Structured matrices in systems theory

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Goal of this lecture

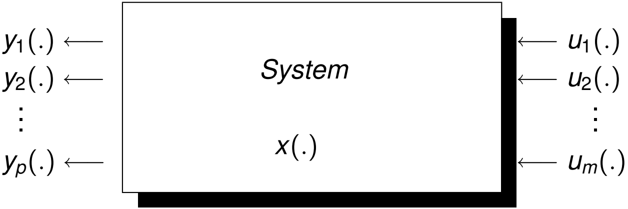
Structured matrix problems in systems and control

- Identification
- Structured eigenvalue problems
- Hankel and Toeplitz solvers
- Positive polynomial matrices

which are all problems with special structure that can be exploited

Structured problems in identification

We assume multi-input multi-output (MIMO) systems of the form



Try to identify the model coefficients of the discrete-time system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned}$$

Impulse response identification

The transfer function is given by

$$\begin{aligned} H(z) &= D + CBz^{-1} + CABz^{-2} + CA^2Bz^{-3} + CA^3Bz^{-4} + \dots \\ &= H_0 + H_1 z^{-1} + H_2 z^{-2} + H_3 z^{-3} + H_4 z^{-4} + \dots \end{aligned}$$

We thus need to factorize the Hankel map as follows

$$\begin{bmatrix} H_1 & H_2 & H_3 & H_4 & \dots \\ H_2 & H_3 & H_4 & \iddots & \dots \\ H_3 & H_4 & \iddots & \iddots & \dots \\ H_4 & \iddots & \iddots & \iddots & \\ \vdots & \vdots & \vdots & & \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} B & AB & A^2B & A^3B & \dots \end{bmatrix}$$

to recover $\{A, B, C, D\}$ from the impulse response $\{H_i,\ i = 1, \dots\}$

There exist efficient rank factorizations for block Hankel matrices

Input-output data identification

From the state-space equations one retrieves

$$\begin{bmatrix} x_2 & x_3 & \cdots & x_N \\ y_1 & y_2 & \cdots & y_{N-1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_{N-1} \\ u_1 & u_2 & \cdots & u_{N-1} \end{bmatrix}$$

from which we can identify the evolution matrix

$$\mathcal{E} := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

provided the sequence $X_{1,N} := [\, x_1 \; x_2 \; \ldots \; x_N \,]$ can be reconstructed

Define the block Hankel matrices

$$Y_{k,i,j} := \begin{bmatrix} y_k & y_{k+1} & \cdots & y_{k+j-1} \\ \vdots & \vdots & & \vdots \\ y_{k+i-1} & y_{k+i} & \cdots & y_{k+i+j-2} \end{bmatrix}$$

Subspace approach

and

$$U_{k,i,j} := \begin{bmatrix} u_k & u_{k+1} & \cdots & u_{k+j-1} \\ \vdots & \vdots & & \vdots \\ u_{k+i-1} & u_{k+i} & \cdots & u_{k+i+j-2} \end{bmatrix}$$

then one retrieves $X_{k,j} := [\, x_k \; x_{k+1} \; \ldots \; x_{k+j-1} \,]$, from

$$\text{rank}\,(H_{k,i,j}) = \text{rank}\,(U_{k,i,j}) + \text{rank}\,(X_{k,j})\,, \qquad H_{k,i,j} := \left[\frac{Y_{k,i,j}}{U_{k,i,j}}\right]$$

or equivalently from the subspace intersection condition

$$Im\left[X_{k+i,j}^T\right] = Im\left[H_{k,i,j}^T\right] \cap Im\left[H_{k+i,i,j}^T\right]$$

This can be recovered using constrained least squares techniques especially adapted to exploit the block Hankel structure

Covariance data identification

Given a stationary white noise input $E\{u_k\ u_{k-i}^T\} = \delta_{ik} I_m$, and measured output cross correlation matrices

$$E\{y_k\ y_{k-i}^T\} = R_i, \quad i = 0, \ldots$$

a predictor polynomial matrix $A(z^{-1}) := I + \sum_{i=1}^{n} A_i z^{-i}$ can be constructed from the decomposition of the block Toeplitz matrix

$$T_{n+1} := \begin{bmatrix} R_0 & R_1 & \ldots & R_n \\ R_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & R_1 \\ R_n & \ldots & R_1 & R_0 \end{bmatrix}$$

