gamma_o^2 I_n/\xi_o & A - j\omega I_n \\ A^* + j\omega I_n & -\gamma_o^{-2} I_n/\xi_o \end{bmatrix} - \begin{bmatrix} B \\ C^* \end{bmatrix} (D + D^* - \frac{\gamma_o^2 + \gamma_o^{-2}}{\xi_o} I)^{-1} \begin{bmatrix} B^* & C \end{bmatrix}$$



The (solid) curve is the lower envelope of the dotted curves and each one is tangent to it in one frequency ω_o

Convergence is quadratic or cubic (Overton-V)

Real passivity

An equivalent problem has been formulated (Hu-Qiu) but only a bound can be obtained in general

It is still an open problem if this bound is always met for the passivity radius.

This bound can be computed using a level set method involving two parameters γ_1 and γ_2

The optimization problem over γ_1 and γ_2 is simple (quasi-convex)

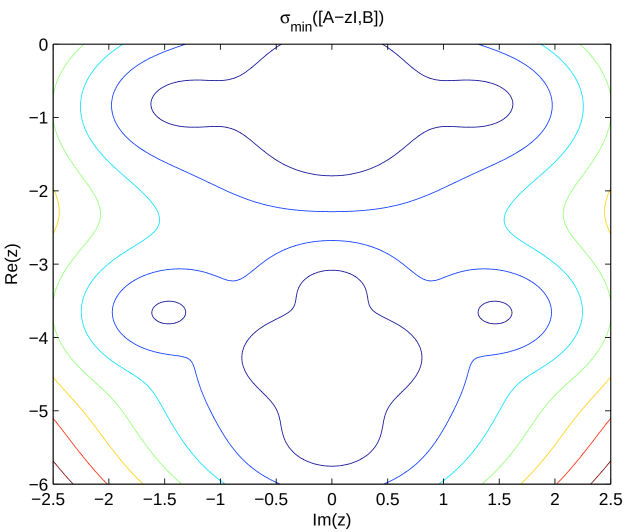
Minimality

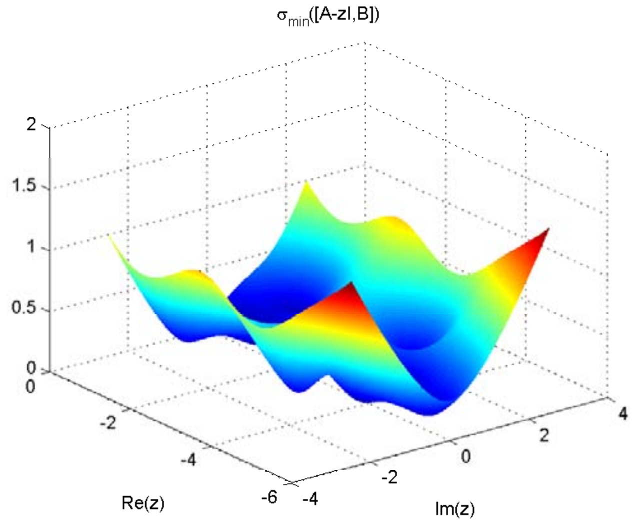
A system remains controllable unless it is perturbed by the amount

$$mr_{\mathbb{C}}(A,B) := \min\{\|[\Delta_A \mid \Delta_B]\|, (A+\Delta_A, B+\Delta_B) \text{ is uncontrollable } \}$$

For complex perturbations this is non-convex minimization (Eising)

$$mr_{\mathbb{C}}(A,B) := \tau_{\mathbb{C}}(A,B) := \min_{\lambda \in \mathbb{C}} \sigma_{\min}\left[A - \lambda I \mid B\right]$$





Bounded derivative

The slope of the singular value plot is bounded :

