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Spectral Functions and Smoothing Techniques on Jordan Algebras

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Contents

List of notation

Basic sets

$\mathbb{N}$	Set of nonnegative integers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of nonnegative real numbers
$\mathbb{R}_{++}$	Set of positive real numbers
$\mathbb{L}^n$	Second-order cone (see p. 11)
$\mathbb{S}^n$	Set of $n \times n$ real symmetric matrices
$\mathbb{S}^n_+$	Set of $n \times n$ real symmetric positive semidefinite matrices
$\mathbb{R}^n_\downarrow$	Set of n-real dimensional vectors with decreasingly ordered components (see p. 105)
Δ_n	n-dimensional simplex (see p. 187)
$\mathcal{P}$	Set of $r \times r$ permutation matrices (see p. 105)
K^*	Dual cone of the cone K (see p. 9)
$\mathrm{conv}(A)$	Convex hull of the set A
$\mathcal{SC}(\lambda)$	Permutahedron generated by λ (see p. 106)

Basic elements and functions

$\mathbf{1}_p$	r-dimensional vector $(1, \cdots, 1, 0 \cdots, 0)^T$, with p "1" and $r - p$ "0" (see p. 105)
$\mathbf{1}$	All-one r-dimensional vector
$s_p(\lambda)$	(for $1 \leq p \leq r$) Sum of the p largest components of the r-dimensional vector λ (see p. 106)
$\mathrm{Det}(A)$	Determinant of the matrix A
$\mathrm{Tr}(A)$	Trace of the matrix A
$\mathcal{O}$	Big-Oh asymptotic notation (see p. 3)
Θ	Theta asymptotic notation (see p. 3)

Functional analysis

$\mathrm{epi}\,(f)$	Epigraph of the function f
$\nabla_x^h f(x)$	Directional derivative of f in x, in the direction h
$\nabla_x f(x)$	Differential of f in x (see p. 132)
$f'(x)$	Gradient of f in x (see p. 132)
$\partial f(x)$	Subdifferential of f in x (see p. 132)
$\partial_B f(x)$	Bouligand subdifferential of f in x (see p. 152)
$\partial_C f(x)$	Clarke subdifferential of f in x (see p. 133)

Algebra

$\mathcal{J} \bigotimes_F R$	Extension of the base field F of the algebra $\mathcal{J}$ with the ring R (see p. 31)
$[\cdot;\cdot]$	Commutator (see p. 33)
$\langle\cdot,\cdot\rangle_{\mathcal{J}}$	Jordan scalar product (see p. 79)
$L(x)$	Operator of multiplication by x (see p. 32)
$\mathcal{H}(\mathcal{J})$	Subalgebra of self-adjoint elements of $\mathcal{J}$ (see p. 37)
$\mathcal{S}_n$	nth Jordan spin algebra (see p. 37)
F_ε	Ring of dual numbers built from F (see p. 51)
ε	Nonzero element of F_ε, whose square is null (see p. 51)
μ_u	Minimal polynomial of u (see p. 39)
g_u	Reduced minimal polynomial of u (see p. 39)
$f(\tau;x)$	Characteristic polynomial (see p. 48)
$\lambda(u)$	Ordered vector of eigenvalues of u (see p. 49 and p. 79)
$\lambda(u;\mathcal{J}')$	Ordered vector of eigenvalues of u, considered in the subalgebra $\mathcal{J}'$ (see p. 79)
$\det(u)$	Determinant of u (see p. 50)
$\mathrm{tr}(u)$	Trace of u (see p. 50)
$\mathrm{detr}_j(u)$	jth dettrace of u (see p. 49)
$\lambda_i'(u,h)$	Short writing for the directional derivative of λ_i in u, in the direction h (see p. 162)
Q_u	Quadratic operator (see p. 58)
$Q_{u,v}$	Polarized quadratic operator (see p. 60)
$\mathcal{J}_1(c), \mathcal{J}_{1/2}(c), \mathcal{J}_0(c)$	Pierce subspace of $\mathcal{J}$ with respect to c (see p. 65)
$\mathcal{E}_i$	Set of subalgebras $\mathcal{J}_1(c)$ of $\mathcal{J}$, where the trace of c equals i (see p. 113)
$\mathcal{E}_i(\mathcal{J}')$	Set of subalgebras $\mathcal{J}_1'(c)$ of $\mathcal{J}'$, where the trace of c equals i (see p. 113)
$\mathcal{K}_{\mathcal{J}}$	Cone of squares (see p. 86)
$l_p(u), u_p(u)$	Index numbers related to the spectral decomposition of u (see p. 143)
$\mathsf{f}_p(u), \mathsf{f}_p'(u), \mathsf{f}_p''(u)$	Idempotents related to the spectral decomposition of u (see p. 144)
$G(\mathcal{J})$	Set of invertible linear operators from $\mathcal{J}$ to $\mathcal{J}$ (see p. 90)
$\mathcal{A}(\mathcal{J})$	Set of automorphisms of $\mathcal{J}$ (see p. 90)
$\Gamma(\mathcal{J})$	Structure group of $\mathcal{J}$ (see p. 91)

CHAPTER 1

Introduction and preliminaries

Human beings have a natural desire to do things as well as possible. And when there are several ways of doing something, they try to choose the best alternative they can afford. *Operations research* is an attempt to formalize their decision problems into a mathematical framework.

This mathematical modeling is usually based on the definition of several parameters of the problem, the *decision variables*. The quality of a decision is described by a numerical function depending on these variables, the *objective function*, that has to be maximized or minimized. Finally, the decision variables are typically subject to various *constraints*, which have to be identified.

The generic optimization problem can be stated as follows. We are given a real-valued objective function f defined on a set E and a subset $X \subseteq E$ of constraints. We need to solve:

$$\begin{aligned} f^* = \; & \min f(x) \\ & \text{subject to } x \in X, \end{aligned}$$

where x represents the decision variables. The points of X are called *feasible points*. An *optimal point* x^* is a feasible point where f achieves its minimum. The constraints X are often described by a finite number of functional inequalities: $X = \{x \in E | f_i(x) \leq 0, 1 \leq i \leq m\}$, where $\{f_i | 1 \leq i \leq m\}$ is a set of real-valued functions defined on E. The subject matter of *Optimization* consists in devising and studying efficient procedures to solve the above mathematical problem.

Solving an optimization problem amounts to finding an optimal point or to proving that such a point does not exist. On a finite-arithmetic computer, this goal is typically unreachable, and we content ourselves with an *approximation* of the optimal point in one of the following senses. Given a tolerance $\epsilon > 0$,

(Absolute tolerance on objective's value) we are looking for a feasible point $\hat{x}$ for which $f(\hat{x}) - f^* < \epsilon$,

(Relative tolerance on objective's value) or, provided that $f^* > 0$, we want to find a feasible point $\hat{x}$ satisfying $f(\hat{x}) - f^* < \epsilon f^*$,

(Absolute tolerance on the minimizer) or, given a distance d on E, we need a point $\hat{x}$ such that $d(\hat{x}, x^*) < \epsilon$, and $d(\hat{x}, X) < \epsilon$. This last condition means that the point $\hat{x}$ lies within a distance of ϵ from the feasible set X, and is called *ϵ-feasibility*.

In order to solve optimization problems, two approaches can be considered. The first one consists in creating a universal algorithm, which is able to solve *every* optimization problem. Unfortunately, this achievement is out of reach. Indeed, it has been proved (see e.g. [Nes03], Chapter 1) that such a method would require *at least* 500 times the estimated age of the Universe to solve up to an accuracy of 0.01 on objective's value an optimization problem involving 15 decision variables on a computer that can evaluate the objective function four billion times a second.

For this reason, the second approach is generally preferred. It involves *restricting* the set of optimization problems we aim to solve to a specific class, in the hope of using its particular features to design more efficient algorithms. The research undertaken in this thesis fits naturally this scope, as we focus on some optimization problems defined in the framework of formally real Jordan algebras.

In the next sections, we give a brief historical account on Convex Optimization in order to exhibit the main guidelines that motivate our work. In Section 1.1, we explain how two methods, designed for solving the same class of problems, can be compared with respect to their efficiency. In Sections 1.2 to 1.5, we review the main optimization strategies that have been developed to solve the classes of problems of interest for this thesis, and we show how formally real Jordan algebras have turned out to be a natural framework of investigation for these classes.

1.1 Comparing algorithms

The theoretical performance of an algorithm A on a problem P is usually measured by the amount of simple arithmetic operations (such as comparisons, addition, subtraction, multiplication, and division) needed to transform its input data into the desired output. This number is called the *complexity of A for solving P*, and is denoted by $C_A(P)$. Since simple arithmetic operations take roughly a constant processing time on a computer, the complexity is proportional to the time needed for its resolution.

Let $I(L)$ be a class of instances that A can solve, and that have the same size L (that is, their data input needs L computer memory units). The *worst-case complexity* of an algorithm A is defined as $C^w_A(L) := \max\{C_A(P)|P \in I(L)\}$. This definition can be refined in the sense that the class $I(L)$ may be characterized by other features than L. For instance, for optimization problems, the number of decision variables and of functional constraints, the required tolerance, or a measure of the regularity of the objective function can be considered as well. Of course, an algorithm A is theoretically better than an algorithm A' on $I(L)$ if $C^w_{A'}(L) \geq C^w_A(L)$.

However, this formal characterization is sometimes not sufficiently precise, because a very small set of instances of $I(L)$ might influence dramatically the worst-case complexity of an algorithm, although they occur very rarely in practice. A first possibility is to consider an averaged complexity over the set of instances, given an appropriate distribution function on it. However, this estimation can be very inaccurate, because random inputs may fail to resemble those encountered in real-life problems. Some efforts to deal with this delicate issue have been initiated by Daniel Spielman and Shang-Hua Teng [ST04]. However, these new techniques are quite laborious to develop, and we will only consider worst-case complexity in the scope of this thesis.

The exact expression of the worst-case complexity function is often abbreviated with the aid of the so-called *Big-Oh* and *Big-Theta* notation: given a function $g : \mathbb{N} \to \mathbb{R}$, we say that g *is in* $\mathcal{O}(\hat{g}(n))$ for a function $\hat{g} : \mathbb{N} \to \mathbb{R}$ if there exist a constant $c > 0$ and a number N for which $g(n) \leq c\hat{g}(n)$ when $n \geq N$. Further, we say that g *is in* $\Theta(\hat{g}(n))$ if g is in $\mathcal{O}(\hat{g}(n))$ and if there exists a constant $d > 0$ and a number M for which $g(n) \geq d\hat{g}(n)$ when $n \geq M$. This notation allows us to compare the performance of two algorithms without focusing on details that have no essential impact. It is naturally extended to functions of several variables.

1.2 Linear Programming

Linear Programming is the first class of optimization instances that has been thoroughly investigated, and it is still representing an active research area. Many practical problems, such as production planning or bond portfolio selection for instance, can be represented in this framework.

In Linear Programming, the set E is a finite-dimensional real vector space, the objective function is linear, and the constraints X are described by a finite set of affine functional inequalities. Every linear programming problem can be reformulated into the following standard version:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. }\ & Ax = b \\ & x \geq 0, \end{aligned} \tag{1.1}$$

where A is a full row-rank real matrix of dimensions m by n, and the column-vectors b and c are of dimension m and n respectively. The vector x represents the n decision variables of the problem, and the notation $x \geq 0$ means that each of its components has to be nonnegative. The feasible set is a *polytope*, and the optimal point, if it exists, is one of its vertices.

A very well studied *duality theory* has been initiated by John Von Neumann in 1947, and developed by David Gale, Harold Kuhn and Albert Tucker [GKT51]. Associated with the linear optimization problem (1.1), which is called the *primal* problem, comes its *dual*:

$$\begin{aligned} \max\ & b^T y \\ \text{s.t. }\ & A^T y + s = c \\ & s \geq 0. \end{aligned} \tag{1.2}$$

These two optimization problems are closely related to each other, as following theorem states. A proof can be found for example in [Lue84] (Section 4.2).

Theorem 1.2.1 (Linear duality Theorem) *Suppose that* $x, s \in \mathbb{R}^n_+$ *and* $y \in \mathbb{R}^m$ *satisfy* $Ax = b$ *and* $A^T y + s = c$.

Weak duality. *The quantity* $b^T y - c^T x$ *is nonnegative. It is called the* duality gap *of* x, y, s, *and equals* $x^T s$.

Strong duality. *The vectors* x, y, s *are optimal points of the primal problem (1.1) and of the dual problem (1.2) and if and only if their duality gap is null.* ■

This duality theorem plays a central role in the development of Linear Programming algorithms. Its generalization to Conic Programming stated in Theorem 1.3.12 is also a crucial result in Optimization.

About sixty years ago, George Dantzig created the *simplex algorithm*, which is the first powerful method dedicated to solving linear optimization problems [Dan63]. Roughly speaking, this scheme moves from a vertex of the feasible polytope to an adjacent one that decreases the value of the objective function, until either an optimal point is reached or it is established that no solution exists.

Since practitioners had at their disposal a very efficient method for Linear Programming, they managed to model their problems as linear optimization instances. As a result, the range of applications of Linear Programming has grown extremely rapidly.

The simplex algorithm is still widely used, especially when practitioners need to solve a bunch of very similar problems (e.g. a set of instances that differ only by a few linear constraints). In spite of its excellent practical behavior, it can be proved that its execution time can grow exponentially with respect to the number of decision variables [KM72].

In 1984, Narendra Karmarkar developed a new strategy to tackle linear optimization problems, namely the *projective method* [Kar84]. His algorithm has a provable worst-case complexity of $\mathcal{O}(n^{3.5}L)$, where L is the bit-size of the problem's data, and n the number of decision variables. Moreover, experiments have demonstrated its excellent numerical behavior. His paper has initiated an important research activity, and many variants of his original algorithm have been proposed afterwards. These methods are known under the name of *interior-point methods*. They generate *in the relative interior* of the feasible polytope a sequence of points that converges to an optimal point. These interior-point methods are theoretically *and* numerically very efficient. They have enabled a considerable increase the size of practically solvable problems – it is now usual to deal with instances of *millions* of variables, provided that the matrix A is sparse (i.e. contains many zero entries). Several monographs are dedicated to interior-point methods for Linear Programming, among which we can mention [Wri96], [RTV97] or [Van96]. A more detailed exposition on interior-point methods is given in Section 1.5.

It was soon realized that the interior-point paradigm could be extended to a broader family of optimization problems, namely to a large class of convex optimization problems. A brief account on this evolution is given in the next sections. Actually, it turns out that many major breakthroughs in Convex Programming have been initiated by new ideas in

Linear Programming. The main goal of the present thesis is to provide technical tools meant to accomplish *automatically* some of these extensions. These technical tools allow us to extend a new technique designed for solving efficiently a class of linear optimization problems.

This new technique has been proposed by Yurii Nesterov in [Nes05a]. It addresses to linear optimization problems with the following structure:

$$\begin{array}{rl} \min & \max_{1\leq i\leq m}[Ax-b]_i \\ \text{s.t.} & \mathbf{1}^T x = 1 \\ & x \geq 0, \end{array}$$

where $\mathbf{1}$ is the n-dimensional vector whose components are all equal to 1. Here, the real matrix A has dimensions $m \times n$, and the column vector b is of dimension m. The decision variables are represented by the n-dimensional vector x. This instance is indeed a linear optimization problem, as it can be reformulated in the standard form (1.1) as follows:

$$\begin{array}{rl} \min & t_+ - t_- \\ \text{s.t.} & Ax + s - (t_+ - t_-)\mathbf{1} = b \\ & \mathbf{1}^T x = 1 \\ & x, s, t_+, t_- \geq 0. \end{array}$$

The decision variables x and s are of dimension n and m respectively, while t_+ and t_- are scalar variables. It includes as particular instances *matrix games* (when the vector b is null), which can be used, among other applications, to model some portfolio optimization problems. The technique of Yurii Nesterov consists in using an optimization scheme traditionally dedicated to Convex Optimization in order to solve an *approximation* of the above problem. It turns out to be very efficient both theoretically and practically for very large-size instances, when the required precision is not too small.

In Chapter 6, we show how the same technique can be adapted to design very efficient procedures for solving some very large-scale *nonlinear* problems.

1.3 Convex Programming

Extremely few phenomena in the world can be very accurately described by a linear model. However, practitioners have simplified a lot of real-life problems to represent them as linear programming instances, in order to have at least a rough approximation of the exact solution. As a generalization of Linear Programming, *Convex Programming* appears as a natural alternative for a more accurate modeling. The last few years have witnessed major breakthroughs in Convex Programming, that have considerably extended the set of tractable nonlinear optimization problems. Nowadays, powerful optimization software packages allow practitioners to deal with nonlinear models efficiently, provided that they are sufficiently well-structured to match the set of tractable instances.

In order to define precisely the class of instances of interest in Convex Programming, let us recall a few basic definitions. We assume henceforth in this chapter that E is a finite-dimensional real vector space.

Definition 1.3.1 *A set $S \subseteq E$ is* convex *if, for every x and y of S, and every λ of $[0, 1]$, the point $\lambda x + (1 - \lambda)y$ also belongs to S.* ■

The *domain* of a function $f : E \to \mathbb{R} \cup \{+\infty\}$ is the set of points x in E where $f(x) < +\infty$; this set is denoted by $\operatorname{dom} f$. The *epigraph* of a function f is the set:

$$\operatorname{epi}(f) := \{(t, x) \in \mathbb{R} \times \operatorname{dom} f | t \geq f(x)\}.$$

Definition 1.3.2 *A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is* convex *if its epigraph is a convex set.* ■

The class of convex programming problems is the set of minimization problems that have a convex objective function and a convex set of constraints. As stressed above, a very broad range of practical problems fall into that class. The monumental monographs of Aaron Ben-Tal and Arkadi Nemirovski [BTN01], and of Stephen Boyd and Lieven Vandenberghe [BV04], display a large amount of application examples in such various fields as electronic chip conception, metallic structure design, or consumer's preference prediction. Other examples include cosmology [BFH+03] and medical imaging [BTMN01]. Of course, this list of application fields is far from being exhaustive.

The interest for *convexity* in optimization lies in the following fact. In contrast with other functions, it is easy to certify that a point is the global minimizer when the considered function is convex. Indeed, it can be easily proved that every local minimum of a convex function (that is, a point of the domain of the function that minimizes it in a suitable neighborhood) is also a global minimizer. Hence, an algorithm designed to find a local minimum of a function always finds a global minimum if the considered function is convex.

The first methods that have been developed for solving convex optimization problems where the so-called *gradient methods and subgradient methods* (see [Pol87], Chapter 2 and 3 of [Nes03], or [Sho85] for a thorough exposition). Gradient methods are dedicated to convex optimization problems with a differentiable objective, while subgradient methods are designed to solve non-differentiable convex optimization problems.

In the context of convex functions, subgradients represent a natural generalization of the concept of gradient. We denote by $\langle \cdot, \cdot \rangle$ a scalar product on E.

Definition 1.3.3 *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex function, and let x be a point in the domain of f. The* subdifferential *of f in x is:*

$$\partial f(x) := \{g \in E | f(y) \geq f(x) + \langle g, y - x \rangle \text{ for every } y \in \operatorname{dom} f\}.$$

Every element of the subdifferential of f in x is called a subgradient *of f in x.* ■

In other words, a subdifferential is the set of all the possible slopes of an affine hyperplane that is tangent to the epigraph of a function at a considered point. Its basic properties are summarized in the proposition below. Its proof can be found in [Roc70], Theorem 23.5 and Theorem 25.1.

Proposition 1.3.4 (Basic properties of subgradients and subdifferential) *The subdifferential of a convex function is never empty on its domain. Moreover, if a convex function f is differentiable at x, then the set $\partial f(x)$ contains only one element, which is the gradient of f at x. Finally, the function f reaches its minimum at a point x^* of its domain if and only if $0 \in \partial f(x^*)$.* ■

Subgradient algorithms generate a sequence of feasible points that converges to an optimal point. In order to construct this sequence, the value of the objective function and of one of the elements of its subdifferential at every previously generated point are solely available. A procedure that projects an unfeasible point on the set of constraints is also required if necessary.

Many variants of subgradient methods exist, but Arkadi Nemirovski and David Yudin have proved that an algorithm that only use the aforementioned piece of information cannot behave better than a certain performance [NY83]. Their proof is based on the concept of *oracle*.

The oracle is simply the routine that delivers needed information for an input point. For subgradient algorithms, the oracle is said to be of *first order*, because it only provides the value of the objective function and the value of one of its subgradient. First-order oracles have typically a relatively low complexity.

Optimization algorithms involving an oracle are called *black-box methods*, because the only access to the specific instance to be solved consists in asking to the oracle several of local characteristics of the problem, exactly as if it is hidden in a black box. The *complexity of a black-box method* is defined as the number of times that the oracle should be invoked in order to obtain a solution within the desired accuracy. The actual worst-case complexity, as defined in Section 1.1, can be immediately estimated by multiplying this number by the worst-case complexity of the oracle itself. Nemirovski and Yudin were able to determine *lower* complexity bounds of subgradient methods for several important classes of convex optimization problem. Their idea consists in constructing a family of problems that all have a different optimal point and a different optimal value. This family is *difficult* in the sense that many oracle calls are needed before recognizing which of its member is considered, and consequently which optimal point should be returned.

The motivations for new smoothing techniques of Nesterov originates in two of their lower complexity bound results.

Proposition 1.3.5 (See, for example, Theorem 3.2.1 in [Nes03]) *For every subgradient method, there exists a convex function $f : E \to \mathbb{R}$ for which at least $\Theta(1/\epsilon^2)$ calls of the oracle are needed to obtain an approximation of the optimal point with an ϵ absolute tolerance on the objective's value.* ■

For the subclass of *smooth* convex optimization problems, the picture is much more favorable provided that the objective function is sufficiently smooth, as the following proposition states. We first need a definition to quantify the smoothness of a function.

Definition 1.3.6 *Let $||\cdot||$ be a norm of E. Its* dual norm *is defined as:*

$$||y||_* := \sup_{||x||=1} \langle y, x \rangle$$

for every element y of E. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a differentiable function. We say that the gradient of f is Lipschitz continuous *with respect to the norm $||\cdot||$ if there exists a constant $L > 0$ such that, for every x and y in the domain of f, we have:*

$$||f'(y) - f'(x)||_* \leq L||y - x||.$$

The constant L is called the Lipschitz constant of the gradient of f. ■

Proposition 1.3.7 *Let $||\cdot||$ be a norm of E. For every gradient method, there exists a convex function $f : E \to \mathbb{R}$ with a Lipschitz constant of its gradient of L, with respect to the norm $||\cdot||$, for which at least $\Theta(\sqrt{L/\epsilon})$ calls of the oracle are needed to obtain an approximation of the optimal point with an ϵ absolute tolerance on the objective's value.* ■

There exist optimal gradient and subgradient methods for problems considered in Proposition 1.3.5 (see Theorem 3.2.3 in [Nes03]), and for problems of Proposition 1.3.7 (see Theorem 2 of [Nes05a]). These methods reach the respective lower-bound for worst-case complexity for the corresponding classes of instances.

The black-box approach for Convex Programming contains an internal contradiction. While convexity, which has to be ascertained before choosing an optimization procedure, is a *global* property of a problem, a black-box method can only use the *local* information given by the oracle. This contradiction has been solved in the framework of interior-point methods.

As mentioned earlier, the efficient projective method of Narendra Karmarkar revolutionized Linear Optimization. In contrast to the simplex algorithm, the subsequent interior-point methods inspired by Karmarkar's approach could be extended to Convex Optimization. This task has been achieved by Yurii Nesterov and Arkadi Nemirovski in their important monograph [NN93]. Introducing the concept of *self-concordant barrier* (a precise definition is given in Section 1.5), they have established firm theoretical foundations for the design and the study of interior-point methods for Convex Programming. Their techniques provide a systematic procedure to construct an algorithm for solving convex optimization problems by inspecting their structure. They have achieved its full complexity analysis, which shows that their methods are very efficient for many important classes of problems. Since then, thousands of research papers have been published on interior-point methods for Convex Optimization.

It is sometimes useful to reformulate a convex optimization problem as a *conic* optimization problem. Actually, a vast majority of efficient interior-point methods are best described in a conic setting.

Definition 1.3.8 *A set $\mathcal{K}$ of $\mathbb{R}^n$ is a* cone *if, for every point x of $\mathcal{K}$ and every positive real λ, the point λx belongs to $\mathcal{K}$.* ■

A *convex conic optimization problem* is an optimization problem of the following standard form (note the similarity with the standard form (1.1) for linear optimization problems):

$$\begin{aligned} \min\ & \langle c, x\rangle_n \\ \text{s.t. } & Ax = b, \\ & x \in \mathcal{K}, \end{aligned} \tag{1.3}$$

where A is a full row-rank real matrix of dimensions m by n, and the column vectors b and c are of dimension m and n respectively. The set $\mathcal{K}$ of $\mathbb{R}^n$ denotes a closed convex cone, which concentrates all the nonlinearities of the problem. Finally, the brackets $\langle \cdot, \cdot \rangle_n$ represent a scalar product on $\mathbb{R}^n$.

The general convex optimization problem:

$$\begin{array}{l} \min\ f(x) \\ \text{s.t. } x \in X, \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $X \subseteq \mathbb{R}^n$ are convex, can be reformulated as a conic instance (we do not discuss its practical efficiency here). Indeed, we have successively:

$$\begin{array}{lclcl} \min\ f(x) & = & \min\ t & = & \min\ t \\ \text{s.t. } x \in X & & \text{s.t. } x \in X & & \text{s.t. } t' = 1 \\ & & \quad (t, x) \in \text{epi}\,(f) & & \quad (t', t, x) \in \mathcal{K}, \end{array}$$

with

$$\mathcal{K} := \{(t', t, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n | (t, x) \in t'\text{epi}\,(f),\ x \in t'X,\ t' \geq 0\}.$$

The main interest for the conic formulation of convex optimization problems resides in the existence of a well-established duality theory for conic programming, initiated in [ET76]. The interested reader can find an interesting account in Section 3.2 of [Ren01].

Definition 1.3.9 *Let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathbb{R}^n$. The* dual *of a cone $\mathcal{K}$ of $\mathbb{R}^n$ with respect to the scaler product $\langle \cdot, \cdot \rangle$ is defined by:*

$$\mathcal{K}^* := \{y \in \mathbb{R}^n | \langle y, x \rangle \geq 0 \textit{ for every } x \in \mathcal{K}\}.$$

■

Given a scalar product $\langle \cdot, \cdot \rangle_m$ of $\mathbb{R}^m$, the dual problem associated with (1.3) has the following form:

$$\begin{array}{rl} \max & \langle b, y \rangle_m \\ \text{s.t.} & A^* y + s = c \\ & s \in \mathcal{K}^*. \end{array} \tag{1.4}$$

We denote here by A^* the *adjoint operator* of A with respect to the scalar products $\langle \cdot, \cdot \rangle_n$ and $\langle \cdot, \cdot \rangle_m$. That is, for every $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^m$, the equality $\langle Ax, y \rangle_m = \langle A^* y, x \rangle_n$ holds.

The following theorem contains the most useful results of conic duality theory for the purposes of this exposition. Its proof can be found in [Ren01], or in Section 4.2.2 of [NN93].

Definition 1.3.10 *Let $x \in \mathcal{K}$, $s \in \mathcal{K}^*$, and $y \in \mathbb{R}^m$ be three vectors satisfying $Ax = b$ and $A^* y + s = c$. The quantity $\langle c, x \rangle_n - \langle b, y \rangle_m = \langle A^* y, x \rangle_n + \langle s, x \rangle_n - \langle Ax, y \rangle_m = \langle s, x \rangle_n$ is called the* duality gap *of (x, y, s).* ■

Definition 1.3.11 *We say that the primal instance is* strictly feasible *if there exists a point $\hat{x}$ in the interior of $\mathcal{K}$ such that $A\hat{x} = b$. Likewise, we say that the dual instance is* strictly feasible *if there is a point $\hat{s}$ in the interior of $\mathcal{K}^*$ and a point $\hat{y}$ in $\mathbb{R}^m$ for which $A^* \hat{y} + \hat{s} = c$.*

Theorem 1.3.12 (Conic duality Theorem) *Let $x \in \mathcal{K}$, $s \in \mathcal{K}^*$, and $y \in \mathbb{R}^m$ be three vectors satisfying $Ax = b$ and $A^*y + s = c$.*

Weak duality. *The duality gap of (x, y, s) is nonnegative.*

Strong duality. *Suppose that the primal instance and the dual instance are both strictly feasible. Then the primal and the dual instances have an optimal solution, and their respective sets of solutions are bounded. Moreover, a feasible point (x, y, s) is a solution to (1.3) and (1.4) if and only if its duality gap is null.*

■

In view of the above theorem, a conic optimization problem can be restated in the following *primal-dual* form:

$$\begin{aligned} \min\ & \langle c, x\rangle_n - \langle b, y\rangle_m \\ \text{s.t. }\ & Ax = b \\ & A^*y + s = c \\ & x \in \mathcal{K} \\ & s \in \mathcal{K}^*. \end{aligned} \tag{1.5}$$

The optimal value equals zero if both primal and dual optimization problems are strictly feasible. This opens a possibility to measure the quality of an approximated solution. The most powerful interior-point methods solve this primal-dual formulation instead of the sole primal, because they generally show a better practical behavior. Up to our knowledge, this phenomenon has only partial theoretical explanations. Nesterov, Nemirovski, and Todd have recently found some evidences from geometrical aspects of interior-point methods [NT02, NN03].

1.4 Self-scaled Optimization, and formally real Jordan algebras

Michael Todd and Yurii Nesterov have discovered in 1994 a subclass of convex optimization problems for which they have designed interior-point methods that are theoretically and practically very efficient. They have called this class the *self-scaled* optimization problems [NT97, NT98]. It consists in conic optimization problems:

$$\begin{aligned} \min\ & \langle c, x\rangle_n \\ \text{s.t. }\ & Ax = b \\ & x \in \mathcal{K}, \end{aligned}$$

where the cone $\mathcal{K} \subseteq \mathbb{R}^n$ is *symmetric*, i.e. is a closed convex cone which has the following properties[1].

[1]The original terminology for those cones is "self-scaled" [NT97]. In their paper, Nesterov and Todd formulate its definition differently, namely in terms of properties of a barrier for these cones (see in Section 1.5). However, Theorem 3.1 in [NT97], combined with results obtained in [Gül96], shows that both definitions are equivalent.

Self-duality: the set $\mathcal{K}$ coincides with its dual $\mathcal{K}^*$ with respect to the scalar product $\langle \cdot, \cdot \rangle_n$; this implies in particular that $\mathcal{K}$ has a nonempty interior and does not contain any straight line.

Homogeneity: for every pair x, y of points in the interior of $\mathcal{K}$, there exists an invertible linear operator A for which $A\mathcal{K} = \mathcal{K}$ and $Ax = y$.

In particular, linear programming problems fall into that class, as well as the two following non-trivial examples.

Example 1.4.1 (Second-Order Programming) *For every integer $m \geq 1$, we define the $m+1$-dimensional* second-order cone $\mathbb{L}^m$ *as follows:*

$$\mathbb{L}^m := \left\{ (t, x) \in \mathbb{R} \times \mathbb{R}^m \,\middle|\, t \geq \sqrt{x_1^2 + \cdots + x_m^2} \right\}.$$

This cone is also called Lorentz cone of dimension $m+1$. *The class of* second-order programming problems *consists in conic optimization problems for which the cone $\mathcal{K}$ is a Cartesian product of second-order cones. It can be proved – all the necessary elements are given in Section 2.8 – that such cones $\mathcal{K}$ are self-scaled.*

As an important example, the class of second-order programming problems contains the set of optimization problems with a convex quadratic objective function and constraints described by a finite set of convex quadratic inequalities. Another particular case of second-order programming consists in robust *linear programming, that is, linear optimization problems (1.1) for which the data set (A, b, c) is known only up to a certain accuracy, modeled by an ellipsoid region. The interested reader can find more details in Section 3.4.2 of [BTN01]. Other applications can be found in the surveys [LVBL98] and [AG03].* ■

Example 1.4.2 (Semidefinite Programming) *Semidefinite Programming involves conic optimization problems that are defined on the cone $\mathbb{S}^n_+$ of $n \times n$ positive semidefinite real symmetric matrices. This cone can also be proved to be self-scaled. Since the implementation of interior-point algorithms for Self-Scaled Optimization in powerful optimization software (for instance* SeDuMi*, which runs on* Matlab *[Stu99]), practitioners became aware of the practical efficiency of semidefinite modeling, and many applications were discovered in such various fields as finance [d'A03], control theory [VB99a], or design of electrical circuits [VBG98]. More applications can be found in the survey paper [VB99b] and in Chapter 4 of [BTN01].* ■

Osman Güler [Gül96] realized *symmetric cones* have already been studied more than thirty year before by the algebraist Ernest Vinberg in his long article [Vin63]. In his work, Vinberg has studied and completely characterized the class of symmetric cones using the so-called T-algebras. Almost simultaneously, Charlotte Hertneck has performed a similar classification using the elegant theory of formally real Jordan algebras [Her62]. Leonid Faybusovich was the first optimizer who has exploited the advantages given by the Jordan algebraic setting in the study of Self-Scaled Programming. He has started his in-depth research by a study of non-degeneracy conditions for Self-Scaled Programming in [Fay97b]. Subsequently, he has analyzed various interior-point strategies for Self-Scaled Programming in [Fay97a, Fay02], where Jordan algebras have played a crucial role. The ideas of Faybusovich have been followed by many optimizers. For instance, Jos Sturm has presented the theoretical basis of his SeDuMi software in terms of Jordan algebras [Stu00]. Later, Masakazu Muramatsu [Mur02], Stefan Schmieta and Farid Alizadeh [SA03] have used this setting in further studies of interior-point methods for Self-Scaled Optimization.

The present thesis lies within this scope: we use the formalism of formally real Jordan algebras, substantially presented in Chapter 2, in order to extend some optimization algorithms, and to study their performance. Following preliminary ideas of Schmieta and Alizadeh [AS00], we show how these extensions can be constructed in many cases in a systematic way, with the help of our study of the so-called *spectral functions and spectral mappings* (see Chapters 4 and 5). In order to support this point of view, let us briefly explain how interior-point methods work in convex optimization, and how their specification to Linear Programming has been (implicitly at first) extended to Self-Scaled Programming.

1.5 A closer look at interior-point methods

A large variety of interior-point methods have been proposed up to now in the literature, and a complete exposition would bring us too far from the scope of this thesis. In this brief introduction, we concentrate on *path-following interior-point methods*, in order to illustrate how Jordan-algebraic extensions work.

1.5.1 Newton's Algorithm: solving unconstrained problems

Newton's Algorithm is a well-known procedure to solve a system of nonlinear equations. Consider a differentiable linear function $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$. Its first-order Taylor expansion around a given point x_0 of D can be written as follows:

$$F(x_0 + h) = F(x_0) + F'(x_0)h + E_{x_0,F}(h),$$

where, according to Taylor's Theorem, we have $\lim_{||h||\to 0} E_{x_0,F}(h)/||h|| = 0$ (see for instance [Apo69]). In order to find a vector $h^* \in \mathbb{R}^n$ for which $F(x_0 + h^*) = 0$, we can approximate $F(x_0 + h)$ by $F(x_0) + F'(x_0)h$, and solve the following linear system of equations:

$$0 = F(x_0) + F'(x_0)h,$$

yielding a vector h_1. Then, the process is repeated with $x_1 := x_0 + h_1$ instead of x_0, and so on. We can construct the sequence:

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k) \quad \text{for } k \geq 0,$$

provided that the differential matrix $F'(x_k)$ remains invertible for every k and that the successive iterates belong to D. Assuming several regularity assumptions on F, it is possible to prove that the sequence $(x_k)_{k\geq 0}$ converges to a root x^* of the system of equations if x_0 is not too far from x^* (see for instance Theorem 6.14 in [Kre98]).

According to Proposition 1.3.4, a differentiable convex function f is minimized at every point where its differential vanishes. In other words, solving $\min_{x\in\mathbb{R}^n} f(x)$, where f is convex and differentiable, amounts to solving the system $f'(x) = 0$. If the function f is twice differentiable, Newton's Algorithm can be used if, of course, the Hessian of f is invertible at every point of the generated sequence. For the sake of numerical stability of the method, it can be very useful to guarantee that the Hessian of f lies sufficiently far away from the set of non-invertible matrices.

1.5.2 Barrier methods: dealing with constraints

Let us now turn our attention to *constrained* convex minimization problems:

$$\begin{array}{l} \min\ f(x) \\ \text{s.t. } x \in X, \end{array}$$

where X is a closed convex subset of $\mathbb{R}^n$ with a nonempty interior. We can expect that optimization problems are significantly more difficult to solve if they are constrained. A popular strategy to tackle this complication consists in replacing a constrained problem P by a sequence $(P_\mu)_{\mu>0}$ of unconstrained instances that approximate P. This sequence is usually constructed by means of a *barrier function for* X.

Definition 1.5.1 *Let X be a convex set with a nonempty interior. A real-valued, below bounded, convex function F is a* barrier for X *if its domain is* $\operatorname{int} X$*, and if $F(x)$ converges to $+\infty$ as x approaches the boundary of X. The value of F outside* $\operatorname{int} X$ *is taken equal to $+\infty$.* ■

Instead of solving the constrained optimization problem $\min_{x\in X} f(x)$, the idea is to solve with Newton's Algorithm several unconstrained problems:

$$x(\mu) := \min_{x\in\mathbb{R}^n} f(x) + F(x)/\mu \tag{P_μ}$$

for well-chosen values of $\mu > 0$. The curve $\mu \mapsto x(\mu)$ is called the *central path of the problem associated with the barrier F*. When μ tends to infinity, the optimal point $x(\mu)$ of the corresponding unconstrained problem approaches the solution of the original problem (see for instance Theorem 1.3.2 in [Nes03]). Procedures that exploit this trick are known in the literature as *barrier methods.*

1.5.3 Choosing an appropriate barrier

We are now left with the following important issue: what is the most appropriate barrier F for a given cone $\mathcal{K}$? This barrier should be easy to compute, and easy to differentiate. Moreover, as the problem (P_μ) is to be solved with Newton's Algorithm, it is desirable that the gradient $f'(x) + F'(x)/\mu$ is well-defined and invertible everywhere in the interior of $\mathcal{K}$. It would also be extremely useful to have a simple test for determining if a point x could serve as a a starting point for Newton's Algorithm, that is, a point for which the method converges to the desired solution $x(\mu)$. And finally, once Newton's Algorithm has found a sufficiently accurate approximation $\hat{x}_\mu$ of $x(\mu)$, we need to define a strategy for increasing the value of μ. This strategy should counterbalance a twofold trend. On the one hand, we have to increase μ as much as possible, to get closer to the solution of the original problem. On the other hand, the approximation $\hat{x}_\mu$ should be sufficiently close to the new target point $x(\mu_+)$ to ensure the convergence of Newton's Algorithm.

Nesterov and Nemirovski have solved all these issues in [NN93], essentially by introducing the class of *self-concordant barriers*, and by studying its properties. In the following definition, the notation $F^{(n)}(x)[h,\ldots,h]$ indicates the nth derivative of the function $F(x+th)$ with respect to t, at $t=0$.

Definition 1.5.2 *Let $X \subseteq \mathbb{R}^n$ be a closed convex set with nonempty interior. A barrier F for X is a ν-self-concordant barrier if it is three times continuously differentiable, and for every $x \in \operatorname{int} X$ and every $h \in \mathbb{R}^n$ we have:*

- ◇ $|F'''(x)[h,h,h]| \leq 2F''(x)[h,h]^{3/2}$, *and*
- ◇ $(F'(x)[h])^2 \leq \nu F''(x)[h,h]$. ■

If the set X does not contain any straight line, it can be proved that the Hessian of each of its ν-self-concordant barriers is invertible at every point of $\operatorname{int} X$ (see for instance Theorem 4.1.3 of [Nes03]). The second inequality above can then be replaced by the following equivalent form:

$$\langle F'(x), F''(x)^{-1}F'(x)\rangle \leq \nu.$$

A simple but important example of such a function is $F(x) := -\ln(x)$ for the set $X := \mathbb{R}_+$; its parameter ν equals 1.

There is a remarkable existence result of Nesterov and Nemirovski, which states that every convex bounded set $X \subseteq \mathbb{R}^n$ with a nonempty interior has a $\mathcal{O}(n)$-self-concordant barrier (see Theorem 2.5.1 in [NN93]). Unfortunately, an evaluation of this function at a point of its domain takes in general a time that is *exponential* in the dimension n. In several cases however, this barrier can be explicitly constructed.

The following proposition is central in the theory of self-concordance (see Theorem 4.1.5 in [Nes03]). It indicates how the convex set X can be locally replaced by a much simpler set, namely an ellipsoid, on which optimization problems are considerably easier to solve. As the technique used in the proof is important for a forthcoming illustration of the interest of Jordan algebras in optimization, we include it below.

Proposition 1.5.3 *Let F be a ν-self-concordant barrier for the convex set X. We assume that X does not contain any straight line. If x belongs to the interior of X, then the open ellipsoid*

$$W(x,1) := \{y \in \mathbb{R}^n | \langle F''(x)(y-x), y-x\rangle < 1\}$$

is included in the interior of X.

Proof

Let us fix $x \in \operatorname{int} X$ and $h \in \mathbb{R}^n$. Consider the following function:

$$\phi(t) := \frac{1}{\langle F''(x+th)h, h\rangle^{1/2}} \tag{1.6}$$

We denote by D the domain of this function. Since F is a barrier function, D is the set of real numbers t for which $x + th \in \operatorname{int} X$. Observe that, for every $t \in D$:

$$\phi'(t) = -\frac{F'''(x+th)[h,h,h]}{2\langle F''(x+th)h,h\rangle^{3/2}}.$$

In view of Definition 1.5.2, we have $|\phi'(t)| \leq 1$ for all these t. Consequently, the domain D contains the interval $]-\phi(0),\phi(0)[$. Therefore, the point $x + \alpha\phi(0)h$ belong to the interior of X for every α in $]-1,1[$, which is what we needed to prove. ■

Observe that the second condition of the Definition 1.5.2 of self-concordance was not used in this proof. This proposition remains true independently on the value of ν.

The following theorem constitutes one of the main results of the self-concordant barriers theory (it represents a slight modification of Theorem 4.2.9 in [Nes03]).

Theorem 1.5.4 (Self-concordant barriers and path-following algorithms) *Let $X \subseteq \mathbb{R}^n$ be a closed convex set, with a nonempty interior, and let c be an n-dimensional vector. We aim to solve the following optimization problem:*

$$\min_{x \in X} \langle c, x\rangle_n.$$

We assume that there exists a ν-self-concordant barrier F for the set X. Consider the following algorithm.

1. *Choose an accuracy $\epsilon > 0$, a positive value of μ_0, and a starting point x_0 for which*

$$\langle F''(x_0)^{-1}(\mu_0 c + F'(x_0)), \mu_0 c + F'(x_0)\rangle \leq 1/9.$$

2. *For $k \geq 0$, set*

$$\mu_{k+1} := \mu_k(1 + 0.1/\sqrt{\nu})$$

and

$$x_{k+1} := x_k - F''(x_k)^{-1}(\mu_{k+1}c + F'(x_k)).$$

No more than $N = \mathcal{O}(\sqrt{\nu}\ln(\nu/\mu_0\epsilon))$ iterations of this algorithm are enough to find a point x_N in X for which $\langle c, x_N\rangle_n - \langle c, x^*\rangle_n \leq \epsilon$. ■

A practical limitation of this result resides in the fact that the iterate x_{k+1} lies in the ellipsoid $W(x_k, 0.1)$, which might be small, yielding a very slow convergence. Hence, every result that can improve Proposition 1.5.3 would significantly ameliorate the efficiency of optimization strategies based on self-concordant barriers. Essentially, Self-Scaled Programming relies on a considerable improvement of Proposition 1.5.3.

Finally, the self-concordant barriers theory fits particularly well the conic setting of Convex Optimization. However, an extra homogeneity property is required for the ν-self-concordant barrier F of a cone $\mathcal{K}$, namely its *logarithmic homogeneity*: for every $x \in \operatorname{int}\mathcal{K}$, and every $\lambda > 0$, we have $F(\lambda x) = F(x) - \nu\ln(\lambda)$. If such a barrier F exists for the cone

$\mathcal{K}$, the above scheme can be applied to solve the problem (here the $n \times m$ matrix A is surjective):

$$\begin{aligned} \min\ & \langle c, x\rangle_n \\ \text{s.t.}\ & Ax = b \\ & x \in \mathcal{K}, \end{aligned}$$

with the restriction of F to the affine subspace $\{x \in \mathbb{R}^n | Ax = b\}$ as ν-self-concordant barrier. In order to deal with the linear constraints in Newton's Algorithm, the celebrated Karush-Kuhn-Tucker Theorem is invoked (for a proof, see for example [Lue84], Section 10.8). The statement we give here is much more general than what is needed for the above application, but this theorem will be subsequently used for some more sophisticated situations.