Both the Levinson and the Schur algorithm solve this in $n^2 m^3$ flops instead of $n^3 m^3$ for a general method for decomposing T_{n+1}

Periodic problems

One can have time-varying state-space models

$$\left\{ \begin{array}{rcl} E_k x_{k+1} & = & A_k x_k + B_k u_k \\ y_k & = & C_k x_k + D_k u_k, \end{array} \right.$$

where the matrices A_k, B_k, C_k, D_k, E_k vary with a period K

The following “cyclic" pencil

$$\lambda\mathcal{E} - \mathcal{A} \doteq \begin{bmatrix} -A_1 & \lambda E_1 & & \\ & \ddots & \ddots & \\ & & -A_{K-1} & \lambda E_{K-1} \\ \lambda E_K & & & -A_K \end{bmatrix}$$

plays an important role in this context.

Periodic problems

The periodic Schur decomposition

$$\text{diag}\{Q_1,\dots Q_K\}^T(\lambda\mathcal{E}-\mathcal{A})\,\text{diag}\{Z_1,\dots Z_K\} =$$

$$\begin{bmatrix} -\hat{A}_1 & \lambda\hat{E}_1 & & \\ & \ddots & \ddots & \\ & & -\hat{A}_{K-1} & \lambda\hat{E}_{K-1} \\ \lambda\hat{E}_K & & & -\hat{A}_K \end{bmatrix},$$

solves the underlying eigenvalue problem of the monodromy matrix

$$\hat{\Phi}_{K,1} := \hat{E}_K^{-1}\hat{A}_K\cdots\hat{E}_1^{-1}\hat{A}_1 = Z_1^T(E_K^{-1}A_K\cdots E_1^{-1}A_1)Z_1 = Z_1^T\Phi_{K,1}Z_1$$

which solves basic periodic control problems in $O(Kn^3)$ flops

The periodic Schur algorithm was shown to be backward stable

Optimal problems

Minimizing a quadratic cost for the response of the dynamical systems

$$J = \int_0^\infty \{x^T(t)Qx(t) + u^T(t)Ru(t)\}dt, \quad \dot{x}(t) = Ax(t) + Bu(t)$$

and

$$J = \sum_0^\infty x_k^T Q x_k + u_k^T R u_k, \quad x_{k+1} = Ax_k + Bu_k,$$

passes via the structured differential and difference equations

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

and

$$\begin{bmatrix} x_{k+1} \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} I & BR^{-1}B^T \\ 0 & A^T \end{bmatrix}^{-1} \begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_k \end{bmatrix}$$

Hamiltonian and Symplectic eigenvalue problems

They are reliably solved via the para-hermitian eigenvalue problems

$$s \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ A^T & Q & 0 \\ B^T & 0 & R \end{bmatrix}$$

and

$$z \begin{bmatrix} 0 & I & 0 \\ A^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ I & Q & 0 \\ 0 & 0 & R \end{bmatrix}$$

These (generalized) eigenvalue problems have a Hamiltonian and Symplectic structure and can be solved in a structurally stable manner

Hankel and Toeplitz solvers

We consider Toeplitz and Hankel matrices

$$T := \begin{bmatrix} t_0 & t_1 & \dots & t_{n-1} \\ t_{-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_1 \\ t_{-(n-1)} & \dots & t_{-1} & t_0 \end{bmatrix}, \quad H := \begin{bmatrix} h_0 & h_1 & \dots & h_{n-1} \\ h_1 & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \vdots \\ h_{n-1} & \dots & \dots & h_{2n-2} \end{bmatrix}$$

where the $t_i,\ h_i$ are scalars or $k \times \ell$ matrices

These matrices have a low rank “displacement" $\nabla T := T - Z^T TZ$

$$Z := \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 0 \end{bmatrix}, \quad \nabla T = \begin{bmatrix} t_0 & t_1 & \dots & t_{n-1} \\ t_{-1} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ t_{-(n-1)} & 0 & \dots & 0 \end{bmatrix}.$$