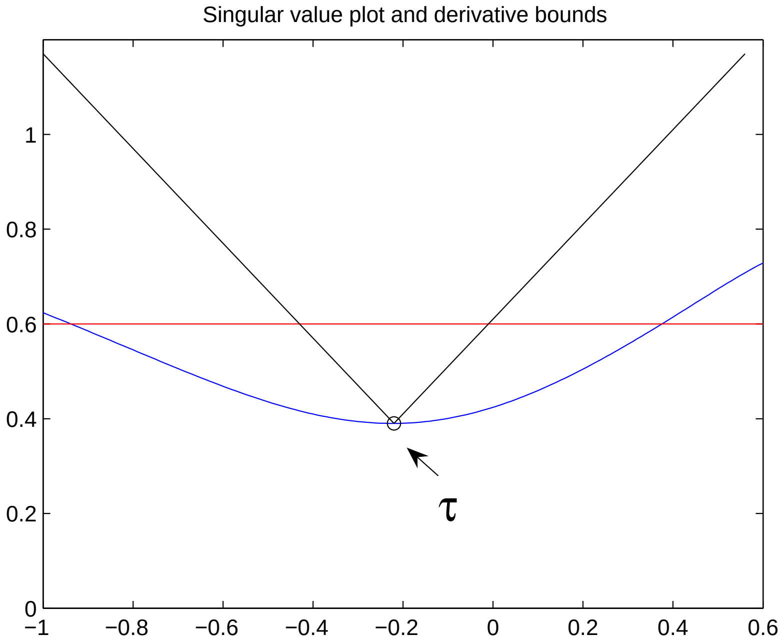
$$|\frac{\partial \sigma_i}{\partial \lambda}| = |u^*[I_n \mid 0]v| \leq 1$$

We have the following (Gu)

Theorem For any $\delta > \tau(A, B)$ and for any $\eta \in [0, 2(\delta - \tau)]$ there exist two pairs of real numbers α, β such that

$$\delta \in \sigma[A - (\alpha + \beta i)I, B]$$

$$\delta \in \sigma[A - (\alpha + \eta + \beta i)I, B]$$



Bisection - trisection

The above theorem yields one of the following bounds (Mengi et al)

$$\delta \geq \tau, \quad or \quad \tau > (\delta - \eta)/2$$

This yields a bisection (or trisection) algorithm to estimate $\tau_{\mathbb{C}}(A, B)$ within a factor 2 at most.

The algorithm is based on the solution of a Sylvester equation to test common roots of two eigenvalue problems

The complexity is $O(n^6)$ for the basic algorithm but between $O(n^5)$ and $O(n^4)$ using sparse techniques

No guaranteed accurate computation unless one can find isolated quasi-convex regions

Can be extended to the estimation of $\tau_{\mathbb{R}}(A, B)$

Other uses of level sets

- An upper bound for μ (Lawrence-Tits-V)
- Radius of hyperbolicity and ellipticity of second order polynomial matrices $P(\lambda) := \lambda^2 A + \lambda B + C$ (Hachez-V)
- Radius of definiteness of pencils $\lambda B - A$ (Crawford number) (Higham-Tisseur-V)

Flexible and fast method for problems with a characterization in terms of eigenvalues or singular values

Depending on frequency and in a quasi-convex way of a few parameters

Some references with level set ideas

R. BYERS, A bisection method for measuring the distance of a stable matrix to the unstable matrices, *SIAM J. Sci. Stat. Comput.*, vol. 9, pp. 875–881, Sept 1988.

S. BOYD, V. BALAKRISHNAN, AND P. KABAMBA, A bisection method for computing the H_∞ norm of a transfer matrix and related problems, *Mathematics of Control, Signals, and Systems*, vol. 2, pp. 207–219, 1989.

D. HINRICHSEN, B. KELB, AND A. LINNEMANN, An algorithm for the computation of the complex stability radius, *Automatica*, vol. 25, pp. 771–775, 1989.

N. BRUINSMA, AND M. STEINBUCH, A fast algorithm to compute the H_∞ norm of a transfer function matrix, *Systems & Control Letters*, vol. 14, pp. 287–293, 1990.

J. SREEDHAR, P. VAN DOOREN, AND A.L. TITS, A fast algorithm to compute the real structured stability radius, in *Int. Series of Numer. Math.*, Birkhauser, pp. 219–230, 1996.

Y. GENIN, P. VAN DOOREN, AND V. VERMAUT. Convergence of the calculation of H_∞ norms and related questions. in *Proceedings MTNS-98*, pp. 429–432, July 1998.

C. LAWRENCE, A. TITS, AND P. VAN DOOREN, A fast algorithm for the computation of an upper bound on the μ-norm, *Automatica*, vol. 36, pp. 449–456, 2000.

M. OVERTON, AND P. VAN DOOREN, On computing the complex passivity radius, in *Proceedings 44th IEEE Conference on Decision and Control*, pp. 7960–7964, Spain, 2005.