Theorem 1.5.5 (Karush-Kuhn-Tucker optimality conditions Theorem) *Let us assume that the functions $f, g_1, \ldots, g_m, h_1, \ldots, h_k$ from $\mathbb{R}^n$ to $\mathbb{R}$ are continuously differentiable. We consider the following optimization problem:*

$$\begin{aligned} \min\ & f(x) \\ \text{s.t.}\ & g_i(x) \leq 0 \quad \textit{for } 1 \leq i \leq m \\ & h_j(x) = 0 \quad \textit{for } 1 \leq i \leq k. \end{aligned} \tag{1.7}$$

If a feasible point x^* is a solution of the problem (1.7), and satisfies the following property:

$$\begin{aligned} &\textit{the gradients } \{g_i'(x^*), h_j'(x^*) \,|\, g_i(x^*) = 0,\ 1 \leq i \leq m,\ 1 \leq i \leq k\} \\ &\textit{are linearly independent}, \end{aligned} \tag{1.8}$$

then there exist a vector $\lambda \in \mathbb{R}^k$ and a vector $\mu \in \mathbb{R}^m_+$ such that:

$$\begin{aligned} \frac{d}{dx_i} f(x^*) + \sum_{j=1}^{k} \lambda_k \frac{d}{dx_i} h_k(x^*) + \sum_{j=1}^{m} \mu_j \frac{d}{dx_i} g_j(x^*) \;&=\; 0 \qquad \textit{for all } 1 \leq i \leq n \\ \mu^T g(x^*) \;&=\; 0. \end{aligned}$$

■

In the approximated conic problem

$$\begin{aligned} \min\ & \langle c, x\rangle_n + F(x)/\mu \\ \text{s.t.}\ & Ax = b, \end{aligned}$$

the constraints are linear, yielding, in view of Karush-Kuhn-Tucker's conditions, the following nonlinear system in x:

$$\begin{aligned} c + F'(x)/\mu + A^T\lambda \;&=\; 0 \\ Ax - b \;&=\; 0. \end{aligned}$$

Observe that the solution point x^* complies with the property (1.8) on constraints, because the matrix A is surjective by assumption.

1.5.4 Path-following interior-point methods for Linear Programming

In this subsection, we present two optimization schemes based on the theory of self-concordant barriers. They rely on the following fundamental proposition, which results from a trivial computation of derivatives.

Proposition 1.5.6 *We denote by $\mathbb{R}^n_{++}$ the set of n-dimensional vectors with positive coefficients. The function*

$$\begin{aligned} F : \mathbb{R}^n_{++} &\to \mathbb{R} \\ x &\mapsto F(x) := -\sum_{i=1}^{n} \ln(x_i) \end{aligned}$$

is an n-self-concordant barrier for $\mathbb{R}^n_{++}$. ■

This proposition immediately suggests a scheme for solving

$$\begin{aligned} \min\ & \langle c, x\rangle_n \\ \text{s.t. } & Ax = b \\ & x \geq 0. \end{aligned}$$

As described in the previous subsections, it suffices to incorporate the constraint $x \geq 0$ into the objective function by means of the above self-concordant barrier:

$$\begin{aligned} \min\ & \langle c, x\rangle_n - \textstyle\sum_{i=1}^n \ln(x_i)/\mu \\ \text{s.t. } & Ax = b. \end{aligned}$$

The corresponding interior-point algorithm is referred to as *the short-step primal path-following method for Linear Programming* in the literature, and has a theoretical worst-case complexity of $\mathcal{O}(\sqrt{n}\ln(n/\epsilon))$ iterations.

Practical experiments tend to show that solving the primal-dual version of this problem, namely:

$$\begin{aligned} \min\ & \langle c, x\rangle_n - \langle b, y\rangle_m \\ \text{s.t. } & Ax = b \\ & A^*y + s = c \\ & x, s \geq 0, \end{aligned}$$

provides a better algorithm. The barrier approximation of this problem is then:

$$\begin{aligned} \min\ & \langle s, x\rangle_n - \textstyle\sum_{i=1}^n \ln(x_i)/\mu - \textstyle\sum_{i=1}^n \ln(s_i)/\mu \\ \text{s.t. } & Ax = b \\ & A^*y + s = c. \end{aligned}$$

It suffices to use Karush-Kuhn-Tucker's conditions to determine the linear system to be solved at each iteration. This algorithm is called in the literature *the short-step primal-dual path-following method for Linear Programming*, and its theoretical worst-case complexity is also in $\mathcal{O}(\sqrt{n}\ln(n/\epsilon))$ iterations.

1.5.5 Path-following interior-point methods for Self-Scaled Programming

In their seminal paper [NT97], Nesterov and Todd have defined the class of self-scaled cones by a set of properties owned by their self-concordant barriers. Particular cases of these *self-scaled barriers* include the following ones.

Second-Order Programming. As defined in Section 1.4, Second-Order Programming consists in conic problems for which the feasible cone $\mathcal{K}$ is a Cartesian product of second-order cones:

$$\mathcal{K} := \mathbb{L}^{m_1} \times \cdots \times \mathbb{L}^{m_k}.$$

The standard self-scaled barrier for the $(n+1)$-dimensional second-order cone $\mathbb{L}^n$ has the following form:

$$F_n(t, x) := -\ln(t^2 - x_1^2 - \cdots - x_n^2).$$

Its parameter ν equals 2. Now, the self-scaled barrier for $\mathcal{K} = \mathbb{L}^{m_1} \times \cdots \times \mathbb{L}^{m_k}$ is simply:

$$F(t^{(1)}, x^{(1)}, \ldots, t^{(k)}, x^{(k)}) := F_{m_1}(t^{(1)}, x^{(1)}) + \cdots + F_{m_k}(t^{(k)}, x^{(k)}).$$

The parameter of the barrier F is equal to $2k$. ■

Semidefinite Programming. Here, the feasible cone $\mathcal{K}$ is the set of $n \times n$ positive semidefinite matrices. The corresponding self-scaled barrier is $F(X) := -\ln(\mathrm{Det}(X))$. ■

In the framework of formally real Jordan algebras, *all* the self-scaled barriers have, up to an additive constant and up to a judicious choice of the scalar product, the *same* expression [Sch00]:

$$F(x) = -\ln \det(x),$$

where the function det should be seen in this introductory exposition as a natural generalization of the determinant for real symmetric matrices (more details are given in our exposition on Jordan algebras in Chapter 2); their domain should be considered as a suitable extension of the set of positive-definite matrices. This interesting feature considerably simplifies the analysis of interior-point algorithms in Self-Scaled Programming.

In fact, the self-scaled barrier $F(x) = -\ln(\det(x))$ is closely related to the self-concordant barrier for Linear Programming $f(x) := -\sum_{i=1}^n \ln(x_i)$. This link can be seen through the concept of *eigenvalues* in Jordan algebras. These real-valued functions $\lambda_1(u), \ldots, \lambda_n(u)$ are precisely defined in Section 2.7. The reader is invited in this introduction to view them as a natural generalization of eigenvalues of symmetric matrices. The Jordan algebraic function det is in fact the product of the eigenvalues of its argument, exactly as in the context symmetric matrices. Thus, the barrier

$$F(x) = -\ln(\det(x)) = -\ln(\lambda_1(x) \cdots \lambda_n(x)) = -\ln(\lambda_1(x)) - \cdots - \ln(\lambda_n(x)) = f(\lambda(x)),$$

defined where $\lambda_i(x) > 0$ for every $1 \leq i \leq n$, corresponds to its Linear Programming counterpart, *where the components of the argument have been changed into the eigenvalues* *of its argument.* The scalar product that should be used in the Jordan-algebraic framework is the so-called *Jordan algebraic scalar product* $\langle\cdot,\cdot\rangle_{\mathcal{J}}$ (see in Section 2.7.6), which coincides with the dot product for linear programming instances and with the Frobenius product $\langle A, B\rangle_F := \sum_{i,j=1}^n A_{ij}B_{ij}$ of $n \times n$ symmetric matrices used in semidefinite programming.

Many interior-point algorithms for self-scaled programming can be produced and analyzed by a more or less systematic application of the above construction. This recipe has already been proposed for Semidefinite Programming by Farid Alizadeh in [Ali95]. For instance, the barrier method applied to Self-Scaled Programming consists simply in considering the following approximated problems:

$$\begin{array}{ll} \min & \langle c, x\rangle_{\mathcal{J}} - \ln(\det(x))/\mu \\ \text{s.t.} & Ax = b. \end{array}$$

In Theorem 4.4.10, we show that essentially the same systematic replacement rule holds for computing the differential of this barrier, yielding a short-step primal path-following method for Self-Scaled Optimization.

The idea is very similar for primal-dual instances, with just one mild technical difficulty. The barrier version of the linear primal-dual instance can be written as follows:

$$\begin{array}{ll} \min & \langle s, x\rangle_n - \sum_{i=1}^n \ln(x_i s_i)/\mu \\ \text{s.t.} & Ax = b \\ & A^* y + s = c. \end{array}$$

The self-scaled version also amounts to replacing the components $x_i s_i$ by the eigenvalues of an appropriate combination of x and s. In the context of semidefinite programming, this combination reduces to $s^{1/2}xs^{1/2}$ when x and s are symmetric matrices.

Many other algorithms for Self-Scaled Optimization are implicitly constructed from linear optimization algorithms following this recipe. These algorithms rely on several important technical ingredients. A non-exhaustive list of them would include the following ones.

- ◇ The Tanabe-Todd-Ye potential function, defined for linear programming in [Tan88, TY90], has been extended by Faybusovich [Fay02].
- ◇ The long-step centrality measure for path-following methods, introduced for Linear Programming in [KMY89], has been extended in the Jordan algebraic framework in [Mur02, SA03].
- ◇ The theory of self-regular functions has been created for Linear Programming by Jiming Peng, Cornelius Roos, and Tamás Terlaky in [PRT02]. They subsequently extended to Second-Order Programming and Semidefinite Programming separately, using implicitly the aforementioned construction. However, the unified treatment of this theory using the Jordan algebraic framework is not accomplished yet.

A major part of this thesis consists in providing technical tools designed to ease these extensions. Chapters 4 and 5 focus on spectral functions/mappings on Jordan algebras. We investigate on the properties that a symmetric function of its arguments transfers to the function constructed by replacing these arguments by eigenvalues.

1.6 Smoothing techniques

Interior-point methods suffer from an important drawback. In spite of their excellent complexity in terms of number of iterations, each of these iterations can be prohibitively expensive. In contrast, the iteration cost of gradient and subgradient methods is typically small, but a lot of them are needed to reach the solution of a problem. This limitation is essentially due to the fact that these methods do not use the structure of the instance to be solved.

According to Proposition 1.3.5, the situation is especially unfavorable for non-smooth convex problems, that is, with a convex objective function that is non-differentiable. With smoothing techniques, Nesterov proposes an efficient way to deal with some of these instances, by converting them into smooth approximations [Nes05a]. In view of Proposition 1.3.7, gradient methods can have a much better complexity, provided that the Lipschitz constant of the gradient of the objective function is not too high. The idea of smoothing techniques consists in restricting the class of non-smooth problems under consideration to instances in which non-differentiability enters in a very precise way in the objective function. In other words, it amounts to taking the structure of non-smoothness explicitly into account.

We are given two closed bounded convex sets Q_1 and Q_2, contained in the Euclidean vector spaces E_1 and E_2 respectively. The norms of these spaces are denoted by $||\cdot||_{E_1}$ and $||\cdot||_{E_2}$ respectively. The objective function, to be minimized over Q_1, is supposed to have the following form:

$$f(x) = \hat{f}(x) + \max_{u \in Q_2} [\langle Ax, u\rangle - \hat{\phi}(u)],$$

where $\hat{f}$ and $\hat{\phi}$ are continuously differentiable convex functions, and A is a linear operator from E_1 to E_2^*. We assume that an evaluation of f is not too expensive, that is, that the maximization of $\langle Ax, u\rangle - \hat{\phi}(u)$ over Q_2 can be performed very efficiently, or even that a closed form of the solution is available.

Now, the objective function is replaced by a smooth approximation of it, with the help of a so-called *prox-function of* Q_2.

Definition 1.6.1 *Let* $||\cdot||$ *be a norm of* $\mathbb{R}^n$. *A* prox-function d *of a set* $Q \subseteq \mathbb{R}^n$ *is a twice continuously differentiable function* $d : Q \to \mathbb{R}$ *that is strongly convex on* Q*:*

$$\text{for every } u \in Q \text{ and } h \in \mathbb{R}^n, \quad \langle d''(u)h, h\rangle \geq \sigma||h||^2.$$

Moreover, this function is supposed to attain its minimum in the relative interior of Q*, and its minimal value is* 0. *The constant* σ *is called the* strong convexity constant of d with respect to the norm $||\cdot||$. ■

Let d_2 be a prox-function of Q_2, and let σ_2 be its strong convexity constant with respect to the norm $||\cdot||_{E_2}$. For each parameter $\mu > 0$, we define the function:

$$f_\mu(x) := \hat{f}(x) + \max_{u \in Q_2} [\langle Ax, u\rangle - \hat{\phi}(u) - \mu d_2(u)].$$

We choose a norm $||\cdot||_{E_1}$ of E_1 and define:

$$||A||_{E_1,E_2} := \max\{\langle Ax, u\rangle : ||x||_{E_1} \leq 1, ||u||_{E_2} \leq 1\}.$$

This family of functions approaches f from below as μ goes to 0, and each of them has a Lipschitz continuous gradient. It can be proved (see Theorem 1 in [Nes05a]) that the Lipschitz constant of f'_μ equals $L_\mu := ||A||^2_{E_1,E_2}/(\mu\sigma_2)$. Hence, the much more favorable complexity result of Proposition 1.3.7 can be applied, provided that L_μ is not too large.

This gradient-like scheme requires a prox-function d_1 of Q_1, with a strong convexity constant for the norm $||\cdot||_{E_1}$ denoted by σ_1 and a minimizer denoted by x_0. Letting $D_1 := \max_{x\in Q_1} d_1(x)$ and $D_2 := \max_{x\in Q_2} d_2(x)$, we put $\mu := \epsilon/(2D_2)$.

Theorem 1.6.2 (Theorem 3 in [Nes05a]) *The smoothing algorithm (reproduced at p. 186 as Algorithm 6.2.1) generates a sequence $(y_k)_{k\geq 0}$ for which $f(y_N) - f^* \leq \epsilon$ as soon as:*

$$N + 1 \geq 4||A||_{E_1,E_2}\sqrt{\frac{D_1 D_2}{\sigma_1\sigma_2}} \cdot \frac{1}{\epsilon} + \sqrt{\frac{4\hat{L}D_1}{\sigma_1\epsilon}},$$

where $\hat{L}$ is the gradient Lipschitz constant of $\hat{f}$ corresponding to the norm $||\cdot||_{E_1}$. ■

Observe that this complexity result concerns the actual non-smooth problem, and not its smoothed approximation.

In order to apply these techniques, we have to find good prox-functions, that is, prox-functions for which the ratio ”diameter of the set” over ”strong convexity constant” is low. It is also crucial to use norms with small unit balls, in order to influence favorably the matrix norm $||\cdot||_{E_1,E_2}$.

An advantageous prox-function is known for the n-dimensional simplex:

$$\Delta_n := \{x \in \mathbb{R}^n_+ | x_1 + \cdots + x_n = 1\}.$$

This function is the so-called entropy function:

$$\begin{aligned} d_{\text{entr}} : \Delta_n &\to \mathbb{R} \\ x &\mapsto d_{\text{entr}}(x) := \sum_{i=1}^n x_i \ln(x_i) + \ln(n). \end{aligned}$$

For the norm $||x|| := \sum_{i=1}^n |x_i|$, its strong convexity constant on Δ_n equals 1.

In the light of the discussion developed in the previous section, it is natural to consider as a good prox-function for corresponding problems in formally real Jordan algebras the prox-function $d_{\text{entr}} \circ \lambda$, where λ is the eigenvalue mapping. In Chapter 6, we show that this function inherits from d_{entr} its advantageous strong convexity characteristics (see Corollary 6.4.5).

More generally, it could be extremely useful to have a general result linking the strong convexity constant of a function f with respect to a given norm $w(x) := ||x||$, and the strong convexity constant of the function $f \circ \lambda$ with respect to the norm $w \circ \lambda(x) = ||\lambda(x)||$.

However, this assertion remains conjectural. In this thesis, we develop some efforts towards a general proof, by considering several interesting particular cases.

In Chapter 3, Theorem 3.6.4 proves that $w(\lambda(v-u)) \geq w(\lambda(v)-\lambda(u))$ for every element u and v in the formally real Jordan algebra corresponding to the eigenvalues function λ, and for every gauge function w - gauge functions constitute a particular class of symmetric norms. This result represents a generalization of Mirski's inequality to formally real Jordan algebras.

In Chapter 4, Theorem 4.4.13 shows that the strong convexity constants of f and of $f \circ \lambda$ coincide, if the considered norm w is the Euclidean norm.

1.7 Eigenvalues in Jordan algebra make it work: more applications

In this section, we briefly sketch how two delicate issues in Self-Scaled Optimization can be quite easily solved using the aforementioned Jordan algebraic formalism, and eigenvalues in Jordan algebras. We leave the proofs for the end of Chapter 2, where the needed material on formally real Jordan algebras is developed.

1.7.1 A concavity result

The first problem concerns the function ϕ introduced in (1.6). Let $Q \subseteq \mathbb{R}^n$ be a convex set with a nonempty interior, and assume that a self-concordant barrier F is known for Q. We recall that for every point $x \in \operatorname{int} Q$ at which the Hessian of F is non-degenerate, and for every nonzero $h \in \mathbb{R}^n$, the function ϕ is constructed as follows:

$$\phi(t) := \frac{1}{\langle F''(x+th)h, h\rangle^{1/2}}$$

for every t such that $F''(x+th)$ exists and is non-degenerate. We have insisted on the fact that the more properties of this function are known, the better the interior-point algorithms can be designed.

In Proposition 2.11.3, we consider a self-scaled barrier F for the symmetric cone $\mathcal{K} \subseteq \mathbb{R}^n$. We have already mentioned in Section 1.4 that this barrier can be written as $F(x) = -\ln\det(x)$ for the determinant function of an appropriate Jordan algebra. We denote by $\langle \cdot, \cdot, \rangle_{\mathcal{J}}$ the corresponding Jordan algebraic scalar product.

Given a point $x \in \operatorname{int} \mathcal{K}$ and a nonzero vector $h \in \mathbb{R}^n$, we can construct the function

$$\phi(t) := \frac{1}{\langle F''(x+th)h, h\rangle_{\mathcal{J}}^{1/2}}$$

We show in Lemma 2.11.1 that the domain of this function is nonempty because $F''(x)$ is invertible. In Proposition 2.11.3, we prove that ϕ is *concave* on its domain. Moreover, $\phi(t)$ tends to 0 when $x + th$ converges to the boundary of $\mathcal{K}$ as a consequence of Lemma 2.11.2.

In a second statement, we prove that for every $x \in \operatorname{int} \mathcal{K}$, and every nonzero h and p of $\mathbb{R}^n$, the function

$$t \mapsto \frac{1}{\langle F''(x+th)p, p\rangle_{\mathcal{J}}^{1/2}}$$

is also concave. This result has the following corollary, which is central to the theory of self-scaled barriers. This result is similar to Theorem 4.1 of [NT97].

For every $x \in \operatorname{int} \mathcal{K}$ and every nonzero $h \in \mathbb{R}^n$, we define

$$\sigma_x(h) := \sup\{t > 0 | x - th \in \operatorname{int} \mathcal{K}\}^{-1},$$

so that if $\sigma_x(h) \neq 0$, the point $x - h/\sigma_x(h)$ belongs to the boundary of $\mathcal{K}$.

Corollary 1.7.1 *For every $t \in [0, 1/\sigma_x(h)[$ and every $p \in \mathbb{R}^n$, we have:*

$$\langle F''(x)p, p\rangle_{\mathcal{J}} \geq (1 - \sigma_x(h)t)^2 \langle F''(x-th)p, p\rangle_{\mathcal{J}}. \tag{1.9}$$

Proof
Suppose first that $\sigma_x(h) \neq 0$, and let $T := \sigma_x(h)^{-1}$. Since the function

$$\phi_p(t) := \langle F''(x-th)p, p\rangle^{-1/2}$$

is concave, we have for every $t \in [0, T]$:

$$\phi_p(t) \geq \phi_p(0)\frac{T-t}{T} + \phi_p(T)\frac{t}{T} = \phi_p(0)\frac{T-t}{T},$$

which is equivalent to the desired inequality. Otherwise, if $\sigma_x(h) = 0$, the function ϕ has no root on $\mathbb{R}_-$ because the Hessian of F is non-degenerate in the interior of $\mathcal{K}$. Since ϕ is concave, we deduce that it is an decreasing function. Therefore, the inequality holds in this case as well. ■

1.7.2 Augmented barriers in Jordan algebras

Augmented barriers were first introduced by Yurii Nesterov and Jean-Philippe Vial in [NV04]. These functions form a new class of barriers for conic optimization, and allowed Nesterov and Vial to create nontrivial optimization problems that can be solved with a complexity that *does not* depend on the particular data of an instance apart from its size.

Definition 1.7.2 *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a closed convex cone with nonempty interior, and let F be a ν-self-concordant barrier for $\mathcal{K}$ that is logarithmically homogeneous. Let M be a positive definite matrix of dimension $n \times n$. The augmented self-concordant barrier constructed by F and M is:*

$$\psi_M(x) := \frac{1}{2}\langle Mx, x\rangle + F(x).$$

■

An augmented self-concordant barrier ψ_M is *not necessarily* a self-concordant function, given that $\langle \psi''_M(x)^{-1}\psi'_M(x), \psi'_M(x)\rangle$ is not necessarily bounded from above. However, Nesterov and Vial have considered the path-following algorithm displayed in Theorem 1.5.4 for minimizing such functions, for $c := \psi'_M(x_0)$. In doing so, they have obtained the following result (see Theorem 4 of [NV04]).

Theorem 1.7.3 (Minimizing augmented self-concordant barriers) *Let x_0 be the starting point of the path-following algorithm for minimizing the augmented barrier ψ_M. We define the constants $\gamma_l(x_0)$ and $\gamma_u(x_0)$ such that:*

$$\gamma_l(x_0)\langle F''(x_0)x, x\rangle \leq \langle Mx, x\rangle \leq \gamma_u(x_0)\langle F'(x_0), x\rangle^2.$$

Then, the path-following algorithm does not take more than

$$\mathcal{O}\left(\sqrt{\nu}\left(\ln \frac{\nu\gamma_u(x_0)}{\gamma_l(x_0)}\right)\right)$$

iterations to enter in the quadratic convergence zone of Newton's Algorithm for nonlinear systems of equations applied to ψ'_M. ■

As an application, they have obtained the following constant complexity result (see Section 5.7 of [NV04]).

Theorem 1.7.4 *Let $\mathcal{K}$ be the set of $n \times n$ symmetric matrices. Let $A_1, \ldots, A_q$ be matrices of $\mathcal{K}$, such that $A_1 + \cdots + A_q$ is invertible. The operator $M : \mathcal{K} \to \mathcal{K}$ is defined as follows:*

$$X \mapsto M(X) := A_1XA_1 + \cdots + A_qXA_q.$$

We construct the augmented barrier $X \mapsto \psi_M(X) := \langle M(X), X\rangle_F/2 - \ln \mathrm{Det}(X)$, where the scalar product $\langle\cdot,\cdot\rangle_F$ is the Frobenius scalar product, and the function Det *is the standard matrix determinant.*

It takes $\mathcal{O}(\sqrt{n}\ln(nq))$ iterations of the path-following algorithm starting from $(A_1 + \cdots + A_q)^{-1}$ to enter in the convergence zone of Newton's Algorithm for nonlinear systems of equations applied to ψ'_M. ■

We show in Section 2.11.2 how it is possible to generalize this result to the framework of a Jordan algebra $\mathcal{J}$, by introducing an appropriate class of linear operators for M, denoted by $\mathcal{Q}(\mathcal{J})$. This set $\mathcal{Q}(\mathcal{J})$ is defined as the conic hull of a set of elementary linear operators Q_u where $u \in \mathcal{J}$. These elementary operators can be seen in this introduction as a natural generalization of operators of the type $\mathbb{S}^n \to \mathbb{S}^n$, $X \mapsto AXA$, where A is a symmetric $n \times n$ matrix. The notion of *rank* of a Jordan algebra can be seen in the introduction as a natural extension of the dimension n of $n \times n$ symmetric matrices (more details are given in Section 2.3).

Our final result is stated as follows.

Proposition 1.7.5 *Let F be a self-scaled barrier, and let $\mathcal{K}$ be its corresponding symmetric cone. We denote by $\mathcal{J}$ the associated formally real Jordan algebra, by r its rank and by*

$\langle\cdot,\cdot\rangle_{\mathcal{J}}$ its Jordan algebraic scalar product. Let $M := \sum_{j=1}^{q} Q_{u_j} \in \mathcal{Q}(\mathcal{J})$ where $\sum_{j=1}^{q} u_j \in \operatorname{int}\mathcal{K}$.

We construct the augmented barrier $\psi_M : \operatorname{int}\mathcal{K} \to \mathbb{R}$, $x \mapsto \psi_M(x) := \langle Mx, x\rangle_{\mathcal{J}}/2 - F(x)$.

It takes $\mathcal{O}(\sqrt{r}\ln(rq))$ iterations of the path-following algorithm starting to enter in the convergence zone of Newton's Algorithm for nonlinear systems of equations applied to ψ'_M.
■

This proposition exactly matches Theorem 1.7.4 for the particular case where the considered cone is a cone of positive semidefinite matrices.

1.8 Overview of the thesis and research summary

This thesis develops and uses Jordan algebraic techniques to solve various theoretical questions in Convex Optimization. The present chapter introduces the scientific context of the thesis in order to motivate our research on spectral functions on Jordan algebras. As an illustration, two original application examples are sketched in Section 1.7. The first one is a concavity result, which turns out to be one of the roots of the whole theory of Self-Scaled Optimization [NT97]. The second one extends very naturally to the Jordan algebraic framework an intriguing complexity result in the theory of augmented barriers [NV04]. Detailed proofs are given in Section 2.11

Chapter 2 contains a self-contained exposition on formally real Jordan algebras.

A significant part of the proofs have been recast to fit the framework of our work, and some of them are original. In particular, the discussion on minimal idempotents in Section 2.7.4 is original. The complete spectral decomposition theorem (Theorem 2.7.25), Proposition 2.7.29 on operator commutativity and its corollary, and Lemma 2.10.10 on transformations of minimal idempotents are the main results for which we give original proofs. A more detailed account of our contributions in this chapter are given in Section 2.12.

The research work starts from Chapter 3, where we derive several variational characterizations of eigenvalues in Jordan algebras, which are indispensable tools in the sensitivity analysis of numerical methods. The extension of the celebrated Courant-Fischer inequalities to formally real Jordan algebras has already been achieved by Hirzebruch in [Hir70]. However, its formulation is not flexible enough for further development. We propose a novel expression in Theorem 3.4.1, which allows us to prove the extension of Wielandt's variational formulation in formally real Jordan algebras (see Theorem 3.5.5 and Theorem 3.5.6). Several important inequalities follow from this result, namely Lidski and Mirski's inequalities.

Spectral functions play an important role in the development of optimization algorithms with the help of the Jordan algebraic framework. Their properties are investigated in Chapter 4. Our main contributions include a new proof of Von Neumann's inequalities for non-simple formally real Jordan algebras, for which we introduce the simplifying concept of *similar joint decomposition* (Definition 4.2.1). We also obtain a characterization of different types of spectral functions convexities (Theorem 4.4.13), and of various first-order differentiability types, from the most classical definition (Theorem 4.4.10) to the more sophisticated Bouligand and Clarke subdifferentiability (Theorem 4.5.4 and Theorem 4.5.5), which play an important role in subgradient techniques. We also derive a closed form expression of directional differentiability for eigenvalues (Theorem 4.4.8), solving a conjecture proposed in [SS04]. The material of Sections 3.2, 4.2, 4.3, and Section 4.4 – Subsection 4.4.2 excluded – has been published as a CORE Discussion Paper [Bae04].

We define the concept of spectral mapping in Chapter 5, which naturally extends gradients of differentiable spectral functions. Our main result is a closed formula for the Jacobian of such functions, which enables us to compute the Hessian of spectral functions. This result settles an open question of Hristo Sendov [Sen00]. We use this closed formula to extend two smoothing methods for the convex complementarity problem [FLT01], namely the Fischer-Burmeister and the Chen-Mangasarian smoothings (see Section 5.7). We ensure that these schemes are well-defined and that they converge. The results of this chapter – Section 5.7 excluded – as well as of Subsection 4.4.2 constitute the spinal chord of a paper that has been submitted to Linear Algebra and Applications [Bae05].

In Chapter 6, we extend the smoothing techniques of Yurii Nesterov to a class of self-scaled instances. As shown in Section 6.5, the problem of minimizing a sum of Euclidean norms falls into this class. We perform its complexity analysis, and we support our theoretical conclusions by computational experiments. We have compared the practical behavior of our smoothing technique with the best interior-point method available for the sum-of-norms problem [XY97]. In most of the large-scale instances, it appears that our algorithm, or a heuristic variant of it, performs better than this interior-point method when the required accuracy is not too small. This chapter has been published as a CORE Discussion Paper [Bae06].

CHAPTER 2

Jordan algebras

HISTORICALLY, Jordan algebras were conceived in an attempt to discover a new algebraic setting for Quantum Mechanics. Surprisingly enough, they turned out to have a very large spectrum of applications. They play a significant role in differential geometry, in the theory of Lie algebras and in projective geometry. They also have many interesting applications in such various topics as differential equations, probability, statistics and conic optimization, our main subject of interest. Indeed, some Jordan algebras were proved a decade ago to be an indispensable tool in the unified study of efficient interior-point algorithms for linear, quadratic and semidefinite optimization problems. They allowed optimizers to extend quite easily some schemes designed for linear programming to schemes for quadratic and semidefinite programming.

This chapter is a survey of the main technical tools that are of a constant use in the study of Jordan algebras, that is, the characteristic polynomial from the viewpoint of generic elements, the Pierce decomposition theorems, and the spectral decomposition theorems. We apply these technical tools to explore more specific questions, such as the development of a differential calculus in Jordan algebras.

2.1 The birth of Jordan algebras

> *Many algebraic structures are famous for 15 minutes, then disappear from the action, but others go on to feature in a variety of settings (keep on ticking, like the bunny) and prove to be an enduring part of the mathematical landscape.*
>
> From K. McCrimmon [McC03], p. 7.

Jordan algebras were initially created to remedy some severe drawbacks in the first algebraic description of observables in Quantum Mechanics.

In the Copenhagen interpretation of Quantum Mechanics, physical observable quantities such as position, speed, energy, momentum, and so on, are represented by self-adjoint linear operators on a complex Hilbert space. Their finite-dimensional approximations are Hermitian matrices. In the language of Quantum Mechanics, these operators are called *observables*, while non-self-adjoint operators are called *unobservables*. A precise description of this mathematical setting is given in the two first chapters of John von Neumann's famous study [vN55].

The standard basic operations on complex linear operators are:

- the multiplication by a complex scalar;
- the addition;
- the composition of operators, which reduces to matrix multiplication in finite dimension;
- the construction of the adjoint operator, which reduces to the conjugate transpose matrix in finite dimension.

These operations are defined for every operator on a Hilbert space, independently of the fact that these operators have to be self-adjoint. Let us check what they specifically produce on self-adjoint objects. The multiplication of a self-adjoint operator H by a complex scalar λ produces another self-adjoint operator λH only when λ is a real number. The sum of two self-adjoint operators is unconditionally self-adjoint. The composition of two self-adjoint operators H and H' produces a self-adjoint operator HH' only when they commute (or, in the language of Quantum Mechanics, when they are *simultaneously observable*; in view of Heisenberg's uncertainty principle, non-commutative operators correspond to physical observables that are not simultaneously measurable with an arbitrary precision – more details are given in [LL77], pages 45-49). Finally, the conjugate transformation is irrelevant for self-adjoint operators, as it reduces to the identity transformation.

The operations that are likely to produce self-adjoint objects from self-adjoint objects (that is, observables from observables) are the following:

- multiplication by a real scalar,
- addition, and
- repeated composition of an operator with itself.

Hence, the only authorized operation on observables seems to be that of forming polynomials with real coefficients; however, this class of transformations is far too restricted for two reasons. First, this class is small only because the basic mathematical operations we have decided to use behave badly with respect to observability; this is a strong clue that an *unappropriate mathematical setting* has been used to describe phenomena. And second, there are practical difficulties for lots of important issues, for instance when one attempts to apply Quantum Mechanics to explain relativistic and nuclear phenomena.

The essential flaw in the standard set of operations comes from the multiplication. Because of it, we cannot consider more sophisticated combinations than powers of matrices. In 1932, Pascual Jordan has proposed to replace it by a *symmetrized* matrix multiplication (this operation is often called the *anticommutator*) [Jor32]:

$$A \circ B := \frac{AB + BA}{2}. \tag{2.1}$$

Strong advantage: when A and B are both self-adjoint, so is $A \circ B$. Conceptual drawback: this new multiplication is defined *after* having specified an unobservable multiplication, that is, devoid of any physical meaning.

As a further step of abstraction, Pascual Jordan tried to formulate the minimal set of axioms that a multiplication suitable for observables in Quantum Mechanics should verify, without any reference to an underlying unobservable multiplication. In other words, he wanted to extract the few main features that his symmetrized multiplication has to satisfy.

After several mathematical experiments, he decided to fix on the following set of axioms. Given a real vector space $\mathcal{J}$ of dimension N (the set of observables), the multiplication $\circ : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is a bilinear operation that satisfies for each $u, v \in \mathcal{J}$:

- ◇ $u \circ v = v \circ u$. A major property of the symmetrized multiplication should indeed be its symmetry !
- ◇ $(u \circ u) \circ (u \circ v) = u \circ ((u \circ u) \circ v)$. This relation of degree four allowed him to prove that the powers of elements are well-defined. Some mathematical experiments led Jordan to be convinced that this relation is the key law governing the symmetrized multiplication.
- ◇ $u \circ u + v \circ v = 0$ implies that $u = v = 0$. This axiom translates the non-degeneracy of nature.

Such algebras (i.e. vector spaces with a multiplication) are currently called *formally real Jordan algebras of finite dimension*, or *Euclidean Jordan algebras* by some authors.

Of course, the first question Pascual Jordan addressed to is whether there exists an algebra for which there is no hidden unobservable multiplication governing the observable operator $\circ$ via a symmetrization of the type (2.1). These algebras are called *exceptional* algebras. In a brilliant paper [JvNW34] of 1934, Jordan, von Neumann and Wigner have determined the complete list of all the simple finite-dimensional algebras satisfying these axioms. As it was proved by Albert in 1950 [Alb50], there is only one of them with no hidden symmetrization: the algebra of 3×3 Hermitian matrices of octonions. The complete set of finite-dimensional formally real Jordan algebras is given by all the Cartesian products of the simple algebras from the list of Jordan, von Neumann and Wigner (more details are given in Section 2.9).

Algebraists have studied intensively the axioms of Pascual Jordan, trying to generalize them, to reformulate them or to discard one or two of them. A very complete survey can be found in [McC03]. For instance, one can consider Jordan algebras in another ring that $\mathbb{R}$. Another well-studied extension consists in replacing Jordan's fourth degree relation by the weaker requirement that powers of elements are well-defined. A considerable amount of work has been done in the last decades to study *infinite-dimensional* Jordan algebras; a complete classification of them has been carried out by Zelmanov in 1982. However, our work does not discuss this very delicate theory. Part IV of [McC03] contains a thorough development of Zelmanov's approach.

In this expository review on Jordan algebras, we do exactly the contrary of what happened in the history of mathematics: we start from a relatively general setting, and, strengthening the axioms little by little, we arrive at the set that Pascual Jordan has created. Doing this way, the proofs of many facts on Jordan algebras that we need are easier to understand.

For further readings, the reader might consult some authoritative works on this topic. Here are some references that have inspired this introduction to Jordan algebras: [BK66, Koe99, Jac68, FK94, McC03].

2.2 Algebras and Jordan algebras

In this section, we present some standard algebraic objects and constructions that are often used in this exposition. We define Jordan algebras in the most general setting for our purposes.

We assume that the reader is familiar with the classical algebraic notions of ring, field, vector space and module (one of the numerous references that the reader might consult is [BNJ94]). Henceforth, the letter F designates a field and the letter R a commutative ring. The extended notation $(R, +, \times, 0, 1)$ refers to the ring R, where the additive operator is denoted by the symbol "+" and the multiplication is written "×", when this symbol is not dropped. The unit element of the addition is 0 and the unit element of the multiplication is 1.

The set of polynomials in the variables $t_1, \ldots, t_n$ with coefficients in R is denoted by $R[t_1, \ldots, t_n]$, while $R(t_1, \ldots, t_n)$ designates the set of rational functions f/g, where f and $g \neq 0$ belong to $R[t_1, \ldots, t_n]$.

2.2.1 Extensions of vector spaces

Definition 2.2.1 *Let $(R, +, \times, 0, 1)$ be a commutative ring. The commutative ring*

$$(R', +', \times', 0', 1')$$

is an extension ring of R *if:*

- ◇ *$R \subseteq R'$;*
- ◇ *$0 = 0'$ and $1 = 1'$;*
- ◇ *for every $x, y \in R$, $x +' y = x + y$ and $x \times' y = x \times y$.* ■

From the definition, one can easily check that an extension ring of R can be seen in particular as a *module* on R. Of course, an extension ring of a field F can be seen as a vector space on F.

We recall below the definition of *basis of a vector space.*

Definition 2.2.2 *Let $\mathcal{J}$ be a vector space over the field F. A* basis *of $\mathcal{J}$ is a set of elements $B = \{b_\alpha | \alpha \in A\}$, where A is a set of indices, satisfying the two following properties.*

- ◇ *The elements of every finite subset of B are linearly independent.*
- ◇ *For every element $x \in \mathcal{J}$, there exists a set $\{\lambda_\alpha | \alpha \in A\}$ included in F such that there is only a finite number of elements λ_α that are different form 0, and for which $x = \sum_{\alpha \in A} \lambda_\alpha b_\alpha$.* ■

The existence of a basis in every vector space is ensured by the Axiom of Choice (see [Ble64]). This observation ensures that the definition below does not depend on the chosen basis.

Definition 2.2.3 *Suppose that $\mathcal{J}$ is a vector space over the field F, and let $\{b_\alpha | \alpha \in A\}$ be one of its bases. We denote by R an extension ring of F. The* extension of $\mathcal{J}$ by R *is the set:*

$$\mathcal{J}' = \mathcal{J} \bigotimes_F R := \Big\{ x = \sum_{\alpha \in A} \lambda'_\alpha b_\alpha \mid \lambda'_\alpha \in R, \textit{ and there is a finite number of } \lambda'_\alpha \textit{ different from } 0 \Big\},$$

endowed with the operations:

$$\begin{aligned} + \ : \mathcal{J}' \times \mathcal{J}' &\to \mathcal{J}', \\ (\textstyle\sum \lambda'_\alpha b_\alpha, \sum \mu'_\alpha b_\alpha) &\mapsto \textstyle\sum (\lambda'_\alpha + \mu'_\alpha) b_\alpha \end{aligned}$$

and

$$\begin{aligned} \cdot \ : R \times \mathcal{J}' &\to \mathcal{J}', \\ (\mu, \textstyle\sum \lambda'_\alpha b_\alpha) &\mapsto \textstyle\sum (\mu \lambda'_\alpha) b_\alpha, \end{aligned}$$

where the sums are taken over the elements α of A. ■

The fact that $\mathcal{J}'$ is a module over the ring R follows directly from the definition.

As we have claimed above, this module is independent of the specific basis we have chosen for $\mathcal{J}$. Denote by $\mathcal{J}'_{(v)}$ the module created in the definition using another basis $\{v_\beta | \beta \in B\}$ of $\mathcal{J}$ instead of $\{b_\alpha | \alpha \in A\}$. Then, there exists a set $\{\lambda_{\beta\alpha} | \beta \in B, \alpha \in A\} \subseteq F$ such that $v_\beta = \sum_{\alpha \in A} \lambda_{\beta\alpha} b_\alpha$; moreover, for every $\beta \in B$, a finite number of $\lambda_{\beta\alpha}$ are nonzero.

If $x = \sum_{\beta \in B} \mu_\beta v_\beta \in \mathcal{J}'_{(v)}$, where again only a finite number of $\mu_\beta \in R$ are nonzero, then $x = \sum_{\alpha \in A} \sum_{\beta \in B} \mu_\beta \lambda_{\beta\alpha} b_\alpha$. The sums in B are all finite – they are thus well-defined – and only a finite number of them are nonzero, implying that x is in $\mathcal{J}'_{(b)}$. Exchanging the roles of $\mathcal{J}'_{(b)}$ and $\mathcal{J}'_{(v)}$ proves $\mathcal{J}'_{(b)} = \mathcal{J}'_{(v)}$.

Here are two simple illustrative applications of this construction. Consider the set $\mathbb{R}[x, y]$ of real polynomials over the two variables x and y; then:

$$\mathbb{R}[x, y] \bigotimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[x, y] \quad \text{and} \quad \mathbb{R}[x, y] \bigotimes_{\mathbb{R}} \mathbb{R}[z] = \mathbb{R}[z][x, y] = \mathbb{R}[x, y, z].$$

The next remark shows that nothing unfortunate could happen with extensions of subspaces.

Remark 2.2.4 *Let $\mathcal{J}$ be a vector space over F and let R be an extension ring of F. We put $\mathcal{J}' := \mathcal{J} \bigotimes_F R$. Suppose that M is a subspace of $\mathcal{J}$ and N a submodule of $\mathcal{J}'$ such that $M \subseteq N$. Then $M' := M \bigotimes_F R \subseteq N$. Indeed, one can decompose each $x \in M'$ into $x = \sum_{\alpha \in A} \lambda_\alpha b_\alpha$ where $\lambda_\alpha \in R$ and $\{b_\alpha | \alpha \in A\}$ is a basis of M. Since $b_\alpha \in N$ and since N is a module over R, we conclude that $x \in N$.* ■

2.2.2 Jordan algebras

This subsection defines the notions of *algebra* and *Jordan algebra.*

Definition 2.2.5 *Let $\mathcal{J}$ be a module over the commutative ring R. If there exists an operator $\circ : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$, $(x, y) \mapsto x \circ y$ that is* bilinear *over R, we say that $(\mathcal{J}, \circ)$ is an R-*algebra. *We call $\circ$ the* multiplication *of $\mathcal{J}$. We often abbreviate the notation $(\mathcal{J}, \circ)$ to $\mathcal{J}$ when there is no possible confusion on the multiplication we use for the algebra $\mathcal{J}$.* ■

In this exposition, we will mostly deal with *vector spaces* $\mathcal{J}$ over a field F. The only algebras over rings that are not fields will be built via the construction given in the Definition 2.2.3. This is an important restriction.

In the context of algebras $\mathcal{J}$ defined on vector spaces over a field F, the bilinearity of the multiplication implies that this operation is completely specified by its action on all pairs of basis elements of $\mathcal{J}$. Every such algebra $\mathcal{J}$ can be easily extended to $\mathcal{J}' := \mathcal{J} \bigotimes_F R$ for every extension ring R of F, making $\mathcal{J}'$ an algebra too. When we deal with this extension, we use the same notation for the multiplication in $\mathcal{J}$ and in $\mathcal{J}'$. By the way, it is easily seen that these two multiplications have many important properties in common. For instance, if the multiplication of $\mathcal{J}$ is associative, or commutative, so is its extension to $\mathcal{J}'$.

Since the operator $\circ$ is bilinear, one can represent the left multiplication by $x \in \mathcal{J}$ by a linear operator $L(x) : \mathcal{J} \to \mathcal{J}$, $y \mapsto L(x)y := x \circ y$; the operator $L(x)$ is also linear in x. The right multiplication by x is standardly denoted by $R(x)$. In order to avoid a possible confusion between the operator $R(x)$ and the set of rational function of x with coefficients in R, we denote the right multiplication operator as $R_{\text{mult}}(x)$ in this exposition.