This low rank is exploited to yield fast algorithms

Schur algorithm

The Cholesky factorization of a scalar Toeplitz matrix $T = U^T U$

$$T = \begin{bmatrix} t_0 & t_1 & \dots & t_{n-1} \\ t_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_1 \\ t_{n-1} & \dots & t_1 & t_0 \end{bmatrix}, \quad U := \begin{bmatrix} u_{1,1} & u_{1,2} & \dots & u_{n,n} \\ & u_{2,2} & \dots & u_{2,n} \\ & & \ddots & \vdots \\ & & & u_{n,n} \end{bmatrix}$$

can be retrieved from the rank 2 factor G of the indefinite ∇T matrix

$$\nabla T = G^T \Sigma G, \quad \Sigma := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad G := \begin{bmatrix} x_0 & x_1 & \dots & x_{n-1} \\ 0 & y_1 & \dots & y_{n-1} \end{bmatrix}$$

A sequence of $n-1$ hyperbolic rotations $\Theta^T \Sigma \Theta = \Sigma$ applied to the rows of G construct the rows of U, one at a time

QR factorization

The Schur algorithm has a block variant and an incomplete variant for positive semidefinite matrices, that can be used for the factorizations

$$T = QR, \quad H = QR$$

Since H is just a column permutation of T we only consider

$$T^T.T = R^T Q^T.QR = R^T.R$$

which implies (if Q is square)

$$M = \begin{bmatrix} T^T T & T^T \\ T & I \end{bmatrix} = \begin{bmatrix} T^T \\ I \end{bmatrix} \begin{bmatrix} T & I \end{bmatrix} = \begin{bmatrix} R^T \\ Q \end{bmatrix} \begin{bmatrix} R & Q^T \end{bmatrix}$$

The generator of M is easy to obtain from T

The algorithm was shown to be stable (in a weak sense)

Numerical examples

Comparison with LAPACK for $T_{mn} = QR$ with $k \times \ell$ blocks

$$\begin{aligned} e_R &= \|T^T T - R^T R\| / \|T^T T\|, \\ e_{QR} &= \|T - QR\| / \|T\|, \\ e_Q &= \|I - Q^T Q\|, \\ e_{\mathrm{qr}} &= \|T - QR\| / \|T\|, \quad \text{(using qr of Matlab).} \end{aligned}$$

k	ℓ	m	n	e_R	e_{QR}	e_Q	e_{qr}
1	1	1000	1000	5.38e-15	3.07e-15	1.99e-09	3.19e-15
10	100	100	10	1.62e-15	2.28e-15	5.00e-09	1.57e-15
100	10	10	100	3.87e-15	1.33e-15	5.94e-08	3.39e-15
100	100	10	10	3.14e-15	2.83e-15	2.07e-10	2.83e-15

Timings of both approaches

t_{TOEPQR}	t_{LAPACK}
3.36s	56.67s
20.40s	55.39s
28.47s	56.49s
39.62s	56.65s

References

Toeplitz and Hankel solvers are described in

Kailath, Sayed (Eds), *Fast Reliable Algorithms for Matrices with Structues*, SIAM, 1999.

Linear Algebra Problems in Systems and Control are surveyed in

Van Dooren, *Graduate Course on Numerical Linear Algebra for Signals Systems and Control*,
`http://www.inma.ucl.ac.be/∼vdooren/grad.html`

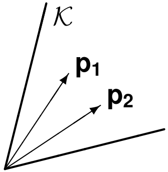
Optimization over positive polynomials

Consider the set of real positive polynomials $\mathcal{K}$:

$$\mathcal{K} = \{\ \mathbf{p} \in \mathbb{R}^{2n+1}\ :\ p(x) = \sum_{\ell=0}^{2n} p_\ell x^\ell \geq 0, \quad \forall x \in \mathbb{R}\ \}$$

$\mathcal{K}$ is a convex cone :

- $\mathbf{p_1} \in \mathcal{K},\ \alpha \geq 0 \quad \Rightarrow \quad \alpha \mathbf{p_1} \in \mathcal{K}$
- $\mathbf{p_1}, \mathbf{p_2} \in \mathcal{K} \quad \Rightarrow \quad \mathbf{p_1} + \mathbf{p_2} \in \mathcal{K}$



Several problems can be formulated as a convex optimization problem with constraints over $\mathbf{p} \in \mathcal{K}$.

The problem formulation

Consider the standard *primal* formulation of a convex problem :

$$\min_{\mathbf{p}} \{ \langle \mathbf{c}, \mathbf{p} \rangle \ : \ A\mathbf{p} = \mathbf{b}, \ \mathbf{p} \in \mathcal{K} \subset \mathbb{R}^{2n+1} \}$$

linear or convex cost

k linear constraints
$A \in \mathbb{R}^{k \times 2n+1}$

convex cone of
positive polynomials

$\langle \cdot, \cdot \rangle$ is the standard inner product :

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

$$\langle X, Y \rangle = \sum_{i,j} X_{ij} Y_{ij}, \quad X, Y \in \mathbb{R}^{m \times n}$$

How to describe the cone $\mathcal{K}$?
How to solve this problem efficiently ?

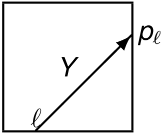
Parametrization of the cone $\mathcal{K}$ of positive polynomials

$$p(x) = \sum_{i=0}^{2n} p_\ell x^\ell \geq 0, \; x \in \mathbb{R}$$

p_ℓ real (Hermitian)

$$\exists Y \succeq 0, \; p_\ell = \sum_{i+j=\ell} Y_{i,j}$$

$$p_\ell = \langle Y, H(e_\ell) \rangle$$



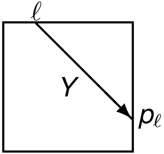
Hankel matrices

$$p(z) = \sum_{i=-n}^{n} p_\ell z^\ell \geq 0, \; |z| = 1$$

$p_{-\ell} = \overline{p}_\ell$ (Hermitian conjugate)

$$\exists Y \succeq 0, \; p_\ell = \sum_{i-j=\ell} Y_{i,j}$$

$$p_\ell = \langle Y, T(e_\ell) \rangle$$



Toeplitz matrices

Proof based on spectral factorization or positive real lemma

Positive polynomials, Hankel and Toeplitz matrices

Let $\pi(x) = [1 \; x \; \ldots \; x^n]^T$ be a polynomial vector, then $\langle \pi(x), Y\pi(x) \rangle$ is a polynomial of degree $2n$ and it is clearly positive if $Y \succ 0$

It then follows for polynomials on the real line that

$$\langle \pi(x), Y\pi(x) \rangle = \langle \pi(x)\pi(x)^T, Y \rangle = \sum_{i=0}^{2n} \langle H(e_i), Y \rangle x^i$$

Similarly, for polynomials on the unit circle, we have that

$$\langle \pi(z^{-1}), Y\pi(z) \rangle = \langle \pi(z^{-1})\pi(z)^T, Y \rangle = \sum_{i=-n}^{n} \langle T(e_i), Y \rangle z^i$$

The necessity condition is harder and follows from the KYP lemma

Dual formulation

Using $p_\ell = \langle Y, H(\mathbf{e}_\ell)\rangle,\ \ell \in 0:2n$, the primal problem is :

$$\min_{Y \in \mathcal{S}^{n\times n}} \{\langle H(\mathbf{c}), Y\rangle :\ \langle H(\mathbf{a}_i), Y\rangle = b_i,\ i \in 1:k,\ Y \succeq 0\ \}.$$

The dual problem is [Nesterov] :

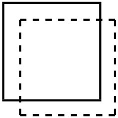
$$\max_{\mathbf{s},\mathbf{y}} \{\langle \mathbf{b}, \mathbf{y}\rangle :\ \mathbf{s} + A^T\mathbf{y} = \mathbf{c},\ H(\mathbf{s}) \succeq 0\ \}.$$

Substitute $\mathbf{s} = \mathbf{c} - A^T\mathbf{y}$ to get

$$\max_{\mathbf{y} \in \mathbb{R}^k} \{\langle \mathbf{b}, \mathbf{y}\rangle :\ H(\mathbf{c} - A^T\mathbf{y}) \succeq 0\ \}.$$

Use the dual formulation to get structured matrices.
$\Rightarrow$ Easy to exploit Hankel/Toeplitz matrices now !