Definition 2.2.6 *We say that two elements x and y of an R-algebra $\mathcal{J}$* left-operator commute [*resp.* right-operator commute] *if the linear operators*

$$L(x)L(y) \text{ and } L(y)L(x) \quad [\textit{resp. } R_{\text{mult}}(x)R_{\text{mult}}(y) \text{ and } R_{\text{mult}}(y)R_{\text{mult}}(x)]$$

are the same. If x and y left- and right-operator commute, we say simply that they operator commute. ■

The *commutator* between two operators A and B is denoted by:

$$[A; B] := AB - BA.$$

With this notation, x and y operator commute if and only if

$$[L(x); L(y)] = [R_{\text{mult}}(x); R_{\text{mult}}(y)] = 0.$$

In fact, we only deal with *commutative* algebras, so that we always have $L(x) = R_{\text{mult}}(x)$. For the sake of notational simplicity, we subsequently use $L(x)$ for the multiplication operator, instead of $R_{\text{mult}}(x)$.

Definition 2.2.7 *The F-algebra $(\mathcal{J}, \circ)$ is said to be* unitary *if it contains a* unit element *e for the multiplication i.e. an element e satisfying $x \circ e = e \circ x = x$ for every $x \in \mathcal{J}$.* ■

It is easily seen, by linearity of multiplication, that every extension of a unitary algebra is unitary as well, *with the same unit element.*

We conclude this subsection by providing the definition of Jordan algebra that we will use throughout this introductory exposition.

Definition 2.2.8 (Jordan algebra) *Let F be a field. The F-algebra $(\mathcal{J}, \circ)$ is a* Jordan algebra *if it is unitary and if the multiplication satisfies the following identities.*

Commutativity: *for all $x, y \in \mathcal{J}$, $x \circ y = y \circ x$;*

Jordan's Axiom: *for all $x, y \in \mathcal{J}$, $(x \circ x) \circ (x \circ y) = x \circ ((x \circ x) \circ y)$.* ■

Denoting $x^2 := x \circ x$, Jordan's Axiom takes the following form in the L-operator notation:

$$L(x^2)L(x) = L(x)L(x^2), \tag{2.2}$$

that is, x^2 and x operator commute, or $[L(x^2); L(x)] = 0$. Note that, by linearity of L in its argument, this relation also holds for every $x \in \mathcal{J} \bigotimes_F R$, whatever is the extension ring R of F. Note that Jordan algebras are ***not*** supposed to be associative.

It is common to make the following assumption on R, even though we will considerably strengthen it in the next sections.

Hypothesis 2.2.1 *We assume 2 is invertible in R. If R is a field, this is equivalent to the assumption that its characteristic is not equal to 2, that is $1 + 1 \neq 0$ in R.* ■

The motivation for this assumption is the following. If x, y are two elements of a commutative R-algebra $\mathcal{J}$, one can compute the so-called *linearization of the square*:

$$q(x, y) := (x + y)^2 - x^2 - y^2 = x \circ y + y \circ x = 2x \circ y.$$

If R does not contain $1/2$, that is, if you are not allowed to get something when having twice this something, it is impossible to recover $x \circ y$ from $q(x, y)$. Many authors assume that Hypothesis 2.2.1 holds to avoid this problem.

In view of the historical comments in Section 2.1, the most common way to create a Jordan algebras is the following: given an **associative** F-algebra $(\mathcal{J}, \cdot)$, we endow it with the *Jordan multiplication* $x \circ y := (x \cdot y + y \cdot x)/2$. It is readily shown that the Jordan multiplication $\circ$ is bilinear, commutative and satisfies Jordan's Axiom, hence $\mathcal{J}^+ := (\mathcal{J}, \circ)$ is indeed a Jordan algebra (See also Example 2.2.2).

Definition 2.2.9 (Formally real algebra) *Let* $(\mathcal{J}, \circ)$ *be an* F*-algebra. We say that* $(\mathcal{J}, \circ)$ *is* formally real *if for every* $x_1, \ldots, x_m \in \mathcal{J}$ *the equality* $x_1 \circ x_1 + \cdots + x_m \circ x_m = 0$ *implies* $x_1 = \cdots = x_m = 0$. ■

As mentioned in the previous section, there are very few formally real Jordan algebras over $\mathbb{R}$ that cannot be written like $\mathcal{J}^+$ for an associative algebra $(\mathcal{J}, \cdot)$.

In a commutative algebra, there exist two classes of elements that play a central role in the study of its structural properties, the *nilpotent* elements and the *idempotent* elements.

Definition 2.2.10 *Let* $(\mathcal{J}, \circ)$ *be a commutative* R*-algebra.*

- ⋄ *An element* $x \in \mathcal{J}$ *is a* nilpotent *of* $\mathcal{J}$ *if there exists a natural number* m *such that* $L(x)^m x = \underbrace{L(x)L(x) \cdots L(x)}_{m \text{ times}} x = 0$.
- ⋄ *A nonzero element* $x \in \mathcal{J}$ *is an* idempotent *of* $\mathcal{J}$ *if* $L(x)x = x$.

■

2.2.3 Strictly power-associative algebras

Strictly power-associative algebras are basically algebras where polynomials can be defined in it, as well as in each of their extensions. Jordan algebras form a particular class of them, as stated in Proposition 2.2.13. We have preferred to develop our exposition on them for not to be bothered by the cumbersome Jordan's Axiom.

In a commutative algebra $(\mathcal{J}, \circ)$, we recursively define *powers* of elements as follows: for every $x \in \mathcal{J}$, we write $x^1 := x$ and $x^{n+1} := L(x^n)x$ when $n \in \mathbb{N}$. If $\mathcal{J}$ is unitary, we also set $x^0 := e$, and if x^n has an inverse, we denote it by x^{-n}.

Definition 2.2.11 *An algebra* $(\mathcal{J}, \circ)$ *is* power-associative *if it is unitary, commutative, and if for each element* x *of* $\mathcal{J}$ *and for all positive integers* m, n *we can write* $x^{n+m} = L(x^n)x^m$. ■

Definition 2.2.12 *An F-algebra $(\mathcal{J}, \circ)$ is* strictly power-associative *when for every extension ring R of F, the algebra $\mathcal{J} \bigotimes_F R$ is power associative.* ∎

Henceforth, we omit the $\circ$ symbol to denote the multiplication in an algebra if no confusion is possible. This convention makes the expressions much more readable. With a slight abuse of notation, we denote by $\mathcal{J}$ the algebra $(\mathcal{J}, \circ)$.

Let $\mathcal{J}$ be a Jordan F-algebra. We first transform expression (2.2) into a more useful form by a common linearization trick. (This manipulation is sometimes called the *polarization of an expression*). Let us fix an extension ring R of F. Taking $\alpha, \beta, \gamma \in R$ and $u, v, w \in \mathcal{J} \bigotimes_F R$, we set $x := \alpha u + \beta v + \gamma w$ in (2.2). By linearity of the L-operator, one can expand the following resulting expression:

$$L((\alpha u + \beta v + \gamma w)^2)L(\alpha u + \beta v + \gamma w) = L(\alpha u + \beta v + \gamma w)L((\alpha u + \beta v + \gamma w)^2).$$

Comparing the coefficients of $\alpha^3, \beta^3, \gamma^3$ results in the original form of Jordan's Axiom.

Considering the coefficients of $\alpha^2\beta$ and al., and given the fact that 2 is invertible in R, we obtain expressions of the form:

$$2L(uv)L(u) + L(u^2)L(v) = 2L(u)L(uv) + L(v)L(u^2). \tag{2.3}$$

Finally, comparing the coefficients of $\alpha\beta\gamma$ allows us to write:

$$L(u)L(vw) + L(v)L(wu) + L(w)L(uv) = L(vw)L(u) + L(wu)L(v) + L(uv)L(w). \tag{2.4}$$

Note that (2.3) and (2.4) are completely equivalent: replacing u by $u + w$ in (2.3) and making the few simplifications allowed by (2.3) gives (2.4); taking $w := u$ in (2.4) gives (2.3) back.

We denote the left-hand side operator in (2.4) by $\mathcal{A}_{uvw}$ and the right-hand side by $\mathcal{S}_{uvw}$.

Observe that $\mathcal{S}_{uvw}x = (vw)(ux) + (wu)(vx) + (uv)(wx)$ is a symmetric expression in u, v, w, x. We can consider every possible permutations of u, v, w, x in $\mathcal{A}_{uvw}x$ to get as many valid expressions. For instance, we have:

$$\begin{aligned} \mathcal{A}_{uvw}x &= \mathcal{A}_{xvw}u \\ u((vw)x) + v((wu)x) + w((uv)x) &= (u(vw))x + v(u(wx)) + w(u(vx)), \end{aligned}$$

or:

$$L(u)L(vw)+L(v)L(wu)+L(w)L(uv) = L(u(vw))+L(v)L(u)L(w)+L(w)L(u)L(v). \tag{2.5}$$

This last expression is the key to prove the strict power-associativity of Jordan algebras as stated in the next proposition. Many nice features on Jordan algebras are more or less direct applications of this expression. Maybe this is the best way, although not the most compact one, to state Jordan's Axiom provided that 3 is invertible in R.

Proposition 2.2.13 *Let $\mathcal{J}$ be a Jordan F-algebra and R be an extension ring of F. For all $x \in \mathcal{J} \bigotimes_F R$ and all positive integers m, n, we have $[L(x^m); L(x^n)] = 0$ and $x^{m+n} = x^m x^n$. Moreover, for every $y, z \in R[x]$, we have $[L(y); L(z)] = 0$.* ∎

This statement follows from [JvNW34], Fundamental Theorem 1. A more modern exposition is given in Lemma 2.4.5 in [HOS84] or Proposition II.1.2 in [FK94], although these authors do not consider extension rings of fields. The proof uses the relation (2.5) to show by induction that $L(x^{n+1})$ can be written as a polynomial in $L(x^n), \ldots, L(x)$. It follows that powers of x operator commute. Strict power-associativity is then immediate.

2.2.4 Examples

In all the following examples, we assume that 2 is invertible in the field F.

Example 2.2.1 (The first example in history: real symmetric matrices)
We take $\mathcal{J}$ to be the set of $r \times r$ real symmetric matrices and we define the following symmetrized multiplication: $U \circ V := (U \cdot V + V \cdot U)/2$, where $\cdot$ is the usual matrix product. Commutativity and Jordan's Axiom are readily seen to be satisfied by this new multiplication, and its unit element is the identity matrix I_r. The symmetrized multiplication by a matrix $U \in \mathcal{J}$ can be represented with the following operator:

$$L(U) := \frac{U \otimes I_r + I_r \otimes U}{2},$$

where $\otimes$ denotes the standard Kronecker product of matrices, so that $L(U)V = U \circ V$.

In fact, this algebra is also formally real. Indeed, if $U_1^2 + \cdots + U_m^2 = 0$, we obtain by taking the trace of both sides that $\mathrm{Tr}(U_1^2 + \cdots + U_m^2) = \mathrm{Tr}(U_1^2) + \cdots + \mathrm{Tr}(U_m^2) = 0$. Since the matrices U_i^2 are positive semidefinite, we have $\mathrm{Tr}(U_i^2) \geq 0$, implying $\mathrm{Tr}(U_i^2) = 0$. Henceforth, $U_i^2 = 0$ for every i, and $U_i = 0$.

This algebra does not have any nonzero nilpotent element. In the terminology of Matrix Theory, its idempotent elements are also called *projectors*.

It is interesting to mention that two matrices U and V operator commute in $(\mathcal{J}, \circ)$ if and only if they commute in $(\mathcal{J}, \cdot)$. ■

Example 2.2.2 (Jordan algebras from associative algebras)
As an immediate generalization of the construction studied in the previous example, let us consider a *unitary associative* algebra $(\mathcal{J}, \cdot)$, where $\mathcal{J}$ is a vector space of finite dimension over the field F. The symmetrized multiplication $u \circ v := (u \cdot v + v \cdot u)/2$ has the same unit element as the multiplication "$\cdot$", is commutative and satisfies Jordan's Axiom. Hence, the algebra $\mathcal{J}^+ := (\mathcal{J}, \circ)$ is a Jordan algebra. Since each power of an element is the same in $(\mathcal{J}, \cdot)$ and in $\mathcal{J}^+$, these two algebras have the same set of nilpotents and idempotents.

Examples of unitary associative algebras include $r \times r$ matrices over F – not necessarily symmetric – with the standard matrix product. ■

The Example 2.2.1 can be generalized in another way, by using algebras *with involution*.

A *conjugation operator* over a field F is an operator $\bar{\cdot} : F \to F$ such that for every α and β in F, we have:

$$\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}, \quad \overline{\alpha\beta} = \overline{\alpha}\overline{\beta} \quad \text{and} \quad \overline{\overline{\alpha}} = \alpha.$$

Definition 2.2.14 *The algebra $(\mathcal{J}, \cdot)$ is an F-*algebra with involution *if:*

- ◇ *the field F has a conjugation operator $\bar{\cdot}$;*

◇ *there is an operator* $^*: \mathcal{J} \to \mathcal{J}$ *such that every* u, v *of* $\mathcal{J}$ *and every* α *of* F *satisfy* $(\alpha u)^* = \overline{\alpha} u^*$, $(u^*)^* = u$, *and* $(u \cdot v)^* = v^* \cdot u^*$.

The operator * *is the* involution *of* $\mathcal{J}$. ■

Example 2.2.3 (Jordan algebras from associative algebras with involution)
Suppose that $(\mathcal{J}, \cdot)$ is a unitary finite-dimensional associative F-algebra with an involution *. We denote by $\mathcal{H}(\mathcal{J})$ the set of elements of $\mathcal{J}$ that are *self-adjoint*: $\mathcal{H}(\mathcal{J}) := \{u \in \mathcal{J} | u = u^*\}$. If $\mathcal{J}$ is the algebra of $r \times r$ matrices over F, the set $\mathcal{H}(\mathcal{J})$ is commonly denoted by $\mathcal{H}_r(F)$; however, optimizers frequently denote $\mathcal{H}_r(\mathbb{R})$ by $\mathbb{S}^r$.

Let $\circ$ be the symmetrized multiplication. Then $\mathcal{H}(\mathcal{J})$ is stable for this multiplication. The algebra $(\mathcal{H}(\mathcal{J}), \circ)$ is commutative and satisfies Jordan's axiom. For the same reason as in the previous example, the nilpotents and the idempotents of this algebras are those of $(\mathcal{J}, \cdot)$ that are in the set $\mathcal{H}(\mathcal{J})$. ■

Example 2.2.4 (Jordan algebra from a symmetric bilinear form)
Let X be a vector space over the field F of finite dimension $N \geq 2$, and let e be a nonzero element of X. We are given a symmetric bilinear form $\mu : X \times X \to \mathbb{R}$ such that $\mu(e, e) = 1$. We construct below a Jordan multiplication on X that has e as unit element.

In order to simplify the notation, we define the linear function $\sigma : X \to \mathbb{R}$, $x \mapsto \sigma(x) := \mu(e, x)$. For every u and v of X, we put:

$$u \circ v := \sigma(u)v + \sigma(v)u - \mu(u, v)e.$$

Since μ is bilinear, this operator is bilinear, and $(X, \circ)$ is an algebra. We denote by L the operator of left application of this mapping: $L(u)v := u \circ v$.

Let us check that $(X, \circ)$ is a Jordan algebra with e as unit element. Since μ is symmetric, the multiplication $\circ$ is commutative. Furthermore, we have:

$$u \circ e := \sigma(u)e + \sigma(e)u - \mu(u, e)e = \sigma(u)e + u - \sigma(u)e = u,$$

and e is the unit element of $(X, \circ)$. Finally, we can write $u^2 = 2\sigma(u)u - \mu(u, u)e$. By linearity of L, we get $L(u)L(u^2) = 2\sigma(u)L(u)^2 - \mu(u, u)L(u) = L(u^2)L(u)$, and Jordan's Axiom is satisfied.

It is immediate to check that an element $c \neq e$ is an idempotent if and only if $\mu(c, c) = 0$ and $\sigma(c) = 1/2$.

An element u is invertible if and only if $\mu(u, u) \neq 0$. Here is a proof. Let $\mathrm{Adj}(u) := 2\sigma(u)e - u$. Observe that $u \circ \mathrm{Adj}(u) = \mu(u, u)e$. If $\mu(u, u) \neq 0$, the element u has $\mathrm{Adj}(u)/\mu(u, u)$ as inverse. Otherwise, the element u would be a divisor of zero.

An element u is nilpotent if and only if $\mu(u, u) = 0$ and $\sigma(u) = 0$. The "if" part is immediate, because $u^2 = 0$. For the "only if" part, note that $\mu(u, u) = 0$ because u is not invertible. Hence $\sigma(u^2) = 2\sigma(u)^2$, and $\sigma(u) = 0$ because there is a sufficiently large $M = 2^k$ for which $\sigma(u^M) = \sigma(0) = 0$.

Some authors denote the algebra $(X, \circ)$ by $[X; \mu; e]$. ■

Example 2.2.5 (Jordan spin algebra)
The *Jordan spin algebra*, or *spin factor*, or *quadratic terms algebra* is widely used in applications, ranging from statistics to relativistic mechanics. Optimizers utilize this algebra when they deal with second-order optimization problems (see Example 1.4.1 for a precise definition and examples

of second-order problems). This algebra is a particular instance of the previous example. We deal here with the vector space $X := \mathbb{R}^{n+1}$, where $n \geq 1$. By convention, we denote every vector $\bar{v}$ of X with an overline. The first component of $\bar{v}$ is written v_0, and the n-dimensional vector formed by its other components is written v, so that $\bar{v} = (v_0, v^T)^T$.

Consider an orthogonal basis $\{\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_n\}$ of the vector space. Let $\bar{e} := \bar{b}_0 = (1, 0, \ldots, 0)^T$ and let $\mu(\bar{u}, \bar{v}) := u_0 v_0 - u^T v$. We call the algebra $\mathcal{S}_n := [\mathbb{R}^{n+1}; \mu; \bar{e}]$ the *nth Jordan spin algebra*. Thus, its multiplication is:

$$\bar{u} \circ \bar{v} = \begin{pmatrix} u_0 \\ u \end{pmatrix} \circ \begin{pmatrix} v_0 \\ v \end{pmatrix} := \begin{pmatrix} u_0 v_0 + u^T v \\ u_0 v + v_0 u \end{pmatrix}, \quad \text{or} \quad L(\bar{u}) := \begin{pmatrix} u_0 & u^T \\ u & u_0 I_n \end{pmatrix}.$$

Particularizing our conclusions from Example 2.2.4, we observe that $\mathcal{J}$ has no nonzero nilpotents, and that every idempotent $\bar{c}$ of $\mathcal{J}$ different from $\bar{e}$ has the form

$$\bar{c} = \frac{1}{2} \begin{pmatrix} 1 \\ u \end{pmatrix},$$

where u is an n-dimensional vector of Euclidean norm 1. ■

2.3 Characteristic polynomial

This section is devoted to the study of finite-dimensional F-algebras that are *strictly power-associative*. As stated in Proposition 2.2.13, Jordan algebras form a particular class of them. We assume that F is a field of characteristic zero, in order to get rid of the unnecessary complications induced by considering finite fields. Of course, Hypothesis 2.2.1 is trivially satisfied in this case.

As mentioned above, the only allowed construction in these algebras is that of building polynomials from their elements and from elements of all their possible extensions. It turns out that there is a certain extension field of F for which this construction provides powerful results. We define and study this extension field in the second subsection. In the first subsection, we focus on a simplified case of the algebras we want to study. Interestingly enough, this particular case is not so far from the general situation and already sheds a dim light on some important features we will encounter in the theory of Jordan algebras, especially those concerning the structure of idempotents.

This section is largely inspired by the authoritative book of Hel Braun and Max Koecher [BK66]. Their work is written in German, and, up to our knowledge, has not been translated in English so far. When we refer to a statement of this book, we indicate it with a star *. Thus, the reference Lemma I.3.2* corresponds to Lemma 3.2 in the first chapter of [BK66], and Section II.4* is the fourth section of the second chapter of [BK66].

2.3.1 Minimal polynomial over associative and commutative algebras

We study in this subsection a very particular case, which, in spite of its simplicity, already highlights some interesting features of the so-called *minimal polynomial*, a particular instance of which is the characteristic polynomial.

Unless explicitly stated otherwise, we assume in this subsection that F is a field and $\mathcal{J}$ an F-algebra of dimension $N < +\infty$. Moreover, its multiplication is supposed to be *associative*, *commutative*, and *unitary*, with e as unit element.

Of course, the associativity of $\mathcal{J}$ implies its strict power-associativity. But our assumptions on $\mathcal{J}$ are *not* satisfied by general Jordan algebras. Nevertheless, for every element u of a Jordan algebra, the set $F[u]$ of all the polynomials in u with coefficients in F is a subalgebra that complies with our strong hypothesis.

Let u be an element of $\mathcal{J}$. Since the vectors of $\{e, u, u^2, \ldots, u^N\}$ are linearly dependent, there exists a nonzero polynomial $p(t) \in F[t]$ whose degree is not greater than N and such that $p(u) = 0$. There also exists a nonzero polynomial of smallest degree $\mu_u(t) \in F[t]$ with a leading coefficient equal to 1 (i.e a *monic* polynomial) such that $\mu_u(u) = 0$. We call this polynomial a *minimal polynomial of* u.

Remark 2.3.1 *The minimal polynomial of an element* $u \in \mathcal{J}$ *is unique; because if* μ_u *and* $\hat{\mu}_u$ *are two distinct minimal polynomials of* u*, their difference* $\mu_u - \hat{\mu}_u$ *vanishes in* u *as well. Since* μ_u *and* $\hat{\mu}_u$ *are monic and of the same degree, we have* $\deg(\mu_u - \hat{\mu}_u) < \deg(\mu_u)$*. This contradicts the minimality of* $\deg(\mu_u)$*.* ∎

The *degree* of an element $u \in \mathcal{J}$ is the degree of μ_u, and we denote it by $\deg(u)$. Of course, $\deg(u) \leq N$. Hence, there exists some elements of maximal degree.

Definition 2.3.2 *An element of maximal degree in* $\mathcal{J}$ *is called a* regular element*. The* rank *of* $\mathcal{J}$ *is the degree of a regular element of* $\mathcal{J}$*.* ∎

We also consider nonzero polynomials h that are nilpotent in u, i.e. for which there exists a positive integer m such that $h(u)^m = 0$. Let $g_u(t) \in F[t]$ be the one with smaller degree and with a leading coefficient equal to 1. As stressed in the following remark, this polynomial is also uniquely defined. Observe that its proof relies heavily on the associativity of $\mathcal{J}$.

Remark 2.3.3 *The sum or the difference of two nilpotent elements* a, b *of* $\mathcal{J}$ *is also nilpotent. Indeed, let* m *and* n *be positive integers such that* $a^m = 0$ *and* $b^n = 0$*. Then all the terms in the binomial expansion of* $(a \pm b)^{m+n}$ *are null, implying that* $a \pm b$ *is also a nilpotent element. The uniqueness of the polynomial* g_u *follows now from the same argument as in Remark 2.3.1.* ∎

We call the polynomial g_u the *reduced minimal polynomial* of u.

Remark 2.3.4 *Suppose that* R *is an extension ring of* F *and that* $\mathcal{J}' := \mathcal{J} \bigotimes_F R$*. All the concepts we have defined so far for* $\mathcal{J}$ *can similarly be defined for* $\mathcal{J}'$*. Existence and uniqueness of the minimal polynomial and of the reduced minimal polynomial in* $\mathcal{J}'$ *follow immediately.* ∎

In this subsection, we elaborate on how our strong assumptions on $\mathcal{J}$ allow us discover deep links between g_u and μ_u. Before introducing the final conclusion in Proposition 2.3.13, we describe intermediate steps of its proof that are particularly informative and that will be subsequently exploited.

Proposition 2.3.5 *Let $\mathcal{J}$ be a finite-dimensional associative and commutative algebra over the field F. If there exists a non-nilpotent element in $\mathcal{J}$, then there exists an idempotent in $\mathcal{J}$.*

This result is given in Lemma I.3.2⋆. The argument of its proof is so elegant that we cannot resist to copy it out here.

Proof
Let $u \in \mathcal{J}$ be a non-nilpotent element. Consider the sequence of subspaces:

$$\mathcal{J} \supseteq L(u)\mathcal{J} \supseteq L(u^2)\mathcal{J} \supseteq \cdots \supseteq L(u^m)\mathcal{J} \supseteq \cdots$$

This sequence has a minimal element $L(u^s)\mathcal{J}$, which is not equal to $\{0\}$ since u is not a nilpotent. Consider now the linear application:

$$\begin{aligned} A : L(u^s)\mathcal{J} &\rightarrow L(u^{2s+1})\mathcal{J} = L(u^s)\mathcal{J}, \\ v &\mapsto A(v) := u^{s+1}v. \end{aligned}$$

This application is obviously surjective. Since the dimensions of $\operatorname{Im} A$ and $\operatorname{dom} A$ are the same, A is bijective. Let c be the inverse image of u^{s+1} by A, that is $u^{s+1}c = A(c) = u^{s+1}$. Of course, $c \neq 0$ because u is not a nilpotent. Observe that $A(c^2) = u^{s+1}c^2 = u^{s+1}c = A(c)$. Since A is injective, $c^2 = c$, and c is the idempotent we were looking for. ■

Corollary 2.3.6 *Suppose that $\mathcal{J}$ is a power-associative F-algebra and that $v \in \mathcal{J}$ is not a nilpotent element. Then $L(v^m)F[v]$ has an idempotent element for each $m \geq 1$.*

Proof
It suffices to apply the previous proposition with the non-nilpotent $u := v^{m+1}$ in the algebra $L(v^m)F[v]$. ■

This corollary allows us to settle the case where e is the only idempotent of the algebra $\mathcal{J}$. Its proof can be found in Section I.3.5⋆.

Proposition 2.3.7 *Suppose that $\mathcal{J}$ is a finite-dimensional associative and commutative algebra that contains only one idempotent e. Let $u, v, w \in \mathcal{J}$. Then:*

1. *if vw is nilpotent, then v or w is nilpotent;*
2. *the reduced minimal polynomial g_u is irreducible;*
3. *there exists a positive integer k such that the minimal polynomial μ_u equals $(g_u)^k$.*

Sketch of the proof
Suppose that v is not nilpotent. Then $e \in L(v)F[v]$ by the previous corollary, i.e. there exists $p(t) \in F[t]$ such that $e = vp(v)$, and $w = (vw)p(v)$ is nilpotent. If $g_u(t) = h_1(t)h_2(t)$, put $v := h_1(u)$ and $w := h_2(u)$ to prove the second point. Since there is a positive integer m for which $g_u(u)^m = 0$, we know that μ_u divides $(g_u)^m$. Irreducibility of g_u shows the third point. ■

For the general case, the idea is to decompose the algebra $\mathcal{J}$ in subalgebras that contain only one idempotent.

Definition 2.3.8 *We say that an idempotent c is* minimal *if the only idempotent of $L(c)\mathcal{J}$ is c. Note that this definition holds merely for commutative, associative and unitary algebras; an adaptation is needed in the case of more general algebras (see Definition 2.7.14).* ■

Each idempotent element c defines a set $L(c)\mathcal{J}$. By associativity, this set is trivially a subalgebra of $\mathcal{J}$. Its unit element is c. If $c \neq e$, the subalgebra $L(c)\mathcal{J}$ has a strictly smaller dimension than $\mathcal{J}$. This observation allows us to deduce that *minimal* idempotents exist in every subalgebra of the form $L(c)\mathcal{J}$. If c and c' are two idempotents of $\mathcal{J}$, then the element cc' is either null or an idempotent of $\mathcal{J}$. If c is minimal and $cc' \neq 0$, then $cc' = c$ because $cc' \in L(c)\mathcal{J}$; further, if c' is also minimal, we get $cc' = c = c'$. An elementary basis argument allows us to conclude that there is a finite number of minimal idempotents in $\mathcal{J}$. We denote them by $c_1, \ldots, c_r$. Since $c_i c_j$ equals zero when $i \neq j$, these elements are linearly independent, and $r \leq N$.

Remark 2.3.9 *Suppose that $c \in \mathcal{J}$ is such that $c^2 = c$ and $cc_j = 0$ for every $1 \leq j \leq r$. Then c must be equal to zero. If this was not the case, c would be an idempotent. As $L(c)\mathcal{J}$ contains a minimal idempotent, say c_k, we would reach the contradiction $cc_k = c_k \neq 0$.* ■

The next lemma is proved in Section I.3.2*.

Lemma 2.3.10 *Every idempotent c of $\mathcal{J}$ is the sum of some distinct minimal idempotents.*
Sketch of the proof
Letting $c' := \sum_{i=1}^{r} cc_i$, observe that $c - c'$ equals its square and that $(c - c')c_i = 0$ for every i. By Remark 2.3.9, we have $c = c'$. Conclude with the fact that cc_i is either 0 or c_i. ■

The next proposition puts itself in a slightly more general situation than what we have considered in this subsection: we do not assume that $\mathcal{J}$ is unitary. This statement is Lemma I.3.4*.

Proposition 2.3.11 *Let R be an extension ring of F and let $\mathcal{J}$ be an associative and commutative F-algebra. If $\mathcal{J}' := \mathcal{J} \bigotimes_F R$ has e as unit element, then e is also the unit element of $\mathcal{J}$. What is more, if $\{c_1, \ldots, c_r\}$ is the set of minimal idempotents of $\mathcal{J}$, we have $e = c_1 + \cdots + c_r$.*
Sketch of the proof
First, we can see with Proposition 2.3.5 that $\mathcal{J}$ has an idempotent element; otherwise, e would be a nilpotent itself as a sum of nilpotents of $\mathcal{J}$ weighted by coefficients of R. Second, put $c := c_1 + \cdots + c_r$. Note that $(e - c)c_i = 0$ for every i. We can prove that $e - c$ is then a nilpotent element (see Lemma I.3.3*). As $e - c$ equals its square, we conclude that $e - c = 0$. Observe that we cannot use Remark 2.3.9 to conclude directly that $e - c = 0$, unless we can preliminarily prove that $e \in \mathcal{J}$. ■

Remark 2.3.12 *Let $\{c_1, \ldots, c_r\}$ be the set of minimal idempotents of $\mathcal{J}$. A lower bound on the rank of $\mathcal{J}$ is given by r. Indeed, let $\lambda_1, \ldots, \lambda_r$ be r distinct elements of F, and consider $u := \sum_{i=1}^{r} \lambda_i c_i$. For every natural number m, we have $u^m = \sum_{i=1}^{r} \lambda_i^m c_i$ (even for $m = 0$ in view of the previous proposition). Thus $\mu_u(u) = \sum_{i=1}^{r} \mu_u(\lambda_i) c_i = 0$, implying that $\mu_u(\lambda_i) = 0$ for all i. Hence μ_u has at least r distinct roots, and its degree is at least r.* ■

Proposition 2.3.13 *Let* $\{c_1, \ldots, c_r\}$ *be the set of minimal idempotents of* $\mathcal{J}$ *and* u *be an element of* $\mathcal{J}$*. We set* $u_i := c_i u$*,* $\mu_i := \mu_{u_i}$ *and* $g_i := g_{u_i}$*. Then there exist positive integers* k_i *such that* $\mu_i = (g_i)^{k_i}$*;* μ_u *and* g_u *are respectively the lowest common multiple of* $\mu_1, \ldots, \mu_r$ *and of* $g_1, \ldots, g_r$*.*

Proof
This is shown in Section I.3.6*. We replicate here its full proof, because the argument is typical for results on minimal polynomials, and will be exploited again.

In view of Proposition 2.3.7, the polynomial μ_i is a power of g_i because the subalgebra $L(c_i)\mathcal{J}$ contains only one idempotent. By Proposition 2.3.11, we have $e = c_1 + \cdots + c_r$. Thus $u = c_1 u + \cdots + c_r u = u_1 + \cdots + u_r$; this decomposition of u in the subalgebras $L(c_i)\mathcal{J}$ is unique, since $c_i(c_j a) = 0$ when $i \neq j$ and $a \in \mathcal{J}$.

Note that $u^m = u_1^m + \cdots + u_r^m$ for every $m \geq 1$ and, again, $u^0 = e = c_1 + \cdots + c_r$. Summarizing, we have $u^m = \sum_{i=1}^r u_i^m c_i$ for $m \geq 0$. Hence, we have $h(u) = \sum_{i=1}^r h(u_i) c_i$ for every polynomial $h(t) \in F[t]$. Since the sum $\mathcal{J} = L(c_1)\mathcal{J} + \cdots + L(c_r)\mathcal{J}$ is direct, $h(u)$ equals zero [resp. is a nilpotent] when and only when all the $h(u_i)$ equal zero [resp. are nilpotent] themselves. We deduce that $g_u = \mathrm{lcm}_{1\leq i\leq r}(g_{u_i})$ and $\mu_u = \mathrm{lcm}_{1\leq i\leq r}\left((g_{u_i})^{k_i}\right)$.
■

2.3.2 Characteristic polynomial over strictly power-associative algebras

In this subsection, we take off to worlds of much greater generality. We assume, unless explicitly stated, that the algebra $\mathcal{J}$ we deal with is a *strictly power-associative* F-algebra of finite dimension N. We denote its unit element by e. As we have already stressed it in Proposition 2.2.13, Jordan algebras of finite dimension satisfy these properties, but they are far from being the only ones.

The fact that $\mathcal{J}$ is strictly power-associative allows us to define polynomials on every algebra generated by extensions of $\mathcal{J}$ as constructed in Definition 2.2.3. The notions of minimal polynomial μ_u, of reduced minimal polynomial g_u, of degree of an element, and of regularity naturally extend to this more general setting. We study the properties of these objects in this subsection.

Nathan Jacobson was the first who defined the characteristic polynomial in strictly power-associative algebras using the machinery of *generic elements*, and who investigated its properties [Jac59, Jac60, Jac61, Jac63]. Following his work, we define generic elements in the Subsection called "Generic elements", and we present some properties of their minimal polynomial in the Subsection called "Multiplicative polynomials". The close relations between multiplicative polynomials and minimal polynomials are discussed in the Subsection "Minimal polynomial of a generic element". The first subsection below links the framework of unitary, associative and commutative algebras previously considered with strictly power-associative algebras.

The subalgebra $F[u]$

Observe that for every $u \in \mathcal{J}$, the finite-dimensional subalgebra $F[u]$ is unitary, associative and commutative. It has e as unit element. Thus *all the properties we have shown in the previous subsection remain valid in this subalgebra.*

We have not yet defined the notion of minimal idempotent in the context of power-associative algebras. This point of our development is not appropriate to elaborate on it. We temporarily use the following compromise instead, which is as close as possible to the notion of the previous subsection.

Definition 2.3.14 *Let $u \in \mathcal{J}$. An idempotent $c \in \mathcal{J}$ is called* minimal with respect to $u \in \mathcal{J}$ *if c belongs to the commutative and associative subalgebra $F[u]$ and is a minimal idempotent on $F[u]$ in the sense of Definition 2.3.8, that is, if c is the only idempotent in $L(c)F[u]$.* ■

Note that every idempotent c is minimal with respect to itself as the subalgebra $L(c)F[c]$ reduces to $\{\alpha c | \alpha \in F\}$.

Let us take an element $u \in \mathcal{J}$ and let $\{c_1, \ldots, c_r\}$ be the set of minimal idempotents of $F[u]$. According to the previous subsection, we know that if $u_i := c_i u$ for every i, then the minimal polynomial μ_u of u is the lowest common multiple of $\mu_{u_1}, \ldots, \mu_{u_r}$. Those polynomials are themselves powers of g_{u_i}. The reduced minimal polynomial is similarly the lowest common multiple of $g_{u_1}, \ldots, g_{u_r}$. But the fact that the subalgebra $F[u]$ contains only polynomials of u allows us to refine this description.

Remark 2.3.15 *Since $c_j \in F[u]$, there exists a polynomial $h_j(t) \in F[t]$ with $c_j = h_j(u)$ for $1 \leq j \leq r$. Let us fix two distinct indices i and j. Observe that $0 = c_i c_j = c_i h_j(u) = c_i h_j(u_i)$. Since c_i and $h_j(u_i)$ are in $L(c_i)F[u]$ and since c_i is a minimal idempotent, $h_j(u_i)$ must be a nilpotent in view of the first item of Proposition 2.3.7. Hence g_{u_i} is a factor of h_j. However, $c_i = c_i^2 = c_i h_i(u)$ is not a nilpotent, thus g_{u_i} is not a factor of h_i. In view of the irreducibility of g_{u_i}, no two polynomials g_{u_i} have a common divisor. Hence $g_u = \operatorname{lcm}_{1\leq i\leq r}(g_{u_i}) = \prod_{i=1}^r g_{u_i}$ and $\mu_u = \operatorname{lcm}_{1\leq i\leq r}\left((g_{u_i})^{k_i}\right) = \prod_{i=1}^r (g_{u_i})^{k_i}$.* ■

The minimal polynomial μ_u can be expressed in an alternative way. We know that the operator $L(u)$ is a linear operator from $\mathcal{J}$ to $\mathcal{J}$. For a fixed basis $\{b_1, \ldots, b_N\}$ of $\mathcal{J}$, it can be seen as an $N \times N$ matrix on F. The restriction $L_0(u)$ on $F[u]$ of the operator $L(u)$ is also an $N \times N$ matrix, but one can parameterize it with an $r \times r$ matrix $\hat{L}_0(u)$ on F. Practically, we can define $\hat{L}_0(u)$ as follows. Consider a basis of $\mathcal{J}$ that starts with the vectors $e, u, \ldots, u^{r-1}$. In this basis, the matrix $L_0(u)$ has the following structure:

$$L_0(u) = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix},$$

where M is an $r \times r$ sub-matrix. We can set $\hat{L}_0(u) := M$.

Of course, a polynomial $p(t) \in F[t]$ satisfies $p(\hat{L}_0(u)) = 0$ if and only if $p(L_0(u)) = 0$. Since $e \in F[u]$, we get $L_0(u)e = L(u)e = u$. Hence, we have $p(L_0(u))e = p(u)$ for every polynomial $p \in F[t]$.

In the following proposition, we denote the determinant of a linear operator A by $\mathrm{Det}(A)$. The identity operator of an n-dimensional vector space is represented by I_n.

Proposition 2.3.16 *For every $u \in \mathcal{J}$, we have $\mu_u(t) = \mathrm{Det}(t\,I_r - \hat{L}_0(u))$.*

Proof
Let $h(t) := \mathrm{Det}(t\,I_r - \hat{L}_0(u))$. This monic polynomial satisfies $h(\hat{L}_0(u)) = 0$, hence $h(L_0(u)) = 0$ and $h(L_0(u))e = h(u) = 0$. The polynomial h is then a multiple of μ_u. We know that $F[u]$ has dimension r, and that the polynomial μ_u has a degree of r by minimality. Applying the Cayley-Hamilton Theorem (see Theorem 2.4.2 in [HJ96]) to the $r \times r$ matrix $\hat{L}_0(u)$, we deduce that the polynomial $h(t)$ has a degree of at most r. Thus h and μ_u have the same degree, and $h = \mu_u$. ■

In fact, the characteristic polynomial that we introduce later in this subsection is an attempt to generalize the previous proposition via the generic element approach. The next proposition is important to validate this approach. This result is given in Section I.4.1*. Its five-line demonstration uses the main argument of the proof of Proposition 2.3.13.

Proposition 2.3.17 *Let u be an element of $\mathcal{J}$ whose minimal polynomial in $\mathcal{J}$ is μ_u. The minimal polynomial of u in $\mathcal{J} \bigotimes_F R$ is also μ_u for every extension ring R of F.* ■

Generic elements

Generic elements and their minimal polynomial have been introduced in the context of strictly power-associative algebras in 1959 by Jacobson [Jac59]. They are studied thoroughly in Chapter 2 of [BK66]; an alternative approach is provided in Chapter 6 of [Jac68]. We give here a brief account of the properties needed in order to support the present exposition.

Definition 2.3.18 *Let $\tilde{F}$ be a field and F be a subfield of $\tilde{F}$. We say that the elements $y_1, \ldots y_k$ of $\tilde{F}$ are* algebraically independent over F *if there is no polynomial G with k variables and coefficients in F for which $G(y_1, \ldots, y_k) = 0$.* ■

In practice, given a field F, it is not difficult to construct a field $\tilde{F}$ in which there are k elements that are algebraically independent over F: it suffices to add k independent indeterminates to F, and to specify an associative and commutative multiplication rule between them.

In this subsection, we deal again with a strictly power-associative F-algebra $\mathcal{J}$ of dimension $N < +\infty$. We fix a basis $\mathcal{B} := \{b_1, \ldots, b_N\}$ of $\mathcal{J}$ and N elements $\tau_1, \ldots, \tau_N$ that are algebraically independent over F.

We set $\tilde{F} := F(\tau_1, \ldots, \tau_N)$ and we put $\tilde{\mathcal{J}} := \mathcal{J} \bigotimes_F \tilde{F}$; by definition, every element f of $\tilde{\mathcal{J}}$ can be written as:

$$f(\tau_1, \ldots, \tau_N) = \sum_{i=1}^{N} f_i(\tau_1, \ldots, \tau_N) b_i,$$

where f_i is a rational function of $\tau_1, \ldots, \tau_N$ with coefficients in F. Note that $\tilde{\mathcal{J}}$ has obviously the same unit element e as $\mathcal{J}$.

Definition 2.3.19 *The* generic element *of $\mathcal{J}$ defined by $\mathcal{B}$ and $\tau_1, \ldots, \tau_N$ is* $x := \tau_1 b_1 + \cdots + \tau_N b_N$. *It belongs to $\tilde{\mathcal{J}}$.* ■

If $\mathcal{J}' := \mathcal{J} \bigotimes_F R$, where R is an extension ring of F, every basis of $\mathcal{J}$ over F is also a basis of $\mathcal{J}'$ over R. In particular, a generic element of $\mathcal{J}$ defined by a basis and a set of algebraically independent elements $\tau_1, \ldots, \tau_N$ over F is also a generic element for $\mathcal{J}'$, provided that $\tau_1, \ldots, \tau_N$ are also algebraically independent over R.

An intuitive way to interpret a generic element is to consider it as an N-dimensional variable over F. The next definition clarifies this viewpoint.

Definition 2.3.20 *Let x be the generic element defined by the basis $\mathcal{B}$ and $\tau_1, \ldots, \tau_N$. We let $a = a_1 b_1 + \cdots + a_N b_N \in \mathcal{J}$, where $a_i \in F$. The* specialization *of the generic element x in a, denoted by $x \to a$, corresponds to the substitution for every i of τ_i by a_i, the ith component of a in the basis $\mathcal{B}$.* ■

Let

$$f(\tau_1, \ldots, \tau_N) = \sum_{i=1}^{N} f_i(\tau_1, \ldots, \tau_N) b_i \in \tilde{\mathcal{J}}.$$

The *domain* of f, denoted by $\operatorname{dom} f$, is the set of all $a = a_1 b_1 + \cdots + a_N b_N \in \mathcal{J}$ such that $(a_1, \ldots, a_N)$ is in the domain of each f_i. The *evaluation* of f in $a \in \operatorname{dom} f$ is:

$$f(a) := \sum_{i=1}^{N} f_i(a_1, \ldots, a_N) b_i.$$

This is simply the value of f in the vector a, identified by its component in the basis $\mathcal{B}$, when one considers f as a function from $\operatorname{dom} f$ to $\mathcal{J}$. With this point of view, the object f is independent of the specific choice of $\tau_1, \ldots, \tau_N$. We say that the evaluation of f in $a \in \operatorname{dom} f$ is the *specialization $x \to a$ applied to f*. With a slight abuse of notation, we will simplify the writings $f(\tau_1, \ldots, \tau_N)$ and $f_i(\tau_1, \ldots, \tau_N)$ by $f(x)$ and $f_i(x)$ respectively.

The functional interpretation of specialization allows us to extend this operation for non-rational functions $g(\tau_1, \ldots, \tau_N)$: the specialization $x \to a$ applied to g is $g(a_1, \ldots, a_N)$, or simply $g(a)$, provided that $(a_1, \ldots, a_N)$ is in the domain of g. With the same abuse of notation as above, we denote $g(\tau_1, \ldots, \tau_N)$ by $g(x)$.

Definition 2.3.21 *Let $x^{(i)} = \tau_1^{(i)} b_1 + \cdots + \tau_N^{(i)} b_N$, for $1 \leq i \leq k$, be a set of generic elements which are all defined with respect to the same basis $\{b_1, \ldots, b_N\}$ of $\mathcal{J}$. They are said to be* generically independent *if the elements $\tau_j^{(i)}$ are all algebraically independent over F.* ■

The following proposition defines a toolbox of operations to produce generic elements. The two first items are proved in Section II.2.2*. The third item results trivially from the definition.

Proposition 2.3.22 *Let $x = \tau_1 b_1 + \cdots + \tau_N b_N$ be a generic element of $\mathcal{J}$. We have the following.*

1. *The element x and the operator $L(x)$ are invertible. Moreover, the element x^{-1} equals $L(x)^{-1}e$ and is a generic element.*
2. *For every nonzero integer k, x^k is a generic element.*
3. *Let $W : \mathcal{J} \to \mathcal{J}$ be an invertible linear operator. We denote its natural extension to $\tilde{\mathcal{J}}$ also by W. The element Wx is generic for the same basis as x.* ■

Multiplicative polynomials

Let us fix a basis $\mathcal{B} := \{b_1, \ldots, b_N\}$ of $\mathcal{J}$ and some algebraically independent elements over F denoted by $\tau, \tau_1, \ldots, \tau_N, \sigma_1, \ldots, \sigma_N$. We set two generic elements $x := \tau_1 b_1 + \cdots + \tau_N b_N$ and $y := \sigma_1 b_1 + \cdots + \sigma_N b_N$.

Definition 2.3.23 *A polynomial $p \in F[\tau_1, \ldots, \tau_N]$ is* multiplicative *if:*

- ⋄ *$p(e) = 1$ and*
- ⋄ *we have $p(f(x)g(x)) = p(f(x))p(g(x))$ for every extension ring R of F and every $f, g \in R[x]$.*

It is worth mentioning that $R[x]$ stands here for the set of polynomial in x with coefficient in R and not *for the ring $R[\tau_1, \ldots, \tau_N]$.* ■

The interest of multiplicative polynomials is evident from the definition. By specialization, they allow us to write identities of the type $p(f(u)g(u)) = p(f(u))p(g(u))$ for polynomials f, g and elements $u \in \mathcal{J}$. Moreover, as stated in Proposition 2.3.25, multiplicative polynomials have close links with minimal polynomials. However, this definition might seem so restrictive that it is natural to wonder whether there exist non-trivial multiplicative polynomials.

In fact, the Cayley-Hamilton polynomial given in Proposition 2.3.16 will be extended in the next subsection to the framework of strictly power-associative algebras, and its multiplicativity will be proved.

Lemma 2.3.24 *Let p be a multiplicative polynomial of degree m of $F[\tau_1, \ldots, \tau_N]$ and let R be an extension ring of F. The polynomial p is homogeneous of degree m. For every $\alpha \in R$, we have $p(\alpha e) = \alpha^m$.*

Proof
It suffices to write $p(x) = \sum_{i=0}^m p_i(x)$, where $p_i(x)$ is homogeneous of degree i, so that $p_i(\alpha x) = \alpha^i p_i(x)$. Of course, $p_m \neq 0$. Next, we can write:

$$\begin{aligned} p(\alpha^2 x^2) &= p(\alpha x)^2 = \left(\sum_{i=0}^m \alpha^i p_i(x) \right)^2 \\ p(\alpha^2 x^2) &= \sum_{i=0}^m \alpha^{2i} p_i(x)^2. \end{aligned}$$

It remains now to compare the terms with the same degree in α in the two formulations to deduce gradually that $p_i(x) = 0$ when $i \neq m$. Hence $p(\alpha x) = p_m(\alpha x) = \alpha^m p(x)$. ■

Let $e = e_1 b_1 + \cdots + e_N b_N$ be the decomposition of the unit element e in the basis $\mathcal{B}$. Note that $\tau e - x = \sum_{i=1}^{N} (\tau e_i - \tau_i) b_i$. We denote (not surprisingly...) by $p(\tau e - x)$ the polynomial constructed by replacing every occurrence of τ_i by $\tau e_i - \tau_i$ in a multiplicative polynomial p. The constructed element lies in $F[\tau, \tau_1, \ldots, \tau_N]$.

Suppose that p is a multiplicative polynomial of degree m. We write:

$$p(\tau e - x) = \sum_{i=0}^{m} (-1)^i \chi_i(x) \tau^{m-i},$$

where $\chi_i \in F[\tau_1, \ldots, \tau_N]$. Replacing $\tau e - x$ by $\alpha \tau e - \alpha x$ in this expression, and using Lemma 2.3.24, shows that $\chi_i(x)$ is a homogeneous polynomial of degree i. In particular $\chi_0(x)$ is a constant, and applying the specialization $x \to e$ in $p(\tau e - x)$ yields $p((\tau - 1)e) = (\tau - 1)^m$, so that $\chi_0(e) = 1$, and $\chi_0(x) = 1$. In other words, the polynomial $q(\tau) = p(\tau e - x)$ is monic.

Finally, we fix an element $u \in \mathcal{J}$. We denote by $\bar{F}$ the algebraic closure of F. The specialization $x \to u$ applied to the polynomial $p(\tau e - x)$ gives a polynomial $p(\tau e - u) \in F[\tau]$ that has m roots $\xi_1, \ldots, \xi_m$ in $\bar{F}$. We can decompose this polynomial into:

$$p(\tau e - u) = \prod_{i=1}^{m} (\tau - \xi_i). \tag{2.6}$$

Surprisingly enough, it is possible to relate the roots ξ_i with the roots of the minimal polynomial μ_u of u. The following proposition covers Section II.3.2*, Satz II.3.1* and Satz II.3.2*.

Proposition 2.3.25 *With the notation above, we can say the following:*

1. *$\xi_1, \ldots, \xi_m$ are roots of μ_u.*
2. *For every $q \in \bar{F}[\tau]$ we have $p(q(u)) = \prod_{i=1}^{m} q(\xi_i)$.*
3. *For all $h \in F[\tau]$, we have:*

   $$\chi_i(h(u)) = S_i(h(\xi_1), \ldots, h(\xi_m)),$$

   where $S_i(a_1, \ldots, a_m)$ is the symmetric function of degree i, that is,

   $$S_i(a_1, \ldots, a_m) := \sum_{\sigma} a_{\sigma_1} \cdots a_{\sigma_i},$$

   where the sum is taken on all the subsets $\sigma = \{\sigma_1, \ldots, \sigma_i\} \subseteq \{a_1, \ldots, a_m\}$ of size i.

Sketch of the proof
To prove the second item, we need to decompose the polynomial $q \in \bar{F}[\tau]$ into linear factors: $q(\tau) = \eta \prod_{j=1}^{n} (\beta_j - \tau)$, so that $q(u) = \eta \prod_{j=1}^{n} (\beta_j e - u)$. We have:

$$p\left(\eta \prod_{j=1}^{n} (\beta_j y - x)\right) = \eta^m \prod_{j=1}^{n} p(\beta_j y - x)$$

because $\beta_j y - x \in F(\sigma_1, \ldots, \sigma_N)[x]$. Specializing $y \to e$ and $x \to u$, this is equal to $p(q(u))$. It remains to rewrite the right-hand side using (2.6) to conclude.

We apply this result at $q(\tau) := \rho - h(\tau)$ for a polynomial $h \in F[\tau]$ and various $\rho \in F$. Comparing the terms of same degree in ρ allows us to prove the third claim. The first item can be shown by taking $h := \mu_u$ in the previous expression of q. ∎

Minimal polynomial of a generic element

We have now all the necessary technical material to define rigorously the *characteristic polynomial* of a strictly power-associative algebra.

Let $\mathcal{B} := \{b_1, \ldots, b_N\}$ be a basis of $\mathcal{J}$ over F and let $\tau, \tau_1, \ldots, \tau_N$ be algebraically independent elements over F. We put $\tilde{F} := F(\tau_1, \ldots, \tau_N)$ and $\tilde{\mathcal{J}} := \mathcal{J} \bigotimes_F \tilde{F}$. Since the dimension N of $\tilde{\mathcal{J}}$ is finite, there exists a positive r such that $\{e, x, \ldots, x^{r-1}\}$ are not linearly dependent over $\tilde{F}$, but $\{e, x, \ldots, x^{r-1}, x^r\}$ are. We explicit this linear dependence by:

$$x^r - a_1(x)x^{r-1} + a_2(x)x^{r-2} + \cdots + (-1)^r a_r(x)e = 0, \tag{2.7}$$

where the coefficients a_i are rational functions of $\tau_1, \ldots, \tau_N$. We define the *characteristic polynomial* $f(\tau; x) \in \tilde{F}[\tau]$ as:

$$f(\tau; x) = \tau^r - a_1(x)\tau^{r-1} + a_2(x)\tau^{r-2} + \cdots + (-1)^r a_r(x).$$

Since $x^0 = e$, we have $f(x; x) = 0$.

The characteristic polynomial is unique, as the same reasoning as in Remark 2.3.1 shows. Suppose now that $y := \rho_1 b_1 + \cdots + \rho_N b_N$ is a generic element of $\mathcal{J}$ with the same basis as x. Then, by substituting τ_i with ρ_i, the polynomial $f(\tau; y)$ also satisfies $f(y; y) = 0$. In other words, the characteristic polynomial does not depend on the chosen generic element defined by the basis $\mathcal{B}$.

The next proposition shows that the coefficients $a_i(x)$ are polynomials of $F[\tau_1, \ldots, \tau_N]$ instead of rational functions from $F(\tau_1, \ldots, \tau_N)$. This is a well-known result (see Proposition II.2.1 in [FK94]) based on Gauss's Lemma, which we quote below (see for instance [DF99], Section 9.3 for details).

Lemma 2.3.26 (Gauss's Lemma) *Let R be a factorial ring*[1]*. Denote by* $\mathrm{Frac}(R)$ *the fraction field of R. Let $p \in R[\tau]$ be a monic polynomial and $s \in \mathrm{Frac}(R)[\tau]$ be a divisor of p. Then $s \in R[\tau]$.* ∎

Proposition 2.3.27 *For each $1 \leq i \leq r$, the coefficient $a_i(x)$ is a homogeneous polynomial in $\tau_1, \ldots, \tau_N$ of degree i.*

Proof
By definition, coefficients a_i are *rational* functions of $\tau_1, \ldots, \tau_N$, that is, $a_i \in R(\tau_1, \ldots, \tau_N) = \mathrm{Frac}(R[\tau_1, \ldots, \tau_N])$.

[1]A factorial ring R, or *unique factorization domain*, is a ring on which every element can be decomposed in a unique way (up to permutations of factors) as a product of *irreducible* elements, that is, elements that cannot be written as the product of two elements of R both different from ± 1. According to the fundamental theorem of arithmetics, the ring of natural numbers is a factorial ring.

Consider the Cayley-Hamilton polynomial of the linear operator $L(x)$:

$$F(\tau) := \mathrm{Det}(\tau I_N - L(x)).$$

The coefficients of this monic polynomial belong to $R[\tau_1, \ldots, \tau_N]$ by construction. We have $F(L(x)) = 0$, so that $F(x) = F(L(x))e = 0$. Henceforth, the characteristic polynomial $f(\tau; x)$ is a divisor of $F(\tau)$. Applying Gauss's Lemma, we deduce that $f(\tau; x)$ is in $R[\tau_1, \ldots, \tau_N](\tau)$, that is, that the coefficient $a_i(x)$ are polynomials.

It remains to show the homogeneity of $a_i(x)$. Let α be a nonzero element of F; we let:

$$q_\alpha(\tau) := \frac{f(\alpha\tau; \alpha x)}{\alpha^r} = \tau^r - \frac{a_1(\alpha x)}{\alpha}\tau^{r-1} + \frac{a_2(\alpha x)}{\alpha^2}\tau^{r-2} + \cdots + (-1)^r \frac{a_r(\alpha x)}{\alpha^r}.$$

Since q_α is a monic polynomial of degree r that vanishes in x, it must be equal to $f(\tau; x)$ by uniqueness of the characteristic polynomial. Thus $a_i(\alpha x) = \alpha^i a_i(x)$. ■

As a consequence of this proposition, we get that $\mu_u(\tau)$ divides $f(\tau; u)$ for each $u \in \mathcal{J}$ by specializing $x \to u$ in $f(\tau; x)$.

Definition 2.3.28 *We call the linear polynomial* $\mathrm{tr}(x) := a_1(x)$ *the* generic trace *of* x. ■

Definition 2.3.29 *The polynomial* $\det(x) := a_r(x)$ *is called the* generic norm *or the* determinant *of* x. ■

Definition 2.3.30 *The degree* r *of* $f(\tau; x)$ *is called the* generic rank *of* $\mathcal{J}$. ■

In contrast with the determinant of a linear operator, which we denote by Det, the determinant of a generic element is denoted by det, with a small "d". In fact, there exists a strong link between the two notions.

Remark 2.3.31 *We can use the Proposition 2.3.16 with* $\tilde{\mathcal{J}}$ *instead of* $\mathcal{J}$ *and with the generic element* x *instead of* u. *We write* $\hat{L}_0(x)$ *for an* $r \times r$ *matrix that parameterizes the restriction* $L_0(x)$ *of the operator* $L(x)$ *on* $F[x]$. *We obtain that* $f(\tau; x) = \mathrm{Det}(tI_r - \hat{L}_0(x))$, *and we deduce that:*

$$\det(x) = a_r(x) = (-1)^r f(0; x) = (-1)^r \mathrm{Det}(-\hat{L}_0(x)) = \mathrm{Det}(\hat{L}_0(x)).$$

Denoting by Tr *the trace of a linear operator, we also get that* $\mathrm{tr}(x) = \mathrm{Tr}(\hat{L}_0(x))$. ■

The coefficient $a_j(x)$ of the characteristic polynomial is sometimes called the jth *dettrace* of $\mathcal{J}$ and is denoted by $\mathrm{detr}_j(x)$.

The dettraces are functions of $\tau_1, \ldots, \tau_N$, and one can apply to them the specialization $x \to u$ for every $u \in \mathcal{J}$. We call $\mathrm{tr}(u)$ the *trace* of the element u of $\mathcal{J}$, $\det(u)$ the *determinant* of u. For instance, the specialization $x \to e$ applied to $\det(x)$ yields $\det(e) = \mathrm{Det}(\hat{L}_0(e)) = 1$.

Definition 2.3.32 *We call the roots* $\{\lambda_1(x), \ldots, \lambda_r(x)\}$ *of* $f(t; x)$ *the* eigenvalues *of* x. ■

For every $u \in \mathcal{J}$, the set $\{\lambda_1(u), \ldots, \lambda_r(u)\}$ is called the set of *eigenvalues* of u. We leave their numbering unspecified until Section 2.7, where we focus on formally real Jordan algebras over $\mathbb{R}$, where the eigenvalues of every elements are real.

As an immediate consequence of the definition, we can write for every $1 \leq j \leq r$:

$$\mathrm{detr}_j(x) = S_j(\lambda_1(x), \ldots, \lambda_r(x)),$$

where S_j is the jth elementary symmetric function as defined for Proposition 2.3.25. In particular, we get:

$$\mathrm{tr}(x) = \sum\{\lambda_1(x), \ldots, \lambda_r\} \quad \text{and} \quad \det(x) = \prod\{\lambda_1(x), \ldots, \lambda_r\}.$$

The next remark completes Proposition 2.3.22 on the characterization of the inverse of a generic element.

Remark 2.3.33 *We define the polynomial $q(\tau; x) \in \tilde{F}[\tau]$ as follows:*

$$q(\tau; x) := a_{r-1}(x) - a_{r-2}(x)\tau + \cdots + (-1)^{r-2}a_1(x)\tau^{r-2} + (-1)^{r-1}\tau^{r-1},$$

so that $\det(x) - \tau q(\tau; x) = (-1)^r f(\tau; x)$*. The polynomial $q(x; x)$ is called by some authors the* adjoint polynomial *of x, by analogy with the standard terminology for matrices. It has no common factor with* $\det(x)$ *in view of the minimality of $f(\tau; x)$. The polynomial $g(\tau; x) := q(\tau; x)/\det(x)$ belongs to $\tilde{F}[\tau]$ and $g(x; x) = x^{-1}$. In other words, x^{-1} can be written as a polynomial in x whose coefficients belong to $F(\tau_1, \ldots, \tau_N)$. Suppose that $u \in \mathcal{J}$ is invertible. Specializing $x \to u$, we obtain that u^{-1} is in the vector space in F spanned by $e, u, \ldots, u^{r-1}$, i.e. $F[u]$.* ■

The following proposition is one of the most important in this section. It proves that $\det(x)$ is a multiplicative polynomial, it generalizes the Proposition 2.3.16 and it establishes a clear link between the minimal polynomial of an element and the characteristic polynomial of its algebra. This link will be strengthened later in the context of formally real Jordan algebras. Indeed, it will be proved that, in this very particular framework, the specialization $x \to u$ of the characteristic polynomial is *exactly* the minimal polynomial of u for every regular $u \in \mathcal{J}$. But this fact is far to be evident from now on.

The proof of the following statement is a bit technically involved. The reader can find a complete demonstration of the multiplicativity of det in Theorem VI.1 of [Jac68] or in Satz II.4.3*.

Proposition 2.3.34 *The function* det *is a multiplicative polynomial. What is more, we can write* $\det(\tau e - x) = f(\tau; x)$ *and, for every $u \in \mathcal{J}$, we have* $\det(\tau e - u) = \prod_{i=1}^r(\tau - \xi_i)$*, where ξ_i are in the set of roots of μ_u. The polynomials $f(\tau; u)$ and $\mu_u(\tau)$ have the same set of roots, which can only differ by their multiplicities.*

Sketch of the proof

The proof relies on the following observation. In view of Remark 2.3.31, and using the notation introduced there, we have:

$$\det(y)\det(z) = \mathrm{Det}(\hat{L}_0(y))\mathrm{Det}(\hat{L}_0(z)) = \mathrm{Det}(\hat{L}_0(y)\hat{L}_0(z)) = \mathrm{Det}(\hat{L}_0(yz)) = \det(yz)$$

if $\hat{L}_0(y)$ and $\hat{L}_0(z)$ are two matrices that commute. The fact these matrices commute when they are polynomials of x is proved in Section II.4.3*.

It remains to use Proposition 2.3.25 to get $\det(\tau e - u) = \prod_{i=1}^{r}(\tau - \xi_i)$, where ξ_i are in the set of roots of μ_u. Moreover, since $\mu_u(\tau)$ is a factor of $f(\tau; u)$, they have the same set of roots. ∎

From the previous proposition, we know that for every $u \in \mathcal{J}$, the set $\{\lambda_1(u), \dots, \lambda_r(u)\}$ is the set of roots of μ_u. This set is homogeneous of degree 1 as the following remark states. We cannot say that the eigenvalues themselves are homogeneous because of the numbering convention we have adopted.

Remark 2.3.35 *Let $x = \tau_1 b_1 + \cdots + \tau_N b_N$ be a generic element and τ, α be two elements that are algebraically independent from each other and of all the elements τ_i. We have by the multiplicativity of* det*:*

$$f(\tau; \alpha x) = \det(\tau e - \alpha x) = \alpha^r f(\tau/\alpha; x) = \alpha^r \prod_{i=1}^{r} \left(\frac{\tau}{\alpha} - \lambda_i(x) \right).$$

The roots of $f(\tau; \alpha x)$ are thus $\{\lambda_1(\alpha x), \dots, \lambda_r(\alpha x)\} = \{\alpha\lambda_1(x), \dots, \alpha\lambda_r(x)\}$. ∎

Associativity of the trace

We conclude this section with a proof of an important associativity property of the trace operator that holds in strictly power-associative algebras.

Let x, y, z be three generically independent elements over F. Then:

$$\operatorname{tr}((xy)z) = \operatorname{tr}(x(yz)).$$

∎

This relation is a deep result. We cannot avoid using a slightly technical machinery to demonstrate it.

Definition 2.3.36 *A* derivation *of an algebra $\mathcal{J}$ is a linear mapping $D : \mathcal{J} \to \mathcal{J}$ such that we have $D(uv) = uD(v) + D(u)v$ for all $u, v \in \mathcal{J}$. Equivalently, $L(D(u)) = [D; L(u)]$ for every $u \in \mathcal{J}$.* ∎

Definition 2.3.37 *Let F be a field. The* ring of dual numbers F_ε built from F *is the set $F \times F$ with the standard componentwise addition and a multiplication defined as follows:*

$$\text{for every } (a,b), (c,d) \text{ in } F \times F, \quad (a,b)(c,d) := (ac, ad + bc).$$

The element $(0,1)$ of F_ε is denoted by ε. According to the definition of the multiplication, its square is null. ∎

The standard terminology "dual numbers" for elements of F_ε seems a little bit unfortunate in this thesis, due to the fact that it can be mistaken with the equally standard denomination of "dual" in conic programming. Dual numbers are only used in this chapter, and not in other parts of this work.

We denote the algebra $\mathcal{J} \bigotimes_F F_\varepsilon$ by $\mathcal{J}_\varepsilon$.

Definition 2.3.38 *Let $A : \mathcal{J} \to \mathcal{J}$ be a linear operator and let p be a polynomial of $F[t]$. This polynomial is* Lie-invariant *under A if the function $p : \mathcal{J}_\varepsilon \to \mathcal{J}_\varepsilon$ satisfies:*

$$p(u + \varepsilon A(u)) = p(u) \quad \textit{for every } u \textit{ in } \mathcal{J}.$$

■

The following proposition was first proved by Jacques Tits in [Tit64]. We rewrite his proof here, with a few necessary adaptations to fit the framework of our exposition.

Proposition 2.3.39 *The coefficients of the generic polynomial of $\mathcal{J}$ are Lie-invariant under all derivations of $\mathcal{J}$.*

Proof
Let $\mathcal{B} := \{b_1, \ldots, b_N\}$ be a basis of $\mathcal{J}$ and let $\tau_1, \ldots, \tau_N, \rho_1, \ldots, \rho_N$ be algebraically independent elements over F. We put $x := \tau_1 b_1 + \cdots + \tau_N b_N$ and $y := \rho_1 b_1 + \cdots + \rho_N b_N$. We define the following rings:

$$R_1 := F[\tau_1, \ldots, \tau_N],\ R_2 := F[\tau_1 + \varepsilon\rho_1, \ldots, \tau_N + \varepsilon\rho_N],$$

$$\text{and } \ R_3 := F_\varepsilon[\tau_1, \ldots, \tau_N, \rho_1, \ldots, \rho_N].$$

Note that R_1 and R_2 are two rings contained in R_3.

The element $x + \varepsilon y$ is generic in $\mathcal{J}$. It amounts to proving that the elements $\tau_i + \varepsilon\rho_i$ are algebraically independent over F. So, let p be a nonzero polynomial with coefficients in F and that vanishes in $(\tau_1 + \varepsilon\rho_1, \ldots, \tau_N + \varepsilon\rho_N)$. The specialization $\rho_i \to 0$ for every i shows that $p(\tau_1, \ldots, \tau_N) = 0$. Since the elements τ_i are algebraically independent, we must have $p \equiv 0$, which contradicts the hypothesis.

Comparing the degree of characteristic polynomials. Let $f(\tau; x)$ be the characteristic polynomial of $\mathcal{J}$, that is, the minimal polynomial of x in $\mathcal{J} \bigotimes_F R_1$. Then $f(\tau; x + \varepsilon y)$ is the minimal polynomial of $x + \varepsilon y$ in $\mathcal{J} \bigotimes_F R_2$ because $x + \varepsilon y$ is a generic element of $\mathcal{J}$ with the same basis as x.

Let $f_\varepsilon(\tau; x + \varepsilon y)$ be the minimal polynomial of $x + \varepsilon y$ in $\mathcal{J} \bigotimes_F R_3$. Since $R_2 \subseteq R_3$, the degree of f_ε is lower than the degree of f.

Now, the specialization $y \to 0$ applied to $f_\varepsilon(x + \varepsilon y; x + \varepsilon y) = 0$ gives $f_\varepsilon(x; x) = p_1(x) + \varepsilon p_2(x) = 0$ for two polynomials p_1, p_2 with coefficients in F. We must then have $p_1(x) = p_2(x) = 0$. The degree of these polynomials are thus bounded from below by the degree of f, which is then smaller than the degree of f_ε. Therefore, the polynomials f and f_ε have the same degree.

Proposition 2.3.17 asserts that the minimal polynomial of x in $\mathcal{J} \bigotimes_F R_3$ is the same as the minimal polynomial of x in $\mathcal{J} \bigotimes_F R_1$. The latter divides $f_\varepsilon(\tau; x)$; comparing degrees and using uniqueness of the minimal polynomial, we conclude that $f(\tau; x) = f_\varepsilon(\tau; x)$.

Defining a practical notation. Following the convention of the expression (2.7) of the characteristic polynomial, we denote by $(-1)^k a_k(x + \varepsilon y)$ the kth coefficient of the polynomial f_ε. In the previous item, we showed that $(-1)^k a_k(x)$ is the kth coefficient of the polynomial f. For $0 \leq k \leq r$, we write:

$$(x + \varepsilon y)^k = x^k + \varepsilon\{x, y\}_k \quad \text{and} \quad a_k(x + \varepsilon y) = a_k(x) + \varepsilon \mu_k(x, y).$$

With this writing, the statement reduces to prove that $\mu_k(x, D(x)) = 0$ for every k and every derivation D of $\mathcal{J}$.

A property of derivations. For every derivation D of $\mathcal{J}$ and every nonnegative integer k, we have $D(x^k) = \{x, D(x)\}_k$. This assertion is not difficult to prove by recurrence on k.

Putting everything together. Using the notation introduced above, we can successively write:

$$\begin{aligned}
0 &= f_\varepsilon(x + \varepsilon y; x + \varepsilon y) \\
&= (x + \varepsilon y)^r - a_1(x + \varepsilon y)(x + \varepsilon y)^{r-1} + \cdots + (-1)^r a_r(x + \varepsilon y)e \\
&= x^r + \varepsilon\{x, y\}_r + \sum_{k=1}^{r} (-1)^k \big(a_k(x) + \varepsilon \mu_k(x, y)\big) \left(x^{r-k} + \varepsilon\{x, y\}_{r-k}\right) \\
&= f(x; x) + \varepsilon \left(\{x, y\}_r + \sum_{k=1}^{r} (-1)^k \mu_k(x, y) x^{r-k} + \sum_{k=1}^{r} (-1)^k a_k(x)\{x, y\}_{r-k} \right).
\end{aligned}$$

The ε-component is then equal to zero:

$$0 = \{x, y\}_r + \sum_{k=1}^{r} (-1)^k x^{r-k} \mu_k(x, y) + \sum_{k=1}^{r} (-1)^k \left(a_k(x)\{x, y\}_{r-k}\right). \tag{2.8}$$

On the other hand, we have:

$$\begin{aligned}
0 &= D(0) = D(f(x; x)) = D(x^r) + \sum_{k=1}^{r} (-1)^k a_k(x) D(x^{r-k}) \\
&= \{x, D(x)\}_r + \sum_{k=1}^{r} (-1)^k a_k(x)\{x, D(x)\}_{r-k}.
\end{aligned} \tag{2.9}$$

Note that y is a generic element of $\mathcal{J} \bigotimes_F \mathrm{Frac}(R_1)$; we may then perform the specialization $y \to D(x)$ in (2.8). Subtracting from it the equation (2.9), we are left with:

$$\sum_{k=1}^{r} (-1)^k \mu_k(x, D(x)) x^{n-k} = 0.$$

The degree of this polynomial is equal to $r-1$; as it vanishes in the generic element x of $\mathcal{J}$, it should be null. Thus $\mu_k(x, D(x)) = 0$ for every k. ■

Corollary 2.3.40 *Let $\mathcal{J}$ be a Jordan algebra over F and let x, y, z be three generic elements of $\mathcal{J}$ which are generically independent over F. We have:*

$$\mathrm{tr}((xy)z) = \mathrm{tr}(x(yz)).$$

Proof
It suffices to prove that the operator $D := R(z)L(x) - L(x)R(z) = L(z)L(x) - L(x)L(z) = [L(z); L(x)]$ is a derivation. If this is indeed the case, the previous proposition will allow us to write $\mathrm{tr}(Dy) = \mathrm{tr}((xy)z - x(yz)) = 0$, which yields the desired relation by linearity of the operator tr (see Proposition 2.3.27).

We need to check that $L(D(x)) = [D; L(x)]$. Reemploying the notation of p. 35, the equality $\mathcal{A}_{ayz}x = \mathcal{A}_{ayx}z$ follows from Jordan's Axiom. We can rewrite it as follows:

$$L(z(yx)) + L(y)L(z)L(x) + L(x)L(z)L(y) = L(x(yz)) + L(y)L(x)L(z) + L(z)L(x)L(y),$$

or equivalently, by linearity of L:

$$L(z(yx) - x(yz)) = L(y)L(x)L(z) + L(z)L(x)L(y) - L(y)L(z)L(x) - L(x)L(z)L(y),$$

that is, $L([L(z); L(x)]y) = [[L(z); L(x)]; L(y)]$, as needed. ■

In Satz III.5.6*, there is a proof of a similar associativity property for a class of linear forms that includes the trace. This result is more general than the one we presented above: the authors only needed commutative algebras that satisfy the hypotheses of Proposition 2.3.39 and an additional property, the *homogeneity.* This concept is introduced later, in the framework of Jordan algebras, in order to avoid the technical difficulties that a more general treatment would require.

2.3.3 Examples

Let us particularize the objects defined in this section to the examples given in Subsection 2.2.4.

Example 2.3.1 (Real symmetric matrices: Example 2.2.1 continued)
Let U be an $r \times r$ symmetric matrix. We have mentioned that powers of U in the algebra $(\mathcal{J}, \cdot)$ and in the algebra $(\mathcal{J}, \circ)$ are the same. Hence, the minimal polynomial μ_U is the usual minimal polynomial of the matrix U, whose degree equals the rank of U. Some properties that particularize the results we have described in Subsections 2.3.1 and 2.3.2 can be found in Section 3.3 of [HJ96]. The characteristic polynomial of U is its Cayley-Hamilton polynomial (see Theorem 2.4.2 in [HJ96]), namely $\mathrm{Det}(tI_r - U)$. As its degree is r, the generic rank of $\mathcal{J}$ is r. The generic trace tr, the determinant det and the generic eigenvalue vector λ defined in this section are identical to the standard trace, determinant and eigenvalue vector defined in Matrix Theory. ■

The same comment holds for Example 2.2.2 and Example 2.2.3.

Example 2.3.2 (Jordan algebra from a symmetric bilinear form)
Let $\{b_1, \ldots, b_N\}$ be a basis of X and $\{\tau_1, \ldots, \tau_N\}$ a set of algebraically independent elements. We denote the generic element $\tau_1 b_1 + \cdots + \tau_N b_N$ by x. We have:

$$x^2 = 2\sigma(x)x - \mu(x, x)e.$$

Hence, the generic rank of $[X; \mu; e]$ is equal to 2. It could have been equal to 1 if we would not have excluded the case $N = 1$. The generic trace is $2\sigma(x)$ and the determinant is $\mu(x, x)$. The generic eigenvalues can be computed explicitly:

$$\{\lambda_1(x), \lambda_2(x)\} = \{\sigma(x) + \sqrt{\sigma(x)^2 - \mu(x, x)}, \sigma(x) - \sqrt{\sigma(x)^2 - \mu(x, x)}\}.$$

■

Example 2.3.3 (Jordan spin algebra: Example 2.2.5 continued)
For the particular case of the Jordan spin algebra $\mathcal{S}_n$, we denote the set of algebraically independent elements by $\{\tau, \tau_1, \ldots, \tau_N\}$. We let $x := \tau_1 \bar{b}_1 + \cdots + \tau_N \bar{b}_N$ and $\bar{x} := \tau \bar{b}_0 + x$. The generic trace is $2\sigma(\bar{x}) = 2\tau$, the determinant is $\mu(\bar{x}, \bar{x}) = \tau^2 - x^T x$, and the eigenvalues are:

$$\{\lambda_1(\bar{x}), \lambda_2(\bar{x})\} = \{\tau + \sqrt{x^T x}, \tau - \sqrt{x^T x}\}.$$

■

2.4 Differential calculus in strictly power-associative algebras

This section develops an algebraic differential calculus for *strictly power-associative* algebras. Needless to say, differential calculus has been proved to be extremely fertile in Algebra. In the particular context of Jordan algebras, many important identities are best described as differential relations.

The differential calculus we present here is defined by means of generic elements and dual number rings. The advantage of this formalization is that the practical computations are very easy to perform. We prove in Corollary 2.4.3 that this algebraic differential calculus is equivalent to the standard differential calculus, defined in normed algebras via differential quotients.

Wherever the term *algebra* is used without modifiers in this section, it is understood as *strictly power-associative algebra of finite dimension over an infinite field F.*

Let $\mathcal{J}$ and $\mathcal{J}'$ be two algebras of dimension N and M respectively. We denote their respective unit elements as e and e'. We fix a basis $\mathcal{B} := \{b_1, \ldots, b_N\}$ for $\mathcal{J}$ and a basis $\mathcal{B}' := \{b'_1, \ldots, b'_M\}$ for $\mathcal{J}'$. Let $x := \tau_1 b_1 + \cdots + \tau_N b_N$ and $y := \rho_1 b_1 + \cdots + \rho_N b_N$ be two generic elements of $\mathcal{J}$ that are generically independent. We denote by F_x and $F_{x,y}$ respectively the two fields of rational functions $F(\tau_1, \ldots, \tau_N)$ and $F(\tau_1, \ldots, \tau_N, \rho_1, \ldots, \rho_N)$.

The objects we aim to differentiate are those of $\mathcal{J}'_x := \mathcal{J}' \bigotimes_F F_x$, that is, functions of the form:

$$f(\tau_1, \ldots, \tau_N) = \sum_{i=1}^{M} f_i(\tau_1, \ldots, \tau_N) b'_i, \tag{2.10}$$

where the $f_i = p_i/q_i$ are rational functions with coefficients in F.

Following Definition 2.3.37, we denote by ε the element $(0, 1)$ of the dual numbers ring F_ε. As established in the proof of Proposition 2.3.39, the element $x+\varepsilon y$ is a generic element of $\mathcal{J}$. So, we can replace each τ_j in (2.10) by $\tau_j+\varepsilon\rho_j$. By the specific rules of multiplication in the field $F(\tau_1 + \varepsilon\rho_1, \ldots, \tau_N + \varepsilon\rho_N)$, one can make the following decomposition:

$$f_i(x + \varepsilon y) = \frac{p_i(x) + \varepsilon\hat{p}_i(x, y)}{q_i(x) + \varepsilon\hat{q}_i(x, y)} = \frac{p_i(x)q_i(x) + \varepsilon[\hat{p}_i(x, y)q_i(x) - p_i(x)\hat{q}_i(x, y)]}{q_i(x)^2}, \qquad (2.11)$$

where $\hat{p}_i$ and $\hat{q}_i$ are polynomials in $\tau_1, \ldots, \tau_N, \rho_1, \ldots, \rho_N$ with coefficients in F – with the standard abuse of notation, we have written them as functions of x and y to shorten the expressions. Observe that $\hat{p}_i$ and $\hat{q}_i$ are linear in $\rho_1, \ldots, \rho_N$ in view of the multiplication rules in F_ε. Hence, one can write:

$$f(x + \varepsilon y) = f(x) + \varepsilon p(x, y)$$

for a rational function $p \in \mathcal{J} \bigotimes_F F_{x,y}$. Note that $p(x, y)$ is linear in y. Note also that the domain of $f(x + \varepsilon u)$ for each $u \in \mathcal{J}$ is the same as the domain of $f(x)$.

Definition 2.4.1 *We define the* differential of f in x in the direction y *as* $\nabla^y_x f(x) := p(x, y)$. ■

This operator possesses all the expected properties of a derivation operator.

Proposition 2.4.2 *Let $f, g \in \mathcal{J}'_x$ be two rational functions. The following identities hold.*

1. $\nabla^y_x e = 0$ *and* $\nabla^y_x x = y$.
2. $\nabla^y_x f(x)$ *is linear with respect to* y.
3. $\nabla^y_x (f(x) + g(x)) = \nabla^y_x f(x) + \nabla^y_x g(x)$.
4. $\nabla^y_x (f(x)g(x)) = f(x)[\nabla^y_x g(x)] + [\nabla^y_x f(x)]g(x)$.
5. $\nabla^y_x f(g(x)) = \nabla^w_{g(x)} f(g(x))$, *where* $w = \nabla^y_x g(x)$.
6. $\nabla^y_x (1/f(x)) = -L(f(x))^{-1}L(1/f(x))\nabla^y_x f(x)$.

Proof
The first item is trivial from the definition, as we have $(x+\varepsilon y)^0 = e+\varepsilon 0$. The second item has already been mentioned above. The third item is immediate. The fourth one is easily proved with the properties of the multiplication in F_ε:

$$\begin{aligned} f(x + \varepsilon y)g(x + \varepsilon y) &= (f(x) + \varepsilon\nabla^y_x f(x))(g(x) + \varepsilon\nabla^y_x g(x)) \\ &= f(x)g(x) + \varepsilon(f(x)\nabla^y_x g(x) + [\nabla^y_x f(x)]g(x)); \\ f(x + \varepsilon y)g(x + \varepsilon y) &= f(x)g(x) + \varepsilon\nabla^y_x (f(x)g(x)). \end{aligned}$$

Next, we have:

$$\begin{aligned} f(g(x + \varepsilon y)) &= f(g(x) + \varepsilon\nabla^y_x g(x)) = f(g(x)) + \varepsilon\nabla^w_{g(x)} f(g(x)); \\ f(g(x + \varepsilon y)) &= f(g(x)) + \varepsilon\nabla^y_x f(g(x)), \end{aligned}$$

where $w = \nabla_x^y g(x)$. And finally:

$$[f(x) + \varepsilon\nabla_x^y f(x)][f(x)^{-1} - \varepsilon L(f(x))^{-1} L(f(x)^{-1})\nabla_x^y f(x)]$$
$$= e + \varepsilon[L(f(x)^{-1})\nabla_x^y f(x) - L(f(x)^{-1})\nabla_x^y f(x)] = e.$$

■

As a corollary of this proposition, we check that our definition coincides with the usual notion of directional differential for rational functions in normed spaces.

Corollary 2.4.3 *Suppose that the algebras $\mathcal{J}$ and $\mathcal{J}'$ are normed algebras. Let f be a rational function from $A \subseteq \mathcal{J}$ to $\mathcal{J}'$, where the set A has a non-empty interior. This function f naturally induces a function in $\mathcal{J}_x' = \mathcal{J}' \bigotimes_F F_x$ that we also denote by f. We fix $h \in \mathcal{J}$ and $u \in \operatorname{int} A$. Then:*

$$\lim_{\delta\downarrow 0,\, \delta\in F} \frac{f(u+\delta h) - f(u)}{\delta} = \nabla_x^y f(x)|_{x\to u, y\to h}.$$

Note that the arrows $x \to u$ and $y \to h$ should be interpreted here as specializations.

Proof

The statement is obviously true for $f(x) := x$ and $f(x) := e$. The operations studied in the previous proposition – multiplication, sum, and division – allow us to build every rational function of x. It suffices then to observe that the left-hand side type of directional differential (with differential quotients) behaves identically to the right-hand side type of definition on these three operations. ■

We have mentioned above that $\nabla_x^y f(x)$ is a rational function that is linear in y, for every rational function f of $\mathcal{J}_x'$. Consequently, the operator:

$$\begin{aligned} \nabla_x f(x) : \mathcal{J} &\to \mathcal{J}_x' \\ u &\mapsto [\nabla_x f(x)]u := \nabla_x^y f(x)|_{y\to u} \end{aligned}$$

is linear.

Definition 2.4.4 *We call the operator $\nabla_x f(x)$ the* differential *of f; we can specialize it for every element in the domain of f to get a linear operator from $\mathcal{J}$ to $\mathcal{J}$.* ■

Remark 2.4.5 *Proposition 2.4.2 allows us to write:*

$$\nabla_x x^{k+1} = L(x^k) + L(x)L(x^{k-1}) + L(x)^2 L(x^{k-2}) + \cdots + L(x)^{k-1}L(x^1) + L(x)^k$$

for every $k \geq 0$. This formula can be easily shown by applying recursively the product differentiation formula. ■

2.5 The quadratic operator

At the beginning, there was the $j(x) = -x^{-1}$,
And Koecher said "let there be Q",
And there was Q,
And the Q was $Q_x = (\nabla_x j(x))^{-1}$.

From K. McCrimmon [McC03], p. 525.

2.5.1 Definition and first properties

In this section, we are interested in a *Jordan F-algebra* $\mathcal{J}$, as defined on p. 33. As usual, we assume that F is an infinite field. Recall that this algebra is strictly power-associative in view of Proposition 2.2.13; thus, most of the conclusions of the two last sections hold for $\mathcal{J}$.

As an unexpected guest, we encounter in the routine computation of the differential of some function the true king that governs *all the theory of Jordan algebras*: the *quadratic operator*. This subsection studies some of its elementary properties.

Definition 2.5.1 *Let x be a generic element of $\mathcal{J}$. We define the* quadratic operator *of $\mathcal{J}$ as $Q_x := 2L(x)^2 - L(x^2)$.* ■

Note that this operator is homogeneous of degree 2 in x and that the specialization $x \to e$ gives $Q_e = I_r$, the identity operator on $\mathcal{J}$.

At first sight, this definition may seem a bit clumsy. But this is only because it is written in terms of the multiplication operator, which, as we will see on many further occasions, is not always the most appropriate investigation tool in Jordan algebras. As a matter of fact, it is possible to define a Jordan algebra only in terms of its quadratic operator. One can show that Jordan's Axiom can be rewritten as $Q_{Q_x y} = Q_x Q_y Q_x$ for two generic elements x and y of $\mathcal{J}$ (see the proof of Theorem IV.3 in [Koe99]; it is essentially based on the tedious polarization of the above identity). Moreover, it is possible to recover the multiplication operator from the quadratic operator by means of the following expression:

$$L(x) := \frac{1}{2}(Q_{x+e} - Q_x - I_N). \tag{2.12}$$

The following important theorem is proved in Propositions II.3.1 and II.3.3 of [FK94], or in Section IV.2 of [BK66].

Theorem 2.5.2 *Let x and y be two generic elements of a Jordan F-algebra. We assume that they are generically independent.*

1. *The application Q_x is invertible and $Q_x^{-1} = Q_{x^{-1}}$.*
2. $\nabla_x x^{-1} = -Q_x^{-1}$.
3. $(Q_x y)^{-1} = Q_x^{-1} y^{-1}$.
4. $Q_{Q_x y} = Q_x Q_y Q_x$.

Sketch of the proof
Recall that x^{-1} can be written as a polynomial of x with rational coefficients in view of Remark 2.3.33. Hence $L(x)$, $L(x^2)$ and $L(x^{-1})$ commute (see Proposition 2.2.13). It remains to use the expression (2.5) of Jordan's Axiom, first with $u := x$, $v := x^{-1}$ and $w := x$, and next with $u := x$, $v := x^{-2}$ and $w := x$, to show the first item. As side result, we get:

$$L(x^{-1})Q_x = L(x). \tag{2.13}$$

For the second item, we use the relation $0 = \nabla_x e = \nabla_x(xx^{-1}) = L(x)\nabla_x x^{-1} + L(x^{-1})$ (see Proposition 2.4.2) and (2.13) with x^{-1} instead of x. In order to show the third point, observe first that $\nabla_x^{y^{-1}}\big(x^{-1}Q_x y\big) = e$ in view of (2.13). We can also prove that $\nabla_x^{y^{-1}} Q_x y = 2x$ and $\nabla_x^{y^{-1}} x^{-1} = -Q_x^{-1}y^{-1}$. Now, the desired relation follows from the product differentiation formula. Finally, the fourth point comes from the differentiation in y of $(Q_x y)^{-1} = Q_x^{-1}y^{-1}$. ■

The identity

$$Q_{Q_x y} = Q_x Q_y Q_x \tag{2.14}$$

is called the *Fundamental Identity* by many authors.

Corollary 2.5.3 *Let x be a generic element of $\mathcal{J}$. For every integer n, we have $Q_x^n = Q_{x^n}$.*

Proof
By the first item of the previous theorem, it suffices to show the statement for all $n > 0$. For $n = 1$, this is trivial. Suppose that n is even, say $n = 2k$, and that the statement is true for the integer k. Since x^k is a generic element (see Proposition 2.3.22), we can restate the Fundamental Identity as:

$$Q_{Q_x^k y} = Q_{x^k} Q_y Q_{x^k}. \tag{2.15}$$

Applying the specialization $y \to e$ to this formula gives:

$$Q_{x^n} = Q_{Q_x^k e} = Q_{x^k} Q_{x^k} = Q_x^k Q_x^k = Q_x^n,$$

because $Q_{x^k} e = x^{2k}$.