Displacement of a matrix



Toeplitz matrices : $\nabla_T = T - ZTZ^T$ has low rank (2)

$$T(\mathbf{s}) \doteq \begin{bmatrix} s_0 & \bar{s}_1 & \cdots & \bar{s}_n \\ s_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bar{s}_1 \\ s_n & \cdots & s_1 & s_0 \end{bmatrix}, \quad \mathbf{s} \in \mathbb{R} \times \mathbb{C}^n$$

In fact, $\nabla_T = G\Sigma G^*$ where $G \in \mathbb{C}^{n+1\times 2}$ is easily computed from T.
$\Rightarrow \mathcal{O}(n^2)$ factorization of T and T^{-1} !
$\Rightarrow \mathcal{O}(n(\ln n)^2)$ matrix solve : $T^{-1}x$

Real Hankel matrices : Similar theory and complexity

Convex optimization scheme [Nesterov-Nemirovsky]

A self-concordant barrier for the convex constraint $H(\mathbf{s}) \succeq 0$ (with $\mathbf{s} \doteq \mathbf{c} - A^T \mathbf{y}$):

$$f(\mathbf{y}) = -\ln\det H(\mathbf{s})$$

The gradient is :

$$\frac{\partial f(\mathbf{y})}{\partial y_i} = \langle H^{-1}(\mathbf{s}), H(\mathbf{a}_i)\rangle, \quad i \in 1:k$$

The Hessian is :

$$\frac{\partial^2 f(\mathbf{y})}{\partial y_i \partial y_j} = \langle H^{-1}(\mathbf{s}) H(\mathbf{a}_i) H^{-1}(\mathbf{s}), H(\mathbf{a}_j)\rangle, \; i, j \in 1:k$$

Use displacement rank techniques and convolution to evaluate one "Newton" step in $\mathcal{O}(kn(\ln n)^2 + k^2 n)$ flops (number of Newton steps is $\mathcal{O}(\sqrt{n}\ln\frac{1}{\epsilon})$)

Conclusion

- "Right" description of the problem
  $\Rightarrow$ look at the *dual problem* !
- Fast algorithms
  $\Rightarrow$ use the *structure* of the matrices !
  - Hankel : positive polynomials on the real line $\mathbb{R}$,
  - Toeplitz : positive trigonometric polynomials over unit circle $\mathbb{T}$.
- Total complexity : $\mathcal{O}(kn^{\frac{3}{2}}((\ln n)^2 + k)\ln\frac{1}{\epsilon})$
- Applications : filter design, path planning

Extensions

Positive "polynomials" can be defined over other curves in $\mathbb{C}$:
Unit circle $\Rightarrow$ Hermitian Toeplitz matrices

$$p(e^{j\omega}) \geq 0,\ \omega \in [0, 2\pi]$$

Imaginary axis $\Rightarrow$ j-Hankel matrices

$$p(jx) \geq 0,\ x \in \mathbb{R}$$

"Segments" (see applications)

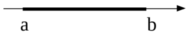
$$p(x) \geq 0,\ x \in [a, b] \subset \mathbb{R} \qquad p(e^{j\omega}) \geq 0,\ \omega \in [\omega_a, \omega_b] \subset [0, 2\pi]$$

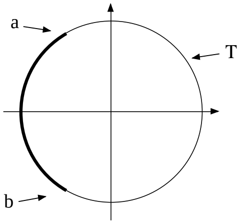
Positivity on a segment

Positive polynomials over $[a, b] \subset \mathbb{R}$ also form a convex cone.
Moreover,