If n is odd, say $n = 2k+1$, and if the statement is true for k, we can rewrite (2.15) by replacing y by x:

$$Q_{x^n} = Q_{Q_x^k x} = Q_{x^k} Q_x Q_{x^k} = Q_x^k Q_x Q_x^k = Q_x^n,$$

and the recurrence is completed. ■

2.5.2 Quadratic operator and determinant

This subsection shows a crucial identity involving the quadratic operator and the determinant in a Jordan algebra. Braun and Koecher dedicate almost the entire third chapter of [BK66] to the study of this relation, presenting many consequences and generalizations. There is no similar property for the operator $L(x)$. This points out the practical importance of the quadratic operator as compared with the linear multiplication operator. This statement is proved for the particular case of *simple* Jordan algebras in Proposition III.4.2 of [FK94] with a different argument. Our proof is inspired by some considerations of Braun and Koecher (see Section II.5 in [BK66]); this reasoning also appears in [Spr73], although in a different context.

Proposition 2.5.4 *Let x, y be two generic elements of $\mathcal{J}$ that are generically independent. Then* $\det(Q_x y) = \det(x^2)\det(y)$.

Proof
We know from Remark 2.3.33 that there exists a polynomial q of degree $r-1$ satisfying $y^{-1} = q(y)/\det(y)$, namely the adjoint polynomial of y. Hence,

$$Q_x^{-1}y^{-1} = Q_x^{-1}\frac{q(y)}{\det(y)} \quad \text{and} \quad (Q_x y)^{-1} = \frac{q(Q_x y)}{\det(Q_x y)},$$

$$\text{or} \quad q(y)\frac{\det(Q_x y)}{\det(y)} = Q_x q(Q_x y).$$

The right-hand side term is a polynomial in y. In view of Remark 2.3.33, this implies that $\det(y)$, the F-valued polynomial in the components of y, divides $\det(Q_x y)$, the F-valued polynomial in the components of y. These two polynomials have the same degree r, thus there exist an F-valued function $\rho(x)$ such that $\det(Q_x y) = \rho(x)\det(y)$. The specialization $y \to e$ finally gives us $\det(Q_x e) = \det(x^2) = \rho(x)$. ■

2.5.3 Polarization of the quadratic operator

As its name indicates, the *polarization* of the quadratic operator results from the polarization $Q_{x+\tau y}$ of the operator Q. We start by giving a formal definition of this new object.

Definition 2.5.5 *Let x, y be two generic elements of $\mathcal{J}$ that are generically independent. The operator*

$$Q_{x,y} := \frac{1}{2}(Q_{x+y} - Q_x - Q_y)$$

is the polarization of the quadratic operator. ■

Some authors call the expression "$Q_{u,v}w$" the *Jordan triple product*, and they denote it as $\{u, w, v\}$.

We observe that:

$$Q_{x,y} = L(x)L(y) + L(y)L(x) - L(xy).$$

Hence, $Q_{x,y}$ is linear in each of its arguments. Furthermore, $Q_x = Q_{x,x}$ and $Q_{e,x} = L(x)$.

It is easy to deduce from the definition a Fundamental Identity for the polarized quadratic operator:

$$2Q_{Q_x y, Q_x z} = Q_{Q_x y+z} - Q_{Q_x y} - Q_{Q_x z} = Q_x(Q_{y+z} - Q_y - Q_z)Q_x = 2Q_x Q_{y,z} Q_x. \quad (2.16)$$

We provide here an alternative description of this operator for the case when one of its arguments is a square.

Proposition 2.5.6 *Let x and y be two generic elements that are generically independent. We put $z := x^2$. Then*

$$Q_{y,z} = Q_x L(Q_{x^{-1}} y) Q_x.$$

Proof
It suffices to use (2.16) with y replaced by e and z replaced by $Q_{z^{-1}}y = Q_z^{-1}y$. ■

2.5.4 Examples

Example 2.5.1 (Real symmetric matrices: Example 2.2.1 continued)
In the framework of real symmetric matrices, the quadratic operator and its polarization take the following form:

$$Q_{U,V} = \frac{U \otimes V + V \otimes U}{2}, \quad \text{and} \quad Q_U = U \otimes U,$$

so that $Q_{U,V}W = (U \cdot W \cdot V + V \cdot W \cdot U)/2$ and $Q_U W = U \cdot W \cdot U$.

The Fundamental Identity is equivalent to the following relation:

$$Q_{Q_U V} W = (U \cdot V \cdot U) \cdot W \cdot (U \cdot V \cdot U) = U \cdot (V \cdot (U \cdot W \cdot U) \cdot V) \cdot U = Q_U Q_V Q_U W.$$

Finally, Proposition 2.5.4 reduces to the well-known formula

$$\mathrm{Det}(U \cdot V \cdot U) = \mathrm{Det}(U^2)\mathrm{Det}(V).$$

■

The same interpretations hold for Jordan algebras built from associative algebras with or without involution (see Example 2.2.2 and Example 2.2.3).

Example 2.5.2 (Jordan algebra from a symmetric bilinear form)
In the context of Jordan algebras of the type $[X; \mu; e]$, the polarized quadratic operator has the following form:

$$Q_{u,v}w = \sigma(v \circ w)u + \sigma(u \circ w)v + \mu(u, v)(w - 2\sigma(w)e).$$

The quadratic operator is thus:

$$Q_u w = 2\sigma(u \circ w)u + \mu(u, u)(w - 2\sigma(w)e).$$

The Fundamental Identity and Proposition 2.5.4 can be directly deduced from the latter formula with lengthy but trivial computations. ■

Example 2.5.3 (Jordan spin algebra: Example 2.2.5 continued)
The polarization of the quadratic operator can be written as:

$$Q_{\bar{u},\bar{v}} = \begin{pmatrix} u_0 v_0 + u^T v & u_0 v^T + v_0 u^T \\ u_0 v + v_0 u & u v^T + v u^T \end{pmatrix} + (u_0 v_0 - u^T v) \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix}.$$

■

2.6 Pierce decompositions

2.6.1 An illustrative example

We present in this section two important decomposition theorems for Jordan algebras, namely the *Pierce decomposition theorems.* They are included amongst the most widely used tools in algebra, and they play a crucial role in our thesis.

In order to understand their nature, let us first describe these decompositions in the framework of real symmetric matrices

Example 2.6.1 (Real symmetric matrices: Example 2.2.1 continued)
Let C be a non-zero projector of rank k. We fix an orthogonal basis of $\mathbb{R}^r$ for which the matrix representation of C has the form

$$C = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}.$$

The first Pierce decomposition of $\mathcal{J} := \mathcal{H}_r(\mathbb{R})$ with respect to the idempotent C consists in splitting this vector space into the three eigenspaces of the operator $L(C)$.

Let us compute these eigenspaces. We partition a symmetric matrix $U \in \mathcal{J}$, represented in this basis, into:

$$U = \begin{pmatrix} U_{11} & U_{21}^T \\ U_{21} & U_{22} \end{pmatrix},$$

where U_{11} is the $k \times k$ upper-left submatrix of U. Note that:

$$\begin{aligned} C \circ U &= \frac{1}{2}\left[\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} U_{11} & U_{21}^T \\ U_{21} & U_{22} \end{pmatrix} + \begin{pmatrix} U_{11} & U_{21}^T \\ U_{21} & U_{22} \end{pmatrix} \cdot \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 2U_{11} & U_{21}^T \\ U_{21} & 0 \end{pmatrix} \\ &= 1 \cdot \begin{pmatrix} U_{11} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 0 & U_{21}^T \\ U_{21} & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 0 & U_{22} \end{pmatrix}. \end{aligned}$$

We denote the eigenspace of $L(C)$ corresponding to the eigenvalue $1, 1/2$, and 0 as $\mathcal{J}_1(C)$, $\mathcal{J}_{1/2}(C)$, and $\mathcal{J}_0(C)$. The components of U in each of these subspaces are respectively:

$$\begin{pmatrix} U_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & U_{21}^T \\ U_{21} & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & U_{22} \end{pmatrix}.$$

It can be shown that $\mathcal{J}_1(C) = Q_C\mathcal{J}$, $\mathcal{J}_{1/2}(C) = Q_{I_r - C, C}\mathcal{J}$, and $\mathcal{J}_0(C) = Q_{I_r - C}\mathcal{J}$. Observe that some multiplication rules occur between these subspaces. For instance, for every $U, V \in \mathcal{J}_1(C)$, we have $U \circ V \in \mathcal{J}_1(C)$. The first Pierce decomposition theorem contains a full description of these very important rules.

Let us turn now our attention to the second Pierce decomposition theorem, which generalizes the first one to *systems* of idempotents.

We fix an orthogonal basis $\{u_1, \ldots, u_r\}$ of $\mathbb{R}^r$ and let $U_i := u_i u_i^T$. Next, we choose some integers k_j such that:

$$0 = k_0 < k_1 < k_2 < \cdots < k_n = r.$$

Letting $M_j := \{k_{j-1} + 1, \ldots, k_j\}$ and $C_j := \sum_{i \in M_j} U_i$, we have formed a *system of idempotents* $\{C_1, \ldots, C_n\}$. This means that the projectors C_i sum up to I_r, and that $C_i \circ C_j = 0$ if $i \neq j$. For the sake of notational simplicity, we assume that $n = 3$ and that the matrices are represented in the chosen basis, so that:

$$C_1 = \begin{pmatrix} I_{d_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{d_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{d_3} \end{pmatrix},$$

where $d_1 := k_1$, $d_2 := k_2 - k_1$, and $d_3 := r - k_2$. Let us now take:

$$V = \begin{pmatrix} V_{11} & V_{21}^T & V_{31}^T \\ V_{21} & V_{22} & V_{32}^T \\ V_{31} & V_{32} & V_{33} \end{pmatrix},$$

where the blocks have the same size as in the decomposition of C_1, C_2, C_3 above. Now, we define $\mathcal{J}_{ij} := Q_{C_i,C_j}\mathcal{J}$, so that the projection of V on $\mathcal{J}_{11}$ is

$$\begin{pmatrix} V_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the projection of V on $\mathcal{J}_{12}$ is

$$\begin{pmatrix} 0 & V_{21}^T & 0 \\ V_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Again, there are multiplication rules between these Pierce subspaces that can easily been deduced. For instance, if $U \in \mathcal{J}_{12}$ and $V \in \mathcal{J}_{22}$, we have $U \circ V \in \mathcal{J}_{12}$. The second Pierce decomposition theorem provides us with a complete description of these rules. ■

2.6.2 Pierce decomposition theorems and first consequences

As in the previous section, we assume that $\mathcal{J}$ is a *Jordan algebra of dimension $N < +\infty$ over the infinite field F*. We denote its unit element by e.

Let us consider an idempotent c of $\mathcal{J}$. Given a generic element x of $\mathcal{J}$, we recall that the multiplication operator $L(x)$ can be expressed in terms of the quadratic operator by:

$$L(x) = \frac{1}{2}(Q_{e+x} - Q_x - I_N).$$

Specializing $x \to -c$ in this identity allows us to write:

$$L(-c) = \frac{1}{2}(Q_{e-c} - Q_c - I_N), \quad \text{or}$$

$$L(c) = \frac{1}{2}(Q_c + I_N - Q_{e-c}) = Q_c + \frac{1}{2}\cdot(2Q_{c,e-c}) + 0\cdot Q_{e-c}, \tag{2.17}$$

because $2Q_{c,e-c} = Q_e - Q_c - Q_{e-c} = I_N - Q_c - Q_{e-c}$. The first Pierce decomposition theorem interprets this relation as a spectral decomposition of the linear operator $L(c)$.

In the statement of the two next theorems the notation $A \circ B$ refers to the set $\{uv | u \in A, v \in B\}$ when the subsets A and B belongs to $\mathcal{J}$. The expression $E = E_1 \oplus E_2$ involving vector spaces E, E_1 and E_2 means that E is the direct sum of E_1 and E_2. A proof of the following result can be found in [Koe99], Theorem III.8. The techniques used in the demonstrations are quite standard in our work, and we include a proof below.

Theorem 2.6.1 (First Pierce decomposition theorem) *Let c be an idempotent of $\mathcal{J}$. We define $\mathcal{J}_1(c) := Q_c\mathcal{J}$, $\mathcal{J}_{1/2}(c) := (I_N - Q_c - Q_{e-c})\mathcal{J} = 2Q_{c,e-c}\mathcal{J}$ and $\mathcal{J}_0(c) := Q_{e-c}\mathcal{J}$. Then:*

1. *$\mathcal{J} = \mathcal{J}_1(c) \oplus \mathcal{J}_{1/2}(c) \oplus \mathcal{J}_0(c)$;*
2. *$\mathcal{J}_\gamma(c) = \{u \in \mathcal{J} | L(c)u = \gamma u\}$ for $\gamma = 1, 1/2, 0$;*
3. *$L(u)$ and $L(c)$ commute if and only if $u \in \mathcal{J}_0(c) \oplus \mathcal{J}_1(c)$;*

4. *$\mathcal{J}_1(c)$ and $\mathcal{J}_0(c)$ are subalgebras of $\mathcal{J}$ and $\mathcal{J}_0(c) \circ \mathcal{J}_1(c) = \{0\}$;*

5. *$\mathcal{J}_{1/2}(c) \circ (\mathcal{J}_0(c) \oplus \mathcal{J}_1(c)) \subseteq \mathcal{J}_{1/2}(c)$;*

6. *$\mathcal{J}_{1/2}(c) \circ \mathcal{J}_{1/2}(c) \subseteq \mathcal{J}_0(c) \oplus \mathcal{J}_1(c)$;*

7. *if $u \in \mathcal{J}_{1/2}(c)$, then $\operatorname{tr}(u) = 0$.*

Proof

By writing the expression (2.5) of Jordan's Axiom with $u = v = w := c$, we get $2L(c)^3 - 3L(c)^2 + L(c) = 0$, or $L(c)Q_c = Q_c$. Since $Q_c^2 = Q_{c^2} = Q_c$ in view of Corollary 2.5.3, we can easily check that the applications $Q^{(1)} := Q_c$, $Q^{(1/2)} := 2Q_{c,e-c}$ and $Q^{(0)} := Q_{e-c}$ satisfy $Q^{(\gamma)}Q^{(\gamma')} = \delta_{\gamma\gamma'}Q^{(\gamma)}$ for $\gamma, \gamma' \in \{0, 1/2, 1\}$, where δ is the Kronecker symbol. In other words, the operators $Q^{(\gamma)}$ are projectors that are orthogonal with respect to each other. It follows that $Q^{(\gamma)}u = u$ when $u \in \mathcal{J}_\gamma(c)$. Since:

$$I_N = Q_c + (I_N - Q_c - Q_{e-c}) + Q_{e-c} = Q^{(1)} + Q^{(1/2)} + Q^{(0)},$$

the first point is settled. We can then decompose every $u \in \mathcal{J}$ into $u = u_1 + u_{1/2} + u_0$, where $u_\gamma \in \mathcal{J}_\gamma(c)$. Note that, by (2.17):

$$cu = Q^{(1)}u + \frac{1}{2}Q^{(1/2)}u + 0 \cdot Q^{(0)}u = u_1 + \frac{u_{1/2}}{2} + 0 \cdot u_0.$$

Hence, $u \in \mathcal{J}_\gamma(c) \Leftrightarrow u = u_\gamma = Q^{(\gamma)}u \Leftrightarrow cu = \gamma u$ and the second point is proved. Let us review the remaining items.

3. Suppose that c and u operator commute. Since:

$$2[L(c); L(u)]c = 2c(cu) - cu - cu = Q_c u - L(c)u,$$

we deduce that $L(c)u = Q_c u$. As $L(c)u = (I_N + Q_c - Q_{e-c})\, u/2 = Q_c u$, we have $u = Q_c u + Q_{e-c}u$, and $u \in \mathcal{J}_0(c) \oplus \mathcal{J}_1(c)$.

We prove now the reverse implication. Let $u \in \mathcal{J}$. The operator equality (2.4) applied to $u := u$ and $v = w := c$ can be written in the following form:

$$2[L(c); L(cu)] + [L(c); L(u)] = 0.$$

Suppose that $u \in \mathcal{J}_0(c)$. Then $L(cu) = L(0) = 0$ and $[L(c); L(u)] = 0$.
If $u \in \mathcal{J}_1(c)$, then $L(cu) = L(u)$ and, again, we have $[L(c); L(u)] = 0$. The statement immediately results by virtue of the linearity of L.

4. Let $u \in \mathcal{J}_1(c)$. Since u and c operator commute, we have:

$$cu^2 = L(c)L(u)u = L(u)L(c)u = u(cu) = u^2.$$

According to item 3, $u^2 \in \mathcal{J}_1(c)$, which implies that $\mathcal{J}_1(c)$ is a subalgebra. The proof is analogous for $\mathcal{J}_0(c)$. Now, let $v \in \mathcal{J}_0(c)$. As u and v operator commute with c, we have:

$$\begin{aligned} 0 &= u(cv) = L(u)L(c)v = L(c)L(u)v \\ &= L(c)L(v)u = L(v)L(c)u = v(cu) = vu. \end{aligned}$$

5. Let $u \in \mathcal{J}_0(c) \oplus \mathcal{J}_1(c)$ and $v \in \mathcal{J}_{1/2}(c)$. Since u and c operator commute, we can write:
$$c(uv) = L(c)L(u)v = L(u)L(c)v = u(cv) = \frac{uv}{2},$$
and $uv \in \mathcal{J}_{1/2}(c)$.

6. Let $u, v \in \mathcal{J}_{1/2}$. Using again expression (2.4), we obtain:
$$\begin{aligned} 0 &= [L(c); L(uv)] + [L(u); L(cv)] + [L(v); L(cu)] \\ &= [L(c); L(uv)] + \frac{[L(u); L(v)] + [L(v); L(u)]}{2} \quad = [L(c); L(uv)]. \end{aligned}$$
Thus c and uv operator commute and $uv \in \mathcal{J}_0(c) \oplus \mathcal{J}_1(c)$ in view of item 3.

7. Suppose that $u \in \mathcal{J}_{1/2}(c)$, i.e. $u = 2Q_{c,e-c}u$. By associativity of the trace (see Corollary 2.3.40), we can write:
$$\begin{aligned} \mathrm{tr}[2Q_{c,e-c}u] &= 2\mathrm{tr}[c((e-c)u)] + 2\mathrm{tr}[(e-c)(cu)] \\ &= 2\mathrm{tr}[(c(e-c))v] + 2\mathrm{tr}[((e-c)c)v] = 0. \end{aligned}$$

■

Definition 2.6.2 *Let c be an idempotent of $\mathcal{J}$. The subspaces $\mathcal{J}_1(c), \mathcal{J}_{1/2}(c)$, and $\mathcal{J}_0(c)$ are called* the Pierce subspaces of $\mathcal{J}$ with respect to c. *The subspace $\mathcal{J}_1(c)$ is called the* Pierce subalgebra of unit c. *The decomposition of an element $u \in \mathcal{J}$ into $u = u_1 + u_{1/2} + u_0$, where $u_\gamma \in \mathcal{J}_\gamma(c)$, is called* the Pierce decomposition of u with respect to c. ■

The first Pierce decomposition theorem displays an important property of the quadratic operator: if c is an idempotent of $\mathcal{J}$, then Q_c is a projector on the Pierce subspace $\mathcal{J}_1(c)$. The following proposition completes this observation.

Proposition 2.6.3 *Let c be an idempotent of $\mathcal{J}$ and let h be an element of the subalgebra $\mathcal{J}_1(c)$. For every $u \in \mathcal{J}$, we can write:*
$$Q_h u = Q_h Q_c u = Q_c Q_h u,$$
and $Q_h u$ belongs to $\mathcal{J}_1(c)$.

Proof
Since $h \in \mathcal{J}_1(c)$, we have $Q_c h = h$, implying, in view of the Fundamental Identity (2.14):
$$\begin{aligned} Q_c Q_h Q_c^2 u &= (Q_c Q_h Q_c)(Q_c u) = Q_{Q_c h} Q_c u = Q_h Q_c u; \\ Q_c Q_h Q_c^2 u &= (Q_c Q_h Q_c) u = Q_{Q_c h} u = Q_h u. \end{aligned}$$
Using the relations above, we finally conclude $Q_h u = Q_c(Q_h Q_c u) = Q_c Q_h u$. ■

As the following remark states, the Pierce decomposition also holds if we replace the field F of the algebra $\mathcal{J}$ by one of its extension rings. This observation will play a crucial role in the proof of the spectral decomposition Theorem.

Remark 2.6.4 *Let R be an extension ring of F and let $\bar{\mathcal{J}} := \mathcal{J} \bigotimes_F R$; for every subspace A of $\mathcal{J}$, we similarly denote by $\bar{A}$ the subspace $A \bigotimes_F R$. Then:*

$$\overline{\mathcal{J}_\gamma(c)} = \bar{\mathcal{J}}_\gamma(c) \quad \text{for} \quad \gamma \in \{0, 1/2, 1\}.$$

Indeed, when $u \in \mathcal{J}_\gamma(c)$, we have $u \in \bar{\mathcal{J}}$ and $cu = \gamma u$, so that $u \in \bar{\mathcal{J}}_\gamma(c)$. Remark 2.2.4 allows us to conclude that $\overline{\mathcal{J}_\gamma(c)} \subseteq \bar{\mathcal{J}}_\gamma(c)$. Conversely, an element u in the Pierce subspace $\bar{\mathcal{J}}_\gamma(c)$ can be decomposed into $u = \sum_\alpha \lambda_\alpha b_\alpha$, where $\lambda_\alpha \in R$ are linearly independent over F and $b_\alpha \in \mathcal{J}$. Observe that $\sum_\alpha \lambda_\alpha c b_\alpha = cu = \gamma u = \sum_\alpha \lambda_\alpha \gamma b_\alpha$. We deduce that $cb_\alpha = \gamma b_\alpha$. Hence, $b_\alpha \in \mathcal{J}_\gamma(c)$, and $u \in \overline{\mathcal{J}_\gamma(c)}$. The reverse inclusion is hereby proved. ■

The second Pierce decomposition Theorem refines the first one in the sense that it splits the algebra $\mathcal{J}$ into more pieces than just three. Moreover, it gives a precise description of the behavior of the multiplication between these pieces. This information will be used intensively in this work.

Definition 2.6.5 *Let $\mathcal{J}$ be a unitary algebra and $\{c_1, \ldots, c_n\}$ be a set of idempotents of $\mathcal{J}$. We say that this set is a* system of idempotents *when $c_i c_j = \delta_{ij} c_i$ and $c_1 + \cdots + c_n = e$.* ■

In Subsection 2.3.1, we have mentioned that the set of all the minimal idempotents of an associative and commutative unitary algebra, such as $F[u]$, is a system of idempotents. In particular, such a system of idempotents exists in $\mathcal{J}$. However, note that we do not require in the definition that the idempotents c_i are minimal idempotents of some subalgebra $F[u]$ of $\mathcal{J}$.

The second Pierce decomposition Theorem that we quote below is Theorem IV.2.1 of [FK94].

Theorem 2.6.6 (Second Pierce decomposition theorem)
Let $\{c_1, \ldots, c_n\}$ be a system of idempotents of $\mathcal{J}$. We put $\mathcal{J}_{ij} := Q_{c_i,c_j}\mathcal{J}$. If $1 \leq i, j, k, l \leq n$, we have:

1. *$\mathcal{J}_{ii} = \mathcal{J}_1(c_i)$ and $\mathcal{J}_{ij} = \mathcal{J}_{1/2}(c_i) \cap \mathcal{J}_{1/2}(c_j) = \mathcal{J}_{ji}$ if $i \neq j$;*
2. *$\mathcal{J} = \bigoplus_{1 \leq i' \leq j' \leq n} \mathcal{J}_{i'j'}$;*
3. *$\mathcal{J}_{ij} \circ \mathcal{J}_{kl} = 0$, if $\{i, j\} \cap \{k, l\} = \emptyset$;*
4. *$\mathcal{J}_{ij} \circ \mathcal{J}_{jk} \subseteq \mathcal{J}_{ik}$ if i, j and k are different;*
5. *$\mathcal{J}_{ij} \circ \mathcal{J}_{ij} \subseteq \mathcal{J}_{ii} + \mathcal{J}_{jj}$;*
6. *$\mathcal{J}_{ii} \circ \mathcal{J}_{ij} \subseteq \mathcal{J}_{ij}$;*
7. *if $u \in \mathcal{J}_{ij}$ and $i \neq j$, then $\mathrm{tr}(u) = 0$.*

Sketch of the proof
This theorem is more or less an immediate consequence of the first Pierce decomposition theorem. Observe that if $1 \leq k \leq n$, we have:

$$\begin{aligned}
\mathcal{J}_1(c_k) &= Q_{c_k}\mathcal{J} = \mathcal{J}_{kk},\\
\mathcal{J}_{1/2}(c_k) &= Q_{c_k,\sum_{i\neq k} c_i}\mathcal{J} = \sum_{i\neq k} Q_{c_k,c_i}\mathcal{J} = \sum_{i\neq k}\mathcal{J}_{ki},\\
\mathcal{J}_0(c_k) &= Q_{\sum_{i\neq k} c_i,\sum_{j\neq k} c_j}\mathcal{J} = \sum_{i\neq k}\sum_{j\neq k} Q_{c_i,c_j}\mathcal{J} = \sum_{i\neq k}\sum_{j\neq k}\mathcal{J}_{ij}.
\end{aligned}$$

These equalities allow us to settle the first item. The second one immediately results from item 1 and the last one from item 7 of the first Pierce decomposition theorem. The proofs of the multiplication rules are very similar to each other. Let us show for instance that $\mathcal{J}_{ij} \circ \mathcal{J}_{kl} = 0$ if i, j, k and l are all distinct numbers. We set $c := c_i + c_j$, which is clearly an idempotent. We have:

$$\mathcal{J}_1(c) = Q_{c_i+c_j}\mathcal{J} = Q_{c_i}\mathcal{J} + 2Q_{c_i,c_j}\mathcal{J} + Q_{c_j}\mathcal{J} = \mathcal{J}_{ii} \oplus \mathcal{J}_{ij} \oplus \mathcal{J}_{jj} \supseteq \mathcal{J}_{ij}.$$

Similarly, $\mathcal{J}_{kl} \subseteq \mathcal{J}_0(c)$. The multiplication rule of item 4 of the first Pierce decomposition theorem allows us to conclude. ■

Remark 2.6.7 *Let $\{c_1, \ldots, c_n\}$ be a system of idempotents of $\mathcal{J}$. We take four integers $1 \leq i, j, k, l \leq n$, with $i \neq j$. Suppose that the element v belongs to $\mathcal{J}_{kl} := Q_{c_k,c_l}\mathcal{J}$. We have:*

1. $Q_{c_i}v = \delta_{ik}\delta_{il}v$;
2. $Q_{c_i,c_j}v = 0$ *when* $k = l$;
3. $Q_{c_i,c_j}v = v/2$ *when* $\{i, j\} = \{k, l\}$;
4. $Q_{c_i,c_j}v = 0$ *otherwise.*

The verifications of these formulas appear to be quite easy. For the first item, the case $i = k = l$ is trivial. When $i \neq k$, we have $Q_{c_i}\mathcal{J}_{kl} \subseteq Q_{c_i}(\mathcal{J}_{1/2}(c_i) + \mathcal{J}_0(c_i)) = \{0\}$.
Next, as Q_{c_i,c_j} and Q_{c_k} commute, we have $Q_{c_i,c_j}Q_{c_k}\mathcal{J} = Q_{c_k}Q_{c_i,c_j}\mathcal{J} = \{0\}$ by the first item.
Recalling that $2Q_{c_i,c_j} = Q_{c_i+c_j} - Q_{c_i} - Q_{c_j}$, we can check the identity $Q^2_{c_i,c_j} = Q_{c_i,c_j}/2$, so that $Q_{c_i,c_j}v = v/2$ when $v \in \mathcal{J}_{ij}$.
For the last case, if $k, l \neq j$, then $v \in \mathcal{J}_{kl} \subseteq \mathcal{J}_0(c_j)$, so that $c_j v = 0$ and $Q_{c_i,c_j}v = 2L(c_i)L(c_j)v = 0$. ■

2.6.3 Further examples

Example 2.6.2 (Jordan algebra from a symmetric bilinear form)
Let u be an element of $\mathcal{J} := [X; \mu; e]$ and let $c \neq e$ be an idempotent of this algebra. We already know that $\sigma(c) = 1/2$ and $\mu(c, c) = 0$. The elements u of $\mathcal{J}_{1/2}(c)$ satisfy:

$$\frac{u}{2} = c \circ u = \frac{u}{2} + \sigma(u)c - \mu(c, u)e, \quad \text{thus} \quad \mathcal{J}_{1/2}(c) = \{u \in \mathcal{J} | \sigma(u) = \mu(c, u) = 0\}.$$

Further, it is not difficult to show that $\mathcal{J}_1(c) = Fc$ and $\mathcal{J}_0(c) = F(e-c)$. ■

Example 2.6.3 (Jordan spin algebra: Example 2.2.5 continued)
Let $u \in \mathbb{R}^n$ be a vector whose Euclidean norm equals 1 and let $\bar{c} := \frac{1}{2}\begin{pmatrix} 1 \\ u \end{pmatrix}$ be an idempotent of $\mathcal{J}$. Particularizing the above example, it is easy to determine the Pierce subspaces corresponding to $\bar{c}$:

$$\mathcal{J}_1(\bar{c}) = \left\{ \alpha \begin{pmatrix} 1 \\ u \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}, \quad \mathcal{J}_{1/2}(\bar{c}) = \left\{ \begin{pmatrix} 0 \\ v \end{pmatrix} \middle| u^T v = 0 \right\}$$
$$\text{and} \quad \mathcal{J}_0(\bar{c}) = \left\{ \alpha \begin{pmatrix} 1 \\ -u \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}.$$

■

2.7 Formally real Jordan algebras and spectral decomposition

The spectral decomposition theorems we study in this section generalize in a natural way well-known results in Linear Algebra. Theorem 2.7.2 below generalizes the *Jordan factorization* of a real square matrix[2] (see Chapter 3 of [HJ96]). Theorems 2.7.13 and 2.7.25 extend the classical spectral theorems for Hermitian matrices (see Chapter 4 of [HJ96]). These results, often used in combination with the Pierce decomposition theorems, will allow us to derive many nice features that have made formally real Jordan algebras an unavoidable framework of investigation in such different domains as conic optimization, statistics, or non-smooth analysis.

2.7.1 Spectral decomposition in power-associative algebras

Just like we did in Section 2.3, we start by considering a more general situation than the case we want to study. The first spectral decomposition result we display is not very refined because our first framework is pretty general.

Unless explicitly stated, we use in this subsection an algebra $\mathcal{J}$ of dimension $N < +\infty$ over the infinite field F. We assume that $\mathcal{J}$ is *strictly power-associative* and we denote its unit element by e. We write r for the degree of its characteristic polynomial. In addition, we assume the following.

Hypothesis 2.7.1 *The field F is sufficiently large to contain all the roots of the minimal polynomial μ_u for every $u \in \mathcal{J}$.* ■

It implies that $\mu_u(t)$ can be decomposed into a product of linear factors in $F[t]$. This hypothesis is relatively restrictive, although it does not necessarily imply that F is algebraically closed, as shown by the following lemma. The proof is an adaptation of Proposition VIII.4.1 of [FK94]. We denote by $\Re(x)$ and $\Im(x)$ the real and imaginary part of a complex vector x, and its conjugate vector by $\bar{x}$.

[2] The Jordan factorization for matrices is named after the French mathematician Camille Jordan, while Jordan algebras where created by the German Pascual Jordan.

Lemma 2.7.1 *Let $\mathcal{J}$ be a formally real power-associative algebra over the field $\mathbb{R}$. The roots of the minimal polynomial μ_u are real for every $u \in \mathcal{J}$.*

Proof
Let $u \in \mathcal{J}$. By definition, all the coefficients of its minimal polynomial μ_u are real. We denote by ξ one of its (complex) roots and we take $p(t) := \mu_u(t)/(t-\xi)$. We need to show that ξ is real. With $v := p(u)$, we have $0 = \mu_u(u) = (u - \xi e)v$. Note that the (complex) vector v cannot be null. Otherwise, the real polynomials $\Re p(t)$ and $\Im p(t)$ would both vanish in u, and they would be both null because their degree is smaller than $\deg(\mu_u)$.

Since $(u-\xi e)v = 0$, we also have $\overline{(u-\xi e)v} = (u-\bar{\xi}e)\bar{v} = 0$, or:

$$uv = \xi v \quad \text{and} \quad u\bar{v} = \bar{\xi}\bar{v}.$$

As $v = p(u) \in \mathbb{C}[u]$ and $\bar{v} = \overline{p(u)} \in \mathbb{C}[u]$, which is an associative algebra, the elements v and $\bar{v}$ operator commute. Hence $v(\bar{v}u) = \bar{v}(vu)$, i.e. $\xi v\bar{v} = \bar{\xi}v\bar{v}$. If $v\bar{v} = (\Re p(u))^2 + (\Im p(u))^2 = 0$, this means that $\Re p(u) = \Im p(u) = 0$ because $\mathcal{J}$ is formally real, a contradiction. Thus $v\bar{v}$ cannot be null, implying $\xi = \bar{\xi}$. ■

Here comes the main result of this subsection, namely the spectral decomposition theorem for strictly power-associative algebras. Its proof follows loosely the observations in Section I.4 of [BK66].

Theorem 2.7.2 *Let us fix an element $u \in \mathcal{J}$. We denote by $\{c_1, \ldots, c_n\}$ the set of minimal idempotents of $F[u]$. Then μ_u has exactly n distinct roots $\xi_1, \ldots, \xi_n$. Up to a renumbering of the roots of μ_u, we have:*

$$u = \sum_{i=1}^{n} \xi_i c_i + v,$$

where v is a nilpotent element of $F[u]$. The nilpotent v is equal to 0 if and only if the elements ξ_i are all simple roots of μ_u. This decomposition is unique in the following sense: if there exist a system of idempotents $\{e_1, \ldots, e_m\} \in \mathcal{J}$, some distinct elements $\eta_1, \ldots, \eta_m$ of F and a nilpotent element v' of $F[u]$ such that $u = \sum_{j=1}^{m} \eta_j e_j + v'$, then $v' = v$, $m = n$ and, up to a renumbering, $\xi_i c_i = \eta_i e_i$.

Sketch of the proof
The proof is entirely built on results presented in Subsection 2.3.1 and on Remark 2.3.15. Hypothesis 2.7.1 on F is essential. Here are the key steps of the demonstration.

Let $u_i := c_i u$. It turns out that its reduced minimal polynomial is $g_{u_i}(t) = t - \xi_i$. From that, we deduce that the scalars ξ_i are all different roots of μ_u. Next $v_i := u_i - \xi_i c_i \in F[u]$ is nilpotent. Thus $v := \sum_{i=1}^{n} v_i = u - \sum_{i=1}^{n} \xi_i c_i$ is nilpotent too and belongs to $F[u]$. It results from the minimality of the idempotents c_i that $v = 0$, iff $v_i = 0$ for all i, iff $\mu_{u_i} = g_{u_i}$ for all i, iff the roots ξ_i are all simple.

Uniqueness can be proved in two steps. First, one assumes that $\{e_1, \ldots, e_m\}$ is a set of idempotents in $F[u]$. The result then follows easily from Lemma 2.3.10. And second, if we do not assume it, we can prove (as in Proposition 2.3.13) that for every polynomial $p \in F[t]$ we have $p(u - v') = \sum_{j=1}^{m} p(\eta_j) e_j$. It suffices then to take $p(t) := \prod_{j \neq k}(t - \eta_j)$ and to multiply the resulting relation by e_k to prove that, for each k, we have $e_k \in F[u]$, which is the situation that we have already covered in the first part of the paragraph. ■

This theorem has numerous consequences. We only mention here those that are used later in this work. The following corollary is given in Section I.4.3 of [BK66]. We include here its simple proof.

Corollary 2.7.3 *An element $u \in \mathcal{J}$ is invertible if and only if in the decomposition $u = \sum_{i=1}^n \xi_i c_i + v$, the scalars ξ_i are all invertible, that is, nonzero. Moreover, if $u = \sum_{i=1}^n \xi_i c_i + v$ is invertible, there exists a nilpotent $v' \in F[u]$ such that $u = \left(\sum_{i=1}^n \xi_i c_i\right)(e + v')$.*

Proof
Suppose that u is invertible. According to Remark 2.3.33, there exists a polynomial $p(t) \in F[t]$ for which $up(u) = e$. That is, the polynomial $h(t) := 1 - tp(t)$ vanishes in u, so it is a multiple of the minimal polynomial $\mu_u(t)$. Since $h(0) \neq 0$, we have $\mu_u(0) \neq 0$ and all its roots ξ_i are nonzero, i.e. they are all invertible in the field F.

Let now $u \in \mathcal{J}$ be such that all the scalars ξ_i are invertible. One can let $v' := \left(\sum_{i=1}^n c_i/\xi_i\right) v$, which is obviously a nilpotent of $F[u]$. Moreover, $u = \left(\sum_{i=1}^n \xi_i c_i\right)(e + v')$. Note that $e + v'$ has an inverse – namely $e - v' + v'^2 - \ldots + (-1)^m v'^m$, where $v'^{m+1} = 0$. Thus u is invertible and everything is shown. ■

Corollary 2.7.4 *Suppose that $u \in \mathcal{J}$ is not invertible. Then* $\det(u) = 0$.

Proof
In view of Proposition 2.3.34, the determinant of an element u is a product of roots of μ_u, and every distinct root of μ_u appears at least once in this product. If u is not invertible, one of these roots is null by the previous corollary. ■

Corollary 2.7.5 *Suppose that the minimal polynomial μ_u of a nonzero element $u \in \mathcal{J}$ has r simple roots. Then there exists a system of at least r idempotents in $F[u]$.*

Proof
The case $r = 1$ being trivial, we assume that $r > 1$. The minimal polynomial has the form $\mu_u(t) = \prod_{i=1}^r (t - \xi_i)$, where the scalars ξ_i are all distinct. We denote $h_j(t) := \prod_{i \neq j}(t - \xi_i)$. Observe that $h_j(u)$ is not a nilpotent element and that $h_j(u)u = h_j(u)\xi_j e$. By associativity in $F[u]$, we have $h_j(u)u^m = h_j(u)\xi_j^m e$ for every nonnegative integer m, and then $h_j(u)p(u) = h_j(u)p(\xi_j)e$ for every polynomial $p(t) \in F[t]$. In particular, $h_j(u)^2 = h_j(u)h_j(\xi_j)e$. As $h_j(u)$ is not nilpotent, $h_j(\xi_j) \neq 0$. We set $c_j := h_j(u)/h_j(\xi_j)$. This element is an idempotent, and $c_j c_i = 0$ when $i \neq j$ because μ_u divides $h_i h_j$. Thus $\{c_1, \ldots, c_r\}$ is a set of orthogonal idempotents. If they do not sum up to e, it suffices to add $c_{r+1} := e - c_1 - \cdots - c_r$ to this set to get the desired system of idempotents. ■

The next proposition is very useful to prove elegantly interesting relations involving the determinant and the quadratic operator. It enables us to take the square root of an invertible element u of the Jordan algebra $\mathcal{J}$ in an appropriate extension of $\mathcal{J}$. Its proof is adapted from Satz I.4.3 in [BK66].

Proposition 2.7.6 *There exists an extension field $\bar{F}$ of F such that for every invertible $u \in \mathcal{J}$, we have an element $w \in \bar{F}[u]$ for which $w^2 = u$.*

Proof
We choose $\bar{F}$ to be the algebraic closure of F. Let $u = \left(\sum_{i=1}^{n} \xi_i c_i\right)(e + v')$ be the spectral decomposition of an invertible $u \in \mathcal{J}$ given in Corollary 2.7.3. In the field $\bar{F}$, each ξ_i has a square root $\xi_i^{1/2}$. We define:

$$a := \sum_{k \geq 0} \binom{1/2}{k} v'^k.$$

Observe that this sum has only a finite number of nonzero terms, since v' is nilpotent. We have $a = (e + v')^{1/2}$ and $a \in F[u]$. Then $w := a \sum_{i=1}^{n} \xi_i^{1/2} c_i$ belongs to $\bar{F}[u]$ and satisfies $w^2 = a^2 \sum_{i=1}^{n} \xi_i c_i = (e + v')\left(\sum_{i=1}^{n} \xi_i c_i\right) = u$. ∎

2.7.2 More properties of the determinant

In this subsection, we exploit Proposition 2.7.6 to deduce a few technical results on the determinant. The context where we put ourselves in this subsection is the following. We assume that $\mathcal{J}$ is a *Jordan algebra* of dimension N over the infinite field F. As in the previous subsection, we also assume that Hypothesis 2.7.1 holds. Due to Jordan's Axiom, our hypotheses on $\mathcal{J}$ are here more restrictive than in the previous subsection. The following proposition can be found in [BK66] Satz III.2.2. Its proof essentially combines Proposition 2.7.6 with the relation $\det(Q_x y) = \det(x^2)\det(y)$ shown in Proposition 2.5.4.

Proposition 2.7.7 *Let F' be an extension field of F and $\mathcal{A} \subseteq \mathcal{J} \bigotimes_F F'$ be an associative and commutative subalgebra. Then for every $u, v \in \mathcal{A}$, we have* $\det(uv) = \det(u)\det(v)$. ∎

This proposition extends Proposition 2.3.34 on multiplicativity of det in the sense that now we can choose every associative and commutative subalgebra $\mathcal{A}$ instead of $F'[u]$. The price we pay is the extra hypothesis that $\mathcal{J}$ is a Jordan algebra.

Corollary 2.7.8 *Let F' be an extension field of F, and let $\mathcal{J}' := \mathcal{J} \bigotimes_F F'$. We denote by r the degree of the characteristic polynomial of $\mathcal{J}$, and we write $F'_1[t]$ for the subset of $F'[t]$ that contains only polynomials with a null constant term. Suppose that u and v are elements of $\mathcal{J}'$ such that $F'_1[u]F'_1[v] = F'_1[v]F'_1[u] = 0$. Then* $\det(\tau e - u)\det(\tau e - v) = \tau^r \det(\tau e - (u + v))$.

Sketch of the proof
It suffices to use the previous proposition with $\mathcal{A} := F'(\tau)[u, v]$. ∎

The following proposition is central in Springer's work [Spr73]. We include here its full proof, because this technique will be reused later to compare eigenvalues of different elements in a Jordan algebra.

Proposition 2.7.9 *For every invertible $u, v \in \mathcal{J}$, we set:*

$$\sigma(u, v) := \det(u)\det(u^{-1} + v).$$

Then σ is a symmetric function, and we can write $\sigma(Q_w u, v) = \sigma(u, Q_w v)$ for all invertible $w \in \mathcal{J}$.

Proof
We actually prove that the statement holds in $\bar{\mathcal{J}} := \mathcal{J} \bigotimes_F \bar{F}$, where $\bar{F}$ is the field provided by Proposition 2.7.6. Let $u, v, w \in \bar{\mathcal{J}}$ be invertible elements. There exists an element $z \in \bar{\mathcal{J}}$ such that $z^2 = u$. By specializing $x \to z$ in Corollary 2.5.3, we know that $Q_u = Q_z^2$. We abbreviate Q_z by Q. Using Proposition 2.5.4, the multiplicativity of det and the Theorem 2.5.2, we obtain:

$$\begin{aligned}
\det(u+v) &= \det(Q(Q^{-1}v+e)) = \det(u)\det(Q^{-1}v+e) \\
&= \det(v)\det(u)\det(v^{-1})\det(Q^{-1}v+e) \\
&= \det(v)\det(Qv^{-1})\det(Q^{-1}v+e) \\
&= \det(v)\det(Qv^{-1}(Q^{-1}v+e)) \\
&= \det(u)\det(v)\det(u^{-1})\det(e+Qv^{-1}) \\
&= \det(u)\det(v)\det(Q^{-1}e+Q^{-1}Qv^{-1}) \\
&= \det(u)\det(v)\det(u^{-1}+v^{-1}).
\end{aligned}$$

It suffices now to replace u by u^{-1} to show the symmetry of σ. Finally, as $Q_w u$ is invertible in view of Theorem 2.5.2, we have:

$$\begin{aligned}
\sigma(Q_w u, v) &= \det(Q_w u)\det(Q_w^{-1}u^{-1}+v) = \det(w^2)\det(u)\det(Q_{w^{-1}}(u^{-1}+Q_w v)) \\
&= \det(w^2)\det(u)\det(w^{-2})\det(u^{-1}+Q_w v) = \sigma(u, Q_w v).
\end{aligned}$$

■

This proposition implies that the elements $Q_u v^2$ and $Q_v u^2$ have the same set of eigenvalues.

Corollary 2.7.10 *Let u, v be two invertible elements of $\mathcal{J}$. Then* $\det(\tau e - Q_u v^2) = \det(\tau e - Q_v u^2)$.
Proof
By multiplicativity of det, we can write $\tau^{-r}\det(\tau e - Q_u v^2) = \det(\tau^{-1}e)\det(\tau e - Q_u v^2) = \sigma(\tau^{-1}e, -Q_u v^2)$. In view of Proposition 2.7.9, we have:

$$\begin{aligned}
\sigma(\tau^{-1}e, -Q_u v^2) &= \sigma(\tau^{-1}Q_u e, -v^2) = \sigma(\tau^{-1}u^2, -v^2) \\
&= \sigma(-v^2, \tau^{-1}u^2) = \sigma(-Q_v e, \tau^{-1}u^2) \\
&= \sigma(-e, \tau^{-1}Q_v u^2).
\end{aligned}$$

That is, $\tau^{-r}\det(\tau e - Q_u v^2) = \det(-e)\det(-e+\tau^{-1}Q_v u^2) = \tau^{-r}\det(\tau e - Q_v u^2)$. ■

2.7.3 Spectral decomposition in formally real Jordan algebras

We assume in this subsection that $\mathcal{J}$ is a *formally real Jordan algebra* $\mathcal{J}$ of dimension $N < +\infty$ over an infinite field F. In Theorem 2.7.13, we particularize within this framework the first spectral decomposition theorem displayed in Theorem 2.7.2.

As the following lemma states, formally real power-associative algebras have a property of importance for our purposes: they do not have nonzero nilpotent. This fact has already been stated in the seminal paper [JvNW34], Theorem I.

Lemma 2.7.11 *Let $\mathcal{J}$ be a power-associative algebra. If $\mathcal{J}$ is formally real, the only nilpotent element of $\mathcal{J}$ is* 0. ■

Consider now the extension $\mathcal{J}' := \mathcal{J} \bigotimes_F F(\tau_1, \ldots, \tau_N)$ of $\mathcal{J}$, where $\tau_1, \ldots, \tau_N$ are algebraically independent over F. This algebra does not have any nonzero nilpotent element. Assume indeed that $f(\tau_1, \ldots \tau_N)^m = 0$ for an element $f \in \mathcal{J}'$ and an integer $m > 1$. Then, for every $u \in \operatorname{dom} f$, the element $f(u)$ is a nilpotent of $\mathcal{J}$, and $f \equiv 0$. The same conclusion holds when $F(\tau_1, \ldots, \tau_N)$ is replaced by its algebraic closure.

This observation immediately leads to the following conclusion.

Remark 2.7.12 *The roots of the characteristic polynomial of a formally real Jordan algebra are all different. Denote by $x = \tau_1 b_1 + \cdots + \tau_N b_N$ a generic element of $\mathcal{J}$ and by $\bar{F}'$ the algebraic closure of $F(\tau_1, \ldots, \tau_N)$.*

Suppose, contrarily to the statement, that $f(\tau; x) = (\tau - \lambda_k(x))^2 q(\tau; x)$, and take $g(\tau; x) := (\tau - \lambda_k) q(\tau; x)$, which is an element of $\mathcal{J} \bigotimes_F \bar{F}'$. Then $g(x; x)^2 = f(\tau; x) q(\tau; x) = 0$ and $g(x; x)$ is a nilpotent. By our previous observation, we have $g(x; x) = 0$. But, in view of Proposition 2.3.17, this contradicts the minimality of $f(\tau; x)$. ■

In the sequel of this work, we restrict our considerations to **algebras over** $\mathbb{R}$. An advantage of this property is that there exists a Euclidean topology on $\mathcal{J} = \mathbb{R}^N$; we can also use the sup-topology it induces on functional spaces of the type $\mathcal{J} \bigotimes_{\mathbb{R}} \mathbb{R}(\tau_1, \ldots, \tau_N)$. However, it will appear later in our exposition that this is not the most natural metric in Jordan algebras. Another advantage of considering algebras over $\mathbb{R}$ is provided by Lemma 2.7.1: if $\mathcal{J}$ happens to be formally real, the field $\mathbb{R}$ is large enough to satisfy Hypothesis 2.7.1. It follows then that we can easily adapt the spectral decomposition Theorem 2.7.2 within this context.

Theorem 2.7.13 (Unique eigenspaces spectral decomposition theorem) *Let $\mathcal{J}$ be a formally real power-associative algebra over $\mathbb{R}$, and let $u \in \mathcal{J}$. Let $\{e_1, \ldots, e_n\}$ be the set of minimal idempotents of $F[u]$. Then the polynomial μ_u is of degree n. Also, all its roots $\xi_1, \ldots, \xi_n$ are real and distinct. Up to a renumbering of these roots, we have:*

$$u = \sum_{i=1}^{n} \xi_i e_i.$$

This decomposition is unique in the following sense: if there exist a system of idempotents $\{e'_1, \ldots, e'_m\} \in \mathcal{J}$ and some distinct elements $\eta_1, \ldots, \eta_m$ of $\mathbb{R}$ such that $u = \sum_{j=1}^{m} \eta_j e'_j$, then $m = n$, and, up to a renumbering, $\xi_i e_i = \eta_i e'_i$.

Proof
As recalled above, we can use Lemma 2.7.1 to confirm that the field $\mathbb{R}$ of the algebra $\mathcal{J}$ satisfies Hypothesis 2.7.1. Moreover, according to Lemma 2.7.11, there is no nonzero nilpotent in $\mathcal{J}$. Observe finally that the existence of idempotent is ensured if $\mathcal{J} \neq \{0\}$ by Corollary 2.3.6. ■

2.7.4 Minimal idempotents

The rest of this section is devoted to the proof of an extremely useful second version of the spectral decomposition theorem and to the presentation of its most immediate consequences. The crucial consequence of this new decomposition is that it shows a strong link between the minimal polynomial of an element of $\mathcal{J}$ and the characteristic polynomial of $\mathcal{J}$, although the latter lives in a much broader algebra than the minimal polynomial. This link will allow us to apply to the algebra $\mathcal{J}$ all the properties we have presented involving generic elements, generic trace, generic norm, and eigenvalues in Section 2.3. In the present approach, this link is demonstrated from the study of minimal idempotents. Most of the proofs in this subsection are original.

Recall that Definition 2.3.14 introduced the notion of *minimal idempotent with respect to an element* u for a strictly power-associative algebra in a somehow artificial manner, given that this characterization depended on a previously chosen element u of the algebra. Here, the minimality of an idempotent is defined as an *intrinsic* concept.

Definition 2.7.14 *Let $\mathcal{J}$ be an F-algebra. An idempotent c of $\mathcal{J}$ is* minimal *if, for every idempotent d of $\mathcal{J}$, we have: $cd = d \;\Rightarrow\; c = d$.* ■

In particular, if $\mathcal{J}$ is associative and commutative, this definition is equivalent to Definition 2.3.8. Suppose indeed that the idempotent c complies with Definition 2.3.8. Then, if $cd = d$ is an idempotent, it must be equal to c since it belongs to $L(c)\mathcal{J}$. Suppose conversely that c is minimal in the sense of our new definition, and that d is an idempotent of $L(c)\mathcal{J}$. Then $d = cd$ because c is the unit element of $L(c)\mathcal{J}$. Thus $d = c$ by hypothesis, and $L(c)\mathcal{J}$ has a single idempotent.

Remark 2.7.15 *In the literature, the minimality of an idempotent is often characterized as follows. The idempotent c is not minimal if and only if there exist two idempotents c_1, c_2 such that $c_1c_2 = 0$ and $c_1 + c_2 = c$. Suppose indeed that c is not minimal. Then, using the idempotent $d \neq c$ for which $cd = d$, it suffices to take $c_1 := d$ and $c_2 := c - d$. Conversely, we can take $d := c_1$ to rule out Definition 2.7.14.* ■

Definition 2.7.16 *A* Jordan frame *is a system of idempotents $\{c_1, c_2, \ldots, c_n\}$ that are all minimal.* ■

Proposition 2.7.17 *Let $\mathcal{J}$ be a formally real Jordan algebra over $\mathbb{R}$. An idempotent $c \in \mathcal{J}$ is a minimal idempotent of $\mathcal{J}$ if and only if $\mathcal{J}_1(c) = c\mathbb{R}$.*

Proof
Let c be an idempotent of $\mathcal{J}$. In view of the first Pierce decomposition theorem, c is the unit element of $\mathcal{J}_1(c)$.

Suppose first that c is a minimal idempotent. It is then the only idempotent of $\mathcal{J}_1(c)$ by definition. As $\mathcal{J}_1(c)$ is a power-associative and formally real algebra, Theorem 2.7.13 gives for every $u \in \mathcal{J}_1(c)$ a decomposition of the form $u = \alpha c$, where $\alpha \in \mathbb{R}$, so that $\mathcal{J}_1(c) \subseteq c\mathbb{R}$. The fact that $\mathcal{J}_1(c)$ is a real nondegenerate vector space allows us to conclude that $\mathcal{J}_1(c) = c\mathbb{R}$.

Conversely, suppose that $\mathcal{J}_1(c) = c\mathbb{R}$. If $d \in \mathcal{J}$ is an idempotent such that $cd = d$, then $d \in \mathcal{J}_1(c)$ and $d = \alpha c$ for a real α. From $d^2 = d$, we necessarily have $\alpha = 1$. Thus $d = c$, and c is minimal in $\mathcal{J}$. ∎

Corollary 2.7.18 *Let $\mathcal{J}$ be a formally real Jordan algebra, and let F be an extension* field *of $\mathbb{R}$. If c is a minimal idempotent in $\mathcal{J}$, then it is also a minimal idempotent in $\mathcal{J}' = \mathcal{J} \bigotimes_{\mathbb{R}} F$, and $(\mathcal{J}')_1(c) = Fc$.*

Proof
As a consequence of the previous proposition and of Remark 2.6.4, we can state that, for every extension field F of $\mathbb{R}$, the equality $\left(\mathcal{J} \bigotimes_{\mathbb{R}} F\right)_1(c) = cF$ holds. Now, if an idempotent $d \in \bar{\mathcal{J}}$ satisfies $cd = d$, then it belongs to $(\mathcal{J}')_1(c)$, and $d = \alpha c$ for some $\alpha \in F$. As $\alpha^2 = \alpha \neq 0$ in the field F, we must have $\alpha = 1$. Hence, the only idempotent of $(\mathcal{J}')_1(c)$ is c and we conclude that c is minimal in $\mathcal{J}'$. ∎

Observe that we cannot use an extension ring R of $\mathbb{R}$ in the previous proposition, except if the equation $\alpha^2 = \alpha$ has only the solutions 0 and 1 in R.

The following characterization, which may appear as an almost insignificant result, represents however the key tool that links the minimal polynomial of a regular element to the corresponding specialization of the characteristic polynomial.

Proposition 2.7.19 *Let c be an idempotent of the formally real Jordan algebra $\mathcal{J}$ over $\mathbb{R}$. Then c is minimal if and only if* $\mathrm{tr}(c) = 1$.

Proof
Let $\mathcal{B} := \{b_1, \ldots, b_N\}$ be a basis of $\mathcal{J}$ and $x = \tau_1 b_1 + \cdots + \tau_N b_N$ be a generic element of $\mathcal{J}$. We write $F := \mathbb{R}(\tau_1, \ldots, \tau_N)$ and $\mathcal{J}' := \mathcal{J} \bigotimes_{\mathbb{R}} F$. We denote by r the generic rank of $\mathcal{J}$, that is, the degree of the minimal polynomial of x in $\mathcal{J}'$.

The trace of idempotent elements. By Proposition 2.3.34, we can write $f(\tau; x) = \det(\tau e - x)$. If we specialize x to e, we get $f(\tau; e) = (\tau - 1)^r$, and thus $\mathrm{tr}(e) = r$. Now, let c be an idempotent of $\mathcal{J}$. Since $\mu_c(\tau) = \tau^2 - \tau$ for $c \neq e$, we have $f(\tau; c) = \tau^{r-k}(1 - \tau)^k$ for an integer $1 \leq k \leq r$ in view of Proposition 2.3.34, so that $\mathrm{tr}(c) = k$.

Let c_1 and c_2 be two idempotents such that $c_1 c_2 = 0$. By Corollary 2.7.8, we have $\det(\tau e - c_1)\det(\tau e - c_2) = \tau^r \det(\tau e - (c_1 + c_2))$. Comparing the coefficient of τ^{2r-1} on both sides of this equality, we obtain $\mathrm{tr}(c_1 + c_2) = \mathrm{tr}(c_1) + \mathrm{tr}(c_2)$.

The "if" part. Hence, if c is an idempotent such that $\mathrm{tr}(c) = 1$, then c cannot be decomposed into the sum of two idempotents i.e. it is minimal. (And, in fact, the previous corollary shows that it is also minimal in $\mathcal{J}'$.)

A system of r idempotents. Let $\bar{F}$ be the algebraic closure of the field F and let $\bar{\mathcal{J}} := \mathcal{J} \bigotimes_F \bar{F}$. Proposition 2.3.17 shows that the minimal polynomial of the generic element x in $\bar{\mathcal{J}}$ is $f(\tau; x)$. All its roots are distinct (see Remark 2.7.12), and they are all in $\bar{F}$. In view of Corollary 2.7.5, there exists a system $\{c_1, \ldots, c_r\}$ of r idempotents in $\bar{\mathcal{J}}$. These idempotents are minimal, and $f(\tau; c_i) = \tau^{r-1}(1 - \tau)$.

Now, let c be a minimal idempotent of $\bar{\mathcal{J}}$; of course, for our purposes, an element c that belongs to $\mathcal{J}$ would be sufficient. We aim to prove that $f(\tau; c) = f(\tau; c_i)$, from which we can conclude that $\operatorname{tr}(c) = 1$. Let us denote $\operatorname{tr}(c)$ by k, so that $f(\tau; c) = \tau^{r-k}(1-\tau^k)$.

Comparing the characteristic polynomials. From Corollary 2.7.18, there exist $\alpha_i \in \bar{F}$ such that $Q_c c_i = \alpha_i c$ for all $1 \leq i \leq r$. Since the idempotents c_i sum up to e, we get $c = \sum_{i=1}^r \alpha_i c$; this implies that one of the coefficients α_i, say α_1, is different from 0. Note that:

$$\det(\tau e - Q_c c_1) = \det(\tau e - \alpha_1 c) = f(\tau; \alpha_1 c). \tag{2.18}$$

Let us now apply Theorem 2.7.2 in the subalgebra $\bar{\mathcal{J}}_1(c_1)$ to $Q_{c_1} c$. Given that there is no nilpotent in $\bar{\mathcal{J}}$, we get a scalar $\beta \in \bar{F}$ such that $Q_{c_1} c = \beta c_1$. We have:

$$\det(\tau e - Q_{c_1} c) = \det(\tau e - \beta c_1) = f(\tau; \beta c_1). \tag{2.19}$$

Now, we prove that $\det(\tau e - Q_{c_1} c) = \det(\tau e - Q_c c_1)$. We cannot apply directly Corollary 2.7.10, because neither c nor c_1 are invertible. To bypass this issue, we define two sequences of $\bar{\mathcal{J}}$. First, we take $u_m := c_1 + (e - c_1)/m$ for every $m \geq 1$, so that the elements u_m are invertible in view of Corollary 2.7.3. With the sup-topology chosen on $\bar{\mathcal{J}}$ (see p. 73), we also have $\lim_{m\to\infty} u_m = c_1$. Similarly, we define $v_m := c + (e - c)/m$ for all $m \geq 1$. The elements v_m are also invertible, and $\lim_{m\to\infty} v_m = c$.

In view of Corollary 2.7.10, we know that $\det(\tau e - Q_{u_m} v_m^2) = \det(\tau e - Q_{v_m} u_m^2)$. Letting m go to $+\infty$, we get by continuity of the function det that:

$$\det(\tau e - Q_{c_1} c) = \det(\tau e - Q_c c_1).$$

According to equations (2.18) and (2.19), we conclude that $f(\tau; \alpha_1 c) = f(\tau; \beta c_1)$. In particular, $\beta \neq 0$. In view of Remark 2.3.35, we have $f(\tau; \alpha_1 c/\beta) = f(\tau; c_1)$, or:

$$\left(\frac{\beta\tau}{\alpha_1}\right)^{r-k}\left(1 - \frac{\beta\tau}{\alpha_1}\right)^k = \tau^{r-1}(1-\tau).$$

Comparing both sides of this relation, we obtain that $\alpha_1 = \beta$ and $k = 1$. ■

As an immediate consequence, a Jordan frame of a formally real Jordan algebra $\mathcal{J}$ over $\mathbb{R}$ contains exactly r idempotents, where r is the generic rank of $\mathcal{J}$.

Corollary 2.7.20 *Let $\mathcal{J}$ be a formally real Jordan algebra. Suppose that the element $u \in \mathcal{J}$ satisfies $\operatorname{tr}(u^2) = 0$. Then $u = 0$.*

Proof

Let $u = \sum_{i=1}^k \xi_i e_i$ be the spectral decomposition of u provided by the unique eigenspaces spectral decomposition theorem. Since $\{e_1, \ldots, e_k\}$ is a system of idempotents, we can immediately compute $u^2 = \sum_{i=1}^k \xi_i^2 e_i$, and $\operatorname{tr}(u^2) = \sum_{i=1}^k \xi_i^2 \operatorname{tr}(e_i)$. From the proof of Proposition 2.7.19, we know that $\operatorname{tr}(e_i) \geq 1$ for every i. Thus $\operatorname{tr}(u^2) = 0$ implies that $k = 1$ and $\xi_1 = 0$. ■

The following lemma plays an important role in our extension of Wielandt's Theorem in Chapter 3. Up to our knowledge, it represents an original result.

Lemma 2.7.21 *Let c, d be two idempotents of the formally real Jordan algebra $\mathcal{J}$ and let d_γ be the projection of d on the Pierce subspace $\mathcal{J}_\gamma(c)$ for $\gamma = 0, 1/2, 1$. If $d_0 = 0$ or $d_1 = 0$, then $d_{1/2} = 0$.*

Proof

Suppose that $d_0 = 0$ without loss of generality (if $d_1 = 0$, it suffices to replace c by $e - c$ in the statement). The relation $d = d^2$ implies:

$$d_1 + d_{1/2} = d_1^2 + 2d_1 d_{1/2} + d_{1/2}^2.$$

According to the first Pierce decomposition theorem, we have $d_1 = d_1^2 + d_{1/2}^2$ by considering the projection on $\mathcal{J}_1(c)$. Hence $d_{1/2}^2 \in \mathcal{J}_1(c)$, i.e. $cd_{1/2}^2 = d_{1/2}^2$. Consequently, we get $\mathrm{tr}(cd_{1/2}^2) = \mathrm{tr}(d_{1/2}^2)$. On the other hand, we have $\mathrm{tr}(cd_{1/2}^2) = \mathrm{tr}((cd_{1/2})d_{1/2}) = \mathrm{tr}(d_{1/2}^2)/2$ by associativity of the trace. Thus $\mathrm{tr}(d_{1/2}^2) = \mathrm{tr}(d_{1/2}^2)/2$, and $\mathrm{tr}(d_{1/2}^2) = 0$. Since $\mathcal{J}$ is formally real, we get that $d_{1/2} = 0$ in view of Corollary 2.7.20. ■

The following statement has been proved by Faybusovich in his recent preprint [Fay05]. As our argument is simpler, we include our proof here.

Proposition 2.7.22 *Let f be an idempotent of the formally real Jordan algebra $\mathcal{J}$ and let c be a minimal idempotent of the subalgebra $\mathcal{J}_1(f)$. Then c is also a minimal idempotent in the algebra $\mathcal{J}$.*

Proof

Suppose that d is an idempotent of $\mathcal{J}$ for which $cd = d$. We need to check that $d = c$. In view of the first Pierce decomposition theorem, we can write $d = d_1 + d_{1/2} + d_0$, where $d_\gamma \in \mathcal{J}_\gamma(f)$. Since $c \in \mathcal{J}_1(f)$, we know that $d_1 c \in \mathcal{J}_1(c)$, $d_{1/2}c \in \mathcal{J}_{1/2}(c)$ and $d_0 c = 0$. The relation $dc = d$ implies that $d_0 = 0$. In view of Lemma 2.7.21, we deduce that $d_{1/2} = 0$. Hence $d = d_1$, and the minimality of c in $\mathcal{J}_1(c)$ implies that $d_1 = c$. ■

2.7.5 A second spectral decomposition theorem for formally real Jordan algebras

The proof of the second version of the spectral decomposition theorem for formally real Jordan algebras over $\mathbb{R}$ is based on the density of regular elements. Before introducing it, we recall a standard concept from algebraic geometry (see [CLO92] for more details).

Definition 2.7.23 *The* Zariski topology *of a finite-dimensional vector space V over a field F is the topology for which a set $A \subseteq V$ is open if and only if there exists a polynomial $p : V \to F$ whose coefficients are in F and whose set of roots is exactly $V \backslash A$.* ■

For instance, a set of $\mathbb{R}$ is Zariski closed in $\mathbb{R}$ either if it is $\mathbb{R}$ itself, or if it contains a finite number of elements. The following statement is proved in [FK94], Proposition II.2.1. It essentially results from the fact that the coefficients of the characteristic polynomial are polynomial themselves (see Proposition 2.3.27).

Proposition 2.7.24 *Let $\mathcal{J}$ be a power-associative algebra. The set of regular elements of $\mathcal{J}$ is a Zariski nonempty open set of $\mathcal{J}$. If $\mathcal{J}$ is an algebra over $\mathbb{R}$, this set is dense in $\mathcal{J}$ for the Euclidean topology.* ■

Given a generic element x, we write $\lambda_i(u)$ for the specialization $x \to u$ applied to $\lambda_i(x)$, the ith root of the characteristic polynomial $f(\tau; x)$.

An alternative proof is given in [FK94], Theorem III.1.2.

Theorem 2.7.25 (Complete spectral decomposition theorem) *Let $\mathcal{J}$ be a formally real Jordan algebra of finite dimension N over $\mathbb{R}$ and of generic rank r. For every $u \in \mathcal{J}$, the quantity $\lambda_i(u)$ is a real number, and we can assume that they are labeled as $\lambda_1(u) \geq \cdots \geq \lambda_r(u)$. If u is a regular element, then $\mu_u(\tau) = f(\tau; u)$.*

For every $u \in \mathcal{J}$, there exists a Jordan frame $\{c_1, \ldots, c_r\}$ such that:

$$u = \sum_{i=1}^{r} \lambda_i(u) c_i.$$

This decomposition is unique in the following sense: if there exist a Jordan frame $\{c'_1, \ldots, c'_r\}$ and real numbers $\eta_1 \geq \cdots \geq \eta_r$ for which $u = \sum_{i=1}^{r} \eta_i c'_i$, then $\eta_i = \lambda_i(u)$ for all i and $\sum_{\{j|\eta_j=\rho\}} c'_j = \sum_{\{j|\eta_j=\rho\}} c_j$ for every real number ρ.

Proof

Let u be a regular element of $\mathcal{J}$. By the first spectral decomposition Theorem 2.7.13, we know that $u = \sum_{i=1}^{n} \xi_i e_i$, where the real numbers ξ_i are the distinct roots of μ_u.

Suppose that the system of idempotents $\{e_1, \ldots, e_n\}$ is not a Jordan frame, i.e. it contains a non-minimal idempotent, say e_n. The idempotent e_n can be decomposed into a sum $e_n = c_n + c_{n+1}$ of two idempotents with $c_n c_{n+1} = 0$. Let $c_i := e_i$ for $1 \leq i \leq n-1$. It is readily seen that $\{c_1, \ldots, c_{n+1}\}$ is a system of idempotents. Now, consider the element $v = \sum_{i=1}^{n+1} i c_i$. Its minimal polynomial should have $\{1, 2, \ldots, n, n+1\}$ as set of roots; hence, the degree of μ_v exceeds the degree of μ_u, contradicting the regularity of u. The system $\{e_1, \ldots, e_n\}$ is then a Jordan frame.

The trace of e equals r because $f(\tau; e) = (\tau - 1)^r$. We know from Proposition 2.7.19 that $\operatorname{tr}(e_i) = 1$. Thus $\operatorname{tr}(e) = \sum_{i=1}^{n} \operatorname{tr}(e_i) = n$, and $n = r$. In other word, the degree of a regular element equals the generic rank of $\mathcal{J}$. And since $\mu_u(\tau)$ divides $f(\tau; u)$, we have $\mu_u(\tau) = f(\tau; u)$.

For regular elements of $\mathcal{J}$, there is nothing more to show: everything follows from the first spectral decomposition theorem.

Consider now a non-regular element u of $\mathcal{J}$. In view of Proposition 2.7.24, the set of regular elements of $\mathcal{J}$ is dense in $\mathcal{J}$. Thus there exists a sequence $(u^{(1)}, u^{(2)}, \ldots, u^{(m)}, \ldots)$ of regular elements in $\mathcal{J}$ that converges to u. By the first part of the theorem, we can perform the decomposition

$$u^{(m)} = \sum_{i=1}^{r} \lambda_i(u^{(m)}) c_i^{(m)},$$

where $\{c_1^{(m)}, \ldots, c_r^{(m)}\}$ is a Jordan frame for every m. Since the converging sequence is a compact set in $\mathcal{J}$, there exists a subsequence of $\{1, 2, \ldots\}$, say $\{m_1, m_2, \ldots\}$, such that the limits $c_i := \lim_{k\to\infty} c_i^{(m_k)}$ exist for every i. Note that $\lambda_i(x)$ are continuous functions, because the roots of a polynomial depend continuously of its coefficients (see [CC89] for instance), which are themselves polynomials in x. Thus $\lambda_i(u) = \lim_{k\to\infty} \lambda_i(u^{(m_k)})$.

The set $\{c_1, \ldots, c_r\}$ is obviously a Jordan frame, since:

$$\lim_{k\to\infty} c_i^{(m_k)} c_j^{(m_k)} = \lim_{k\to\infty} \delta_{ij} c_i^{(m_k)} = \delta_{ij} c_i \quad \text{and} \quad \lim_{k\to\infty} \sum_{i=1}^{r} c_i^{(m_k)} = \lim_{k\to\infty} e = e.$$

Then $u = \sum_{i=1}^{r} \lambda_i(u) c_i$, because $u = \lim_{k\to\infty} u_i^{(m_k)}$. Let us define the integers $s, k_1, \ldots, k_s$ such that $k_s := r$ and:

$$\lambda_1(u) = \cdots = \lambda_{k_1}(u) > \lambda_{k_1+1}(u) = \cdots = \lambda_{k_2}(u) > \cdots \lambda_{k_s}(u).$$

Denote $M_j := \{k_{j-1}+1, \ldots, k_j\}$ (with $k_0 = 0$) and put $e_j := \sum_{i\in M_j} c_i$. Then $\{e_1, \ldots, e_s\}$ is a system of idempotents and $u = \sum_{i=1}^{s} \lambda_{k_j}(u) e_j$. By the Theorem 2.7.13, this decomposition is unique, and everything is shown. ■

In the previous theorem, we have shown that the generic rank of $\mathcal{J}$ equals the degree of regular elements in $\mathcal{J}$. We call this quantity the *rank* of $\mathcal{J}$.

This theorem allows us to consider the determinant, the trace and all the dettraces on $\mathcal{J}$ as the specialization on $\mathcal{J}$ of the generic norm, the generic trace and the generic dettraces introduced in Section 2.3.

Consequently, all results presented so far for these objects immediately apply to $\mathcal{J}$.

Moreover, this theorem allows us to number the eigenvalues of an element, since each of them is a real number. By convention, we assume in the rest of this thesis that for every $u \in \mathcal{J}$, we have $\lambda_1(u) \geq \cdots \geq \lambda_r(u)$.

In this work, we sometimes need to study the spectrum of elements u that belong to a Jordan subalgebra $\mathcal{J}'$ of $\mathcal{J}$. As the vector of eigenvalues of u depends on the algebra in which u is considered, we explicit this dependence by writing $\lambda(u; \mathcal{J}')$ for its ordered eigenvalue vector in $\mathcal{J}'$ and $\lambda(u; \mathcal{J})$ or simply $\lambda(u)$ for its eigenvalue vector in $\mathcal{J}$.

As an example, we can consider in the framework of $r \times r$ real symmetric matrices an idempotent C of trace $k < r$ (that is, a projector of rank k) and the subalgebra $\mathcal{J}' := \mathcal{J}_1(C)$ (see Theorem 2.6.1). We fix an orthogonal basis of $\mathbb{R}^r$ for which the matrix representation of C has the form:

$$C = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{so that } \mathcal{J}' \text{ consists of matrices of the form } \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where A is a $k \times k$ real symmetric matrix. Let us consider a matrix:

$$U = \begin{pmatrix} U' & 0 \\ 0 & 0 \end{pmatrix}$$

in $\mathcal{J}_1(C)$. The eigenvalues of U in $\mathcal{J}_1(C)$ are exactly those of U'. The eigenvalues of U in $\mathcal{J}$ are the ones of U', plus $r - k$ times the eigenvalue 0.

2.7.6 A Euclidean topology in $\mathcal{J}$

From now on, the character $\mathcal{J}$ refers, unless explicitly stated, to a formally real Jordan algebra of rank r and of dimension $N < +\infty$ over $\mathbb{R}$.

Definition 2.7.26 *We define the following bilinear form:* $\langle\cdot,\cdot\rangle_{\mathcal{J}} : \mathcal{J}\times\mathcal{J}\to\mathbb{R}$, $(u,v)\mapsto \langle u,v\rangle_{\mathcal{J}} := \mathrm{tr}(uv)$. ■

Proposition 2.7.27 (A scalar product for $\mathcal{J}$) *The operator* $\langle\cdot,\cdot\rangle_{\mathcal{J}}$ *is a scalar product for* $\mathcal{J}$*. Moreover, it is* associative*: for every* $u,v,w\in\mathcal{J}$*, we have* $\langle uv,w\rangle_{\mathcal{J}} = \langle u,vw\rangle_{\mathcal{J}}$*; in other words, the operator* $L(v)$ *is self-adjoint for every* $v\in\mathcal{J}$ *with respect to* $\langle\cdot,\cdot\rangle_{\mathcal{J}}$.

Proof
By linearity of tr, the operator $\langle\cdot,\cdot\rangle_{\mathcal{J}}$ is bilinear. It is also symmetric, because $\mathcal{J}$ is commutative, and associative by virtue of the associativity of the trace (see Corollary 2.3.40). As it is shown in Corollary 2.7.20, the fact that $\mathcal{J}$ is formally real implies that it is positive definite. ■

The operator $\langle\cdot,\cdot\rangle_{\mathcal{J}}$ is called the *Jordan scalar product* of $\mathcal{J}$. Henceforth, we drop the subscript $_{\mathcal{J}}$ when it is clear from the context that we use the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{J}}$. If we fix a basis $\mathcal{B} := \{b_1,\ldots,b_N\}$, the link between the Jordan scalar product and the standard dot product is provided by the following Gram matrix:

$$G_{\mathcal{B}} := [\mathrm{tr}(b_ib_j)]_{i,j=1}^N.$$

If $u = \sum_{i=1}^N x_ib_i$ and $v = \sum_{i=1}^N y_ib_i$ are two elements of $\mathcal{J}$, we indeed have $\langle u,v\rangle_{\mathcal{J}} = x^TG_{\mathcal{B}}y$.

We denote by $||\cdot||_{\mathcal{J}}$ the norm induced by our scalar product, that is, $||u||_{\mathcal{J}} := \sqrt{\mathrm{tr}(u^2)}$ for every $u\in\mathcal{J}$. The subscript $_{\mathcal{J}}$ may also be deleted when there is no place for confusion on the norm we use. This norm is referred to as the *Jordan norm.*

As $\mathcal{J}$ is finite-dimensional, $(\mathcal{J},\langle\cdot,\cdot\rangle_{\mathcal{J}})$ is an Euclidean space and the Riesz representation Theorem (see for instance in [Sch91], Théorème 2.14.22) applies: for every linear form $l:\mathcal{J}\to\mathbb{R}$, there exists a unique $u\in\mathcal{J}$ such that $l(v) = \langle u,v\rangle_{\mathcal{J}}$ for all $v\in\mathcal{J}$. We elaborate more on the links between the vector space $\mathcal{J}$ and its dual in the next section.

Remark 2.7.28 *With the Jordan scalar product, the subspaces* $\mathcal{J}_1(c),\mathcal{J}_{1/2}(c)$*, and* $\mathcal{J}_0(c)$ *are mutually orthogonal for every idempotent* c*, because* $\mathcal{J}_1(c)\circ\mathcal{J}_0(c) = \{0\}$*,* $\mathcal{J}_{1/2}(c)\circ(\mathcal{J}_1(c)+\mathcal{J}_0(c))\subseteq\mathcal{J}_{1/2}(c)$ *and* $\mathrm{tr}(\mathcal{J}_{1/2}(c)) = \{0\}$ *(see item 7 of Theorem 2.6.1). Let* $\{c_1,\ldots,c_r\}$ *be a Jordan frame of* $\mathcal{J}$*, and let* $\mathcal{J}_{ij} := Q_{c_i,c_j}\mathcal{J}$*. In view of the description of* $\mathcal{J}_{ij}$ *given in the first item of Theorem 2.6.6, we deduce that the subspaces* $\mathcal{J}_{ij}$ *are mutually orthogonal for the Jordan scalar product.* ■

2.7.7 Operator commutativity

The following corollary has been shown in [SA03], Theorem 27. We give here our own proof.

Proposition 2.7.29 (Operator commutativity) *Two elements* u,v *of* $\mathcal{J}$ *operator commute if and only if there exists a Jordan frame* $\{c_1,\ldots,c_r\}$ *for which they can be written as* $u = \sum_{i=1}^r\alpha_ic_i$ *and* $v = \sum_{i=1}^r\beta_ic_i$*, for some* r*-dimensional real vectors* α *and* β.

Proof
The "if" part is easy. We know from Theorem 2.6.1 that, if c is an idempotent, then u operator commutes with c iff $u \in \mathcal{J}_0(c) \oplus \mathcal{J}_1(c)$. Let $u = \sum_{i=1}^{r} \lambda_i(u) c_i \in \mathcal{J}$. The element u obviously operator commutes with every idempotent c_i, because $c_i u = 1 \cdot \lambda_i(u) c_i + 0 \cdot \sum_{j \neq i} \lambda_j(u) c_j$. Moreover, if u operator commutes with a and b, then u operator commutes every linear combination of a and b by linearity of the operator L. Thus, if $v \in \mathcal{J}$ can be written as $v = \sum_{i=1}^{n} \beta_i c_i$ for $\beta \in \mathbb{R}^r$, then u and v operator commute.

The proof of the "only if" part relies on the second Pierce decomposition theorem. Let $u = \sum_{i=1}^{n} \xi_i e_i$ be the unique eigenspaces spectral decomposition of u, and let $\mathcal{J}_{ij} := Q_{e_i,e_j}\mathcal{J}$ be the Pierce subspaces with respect to the system of idempotents $\{e_1, \ldots, e_n\}$. Suppose that u and v operator commute. Then, for every $1 \leq i \leq n$, we have $L(u)L(v)e_i = L(v)L(u)e_i$. Let v_{ij} stand for the projection of v on $\mathcal{J}_{ij}$.

Using the multiplication rules for $\mathcal{J}_{ij}$ subspaces and the convention $v_{ij} = v_{ji}$, we get:

$$v(ue_i) = v\xi_i e_i = \xi_i v_{ii} + \frac{\xi_i}{2} \sum_{l \neq i} v_{il}.$$

On the other hand,

$$\begin{aligned} u(ve_i) &= \left(\sum_{l=1}^{n} \xi_l e_l\right)\left(v_{ii} + \sum_{i \neq j} \frac{v_{ij}}{2}\right) \\ &= \xi_i e_i v_{ii} + \xi_i e_i \left(\sum_{j \neq i} \frac{v_{ij}}{2}\right) + \left(\sum_{l \neq i} \xi_l e_l\right) v_{ii} + \left(\sum_{l \neq i} \xi_l e_l\right)\left(\sum_{j \neq i} \frac{v_{ij}}{2}\right) \\ &= \xi_i v_{ii} + \sum_{j \neq i} \frac{\xi_i v_{ij}}{4} + 0 + \sum_{j \neq i} \frac{\xi_j v_{ij}}{4}. \end{aligned}$$

Thus,

$$v(ue_i) - u(ve_i) = \sum_{j \neq i} \frac{\xi_i - \xi_j}{4} v_{ij} = 0.$$

Since the numbers ξ_j are all different, and since the vectors v_{ij} belong to linearly independent subspaces, the above relation implies that $v_{ij} = 0$ for $i \neq j$. Thus $v \in \mathcal{J}_{11} \oplus \cdots \oplus \mathcal{J}_{nn}$.

Note that the subalgebra $\mathcal{J}_{ii}$, with e_i as unit element and rank $r_i := \mathrm{tr}(e_i)$, is also a formally real Jordan algebra. Applying the second spectral decomposition theorem in $\mathcal{J}_{ii}$ to v_{ii} gives, with the notational convention introduced on p. 79,

$$v_{ii} = \sum_{j=1}^{r_i} \lambda_j(v_{ii}; \mathcal{J}_{ii}) c_j^{(i)}.$$

By the second Pierce decomposition theorem, we deduce $c_j^{(i)} c_l^{(k)} = 0$ whenever $i \neq k$. Thus, $\left\{c_1^{(1)}, \ldots, c_{r_1}^{(1)}, \ldots, c_{r_n}^{(n)}\right\}$ is a Jordan frame of $\mathcal{J}$ for which:

$$u = \sum_{i=1}^{n} \sum_{j=1}^{r_i} \xi_i c_j^{(i)} \quad \text{and} \quad v = \sum_{i=1}^{n} \sum_{j=1}^{r_i} \lambda_j(v_{ii}; \mathcal{J}_{ii}) c_j^{(i)},$$

which concludes the proof. ■

We can formulate this corollary in an alternative way: if the set $\mathcal{A}$ is a commutative and associative subalgebra of $\mathcal{J}$, there must exist a system of idempotents $\{e_1, \ldots, e_n\}$ such that:

$$\mathcal{A} = \left\{ \sum_{i=1}^{n} \alpha_i e_i \,\middle|\, \alpha_1, \ldots, \alpha_n \in \mathbb{R} \right\}.$$

Corollary 2.7.30 *Let u and v be two elements of $\mathcal{J}$. We denote the unique eigenspaces spectral decomposition of u as $u = \sum_{j=1}^{s} \xi_j e_j$. The elements u and v operator commute if and only if v belongs to $\bigoplus_{j=1}^{s} \mathcal{J}_1(e_j)$.*

Proof

Suppose that v belongs to $\bigoplus_{j=1}^{s} \mathcal{J}_1(e_j)$, and denote by v_j the projection of v on $\mathcal{J}_1(e_j)$. Since, in view of item 4 of the first Pierce decomposition theorem, the subspaces $\mathcal{J}_1(e_j)$ are formally real Jordan subalgebras of $\mathcal{J}$, we can apply the complete spectral decomposition theorem to v_j on $\mathcal{J}_1(e_j)$. We obtain $v_j = \sum_{i=1}^{\mathrm{tr}(e_j)} \lambda_{ji} c_{ji}$, where $\{c_{j1}, \ldots, c_{j,\mathrm{tr}(e_j)}\}$ is a Jordan frame of $\mathcal{J}_1(e_j)$. In view of Proposition 2.7.22, the idempotents c_{ij} are minimal in the full algebra $\mathcal{J}$. Thus, the set

$$\{c_{11}, \ldots, c_{1,\mathrm{tr}(e_1)}, c_{21}, \ldots, c_{s,\mathrm{tr}(e_s)}\}$$

is a Jordan frame according to the second Pierce decomposition theorem, and we can write:

$$u = \sum_{j=1}^{s} \sum_{i=1}^{\mathrm{tr}(e_j)} \xi_j c_{ij} \quad \text{and} \quad v = \sum_{j=1}^{s} \sum_{i=1}^{\mathrm{tr}(e_j)} \lambda_{ij} c_{ij}.$$

Using the previous proposition, we conclude that u and v operator commute.

Suppose now that u and v operator commute. Again by the previous proposition, there exist a Jordan frame $\{c_1, \ldots, c_r\}$ and a vector $\beta \in \mathbb{R}^r$ such that:

$$u = \sum_{j=1}^{s} \sum_{i \in M_j} \xi_j c_i \quad \text{and} \quad v = \sum_{i=1}^{r} \beta_i c_i,$$

where $i \in M_j$ if and only if $\lambda_i(u) = \xi_j$. Now, let $v_j := \sum_{i \in M_j} \beta_i c_i$. In order to complete the proof, it remains to observe that $v_j \in \mathcal{J}_1(e_j)$. ■

2.7.8 Eigenvalues of operators

It is convenient to have a spectral description of the most common linear operators that act on a formally real Jordan algebra, namely $L(u)$ and Q_u. This description is highly useful to deduce easily some important inequalities between various quantities.

The spectral description of $L(u)$ is given in Section V.5 of [Koe99]. We include here its proof, because it gives an instructive illustration on how the Pierce and the spectral decomposition theorems interact with each other.

Proposition 2.7.31 (Eigenvalues and eigenspaces of $L(u)$) *Let us write $u = \sum_{i=1}^n \xi_i e_i$ for the decomposition of an element $u \in \mathcal{J}$ given by the unique eigenspaces spectral decomposition Theorem, and let $\mathcal{J}_{ij} := Q_{e_i,e_j}\mathcal{J}$ be the subspaces given by the second Pierce decomposition theorem for the system of idempotents $\{e_1, \ldots, e_n\}$.*

The eigenvalues of $L(u)$ are $\{(\xi_i + \xi_j)/2 | 1 \leq i \leq j \leq n\}$. The corresponding eigenspaces are $\{\mathcal{J}_{ij} | 1 \leq i \leq j \leq n\}$.

Proof

Let $v \in \mathcal{J}$ and v_{ij} be the projection of v on the Pierce subspace $\mathcal{J}_{ij}$ for every $1 \leq i \leq j \leq n$. From the second Pierce decomposition theorem and from Remark 2.6.7, we know that:

- $e_i v_{jj} = \delta_{ij} v_{jj}$,
- $e_i v_{jk} = v_{jk}/2$ if $j < k$ and $i = j$ or $i = k$,
- and $e_i v_{jk} = 0$ when i, j and k are three different numbers.

In view of these relations, we can write:

$$L(u)v = \left(\sum_{i=1}^n \xi_i e_i\right)\left(\sum_{j \leq k} v_{jk}\right) = \sum_{i=1}^n \xi_i v_{ii} + \frac{1}{2}\sum_{j<k}[\xi_j + \xi_k]v_{jk}.$$

In particular, if $v \in \mathcal{J}_{ij}$, we get that $uv = v(\xi_i + \xi_j)/2$. Hence the eigenvalues of $L(u)$ are:

$$\left\{ \frac{\xi_i + \xi_j}{2} \middle| 1 \leq i \leq j \leq n \right\},$$

corresponding to eigenspaces $\{\mathcal{J}_{ij} | 1 \leq i \leq j \leq r\}$ respectively. Of course, their respective multiplicity is equal to the dimension of the corresponding eigenspace. Note that the *set* of eigenvalues of $L(u)$ can also be written as:

$$\left\{ \frac{\lambda_i(u) + \lambda_j(u)}{2} \middle| 1 \leq i \leq j \leq r \right\}.$$

■

This result can further be used for characterizing the spectral decomposition of Q_u. In fact, we deduce here an interesting generalization of this decomposition practically for free.

Let $u, v \in \mathcal{J}$ be two elements that operator commute. From Proposition 2.7.29, we know that there exist a system of idempotents $\{e_1, \ldots, e_n\}$ and real numbers $\xi_1, \ldots, \xi_n$, $\xi'_1, \ldots, \xi'_n$ for which $u = \sum_{i=1}^n \xi_i e_i$ and $v = \sum_{i=1}^n \xi'_i e_i$, where we assume that the pairs (ξ_i, ξ'_i) are different.