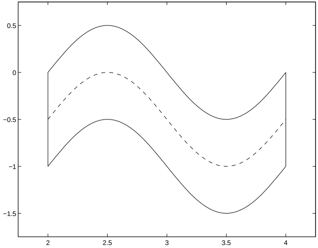
$$
\begin{aligned}
p(x) &= [p_1(x)]^2 + (x - a)(b - x)[p_2(x)]^2 \quad \text{(even degree)} \\
&= (x - a)[p_3(x)]^2 + (b - x)[p_4(x)]^2 \quad \text{(odd degree)}
\end{aligned}
$$

Using this theorem (Markov-Lukacs), we can still use the above results !





Path planning application



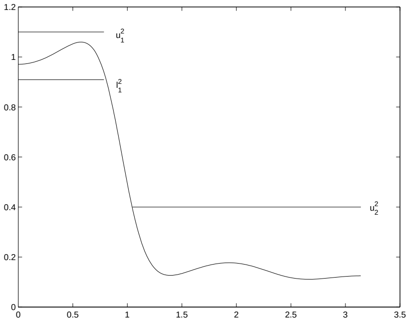
$p_u(x) \geq p(x) \geq p_l(x)$ becomes for $x \in [2, 4]$:

$$\begin{cases} p_u(x) - p(x) & \geq & 0 \\ p(x) - p_l(x) & \geq & 0 \end{cases}$$

We can minimize some convex function of $p(x)$.
Combination of different constraints for each interval can be handled.

Filter design application

Digital bandpass filter $r(z) = \frac{n(z)}{d(z)},\ z \in \mathbb{T}$ of a given degree:

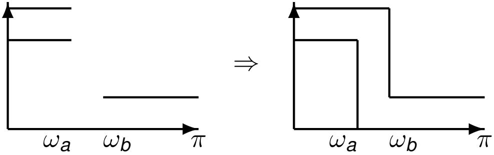


Design constraints (symmetry !)

$$\begin{cases} u_1^2 \geq \dfrac{n(e^{j\omega})n(e^{-j\omega})}{d(e^{j\omega})d(e^{-j\omega})} \geq l_1^2, & \omega \in [0, \omega_a] \\ u_2^2 \geq \dfrac{n(e^{j\omega})n(e^{-j\omega})}{d(e^{j\omega})d(e^{-j\omega})} \geq 0, & \omega \in [\omega_b, \pi] \end{cases}$$

Filter design application

Modify the bounds to avoid any overshoot



New design constraints :

$$\begin{cases} u_1^2.d(e^{j\omega})d(e^{-j\omega}) - n(e^{j\omega})n(e^{-j\omega}) & \geq & 0, & \omega \in [0, \omega_b] \\ n(e^{j\omega})n(e^{-j\omega}) - l_1^2.d(e^{j\omega})d(e^{-j\omega}) & \geq & 0, & \omega \in [0, \omega_a] \\ u_2^2.d(e^{j\omega})d(e^{-j\omega}) - n(e^{j\omega})n(e^{-j\omega}) & \geq & 0, & \omega \in [\omega_b, \pi] \\ n(e^{j\omega})n(e^{-j\omega}) & \geq & 0, & \omega \in [\omega_a, \pi] \Rightarrow [0, \pi] \end{cases}$$

Constraints : 7 Toeplitz matrices defined by 4 complex vectors.

Filter design application

What is our objective function ?

- minimize the passband ripple given stopband attenuation : $\min(u_1^2 - l_1^2)$;
- maximize the stopband attenuation : $\min u_2^2$;
- minimize the degree of the filter.

“Nearly convex” problems $\Rightarrow$ two-steps algorithm :

1. Bisection rule + Feasibility problem;
2. Spectral factorization to get $n(z)$ and $d(z)$.

References

This has been extended to the case of polynomial matrices in

Genin, Hachez, Nesterov, Stefan, Van Dooren, Xu, *Positivity and linear matrix inequalities*, European Journal of Control, 2002.

Genin, Hachez, Nesterov, Van Dooren, *Optimization problems over positive pseudo-polynomial matrices*, SIMAX, 2003.