Corollary 2.7.32 *With the above notation, the operator $Q_{u,v}$ has as eigenvalues*

$$\left\{ \frac{\xi_i \xi'_j + \xi_j \xi'_i}{2} \middle| 1 \leq i \leq j \leq n \right\}.$$

The eigenspace corresponding to $(\xi_i \xi'_j + \xi_j \xi'_i)/2$ is the direct sum of the subspaces $\mathcal{J}_{kl} := Q_{e_k,e_l}\mathcal{J}$ with $\xi_k \xi'_l + \xi_l \xi'_k = \xi_i \xi'_j + \xi_j \xi'_i$.

Proof

Let us fix $1 \le i \le j \le n$. On the subspace $\mathcal{J}_{ij}$, the operator $Q_{u,v}$ reduces to the following:

$$\begin{aligned} Q_{u,v}|_{\mathcal{J}_{ij}} &= [L(u)L(v) + L(v)L(u) - L(uv)]|_{\mathcal{J}_{ij}} = [2L(v)L(u) - L(uv)]|_{\mathcal{J}_{ij}} \\ &= \frac{(\xi_i + \xi_j)(\xi_i' + \xi_j')}{2} I - \frac{\xi_i\xi_i' + \xi_j\xi_j'}{2} I = \frac{\xi_i\xi_j' + \xi_j\xi_i'}{2} I, \end{aligned}$$

where I is the identity operator on $\mathcal{J}_{ij}$. The statement is hereby proved. In particular, if $u = v$, the eigenvalues of Q_u are $\{\xi_i\xi_j : 1 \le i \le j \le n\}$. ■

These spectral decomposition results allow us to demonstrate easily some relations between operators.

Remark 2.7.33 *Let $u = \sum_{i=1}^n \xi_i e_i \in \mathcal{J}$. The eigenspaces of $L(u)$ and Q_u are the same. The eigenvalue of $L(u)^2 - Q_u$ corresponding to $Q_{e_i,e_j}\mathcal{J}$ are:*

$$\left(\frac{\xi_i + \xi_j}{2}\right)^2 - \xi_i\xi_j = \left(\frac{\xi_i - \xi_j}{2}\right)^2,$$

so they are nonnegative. In other words, the operator $L(u)^2 - Q_u$ is positive semidefinite.■

Remark 2.7.34 *Assume that the elements u and v of $\mathcal{J}$ have nonnegative eigenvalues. Then $Q_u \equiv Q_v$ if and only if $u = v$. Here is a simple justification. Let $u = \sum_{i=1}^n \xi_i e_i$ and $v = \sum_{i=1}^m \xi_i' e_i'$ be the two unique eigenspaces spectral decomposition of these elements. If $Q_u \equiv Q_v$, these elements must either have the same number of different eigenvalues; thus $n = m$. Next, the sets of eigenvalues $\{\xi_i\xi_j\}_{1\le i\le j\le n}$ and $\{\xi_i'\xi_j'\}_{1\le i\le j\le n}$ of Q_u and Q_v must coincide. This implies $\xi_i = \xi_i'$ by virtue of their nonnegativity. Finally, the eigenspaces $Q_{e_i}\mathcal{J}$ and $Q_{e_i'}\mathcal{J}$ of Q_u and Q_v respectively must also coincide. In particular, their corresponding unit element e_i and e_i' must be the same.* ■

Below, we give an example of inequality induced by the previous results on spectral decomposition of operators.

Remark 2.7.35 *Let $u, v \in \mathcal{J}$. We have $||uv||_{\mathcal{J}} = ||L(u)v||_{\mathcal{J}} \le ||L(u)||||v||_{\mathcal{J}}$, where $||\cdot||$ is the norm on operators induced by $||\cdot||_{\mathcal{J}}$. Since $||L(u)|| = \max\{\lambda_1(u), -\lambda_r(u)\} \le \sqrt{\sum_{i=1}^r \lambda_i(u)^2} = ||u||_{\mathcal{J}}$, we obtain $||uv||_{\mathcal{J}} \le ||u||_{\mathcal{J}}||v||_{\mathcal{J}}$.* ■

The spectral decomposition theorems open more interesting possibilities, such as the introduction of *spectral functions*, which are real valued functions on $\mathcal{J}$ that only depend on the eigenvalues of the argument. It is also possible to describe the eigenvalues of an element as the optimal value of an optimization problem. This characterization, as well as further generalizations, are presented in Chapter 3.

2.7.9 Examples

Example 2.7.1 (Real symmetric matrices: Example 2.2.1 continued)

We have already observed that the Jordan algebra $\mathcal{H}_r(\mathbb{R})$ is formally real. We can then apply the unique eigenspaces spectral decomposition theorem and the complete spectral decomposition theorem.

Let U be a real symmetric matrix of dimensions $r \times r$. As already mentioned, the eigenvalues of U in the classical sense are equal to the eigenvalues in the sense of Jordan algebras. We denote them by $\lambda_1(U) \geq \cdots \geq \lambda_r(U)$, and the corresponding normed eigenvectors by $u_1, \ldots, u_r$. In view of the classic spectral decomposition theorem for symmetric matrices (see Theorem 4.1.5 of [HJ96]), we can write that $U = \sum_{i=1}^{r} \lambda_i(U) u_i u_i^T$. We define $C_i := u_i u_i^T$; note that C_i is a rank one projector. Since its trace is one, it is a minimal idempotent, and $\{C_1, \ldots, C_r\}$ is a Jordan frame. A complete spectral decomposition of U is given by $U = \sum_{i=1}^{r} \lambda_i(U) C_i$. Suppose now that:

$$\lambda_1(U) = \cdots = \lambda_{k_1}(U) > \lambda_{k_1+1}(U) = \cdots = \lambda_{k_2}(U) > \cdots \lambda_{k_s}(U).$$

Denote $M_j := \{k_{j-1}+1, \ldots, k_j\}$ (with $k_0 = 0$) and $E_j := \sum_{i \in M_j} C_i$, so that E_j is a projector of rank $|M_j|$. We have $U = \sum_{j=1}^{s} \lambda_{k_j}(U) E_j$. This spectral decomposition corresponds to the unique eigenspaces spectral decomposition of U. It turns out that the idempotent E_j is a projector on the eigenspace of U corresponding to the eigenvalue $\lambda_{k_j}(U)$.

The Jordan scalar product is in this context the classical Frobenius scalar product. Corollary 2.7.10 shows that the matrices $U \cdot V^2 \cdot U$ and $V \cdot U^2 \cdot V$ have the same eigenvalues. Proposition 2.7.29 on operator commutativity can be rewritten as the following well-known result: *two symmetric $r \times r$ matrices* commute *if and only if they share a basis of eigenvectors.* ■

Example 2.7.2 (Jordan algebras from associative algebras)
In this framework, we do not necessarily have a formally real algebra. Only Theorem 2.7.2 can be used. If $\mathcal{J}$ is the set of $r \times r$ real matrices, it reduces to the standard Jordan factorization (from *Camille* Jordan, not from *Pascual* Jordan). The nilpotent matrix v in Theorem 2.7.2 has the form $S^{-1}AS$, where S is a nonsingular $r \times r$ matrix, and A a matrix with a superdiagonal $(A_{12}, A_{23}, \ldots, A_{r-1,r})$ of 0 and 1, all the other coefficients being null. ■

Example 2.7.3 (Jordan algebra from a symmetric bilinear form)
It is instructive to note that an algebra of the type $[\mathbb{R}^N; \mu; e]$ is not necessarily formally real, even if it does not contain any nilpotent. Suppose indeed, given a basis $\{b_1, \ldots, b_N\}$ of $\mathbb{R}^N$, that $\mu(u,v) = u^T v$ and $e = (1, 0, \ldots, 0)^T$. Then, in view of the formula for the generic eigenvalues derived in Example 2.3.2, we have $\lambda_1(u) = u_1 + i\sqrt{u_2^2 + \cdots + u_N^2}$. In view of Proposition 2.7.1, $\mathcal{J}$ cannot be formally real, and Hypothesis 2.7.1 does not hold. The spectral decomposition given by Theorem 2.7.2 does not apply either. Indeed, only $v = 0$ makes the determinant $\mu(v,v)$ vanish. In view of our previous observations, it means that our algebra has only e as idempotent, and does not contain any nilpotent. ■

Example 2.7.4 (Jordan spin algebra)
The Jordan spin algebra $\mathcal{S}_n$ is formally real: if $\bar{u}^2 + \bar{v}^2 = 0$, then

$$0 = \mathrm{tr}(\bar{u}^2 + \bar{v}^2) = u_0^2 + u^T u + v_0^2 + v^T v,$$

and $\bar{u} = \bar{v} = 0$.

The rank of $\mathcal{J}$ is equal to 2. Hence, all the idempotents that are different from $\bar{e}$ are minimal. We have for every $\bar{u} \in \mathcal{J}$:

$$\lambda_1(\bar{u}) = u_0 + ||u||_2 \quad \text{and} \quad \lambda_2(\bar{u}) = u_0 - ||u||_2,$$

where $||\cdot||_2$ is the Euclidean norm on $\mathbb{R}^n$. Hence, if $\bar{u}$ is not a multiple of $\bar{e}$, i.e. if $\lambda_1(\bar{u}) \neq \lambda_2(\bar{u})$, we have the following spectral decomposition:

$$\bar{u} = \frac{u_0 + ||u||_2}{2} \begin{pmatrix} 1 \\ u/||u||_2 \end{pmatrix} + \frac{u_0 - ||u||_2}{2} \begin{pmatrix} 1 \\ -u/||u||_2 \end{pmatrix}.$$

The Jordan scalar product is here $\langle \bar{u}, \bar{v} \rangle_{\mathcal{J}} = 2(u_0 v_0 + u^T v)$, and the corresponding norm is $||\bar{u}||_{\mathcal{J}} = \sqrt{2(u_0^2 + ||u||_2^2)}$. ∎

2.8 Cone of squares

The set of square elements, or *cone of squares*, of a formally real Jordan algebra plays a central role not only in the derivation of fundamental algebraic properties, but also in a broad variety of applications, ranging from statistics [MN98] to optimization [Fay97b]. Its geometrical characteristics were investigated by Max Koecher, under the name of "domain of positivity theory" [Koe99], while Vinberg has studied its algebraic properties, under the name of "homogeneous domains".

Optimizers show a special interest in the cone of squares, as optimization problems on Jordan algebras often have as feasible region an intersection of this set with an affine space. Benefiting from the many properties of the cone of squares, we can design powerful optimization algorithms for this kind of problems (see Section 1.4). In this section, we present several basic characteristics of this set.

Definition 2.8.1 *The* cone of squares *of $\mathcal{J}$ is the set $\mathcal{K}_{\mathcal{J}} := \{v | v = u^2 \text{ for some } u \in \mathcal{J}\}$.* ∎

Note that the *sums of squares cone* that one encounters in the study of real or complex polynomials is *not* a cone of squares in the framework of Jordan algebras. The objects we deal with here depend heavily on the specific multiplication of $\mathcal{J}$. The interested reader can consult for further details the preprint [KM04], where such a sums of squares cone is defined and studied in the framework of formally real Jordan algebras.

The following proposition brings together some basic properties of the cone of squares. We omit its proof, as it is essentially a trivial consequence of the spectral decomposition theorems and of the continuity of eigenvalues. More details are given in Section III.2 of [FK94].

Proposition 2.8.2 *The set $\mathcal{K}_{\mathcal{J}}$ introduced in Definition 2.8.1 is a closed cone. An element $v \in \mathcal{J}$ belongs to $\mathcal{K}_{\mathcal{J}}$ if and only if its eigenvalues are nonnegative; in that case, there exists an element $u \in \mathcal{K}_{\mathcal{J}}$ such that $u^2 = v$. The interior of $\mathcal{K}_{\mathcal{J}}$ is the set of all elements that have positive eigenvalues, i.e. the set of all invertible elements of $\mathcal{K}_{\mathcal{J}}$.* ∎

As an immediate consequence of this, we can formulate the following statement.

Remark 2.8.3 *Let $u \in \mathcal{J}$. The element $v := u - \lambda_r(u)e$ belongs to the boundary of $\mathcal{K}_{\mathcal{J}}$. Indeed, its eigenvalues $\lambda_i(u) - \lambda_r(u)$ are all nonnegative, and the last one is equal to 0. Similarly, $\lambda_1(u)e - u$ is also on the boundary of $\mathcal{K}_{\mathcal{J}}$.* ∎

Definition 2.8.4 *The* dual cone *of* $\mathcal{K}_{\mathcal{J}}$ *is the set*

$$\mathcal{K}^*_{\mathcal{J}} := \{v \in \mathcal{J} | \langle v, u\rangle = \mathrm{tr}(uv) \geq 0 \ \textit{for all}\ u \in \mathcal{K}_{\mathcal{J}}\}.$$

■

Since the operator $v \mapsto \mathrm{tr}(L(u)v)$ is continuous, the set $\mathcal{K}^*_{\mathcal{J}}$ is a closed cone. Note that $\mathcal{K}^*_{\mathcal{J}}$ is a set of $\mathcal{J}$ (*not* of the dual $\mathcal{J}$) and is strongly linked with the Jordan scalar product.

Definition 2.8.5 *A cone* $\mathcal{K}$ *is* pointed *if* $\pm u \in \mathcal{K}$ *implies* $u = 0$. ■

Theorem 2.8.6 *The cone of squares* $\mathcal{K}_{\mathcal{J}}$ *is a convex pointed cone that has a non-empty interior. Moreover, it is self-dual:* $\mathcal{K}_{\mathcal{J}} = \mathcal{K}^*_{\mathcal{J}}$.

Sketch of the proof
It is easier to study first the properties of the cone $\mathcal{K}^*_{\mathcal{J}}$. Like every dual set, it is a convex cone. We have $v \in \mathcal{K}^*_{\mathcal{J}}$ iff $\forall u \in \mathcal{J}, \langle v, u^2\rangle = \langle vu, u\rangle = \langle L(v)u, u\rangle \geq 0$ iff $L(v)$ is a positive semidefinite linear operator, and the inclusion $\mathcal{K}_{\mathcal{J}} \subseteq \mathcal{K}^*_{\mathcal{J}}$ follows directly from Proposition 2.7.31. For the other inclusion, let $v = \sum_{i=1}^r \lambda_i(v)c_i \in \mathcal{K}^*_{\mathcal{J}}$. Since $c_i = c_i^2 \in \mathcal{K}_{\mathcal{J}}$ and $\mathrm{tr}(c_i) = 1$, we have $0 \leq \langle v, c_i\rangle = \lambda_i(v)$. From Proposition 2.8.2, $v \in \mathcal{K}_{\mathcal{J}}$. The interior of $\mathcal{K}_{\mathcal{J}}$ is nonempty, because $\lambda_i(e) = 1 > 0$. Finally, $\pm u \in \mathcal{K}_{\mathcal{J}}$ implies $\lambda_i(u) = 0$ for every i. ■

Definition 2.8.7 *A subset* Q *of* $\mathcal{J}$ *is* homogeneous *if, for every pair of elements* u *and* v *of* Q*, there exists an invertible linear application* H *from* $\mathcal{J}$ *to* $\mathcal{J}$ *that maps bijectively* Q *to itself and for which* $Hu = v$. ■

Suppose that a certain point of a set plays a particular role, as for instance the unit e in the cone of squares. If this set is homogeneous, each of its points can play the role of this particular point using an invertible linear application. Hence, all the points of a homogeneous set are undistinguishable from this geometrical viewpoint. The homogeneity of $\mathcal{K}_{\mathcal{J}}$ stated in Theorem 2.8.8 below allowed Koecher and Braun to define the so-called *mutations* of Jordan algebras. These algebras are constructed from a Jordan algebra $\mathcal{J}$ by choosing first an arbitrary point f of $\mathrm{int}\,\mathcal{K}_{\mathcal{J}}$ as the unit point of the mutation, and then by defining its quadratic operator as $Q_u^{(f)} := Q_u Q_f^{-1}$ (see Chapter V in [BK66], Section IV.2 in [Koe99], or, under the name of *isotope algebra*, in [McC03]).

The following theorem indicates another crucial property of the quadratic operator. Its proof is given in Theorem III.2.1 of [FK94]. We rewrite it here, because this reasoning will be reused to establish a similar property for the multiplication operator $L(u)$.

Theorem 2.8.8 *The cone* $\mathcal{K}_{\mathcal{J}}$ *is homogeneous. For every invertible* $u \in \mathcal{J}$*, the application* Q_u *maps bijectively* $\mathcal{K}_{\mathcal{J}}$ *into itself.*

Proof
Let us fix an invertible element u in $\mathcal{J}$. Theorem 2.5.2 shows that the operator Q_u is also invertible. The application $v \mapsto Q_u v$ is continuous, and the set $\mathrm{int}\,\mathcal{K}_{\mathcal{J}}$ is simply connected because it is convex. Hence, $S := Q_u(\mathrm{int}\,\mathcal{K}_{\mathcal{J}})$ is also a simply connected set. The element $Q_u u'$ is invertible for every $u' \in \mathrm{int}\,\mathcal{K}_{\mathcal{J}}$ in view of Theorem 2.5.2. Hence S does not cross

the boundaries of $\mathcal{K}_\mathcal{J}$. Next, $Q_u e = u^2 \in \operatorname{int}\mathcal{K}_\mathcal{J}$. Since S is connected and contains e, it is entirely in $\operatorname{int}\mathcal{K}_\mathcal{J}$. So,

$$Q_u^{-1} S = \operatorname{int}\mathcal{K}_\mathcal{J} \subseteq Q_u^{-1}\operatorname{int}\mathcal{K}_\mathcal{J} = Q_{u^{-1}}\operatorname{int}\mathcal{K}_\mathcal{J} \subseteq \operatorname{int}\mathcal{K}_\mathcal{J}.$$

The last inclusion comes from the same argument developed for u^{-1} instead of for u. Applications of the form Q_u restricted to $\operatorname{int}\mathcal{K}_\mathcal{J}$ are thus bijective.

Now, let $v, w \in \operatorname{int}\mathcal{K}_\mathcal{J}$. Note that $Q_{v^{-1}} v^2 = e$ and $Q_w e = w^2$. Hence the invertible linear application $A := Q_w Q_{v^{-1}}$ maps v^2 on w^2. We showed that A is an automorphism of $\operatorname{int}\mathcal{K}_\mathcal{J}$; by continuity, A is an automorphism of $\mathcal{K}_\mathcal{J}$ and the cone of squares is thus homogeneous. ■

The following remark has been proved by Bharath Rangarajan in Lemma 2.12 of [Ran06], under the name of *Lyapunov Lemma for Euclidean Jordan algebras*. As our argument is very different, we include a proof below.

Remark 2.8.9 *Following the lines of the previous reasoning, we can show that for every invertible $u \in \mathcal{J}$ we have:*

$$L(u^2)^{-1}\operatorname{int}\mathcal{K}_\mathcal{J} \subseteq \operatorname{int}\mathcal{K}_\mathcal{J} \quad \text{and} \quad L(u^2)^{-1}\mathcal{K}_\mathcal{J} \subseteq \mathcal{K}_\mathcal{J}.$$

We already know that $L(u^2)$ is invertible for every invertible u and that it is a continuous function of u. Since $\operatorname{int}\mathcal{K}_\mathcal{J}$ is simply connected, we deduce that the set $S := L(u^2)^{-1}(\operatorname{int}\mathcal{K}_\mathcal{J})$ is also simply connected.

We prove now that $L(u^2)^{-1}u'$ is invertible for each $u' \in \operatorname{int}\mathcal{K}_\mathcal{J}$. Suppose on the contrary that there exists an element $u' \in \operatorname{int}\mathcal{K}_\mathcal{J}$ for which $v := L(u^2)^{-1}u'$ is not invertible. Then, there exist an idempotent c such that $vc = 0$. The contradiction follows from $0 = \operatorname{tr}(c(vu^2)) = \operatorname{tr}(cu') \geq \lambda_r(u') > 0$. In other words, S does not cross the boundaries of $\mathcal{K}_\mathcal{J}$. It remains to note that $L(u^2)^{-1}e = u^{-2} \in \operatorname{int}\mathcal{K}_\mathcal{J}$ to conclude that S is entirely in $\operatorname{int}\mathcal{K}_\mathcal{J}$. A continuity argument proves that $L(u^2)^{-1}\mathcal{K}_\mathcal{J} \subseteq \mathcal{K}_\mathcal{J}$, or $\mathcal{K}_\mathcal{J} \subseteq L(u^2)\mathcal{K}_\mathcal{J}$. In other words, given an element $v \in \mathcal{K}_\mathcal{J}$, the unique solution x of the equation $u^2 x = v$ belongs to $\mathcal{K}_\mathcal{J}$. A continuity argument shows that the inclusion $\mathcal{K}_\mathcal{J} \subseteq L(u^2)\mathcal{K}_\mathcal{J}$ also holds for non-invertible elements u. ■

The next proposition contains two simple results of crucial importance in the design of interior-point methods within the framework of Jordan algebras. The first part has been proven in Lemma 2.2 of [Fay97b].

Proposition 2.8.10 *Let $u, v \in \mathcal{K}_\mathcal{J}$. Then $\operatorname{tr}(uv) \geq 0$. We have $\operatorname{tr}(uv) = 0$ if and only if $uv = 0$.*

Let $x \in \mathcal{K}_\mathcal{J}$ be such that $u = x^2$, and let $(v_m)_{m\geq 0} \subset \mathcal{K}_\mathcal{J}$. Suppose that this sequence satisfies $\lim_{m\to\infty} \operatorname{tr}(uv_m) = 0$. Then $\lim_{m\to\infty} Q_x v_m = 0$.

Proof

We only prove the second part here. We take $w_m := Q_x v_m$ for all $m \geq 0$, so that $(w_m)_{m\geq 0} \subset \mathcal{K}_\mathcal{J}$ in view of Theorem 2.8.8. Hence $\lambda_i(w_m) \geq 0$ for every $1 \leq i \leq r$ and every $m \geq 0$. By hypothesis, $\lim_{m\to\infty} \operatorname{tr}(w_m) = \lim_{m\to\infty} \operatorname{tr}(uv_m) = 0$, i.e. $\sum_{i=1}^r \lambda_i(w_m)$ tends to 0 as m goes to ∞. Thus $\lim_{m\to\infty} \lambda_i(w_m) = 0$ for all $1 \leq i \leq r$, and then $\lim_{m\to\infty} w_m = 0$ in view of the complete spectral decomposition theorem. ■

2.8.1 Examples

The cone of squares of the algebra $\mathcal{H}_r(\mathbb{R})$ of $r \times r$ symmetric matrices is the cone of positive semidefinite matrices, frequently denoted by $\mathbb{S}^r_+$ by optimizers.

In the Jordan spin algebra $\mathcal{S}_n$, the square of an element $\bar{u} = \begin{pmatrix} u_0 \\ u \end{pmatrix}$ is

$$\bar{u}^2 = \begin{pmatrix} u_0^2 + ||u||_2^2 \\ 2u_0 u \end{pmatrix},$$

where $||\cdot||_2$ represents here the Euclidean norm of the n-dimensional vector u.

Since the eigenvalues of an element $\bar{v} = \begin{pmatrix} v_0 \\ v \end{pmatrix}$ are $\lambda_1(\bar{v}) = v_0 + ||v||_2$ and $\lambda_2(\bar{v}) = v_0 - ||v||_2$, the cones of squares of $\mathcal{S}_n$ can be written as $\mathcal{K}_{\mathcal{J}} = \{\bar{v} \in \mathcal{S}_n | v_0 \geq ||v||_2\}$. This cone is also known as the *second-order cone*, the *Lorentz cone*, the *light cone*, or the *ice-cream cone*.

Mmmmmmmmmmmmmmmmmm!! Let us specialize $\mathcal{K}_{\mathcal{J}}$ to a Larive-ice-cream cone. By definition, $\mathcal{K}_{\mathcal{J}}$ lies in Belgium. With z=car belonging to M=the set of technical devices at our disposal and i an element in the set of interested people, i can enter $\mathcal{K}_{\mathcal{J}}$ given z with a speed that is maximal for the subset of interested people given by the 3 musketeers. :-))

2.9 Simple Jordan algebras

Definition 2.9.1 *Let $\mathcal{J}$ be an F-algebra. An* ideal *$\mathcal{I}$ of $\mathcal{J}$ is a vector space of $\mathcal{J}$ such that for every $a \in \mathcal{I}$ and every $u \in \mathcal{J}$, the elements au and ua belong to $\mathcal{I}$.* ■

Definition 2.9.2 *An F-algebra $\mathcal{J}$ is* simple *if it contains only two ideals, namely $\{0\}$ and $\mathcal{J}$, and if $\mathcal{J} \circ \mathcal{J} \neq \{0\}$.* ■

The ideals $\{0\}$ and $\mathcal{J}$ are called the *trivial ideals* of $\mathcal{J}$.

This proposition characterizes simple Jordan algebras using the first Pierce decomposition. We include its proof, from which it results that every Jordan algebra $\mathcal{J}$ can be decomposed into a direct sum of simple algebras of the form $\mathcal{J}_1(c)$, where c is an idempotent of $\mathcal{J}$.

Proposition 2.9.3 *Let F be an infinite field and let $\mathcal{J}$ be a Jordan algebra of finite dimension over the field F. The algebra $\mathcal{J}$ is simple if and only if $\mathcal{J}_{1/2}(c) \neq \{0\}$ for every idempotent $c \neq e$ of $\mathcal{J}$.*

Proof
Suppose that $\mathcal{J}$ is simple and that there exists an idempotent $c \in \mathcal{J}$, different from e, for which $\mathcal{J}_{1/2}(c) = \{0\}$. By Theorem 2.6.1, the subalgebra $\mathcal{J}_1(c)$ is an ideal of $\mathcal{J}$. Hence $\mathcal{J}_1(c) = \mathcal{J}$, and c is a unit element of $\mathcal{J}$, a contradiction.

Conversely, if $\mathcal{J}$ is not simple, it contains a non-trivial ideal $\mathcal{I}$. Let $\mathcal{I}'$ be the complementary vector space of $\mathcal{I}$ in $\mathcal{J}$. An immediate verification shows that $\mathcal{I}'$ is also an ideal and that $\mathcal{I} \circ \mathcal{I}' = \{0\}$. Since $\mathcal{I}$ is a subalgebra, it contains a unit element $c \neq e$. We then have $\mathcal{J}_1(c) = \mathcal{I}$ and $\mathcal{J}_0(c) = \mathcal{I}'$. Thus $\mathcal{J}_{1/2}(c) = \{0\}$. ■

The following theorem has firstly been proved in Fundamental Theorem 2 in the seminal paper [JvNW34]. A modern exposition can be found in Chapter V of [FK94].

Theorem 2.9.4 *Every finite-dimensional formally real Jordan algebra is a direct sum of a finite number of* simple *formally real Jordan algebras. Every simple formally real Jordan algebra is isomorphic to one of these 5 types of algebras.*

1. *The algebra $\mathcal{H}_r(\mathbb{R})$ for $r \geq 1$;*
2. *the algebra $\mathcal{H}_r(\mathbb{C})$ for $r \geq 1$;*
3. *the algebra $\mathcal{H}_r(\mathbb{H})$ for $r \geq 1$;*
4. *the algebra $\mathcal{H}_3(\mathbb{O})$;*
5. *the algebra $\mathcal{S}_n$ for $n \geq 1$.*

The symbol "$\mathbb{H}$" stands for Hamilton's quaternions algebra, and "$\mathbb{O}$" designates the algebra of Cayley's octonions. ■

2.10 Automorphisms

In this section, we present some classical results on linear applications that map a formally real Jordan algebra into itself. More particularly, we focus on an important class of those linear applications, namely on the set of *automorphisms*.

We denote henceforth the set of all invertible linear applications from $\mathcal{J}$ to $\mathcal{J}$ by $G(\mathcal{J})$.

Definition 2.10.1 *An* automorphism *V of $\mathcal{J}$ is a linear application of $G(\mathcal{J})$ such that for every u and v in $\mathcal{J}$, we have $V(uv) = V(u)V(v)$. We denote by $\mathcal{A}(\mathcal{J})$ the set of automorphisms of $\mathcal{J}$.* ■

We can immediately deduce that $\mathcal{A}(\mathcal{J})$ is not empty, as $I_N \in \mathcal{A}(\mathcal{J})$.

Remark 2.10.2 *Let u be an element of $\mathcal{J}$ and V be an automorphism in $\mathcal{A}(\mathcal{J})$. Then $\lambda(u) = \lambda(Vu)$. Here is a short proof.*

We first show that u and Vu have the same set of eigenvalues, regardless of their multiplicities. Note that the set of roots of $f(\tau; u)$ is $\{\lambda_i(u) | 1 \leq i \leq r\}$. Also:

$$\begin{aligned} f(Vu; u) &= (Vu)^r - a_1(u)(Vu)^{r-1} + \cdots + (-1)^r a_r(u)e \\ &= V[u^r - a_1(u)u^{r-1} + \cdots + (-1)^r a_r(u)e] = Vf(u; u) = 0. \end{aligned}$$

Thus, the minimal polynomial $\mu_{Vu}(\tau)$ divides $f(\tau; u)$, so that $\{\lambda_i(Vu) | 1 \leq i \leq r\} \subseteq \{\lambda_i(u) | 1 \leq i \leq r\}$. Since V is invertible, we can exchange the roles of u and Vu, and we conclude that these two elements have the same set of eigenvalues.

Now, if u is a regular element, i.e. if all the eigenvalues of u are distinct, then the eigenvalues of the element Vu are also all distinct. Henceforth, $\lambda(u) = \lambda(Vu)$ for regular elements u. In order to reach the final statement for non-regular elements u, it suffices to use the continuity of eigenvalues and the density of regular elements in $\mathcal{J}$. ■

It turns out that the reciprocal statement is also true for *simple* Jordan algebras (see Corollary 2.10.12 below).

2.10.1 The structure group

The fundamental concept of *structure group* appears to be the spinal chord of the work of Max Koecher (see [BK66], especially Section II.2 and the whole Chapter III) and of Tonny Springer (see [Spr73]). It is possible to define this notion in the much more general context of *non-commutative* algebras, but considerable efforts are needed in this broader framework even to obtain the very first useful fundamental result. In this exposition, we restrict ourselves to a finite-dimensional formally real Jordan algebra $\mathcal{J}$.

Definition 2.10.3 *The* structure group *of $\mathcal{J}$ is the set:*

$$\Gamma(\mathcal{J}) := \{V \in G(\mathcal{J}) \mid \text{ there exists } W \in G(\mathcal{J}) \text{ such that, for all invertible } u \in \mathcal{J}, \text{ we have } (Vu)^{-1} = W^{-1}u^{-1}\}.$$

■

In the context of Jordan algebras, the quadratic operator Q_u belongs to $\Gamma(\mathcal{J})$ for each invertible u of $\mathcal{J}$ in view of Theorem 2.5.2. Moreover, since $(Q_u v)^{-1} = Q_u^{-1} v^{-1}$, we deduce that one can take $W := Q_u$ for the application needed in the above definition for $V := Q_u$.

Note also that if V belongs to $\Gamma(\mathcal{J})$, the corresponding W also belongs to $\Gamma(\mathcal{J})$. Indeed, the relation $W(Vu)^{-1} = u^{-1}$ holds for every invertible u, and, with $v := (Vu)^{-1}$, we obtain $Wv = (V^{-1}v^{-1})^{-1}$. The invertibility of V insures that this equality is satisfied for every invertible v.

Proposition 2.10.5 clarifies the definition of structure group: actually, for every $V \in \Gamma(\mathcal{J})$, the corresponding application W for which $(Vu)^{-1} = W^{-1}u{-}1$ for every invertible $u \in \mathcal{J}$ is uniquely defined. Moreover, it is the adjoint of V with respect to the Jordan scalar product. In order to understand the proof of this proposition, we first need the important lemma below, which follows the lines of Sections II.5.1 and II.5.2 of [BK66].

Lemma 2.10.4 *For each $V \in \Gamma(\mathcal{J})$, there exists a real number κ_V such that* $\det(Vu) = \kappa_V \det(u)$ *for every $u \in \mathcal{J}$. Moreover, $\kappa_{V^{-1}} = \kappa_V^{-1}$ and, for every $W \in G(\mathcal{J})$ that satisfies $(Vu)^{-1} = W^{-1}u^{-1}$ for all invertible u of $\mathcal{J}$, we have $\kappa_V = \kappa_W$.*

Sketch of the proof
The proof of the existence of κ_V uses the same argument as in Proposition 2.5.4.

Taking $u := e$ yields $\kappa_V = \det(Ve)$. Hence, we have:

$$1 = \det(VV^{-1}e) = \kappa_V \det(V^{-1}e) = \kappa_V \kappa_{V^{-1}},$$

so that $\kappa_{V^{-1}} = \kappa_V^{-1}$. Finally, by multiplicativity of the determinant (see Proposition 2.3.34), we can write:

$$1 = \det(Ve(Ve)^{-1}) = \det(Ve)\det((Ve)^{-1}) = \kappa_V \det(W^{-1}e) = \frac{\kappa_V}{\kappa_W},$$

and $\kappa_V = \kappa_W$. ■

The following proposition comes partially from Proposition VIII.5.2 of [FK94]. The proof of the last part is ours.

Proposition 2.10.5 *Let u and v be two arbitrary elements of $\mathcal{J}$. Then $V \in \Gamma(\mathcal{J})$ if and only if there exists an application $W \in G(\mathcal{J})$ such that $Q_{Vu} = VQ_uW$. This application W is unique. We also have $\sigma(Vu, v) = \sigma(u, Wv)$ for the function σ defined in Proposition 2.7.9. Moreover, $\langle u, Wv\rangle = \langle Vu, v\rangle$.*

Sketch of the proof

It is sufficient to prove this statement only for invertible $u, v \in \mathcal{J}$. To prove that $Q_{Vu} = VQ_uW$, it suffices to differentiate the relation $(Vu)^{-1} = W^{-1}u^{-1}$ in u. The converse comes immediately from the relation $Q_v^{-1}v = v^{-1}$.

The application W can be reformulated as $W = Q_u^{-1}V^{-1}Q_{Vu}$. It is thus uniquely defined. With Lemma 2.10.4, the relation $\sigma(Vu, v) = \sigma(u, Wv)$ is a routine two-lines computation.

To prove the last part, consider the extension field $\bar{F}$ of $\mathbb{R}$ given by Proposition 2.7.6 (it is actually $\mathbb{C}$), and the extension $\bar{\mathcal{J}} := \mathcal{J} \bigotimes_{\mathbb{R}} \bar{F}$, where every invertible element of $\mathcal{J}$ has its square root.

Let us fix two invertible elements $u, v \in \bar{\mathcal{J}}$. We aim to prove that, for each $1 \leq i \leq r$, the relation $a_i(Q_u^{1/2}Wv) = a_i(Q_{Vu}^{1/2}v)$ holds, where a_i is the ith coefficient of the characteristic polynomial. By the expression of the characteristic polynomial via the determinant (see Proposition 2.3.16), it suffices to check if $\det(\tau e - Q_u^{1/2}Wv) = \det(\tau e - Q_{Vu}^{1/2}v)$. The verification below relies on Proposition 2.5.4 and on the previous lemma.

$$\begin{aligned}
&\det(\tau e - Q_u^{1/2}Wv)\\
= \;&\det(Q_u^{1/2}(\tau u^{-1} - Wv)) = \det(u)\det(\tau u^{-1} - Wv)\\
= \;&\det(u)\det(W(\tau W^{-1}u^{-1} - v)) = \kappa_W \det(u)\det(\tau(Vu)^{-1} - v)\\
= \;&\kappa_V \det(u)\det(\tau(Vu)^{-1} - v) = \det(Vu)\det(\tau(Vu)^{-1} - v)\\
= \;&\det(Q_{Vu}^{1/2}(\tau(Vu)^{-1} - v)) = \det(\tau e - Q_{Vu}^{1/2}v).
\end{aligned}$$

In particular, when the index i equals 1, we get:

$$\langle u, Wv\rangle = \mathrm{tr}(u(Wv)) = \mathrm{tr}(Q_u^{1/2}Wv) = \mathrm{tr}(Q_{Vu}^{1/2}v) = \mathrm{tr}((Vu)v) = \langle Vu, v\rangle.$$

The second and the fourth equality are justified by the associativity of the trace. ■

The uniqueness of the application W that corresponds in the definition to an application V of $\Gamma(\mathcal{J})$ allows us to introduce the notation: $V^* := W$, so that $(Vu)^{-1} = (V^*)^{-1}u^{-1}$. For instance, the last relation shown in the previous proposition can be written as $\langle Vu, v\rangle = \langle u, V^*v\rangle$. Thus, we can call the application V^* the *Jordan adjoint* of V. If $V = V^*$, we say that V is *self-adjoint.*

Remark 2.10.6 *The Jordan adjoint operator has the following properties: for every $V, W \in \Gamma(\mathcal{J})$, we have:*

- $\diamond$ $VW \in \Gamma(\mathcal{J})$ *and* $(VW)^* = W^*V^*$,
- $\diamond$ $(V^{-1})^* = (V^*)^{-1}$,
- $\diamond$ $(V^*)^* = V$ *and*
- $\diamond$ $Q_u^* = Q_u$ *for every* $u \in \mathcal{J}$.

Indeed, $(VWu)^{-1} = (V^*)^{-1}(Wu)^{-1} = (V^*)^{-1}(W^*)^{-1}u^{-1}$ *shows the first item; if* $v := V^{-1}u$, *we have* $(V^*)^{-1}v^{-1} = (Vv)^{-1} = u^{-1}$ *or* $(V^{-1}u)^{-1} = V^*u^{-1}$, *which shows the second one. Next, if* $v := V^{-1}u^{-1}$, *we have* $(V^*)^{-1}v^{-1} = (Vv)^{-1} = u$ *or* $(V^*u)^{-1} = v = V^{-1}u^{-1}$*; this is exactly the third item. The fourth one comes from the Fundamental Identity* $Q_{Q_u v} = Q_u Q_v Q_u$. ■

2.10.2 Automorphisms of Jordan algebras

As the following proposition shows, automorphisms of Jordan algebras can be described as particular elements of the structure group. Its proof appears in Satz IV.6.1 of [BK66].

Proposition 2.10.7 *A linear application A is in $\mathcal{A}(\mathcal{J})$ if and only if $A \in \Gamma(\mathcal{J})$ and $Ae = e$. In that case, we have $A^* = A^{-1}$.*

Sketch of the proof
Suppose first that $A \in \mathcal{A}(\mathcal{J})$. Trivially, $Ae = e$. It is easy to show that $AQ_u v = Q_{Au}Av$ for every u and v of $\mathcal{J}$. Hence $AQ_u A^{-1} = Q_{Au}$. In view of Proposition 2.10.5, we deduce that $A \in \Gamma(\mathcal{J})$ and that $A^* = A^{-1}$.

Conversely, if $A \in \Gamma(\mathcal{J})$ and $Ae = e$, we have $AQ_u A^* = Q_{Au}$ for every $u \in \mathcal{J}$; taking $u := e$ gives $AA^* = Q_{Ae} = Q_e = I_N$ and $A^* = A^{-1}$. Thus $AQ_u = Q_{Au}A$. Applying this operator to e, we get $AQ_u e = A(u^2) = Q_{Au}Ae = (Au)^2$. It remains to polarize this expression to get $A(uv) = A(u)A(v)$. ■

Remark 2.10.8 *We immediately deduce from this lemma that $Q_w \in \mathcal{A}(\mathcal{J})$ if and only if $w^2 = e$.*

Let us denote by $\mathcal{W}(\mathcal{J})$ the set of all elements for which $w^2 = e$. The element w belongs to $\mathcal{W}(\mathcal{J})$ if and only if its eigenvalues equal ± 1. Hence $w \in \mathcal{W}(\mathcal{J})$ if and only if $w = e$, or there exists an idempotent c such that $w = 2c - e$. ■

We present below the polar decomposition theorem for formally real Jordan algebras. Its matrix version can be formulated as follows (see for instance in [HJ96], Theorem 7.3.2 and Corollary 7.3.3):

> Let A be an $n \times n$ real matrix. There exist an orthogonal $n \times n$ matrix U and a real self-adjoint matrix S such that $A = US$. ■

In the context of linear operators of the structure group of a formally real Jordan algebra, the applications of $\mathcal{A}(\mathcal{J})$ play the role of unitary transformations of $\mathcal{J}$, while applications of type Q_u can be seen as self-adjoint transformations of $\mathcal{J}$.

Theorem 2.10.9 (Polar decomposition theorem) *Let $V \in \Gamma(\mathcal{J})$ be a linear application such that $Ve \in \mathcal{K}_{\mathcal{J}}$. Then there exist an automorphism $A \in \mathcal{A}(\mathcal{J})$ and an element $u \in \operatorname{int} \mathcal{K}_{\mathcal{J}}$ such that $V = Q_u A$.*

Proof

Since V is invertible, the element Ve is also invertible. As it is in $\mathcal{K}_{\mathcal{J}}$, there exists an invertible element $u \in \mathcal{K}_{\mathcal{J}}$ such that $u^2 = Ve$. Note that $Q_u e = Ve$, i.e. $A := Q_u^{-1} V$ satisfies $Ae = e$. The Remark 2.10.6 shows that $Q_u^{-1} = Q_{u^{-1}} \in \Gamma(\mathcal{J})$ and that $\Gamma(\mathcal{J})$ is closed under composition. Hence $A \in \Gamma(\mathcal{J})$ and, by Proposition 2.10.7, $A \in A(\mathcal{J})$. ■

The next lemma makes an important step in the proof of the main theorem of this section, and of the converse of Remark 2.10.2. This proof represents an original contribution, although the idea of considering the subalgebra $\mathbb{R}c+\mathbb{R}d+\mathbb{R}cd$ comes from [BK66], Chapter IV. Unfortunately, our argument is quite technically involved.

Lemma 2.10.10 *Let $c, d \in \mathcal{J}$ be two minimal idempotents for which $\mathcal{J}_{1/2}(c) \cap \mathcal{J}_{1/2}(d) = \{0\}$. There exists an element w of $\mathcal{W}(\mathcal{J})$ for which $Q_w c = d$ if $\operatorname{tr}(cd) \neq 0$ or if $\mathcal{J}$ is a simple Jordan algebra.*

Proof

Let $\alpha := \operatorname{tr}(cd)$; obviously, $0 \leq \alpha \leq 1$. When $\alpha = 1$, there is nothing to prove: we have $c = d$ and we can take $w := e$.

A. The case $\alpha \neq 0, 1$.

1. **The subalgebra $\mathbb{R}c + \mathbb{R}d + \mathbb{R}cd$.** Observe first that in view of Proposition 2.7.17, we know that $Q_c d \in \mathcal{J}_1(c) = \mathbb{R}c$ and $Q_d c \in \mathcal{J}_1(d) = \mathbb{R}d$. Hence $Q_c d = \alpha c$ and $Q_d c = \alpha d$. In order to check that $A := \mathbb{R}c + \mathbb{R}d + \mathbb{R}cd$ is a subalgebra of $\mathcal{J}$, we have to prove that $c(cd), d(cd)$ and $(cd)^2$ are all in A. Actually, we have:
   - ◇ $2c(cd) = \alpha c + cd$,
   - ◇ $2d(cd) = \alpha d + cd$, and
   - ◇ $4(cd)^2 = \alpha(c + d + 2cd)$.

   Indeed, we can write $2c(cd) = Q_c d + cd = \alpha c + cd \in A$. Exchanging the roles of c and d, we get $2d(cd) = \alpha d + cd \in A$. Next, using the relation (2.4) with $u := c$, $v := c$ and $w := d$, we have:

   $$2L(c)L(cd) + L(d)L(c) = 2L(cd)L(c) + L(c)L(d).$$

   If we apply the left-hand side to d, we obtain:

   $$2c((cd)d) + d(cd) = cQ_d c + c(cd) + d(cd) = \alpha cd + \frac{\alpha(c + d) + 2cd}{2},$$

   and equating this to the right-hand side applied to d, we have:

   $$\alpha cd + \frac{\alpha}{2}(c + d) + cd = 2(cd)^2 + cd.$$

Thus
$$4(cd)^2 = \alpha(c + d + 2cd) \in A.$$
The algebra A is a formally real Jordan subalgebra of rank 2. Its unit element is:
$$f := \frac{c+d}{1-\alpha} - \frac{2cd}{1-\alpha}.$$
The element $c = \mu c + \nu d + \rho cd$ is a minimal idempotent if and only if $\rho^2\alpha = 4\mu\nu$ and $\mu + \nu + \rho\alpha = 1$. The idempotent c' for which $\mu = \nu$ is of special interest. Observe that its coefficient ρ satisfies then the quadratic relation $\rho^2\alpha = (\rho\alpha - 1)^2$.

2. An element $w \in \mathcal{W}(\mathcal{J})$ for which $Q_w c = d$. It can be checked that the element $w := 2c' - e$, which belongs to $\mathcal{W}(\mathcal{J})$, satisfies the required relation. Indeed, a straightforward computation yields:
$$c'(c'c) - c'c = -\frac{c}{4} + \left(\mu + \frac{\rho}{2}\right)^2 \frac{\alpha}{2}(c + d).$$
In view of the quadratic relation satisfied by ρ, and since $2\mu + \alpha\rho = 1$, it is not difficult to establish that $\alpha(2\mu + \rho)^2 = 1$. Thus $Q_{2c'-e}c = 8c'(c'c) - 8c'c + c = d$, and w is the element we were looking for.

The case $\alpha = 0$ for simple $\mathcal{J}$. Now, we turn our attention to the case where $\alpha = 0$. We then have $cd = 0$ implying that $d \in \mathcal{J}_0(c)$. We also assume that $\mathcal{J}$ is simple. Hence $\mathcal{J}_0(c)$ is simple, and, in view of Proposition 2.9.3, the subspace $E := \mathcal{J}_{1/2}(c) \cap \mathcal{J}_{1/2}(d) \supseteq (\mathcal{J}_0(c))_{1/2}(d)$ is non-trivial. Let $v \in E$ be an element such that $\operatorname{tr}(v^2) = 2$. From the second Pierce decomposition Theorem, we can deduce that $v^2 \in \mathcal{J}_1(c) + \mathcal{J}_1(d)$, or $v^2 = \gamma c + \delta d$. Note that $\gamma = \operatorname{tr}(v^2 c) = \operatorname{tr}(v(vc)) = \operatorname{tr}(v^2/2) = 1$, and $\delta = 1$. Thus $v^2 = c + d$. Observe also that $Q_v c = 2v(vc) - v^2 c = v^2 - v^2 c = d$. We now construct an element $w \in \mathcal{W}(\mathcal{J})$ from v. Let $e' := e - c - d$; e' is the unit element of the Jordan subalgebra $\mathcal{J}_0(c + d)$. We have:

- ◇ $e'v = (e - c - d)v = v - v/2 - v/2 = 0$;
- ◇ $(e' + v)^2 = e' + 2e'v + v^2 = e - c - d + c + d = e$, i.e. $e' + v \in \mathcal{W}(\mathcal{J})$;
- ◇ $Q_{e'+v}c = Q'_e c + 2Q_{e',v}c + Q_v c = 0 + 2e'(vc) + 0 - 0 + d = e'v/2 + d = d$.

The element $w := e + v - c - d$ complies with the statement.

■

Theorem 2.10.11 *Let $\{c_1, \dots, c_r\}$ and $\{d_1, \dots, d_r\}$ be two Jordan frames of a simple Jordan algebra $\mathcal{J}$. There exists an automorphism A of $\mathcal{A}(\mathcal{J})$ such that $Ac_i = d_i$ for every $1 \le i \le r$.*

Proof

Let $A_1 \in \mathcal{A}(\mathcal{J})$ be such that $A_1 c_1 = d_1$. Note that $d'_2 := A_1 c_2$ is a minimal idempotent such that $d_1 d'_2 = 0$, that is, d'_2 (and d_2 as well) belongs to $\mathcal{J}_0(d_1)$. The previous lemma gives us an element $w \in \mathcal{W}(\mathcal{J}_0(d_1))$ such that $Q_w d'_2 = d_2$. Now, we put $A_2 := Q_{w+d_1}$. By an argument similar to the last part of the proof of the previous lemma, we have:

- $\diamond$ $w = 2c - (e - d_1)$ for an idempotent c from $\mathcal{J}_0(d_1)$; hence $w \in \mathcal{J}_0(d_1)$, and for every $u \in \mathcal{J}_1(d_1)$, we have $wu = 0$.
- $\diamond$ $(w + d_1)^2 = e - d_1 + 2wd_1 + d_1 = e$, i.e. $w + d_1 \in \mathcal{W}(\mathcal{J})$.
- $\diamond$ $Q_{w+d_1}d_2' = Q_w d_2' + 2Q_{w,d_1}d_2' + Q_{d_1}d_2' = d_2 + 0 + 2d_1(wd_2') - 0 + 0 = d_2$, because $wd_2' \in \mathcal{J}_0(d_1)$.
- $\diamond$ $Q_{w+d_1}u = Q_w u + 2Q_{w,d_1}u + Q_{d_1}u = u$ for every $u \in \mathcal{J}_1(d_1)$.

Then A_2A_1 maps c_i on d_i for $i = 1, 2$. A recursive application of this argument allows us to conclude. ■

Corollary 2.10.12 *Suppose that $\mathcal{J}$ is a simple Jordan algebra and that u and v belong to $\mathcal{J}$. Then $\lambda(u) = \lambda(v)$ if and only if there exists an automorphism $A \in \mathcal{A}(\mathcal{J})$ such that $v = Au$.*

Proof
Remark 2.10.2 already proves the "if" part. Now let $\lambda := \lambda(u)$. There exist two Jordan frames $\{c_1, \ldots, c_r\}$ and $\{d_1, \ldots, d_r\}$ for which $u = \sum_{i=1}^r \lambda_i c_i$ and $v = \sum_{i=1} \lambda_i d_i$. It suffices to take the automorphism A that maps c_i on d_i for every i to get $Au = v$. ■

2.11 Jordan algebras make it work: proofs for Section 1.7

In this section, we use the technical tools presented in this chapter to solve the two problems described in Section 1.7. We assume that $\mathcal{J}$ is a formally real Jordan algebra or rank r and dimension N.

2.11.1 A concavity result

We start by recalling a classical result (see Section I.6 of [Koe99]).

Lemma 2.11.1 *Consider the function $F : \operatorname{int}\mathcal{K}_\mathcal{J} \to \mathbb{R}$, $u \mapsto F(u) := -\ln\det(u)$. We have $\nabla_u^h F(u) = \operatorname{tr}(-u^{-1}h)$ and $\nabla_u^2 F(u) = Q_u^{-1}$ for every $u \in \operatorname{int}\mathcal{K}_\mathcal{J}$.*

Proof
The formula for the first derivative follows easily from our differential rules of Section 2.4, and from the property of the quadratic operator established in Proposition 2.5.4. Indeed, let us take two generically independent generic elements x and y. We can write:

$$\begin{aligned}
-\ln\det(x^2 + \varepsilon y) &= -\ln\det(x^2) - \ln\det(e + \varepsilon Q_x^{-1}y) = -\ln\det(x^2) - \ln\prod_{i=1}^r \lambda_i(e + \varepsilon Q_x^{-1}y) \\
&= -\ln\det(x^2) - \sum_{i=1}^r \ln(1 + \varepsilon\lambda_i(Q_x^{-1}y)) = -\ln\det(x^2) - \varepsilon\sum_{i=1}^r \lambda_i(Q_x^{-1}y)) \\
&= -\ln\det(x^2) - \varepsilon\operatorname{tr}(Q_x^{-1}y) = -\ln\det(x^2) - \varepsilon\operatorname{tr}(x^{-2}y).
\end{aligned}$$

This establishes the first formula. The second one coincides with the second item of Theorem 2.5.2. ■

The next result on this function has been proved by Stefan Schmieta in [Sch00], Theorem 2.

Lemma 2.11.2 *The function $F : \operatorname{int}\mathcal{K}_{\mathcal{J}} \to \mathbb{R}$, $u \mapsto F(u) := -\ln\det(u)$ is a r-self-concordant barrier for $\mathcal{K}_{\mathcal{J}}$.*

Proof

First, F is convex on its domain, because the Hessian Q_u^{-1} is positive definite for every $u \in \operatorname{int}\mathcal{K}_{\mathcal{J}}$ (see Corollary 2.7.32). It is also obvious that F is a barrier for $\mathcal{K}_{\mathcal{J}}$. Next, observe that $\langle F''(u)^{-1}F'(u), F'(u)\rangle_{\mathcal{J}} = \operatorname{tr}((Q_u u^{-1})u^{-1}) = \operatorname{tr}(e) = r$ for every $u \in \operatorname{int}\mathcal{K}_{\mathcal{J}}$.

The last property to be checked can be reformulated as follows (see Section 2.5 of [Ren01]).

- ◇ For every $x \in \operatorname{int}\mathcal{K}_{\mathcal{J}}$, the set $B_x := \{y \in \mathcal{J} | \langle F''(x)(y-x), y-x\rangle_{\mathcal{J}} < 1\}$ is included in $\operatorname{int}\mathcal{K}_{\mathcal{J}}$, and
- ◇ for every $x \in \operatorname{int}\mathcal{K}_{\mathcal{J}}$, every $y \in B_x$ and every nonzero $h \in \mathcal{J}$, we have

$$1 - r \le \sqrt{\frac{\langle F''(y)h, h\rangle_{\mathcal{J}}}{\langle F''(y)h, h\rangle_{\mathcal{J}}}} \le \frac{1}{1-r},$$

  where $r := \langle F''(x)(y-x), y-x\rangle_{\mathcal{J}}^{1/2}$.

Let $u \in \mathcal{K}_{\mathcal{J}}$ and $v \in \mathcal{J}$. We have

$$\langle F''(u)(v-u), v-u\rangle_{\mathcal{J}} = \langle Q_u^{-1}(v-u), v-u\rangle_{\mathcal{J}} = ||Q_u^{-1/2}(v-u)||_{\mathcal{J}}^2 = ||z-e||_{\mathcal{J}}^2,$$

where $z := Q_u^{-1/2}v$. If $v \in B_u$, i.e. if $\sum_{i=1}^r (\lambda_i(z)-1)^2 < 1$, then $(\lambda_i(z)-1)^2 < 1$. This implies that $0 < \lambda_i(z)$, and $z \in \operatorname{int}\mathcal{K}_{\mathcal{J}}$. Hence $v \in \operatorname{int}\mathcal{K}_{\mathcal{J}}$ as a consequence of Theorem 2.8.8. Further, if $h \in \mathcal{J}$, we have, with $\tilde{h} := Q_u^{-1/2}h$:

$$\frac{\langle Q_v^{-1}h, h\rangle_{\mathcal{J}}}{\langle Q_u^{-1}h, h\rangle_{\mathcal{J}}} = \frac{\langle Q_u^{1/2}Q_v^{-1}Q_u^{1/2}\tilde{h}, \tilde{h}\rangle_{\mathcal{J}}}{\langle \tilde{h}, \tilde{h}\rangle_{\mathcal{J}}} = \frac{\langle Q_z^{-1}\tilde{h}, \tilde{h}\rangle_{\mathcal{J}}}{\langle \tilde{h}, \tilde{h}\rangle_{\mathcal{J}}}$$

by the Fundamental Identity (2.14). In view of Corollary 2.7.32, this fraction is bounded from above by $1/\lambda_r(z)^2$ and below by $1/\lambda_1(z)^2$. It remains now to relate $\lambda_1(z)^2$ and $\lambda_r(z)^2$ with $\left(1 - \sqrt{\sum_{i=1}^r (\lambda_i(z)-1)^2}\right)^2$. ■

The next proposition is a particular case of the result needed to prove Corollary 1.7.1. As it transparently gives the structure of the proof of the more general Proposition 2.11.5, we leave it in this text.

Proposition 2.11.3 *Let $\mathcal{J}$ be a Jordan algebra. Consider the function $F : \operatorname{int}\mathcal{K}_{\mathcal{J}} \to \mathbb{R}$, $u \mapsto F(u) := -\ln(\det(u))$, and two elements $u \in \operatorname{int}\mathcal{K}_{\mathcal{J}}$ and $h \in \mathcal{J}$, $h \neq 0$. We denote:*

$$\phi(t) := \frac{1}{\langle F''(u+th)h, h\rangle_{\mathcal{J}}^{1/2}}.$$

The function ϕ is concave on its domain.

Proof
Consider first the very special case where $\mathcal{K}_{\mathcal{J}} := \mathbb{R}^r_+$, and $u := \mathbf{1}$, the all-one r-dimensional vector. Letting
$$g(t) := \sum_{i=1}^{r} \left(\frac{h_i}{1+th_i}\right)^2,$$
the function $\phi(t)$ equals $g(t)^{-1/2}$, and its concavity is equivalent to the following inequality:
$$\phi''(t) = -\frac{1}{2}g(t)^{-3/2}g''(t) + \frac{3}{4}g(t)^{-5/2}g'(t)^2 \leq 0,$$
or $3g'(t)^2 \leq 2g(t)g''(t)$. Simple computations show that:
$$g'(t) = -2\sum_{i=1}^{r} \left(\frac{h_i}{1+th_i}\right)^3, \quad \text{and} \quad g''(t) = 6\sum_{i=1}^{r} \left(\frac{h_i}{1+th_i}\right)^4.$$
Denote $b_i := h_i/(1+th_i)$. The concavity of ϕ reduces then to:
$$\left(\sum_{i=1}^{r} b_i^3\right)^2 \leq \left(\sum_{i=1}^{r} b_i^2\right)\left(\sum_{i=1}^{r} b_i^4\right).$$
The latter relation is an immediate consequence of the well-known Cauchy-Schwartz inequality.

We turn now our attention to the general case. We know from Lemma 2.11.1 that $F''(x) = Q_x^{-1}$ for every $x \in \operatorname{int}\mathcal{K}_{\mathcal{J}}$.

Observe that, for every $a, b \in \mathcal{J}$ and for every invertible $p \in \mathcal{J}$, we have:
$$\begin{aligned}\operatorname{tr}(bQ_a b) &= \langle b, Q_a b\rangle_{\mathcal{J}} = \langle Q_p Q_p^{-1} b, Q_a Q_p Q_p^{-1} b\rangle_{\mathcal{J}} \\ &= \langle Q_p^{-1} b, Q_p Q_a Q_p Q_p^{-1} b\rangle_{\mathcal{J}} = \langle \tilde{b}, Q_{\tilde{a}}\tilde{b}\rangle_{\mathcal{J}} = \operatorname{tr}(\tilde{b}Q_{\tilde{a}}\tilde{b}),\end{aligned}$$
with $\tilde{a} := Q_p^{-1}a$ and $\tilde{b} := Q_p^{-1}b$. To prove this homogeneity result, we have used Proposition 2.7.27 and Theorem 2.5.2. Taking in the above expression $a := (u+th)^{-1}$, $b := h$, and $p := u^{1/2}$, using again Theorem 2.5.2, we obtain:
$$\tilde{a} = Q_p a = Q_u^{1/2}(u+th)^{-1} = [Q_u^{-1/2}(u+th)]^{-1} = [e + tQ_u^{-1/2}h]^{-1} = (e+t\tilde{h})^{-1},$$
where $\tilde{h} := Q_u^{-1/2}h$. Now, let $\tilde{h} = \sum_{i=1}^{r}\lambda_i(\tilde{h})c_i$ be a spectral decomposition of $\tilde{h}$. We have:
$$g(t) := \operatorname{tr}(hQ_{u+th}^{-1}h) = \operatorname{tr}(\tilde{h}Q_{e+t\tilde{h}}^{-1}\tilde{h}) = \sum_{i=1}^{r}\left(\frac{\lambda_i(\tilde{h})}{1+t\lambda_i(\tilde{h})}\right)^2.$$
The function $\phi(t)$ equals $g(t)^{-1/2}$, and its concavity can be proved by means of the first part of this demonstration. ■

In order to prove the general concavity statement, we need a technical lemma.

Lemma 2.11.4 *Let $f : I_f \to \mathbb{R}_{++}$ and $g : I_g \to \mathbb{R}_{++}$ be two twice differentiable functions defined respectively on the open intervals I_f and I_g. We assume that I_f and I_g have a nonempty intersection I. Suppose that $f^{-1/2}$ and $g^{-1/2}$ are concave on their respective domains. Then the function $(f+g)^{-1/2}$ is concave on $I_f \cap I_g$.*

Proof
As mentioned in the proof of the previous proposition, the function $f^{-1/2}$ is concave on its domain I_f if and only if for every $t \in I_f$ we have $3f'(t)^2 \leq 2f(t)f''(t)$. Similarly, we have $3g'(t)^2 \leq 2g(t)g''(t)$ for every $t \in I_g$.

Observe that for every $t \in I_f \cap I_g$, we have:

$$3f'(t)g'(t) \leq \sqrt{9f'(t)^2g'(t)^2} \leq \sqrt{4f(t)f''(t)g(t)g''(t)} \leq g(t)f''(t) + f(t)g''(t),$$

because $f''(t) \geq 0$ and $g''(t) \geq 0$.

Now, we can write for every $t \in I_f \cap I_g$:

$$\begin{aligned} 3(f'(t)+g'(t))^2 &= 3f'(t)^2 + 6f'(t)g'(t) + 3g'(t)^2 \\ &\leq 2f(t)f''(t) + 2(g(t)f''(t) + f(t)g''(t)) + 2g(t)g''(t) \\ &= 2(f(t)+g(t))(f''(t)+g''(t)). \end{aligned}$$

Therefore, the function $f+g$ is concave on $I_f \cap I_g$. ■

Proposition 2.11.5 *Let $\mathcal{J}$ be a Jordan algebra. Consider the function $F : \operatorname{int}\mathcal{K}_{\mathcal{J}} \to \mathbb{R}$, $u \mapsto F(u) := -\ln(\det(u))$, and three elements $h, p \in \mathcal{J}$ and $u \in \operatorname{int}\mathcal{K}_{\mathcal{J}}$. We assume that $p \neq 0$. We denote:*

$$\phi_p(t) := \frac{1}{\langle F''(u+th)p, p\rangle_{\mathcal{J}}^{1/2}}.$$

The function ϕ_p is concave on its domain.

Proof
Similarly to the previous proof, we define

$$g(t) := \operatorname{tr}(pQ_{u+th}^{-1}p) = \operatorname{tr}(\tilde{p}Q_{e+t\tilde{h}}^{-1}\tilde{p}),$$

where $\tilde{p} = Q_u^{-1/2}p$ and $\tilde{h} = Q_u^{-1/2}h$. We need to prove that $3g'(t)^2 \leq 2g(t)g''(t)$ for every t. Let $\tilde{h} = \sum_{i=1}^r \lambda_i c_i$ be a complete spectral decomposition of $\tilde{h}$, and let $\tilde{p}_{ij} := Q_{c_i,c_j}\tilde{p}$. In view of the second Pierce decomposition theorem, we can decompose $g(t)$ into:

$$g(t) = \sum_{i,j=1}^r \frac{\operatorname{tr}(\tilde{p}Q_{c_i,c_j}\tilde{p})}{(1+t\lambda_i(\tilde{h}))(1+t\lambda_j(\tilde{h}))} = \sum_{i=1}^r \frac{\operatorname{tr}(\tilde{p}_{ii}^2)}{1+t\lambda_i(\tilde{h})^2} + \sum_{i\neq j}^r \frac{2\operatorname{tr}(\tilde{p}_{ij}^2)}{(1+t\lambda_i(\tilde{h}))(1+t\lambda_j(\tilde{h}))}.$$

In view of the previous lemma, we only need to check that each of these terms represents a function of t which when raised to the power of $-1/2$ is concave.

Let $\phi(t) := c/((1+ta)(1+tb))$, where $c \geq 0$. Every term of the second summation has this form, and it remains to check that $3\phi'(t)^2 \leq 2\phi(t)\phi''(t)$ on the neighborhood of 0 where ϕ is defined. We compute:

$$\begin{aligned}
\phi'(t) &= \frac{-c(2abt+a+b)}{(1+ta)^2(1+tb)^2},\\
\phi''(t) &= \frac{2c(3a^2b^2t^2+3a^2bt+3ab^2t+a^2+b^2+ab)}{(1+ta)^3(1+tb)^3}.
\end{aligned}$$

The simplification of the denominators allows us to transform the inequality $3\phi'(t)^2 \leq 2\phi(t)\phi''(t)$ into:

$$3c^2(2abt+a+b)^2 \leq 4c^2(3a^2b^2t^2+3a^2bt+3ab^2t+a^2+b^2+ab).$$

All the terms in t and in t^2 can be simplified. We end up with:

$$c^2(3a^2+3b^3+6ab) \leq c^2(4a^2+4b^2+4ab),$$

which is obviously true. ■

2.11.2 Augmented barriers in Jordan algebras

Let $\mathcal{J}$ be a formally real Jordan algebra of rank r and dimension N. We define:

$$\mathcal{Q}(\mathcal{J}) := \operatorname{conv}\{Q_u | u \in \mathcal{K}_{\mathcal{J}}\}.$$

This set of linear operators possesses several interesting properties. Among them, we mention the following ones.

- ◇ The set $\mathcal{Q}(\mathcal{J})$ is a convex cone of the set of all linear applications from $\mathcal{J}$ to $\mathcal{J}$. Let $q-1$ be its dimension. By Carathéodory's Theorem (see Theorem 17.1 of [Roc70]), every $M \in \mathcal{Q}(\mathcal{J})$ can be expressed as a convex combination of q quadratic operators. Note that $q-1 \leq N^2$.
- ◇ Let $M = \sum_{i=1}^q \alpha_i Q_{u_i} \in \mathcal{Q}(\mathcal{J})$, with $\sum_{i=1}^q \alpha_i = 1$ and $\alpha_i \geq 0$. By taking $v_i := \sqrt{\alpha_i} u_i$, we can represent M as:
$$M = Q_{v_1} + \cdots + Q_{v_q}.$$
- ◇ For every $M \in \mathcal{Q}(\mathcal{J})$ and every $u, v \in \mathcal{J}$, we have $\operatorname{tr}((Mu)v) = \operatorname{tr}((Mv)u)$, because the basic quadratic operator is self-adjoint (see Proposition 2.7.27).
- ◇ Every $M \in \mathcal{Q}(\mathcal{J})$ maps $\mathcal{K}_{\mathcal{J}}$ into itself. This comes from the convexity of $\mathcal{K}_{\mathcal{J}}$ (see Theorem 2.8.6 and Theorem 2.8.8).

We denote the augmented barrier of $\mathcal{K}_{\mathcal{J}}$ built from $M \in \mathcal{Q}(\mathcal{J})$ by:

$$\begin{aligned}
\psi_M : \operatorname{int}\mathcal{K}_{\mathcal{J}} &\to \mathbb{R}\\
u &\mapsto \psi_M(u) := \frac{\operatorname{tr}(uMu)}{2} - \ln(\det(u)).
\end{aligned}$$

In order to minimize efficiently this function, the results of [NV04] suggest that the path-following algorithm displayed in Theorem 1.5.4 can be very efficient in theory. We prove below that this is the case, by extending almost automatically the argumentation of Nesterov and Vial to the Jordan algebraic framework.

Lemma 2.11.6 *Let $M = \sum_{i=1}^q Q_{v_i} \in \mathcal{Q}(\mathcal{J})$, where $v_i \in \mathcal{K}_\mathcal{J}$, and let $\hat{v} := \sum_{i=1}^q v_i$. For every $u \in \mathcal{K}_\mathcal{J}$, we have:*

$$\frac{\mathrm{tr}(uQ_{\hat{v}}u)}{q} \leq \mathrm{tr}(uMu) \leq \Big(\mathrm{tr}(\hat{v}u)\Big)^2.$$

Proof
Let us fix an element u of $\mathcal{K}_\mathcal{J}$, and consider the function $f : \mathcal{K}_\mathcal{J} \to \mathbb{R}$, $v \mapsto f(v) := \mathrm{tr}(vQ_u v)$. Observe that

$$f(v) = \langle uv, uv\rangle = \mathrm{tr}(uQ_v u),$$

and that the function f takes nonnegative values. Moreover, the function f is convex, as shown by the following development. For every a and b of $\mathcal{J}$, we have by the arithmetic-geometric inequality and by Remark 2.7.35:

$$\begin{aligned}\left(\frac{f(a)+f(b)}{2}\right)^2 &\geq f(a)f(b) = \mathrm{tr}(aQ_u a)\mathrm{tr}(bQ_u b) = ||Q_u^{1/2}a||_\mathcal{J}^2 ||Q_u^{1/2}b||_\mathcal{J}^2 \\ &\geq \Big(\mathrm{tr}[(Q_u^{1/2}a)(Q_u^{1/2}b)]\Big)^2 = \Big(\mathrm{tr}(aQ_u b)\Big)^2.\end{aligned}$$

Therefore:

$$\frac{f(a)+f(b)}{2} \geq \frac{f(a)+f(b)}{4} + \frac{\mathrm{tr}(aQ_u b)}{2} = f\left(\frac{a+b}{2}\right).$$

Now, let $M = \sum_{j=1}^q Q_{v_j} \in \mathcal{Q}(\mathcal{J})$, where $v_j \in \mathcal{K}_\mathcal{J}$ and $\hat{v} := \sum_{j=1}^q v_j$. By convexity of f, we can write $f(\hat{v}/q) \leq \sum_{j=1}^q f(v_j)/q$, i.e. $\mathrm{tr}(uQ_{\hat{v}}u) \leq q\mathrm{tr}(uMu)$. Furthermore, observe that:

$$\mathrm{tr}(uQ_{v_j}u) = ||Q_{v_j}^{1/2}u||_\mathcal{J}^2 = \sum_{i=1}^r \lambda_i^2(Q_{v_j}^{1/2}u) \leq \left(\sum_{i=1}^r \lambda_i(Q_{v_j}^{1/2}u)\right)^2 = \Big(\mathrm{tr}(Q_{v_j}^{1/2}u)\Big)^2 = \Big(\mathrm{tr}(v_j u)\Big)^2.$$

Summing over the indices j, we obtain in view of $\mathrm{tr}(v_j u) \geq 0$ that:

$$\mathrm{tr}(uMu) = \sum_{j=1}^q \mathrm{tr}(uQ_{v_j}u) \leq \sum_{j=1}^q \Big(\mathrm{tr}(v_j u)\Big)^2 \leq \left(\sum_{j=1}^q \mathrm{tr}(v_j u)\right)^2 = \Big(\mathrm{tr}(\hat{v}u)\Big)^2.$$

■

With the notation of the above lemma, we assume now that $\hat{v}$ is invertible. We take $x_0 := \hat{v}^{-1}$ as a starting point for the path-following algorithm in Theorem 1.5.4, where $c := \psi_M'(x_0)$. Reformulating the previous lemma for $F(u) := -\ln(\det(u))$, we derive:

$$\frac{\mathrm{tr}(uF''(x_0)u)}{q} = \frac{\mathrm{tr}(uQ_{x_0}^{-1}u)}{q} \leq \mathrm{tr}(uMu) \leq \Big(\mathrm{tr}(x_0^{-1}u)\Big)^2 = \Big(\mathrm{tr}(F'(x_0)u)\Big)^2,$$

by the formulas for the gradient and the Hessian given in Lemma 2.11.1, so that we can further take:

$$\gamma_l(x_0) := \frac{1}{q}, \quad \gamma_u(x_0) := 1.$$

This yields a final complexity of $\mathcal{O}(\sqrt{r}\ln(rq))$, a result that exactly matches Theorem 1.7.4 for the particular case where the considered cone is a cone of positive semidefinite matrices.

2.12 Conclusion

It is worthy to conclude this long chapter by giving a short list of our personal contributions, and by highlighting their impact on the next chapters.

Most of the material of this chapter has already been discovered since decades by algebraists. But they are still not very familiar to the optimization community. Our contribution consists mainly in introducing them in the most self-contained way as possible. Although we have adapted existing proofs for a large majority of the results, some of them are up to our knowledge original.

The reader might be puzzled by the fact that we define Jordan's Axiom in a relatively general setting, allowing extensions of the algebra. However, this level of generality is needed in the definition of the characteristic polynomial via generic elements. Also, it is indispensable in the demonstration of Proposition 2.3.39, which is a decisive step towards the proof of the associativity of the trace. Our proof of Proposition 2.3.39 is entirely based on the elegant argument of Jacques Tits. Our contribution only consists in adapting his proof to the framework of our work by solving a few preliminary technical issues.

The section on differential calculus is original in its presentation. However, the idea of using the ring of dual numbers in order to define an algebraic differential calculus is well-known.

Proposition 2.6.3 is new, and plays an important role in Chapter 3, as well as Lemma 2.7.21 and Proposition 2.7.22 – however, this last result has been shown in a recent preprint of Faybusovich [Fay05]. These three results provide information on Pierce subspaces and subalgebras of the type $\mathcal{J}_1(c)$. These subalgebras are the key to formulate and to prove powerful variational characterizations of eigenvalues (see Chapter 3).

Our proof for the complete spectral decomposition Theorem is, up to our knowledge, new. It is based on the not-so-trivial fact that an idempotent of a formally real Jordan algebra is minimal if *and only if* its trace equals 1 (Proposition 2.7.19). Our proof is, up to our knowledge, mainly original, as well as several useful preliminary results on extension of Pierce subspaces (Remark 2.6.4, Proposition 2.7.17, and Corollary 2.7.18).

The proof of the two results on operator commutativity (Proposition 2.7.29 and 2.7.30) are original. These results have already been published, with a different proof, in [SA03]. Proposition 2.8.10 is original, and of primal importance in Chapter 5, where we study the limiting behavior of Jordan frames of a converging sequence.

The proof of Lemma 2.10.10 is, up to our knowledge, also original.

Finally, Section 2.11 is completely original, with the exception of Lemma 2.11.2 on self-concordancy of the function $-\ln(\det(x))$, which has been obtained in [Sch00]. The result displayed in Lemma 2.11.1 is well-known, but our proof is original.

CHAPTER 3

Variational characterizations of eigenvalues in Jordan algebras

VARIATIONAL CHARACTERIZATIONS OF EIGENVALUES are included among the most important technical tools to understand how the eigenvalues vary under perturbations of their argument. These characterizations are of primal importance to investigate differentiability properties of functions of eigenvalues. They also lead to many useful inequalities.

In this chapter, we extend Wielandt's variational characterization of partial sums of eigenvalues. Particular cases of this result include Fan's Theorem and Fischer's Theorem. From our theorem, we derive Weyl inequalities, Lidskii's inequalities, and Mirski's inequalities. We also obtain a Lipschitz continuity result for spectral functions in Jordan algebras, with respect to *any* gauge norm.

3.1 Introduction

How does the eigenvalues function of symmetric matrices change when its argument is subject to a perturbation ? This problem is a major question in the Matrix Perturbation Theory, which has now applications in a broad range of Applied Mathematics fields. For instance, in Numerical Analysis, one may model rounding errors in a computer by using such perturbations. In Robust Optimization, uncertainties on the instance of linear optimization problems can also be interpreted as perturbations of this type. The interested reader can find a thorough exposition of Matrix Perturbation Theory in the book of Stewart and Sun [SS90].

In order to have access to an eigenvalue of a matrix, the first idea one might have is simply to multiply this matrix by the corresponding eigenvector. However, this eigenvector is not always available or practical to use. For instance when one needs to compare the spectrum of two matrices, say an original matrix and a perturbed version of this original matrix, there is in general no connection between the corresponding eigenvectors. In order to bypass this problem, variational characterizations of eigenvalues describe an eigenvalue as an extremal value of a certain function over a set of possible eigenvectors, which is *independent* of the considered matrix. In fact, the eigenvector corresponding to the studied eigenvalue pops up as the solution of this optimization problem. As this formulation considerably facilitates the spectral comparison of different matrices, it plays a key role in Matrix Perturbation Theory.

In the framework of this work, we aim at considering how these variational characterizations can be extended to formally real Jordan algebras. More specifically, we focus on an extension of the *maximin Wielandt's Theorem*, which appears to be one of the most powerful results in the field. For Hermitian matrices, this relation is stated as follows in [SS90], Theorem 4.5.

> Let A be an $n \times n$ Hermitian matrix and let us fix an increasing sequence $1 \leq i_1 < \ldots < i_k \leq n$ of integers. Then
>
> $$\sum_{j=1}^{k} \lambda_{i_j}(A) = \max_{\substack{\dim(X_j)=i_j \\ X_1 \subseteq \cdots \subseteq X_k \subseteq \mathbb{R}^n}} \min_{\substack{h_j \in X_j \\ h_i^T h_j = \delta_{ij}}} \sum_{j=1}^{k} h_j^T A h_j. \tag{3.1}$$
>
> ■

In this formulation, the sets X_j are subspaces of $\mathbb{R}^n$ of dimension i_j, the eigenvalues $\lambda_i(A)$ are ordered decreasingly, and the symbol δ_{ij} stands for the Kronecker delta, which equals zero if $i \neq j$ and one otherwise.

Special cases of this characterization are classical results in Matrix Theory. When $k := 1$ for instance, it reduces to the Courant-Fischer maximin Theorem (the original paper of Fischer is [Fis05]. For a modern exposition of this important result, see Theorem 4.2.11 in [HJ96]). And if one takes $i_j := j$ for $1 \leq j \leq k$, one can easily derive Ky Fan's inequalities [Fan49].

When dealing with Jordan algebras, several changes in the above presentation have to be performed. Indeed, in contrast with Hermitian matrices, the elements of a Jordan algebra are not seen as operators on a vector space. The condition $X_j \subseteq \mathbb{R}^n$ has no meaning in this context, and we need to replace it somehow with more appropriate objects. Moreover, the geometric constraint $\dim(X_j) = i_j$ has to be changed with an algebraic constraint, more suited for the Jordan algebraic structure.

As a candidate to replace the i_j-dimensional subspaces X_j, we propose here to consider Jordan subalgebras of the form $\mathcal{J}_1(c)$, where c is an idempotent of trace i_j. To motivate this choice, we observe that for every element $u \in \mathcal{J}$, we have $uv = \lambda_i(u)v$ if and only if v belongs to the eigenspace of $L(u)$ corresponding to the eigenvalue $\lambda_i(u)$. This eigenspace is a subalgebra of the mentioned form (see Proposition 2.7.31).

However, several technical issues need to be solved before presenting and proving the Jordan algebraic version of Wielandt's Theorem, especially concerning the interactions between two Jordan subalgebras of the type $\mathcal{J}_1(c)$. Most of them are solved in the second section of this chapter.

The reader might wonder why we do not prefer to simply apply the Hermitian matrix version of Wielandt's relations to the operator $L(u)$, which has among its eigenvalues all the components of $\lambda(u)$. The problem with this operator is that the rank of one of its eigenvalues of the type $\lambda_i(u)$ depends on the structure of the Jordan algebra, and especially on how it decomposes itself into simple algebras. As some of the desired applications of Wielandt's Theorem, namely the Mirsky's inequalities, explicitly requires to use non-simple Jordan algebras, we have preferred to design a formulation that is more convenient for our purposes.

Throughout this chapter, $\mathcal{J}$ denotes a formally real Jordan of dimension N and of rank r. We do not assume that $\mathcal{J}$ is a simple algebra.

3.2 Majorization and Ky Fan's relations in Jordan algebras

We start our discussion with the presentation of the simplest variational characterization of eigenvalues: the Ky Fan's relations. To ease the description, we first introduce a few notational conventions.

The notation $\mathbb{R}^n_\downarrow$ represents the set $\{\gamma \in \mathbb{R}^n | \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n\}$. We write $\mathcal{P}$ for the set of all permutations of r-dimensional vectors; we view them here as $r \times r$ matrices of 0 and 1. We denote the all-one r-dimensional vector by $\mathbf{1}$. We also write

$$\mathbf{1}_p := (1, \dots, 1, 0, \dots, 0)^T \in \mathbb{R}^r,$$

with p times the value "1" and $(r - p)$ times the value "0", so that $\mathbf{1}_r = \mathbf{1}$.

For the ease of reference, we recall below a classical statement involving the set $\mathcal{P}$, known as the *rearrangement inequality*. Its proof can be found in [HLP67], Theorem 368 or in [Lew96a], Lemma 2.1.

Theorem 3.2.1 (Rearrangement inequality) *Let $\gamma, \lambda \in \mathbb{R}^r_\downarrow$ and $P \in \mathcal{P}$. Then $\gamma^T P\lambda \leq \gamma^T \lambda$.* ∎

For every $\lambda \in \mathbb{R}^r$, we define:

$$\mathcal{P}(\lambda) := \{P\lambda | P \in \mathcal{P}\} \text{ and } \mathcal{SC}(\lambda) := \mathrm{conv}(\mathcal{P}(\lambda)).$$

The notation $\mathrm{conv}(A)$ stands for the convex hull of a set $A \in \mathbb{R}^r$. We can immediately observe that $\mathcal{SC}(\lambda)$ is symmetric with respect to permutations. Actually, this is the smallest symmetric convex subset of $\mathbb{R}^r$ that contains the vector λ. Some authors call this set the *permutahedron generated by* λ. The relation $\gamma \in \mathcal{SC}(\lambda)$ is often denoted by $\gamma \preceq \lambda$ or by "λ majorizes γ". This relation gives rise to an important class of functions, the Schur-convex functions.

Definition 3.2.2 *A function $f : Q \subseteq \mathbb{R}^r \to \mathbb{R}$ is* Schur-convex *if for every $\gamma, \lambda \in Q$ such that $\gamma \preceq \lambda$, we have $f(\gamma) \leq f(\gamma)$.* ■

The study of permutahedron's properties forms an important part of the so-called *theory of Majorization* [MO79]. We provide in Lemma 3.2.5 an alternative description of $\mathcal{SC}(\lambda)$. In order to understand it, let us briefly recall the notion of *support function of a set.*

Definition 3.2.3 *Let $Q \subseteq \mathbb{R}^n$. The* support function *of Q is the function:*

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R} \cup \{+\infty\} \\ u &\mapsto f(u) = \sup_{v \in Q} \langle u, v \rangle. \end{aligned}$$

■

This object is an important tool of investigation in convex analysis. The interested reader can find an introductory survey on support functions in Section 13 of [Roc70]. The support function of a set $Q \subseteq \mathbb{R}^r$ is simply the conjugate of the indicator function of Q, which is null on Q and equals $+\infty$ everywhere else. As the supremum of linear functions, a support function is always convex.

As a simple exercise, let us compute the support function of $\mathcal{SC}(\lambda)$.

Example 3.2.1 (Support function of $\mathcal{SC}(\lambda)$) *Let $\lambda \in \mathbb{R}^r$. We denote the support function of $\mathcal{SC}(\lambda)$ by f. We fix a vector $\mu \in \mathbb{R}^r$. Note that, in view of Hahn-Banach's Theorem (see Corollary 11.5.1 in [Roc70]), we can write $\sup_{x \in \mathrm{conv}(A)} \langle x, y \rangle = \sup_{x \in A} \langle x, y \rangle$ for every bounded set A of $\mathbb{R}^n$. As a consequence of the rearrangement inequality, we have:*

$$f(\mu) = \sup_{\gamma \in \mathcal{SC}(\lambda)} \langle \gamma, \mu \rangle = \sup_{\gamma \in \mathcal{P}(\lambda)} \langle \gamma, \mu \rangle = \gamma^T P^* \mu,$$

where P^* is the permutation that orders the components μ in the same order as the components of γ. ■

Let us denote by s_p the function $s_p : \mathbb{R}^r \to \mathbb{R}$ that maps every vector $\lambda \in \mathbb{R}^r$ to the sum of its p largest components.

Lemma 3.2.4 *For every $1 \leq p \leq r$, the function s_p is convex.*

Proof
Let us specialize the previous example to $\lambda := \mathbf{1}_p$. For each $\mu \in \mathbb{R}^r$, we have $s_p(\mu) = \sup_{\gamma\in\mathcal{SC}(\mathbf{1}_p)}\langle\gamma,\mu\rangle$. The function s_p is thus convex as the support function of some set. ■

We are ready to give the alternative description of majorization.

Lemma 3.2.5 *Let $\lambda, \gamma \in \mathbb{R}^r$. We have:*

$$\gamma \in \mathcal{SC}(\lambda) \ \Leftrightarrow\ s_p(\gamma) \le s_p(\lambda) \quad \text{for all } 1 \le p \le r \quad \text{and} \quad s_r(\gamma) = s_r(\lambda).$$

This is Theorem 4.C.1 of [MO79]. However, the proof we give here is, up to our knowledge, the first one that presents this description as a direct consequence of the linear programming duality theory.

Proof
Observe that the statement to prove does not depend on the ordering of the components of λ. Hence, we can fix without loss of generality λ in $\mathbb{R}^r_\downarrow$. We define:

$$S := \{\gamma \in \mathbb{R}^r | s_p(\gamma) \le s_p(\lambda) \text{ for all } 1 \le p \le r \text{ and } s_r(\gamma) = s_r(\lambda)\}.$$

The set S is trivially a closed and symmetric set that contains λ. This set is also convex, because s_p is a convex function as it is shown in the previous lemma.

We will compare the support function f of $\mathcal{SC}(\lambda)$ with the support function g of S. Since S and $\mathcal{SC}(\lambda)$ are both convex and closed, we get $f^* = \mathrm{Ind}_{\mathcal{SC}(\lambda)}$ and $g^* = \mathrm{Ind}_S$ in view of the duality relations between conjugate functionals (see Theorem 12.2 in [Roc70]). In order to prove that $\mathcal{SC}(\lambda) = S$, it suffices to show that $f = g$.

Let $\mu \in \mathbb{R}^r$ and $P \in \mathcal{P}$ such that $\bar\mu := P\mu \in \mathbb{R}^r_\downarrow$. Note that, by the rearrangement inequality and the symmetry of S, we have:

$$g(\mu) = \max_{\gamma\in S}\langle\gamma,\mu\rangle = \max_{\gamma\in S\cap\mathbb{R}^r_\downarrow}\langle\gamma,\bar\mu\rangle. \tag{3.2}$$

Observe that $g(\mu) \ge \langle\lambda,\bar\mu\rangle$, since $\lambda \in S$. We proceed below to prove the converse inequality.

The optimization problem (3.2) is linear: it can be written as:

$$\begin{array}{rll} g(\mu) = \max & \langle\gamma,\bar\mu\rangle & \\ \text{s.t.} & A_1\gamma \le 0 & (\text{because } \gamma \in \mathbb{R}^r_\downarrow) \\ & \left.\begin{array}{l} A_2\gamma \le b \\ \mathbf{1}^T\gamma = s_r(\lambda) \end{array}\right\} & (\text{because } \gamma \in S) \\ & \gamma \in \mathbb{R}^r, & \end{array}$$

with:

$$A_1 := \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 1 & 0 & & & \\ 1 & 1 & & & \\ \vdots & \vdots & \ddots & \ddots & \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}, \quad b := \begin{pmatrix} s_1(\lambda) \\ s_2(\lambda) \\ \vdots \\ s_{r-1}(\lambda) \end{pmatrix}.$$

The point λ is feasible for this problem and the corresponding objective value equals $\langle \lambda, \bar{\mu} \rangle$. The dual of this problem has the following form:

$$\begin{array}{rrcl} \min & \displaystyle\sum_{p=1}^{r-1} s_p(\lambda) v_p + s_r(\lambda) w & & \\ \text{s.t.} & A_1^T u + A_2^T v + \mathbf{1}^T w & = & \mu \\ & u, v & \geq & 0 \\ & u \in \mathbb{R}^{r-1}, \quad v \in \mathbb{R}^{r-1}, & & w \in \mathbb{R}. \end{array}$$

One can easily check that a dual feasible point is provided by $u := 0$, $v_p := \bar{\mu}_p - \bar{\mu}_{p+1}$ for $1 \leq p < r-1$, and $w := \bar{\mu}_r$; the corresponding objective value is equal to $\langle \lambda, \bar{\mu} \rangle$, which is $f(\mu)$ by the Example 3.2.1. By the fundamental theorem of duality in linear programming (see Theorem 1.2.1), we get $g(\mu) \leq \langle \lambda, \bar{\mu} \rangle = f(\mu)$. Hence $g(\mu) = f(\mu)$, and $S = \mathcal{SC}(\lambda)$. ■

We denote by $\mathcal{S}(\mu)$ the set of elements v of $\mathcal{J}$ that have their eigenvalues in the permutahedron $\mathcal{SC}(\mu)$:

$$\mathcal{S}(\mu) := \{v \in \mathcal{J} | \lambda(v) \in \mathcal{SC}(\mu)\}.$$

Remark 3.2.6 *The previous lemma can be rephrased as follows. For all $\mu \in \mathbb{R}^r$:*

$$v \in \mathcal{S}(\mu) \;\Leftrightarrow\; \sum_{j=1}^{p} \lambda_j(v) \leq s_p(\mu) \quad \text{for all } 1 \leq p \leq r \quad \text{and} \quad \mathrm{tr}(v) = s(\mu).$$

■

The next proposition is a generalization to Jordan algebras of a well-known variational description of the sum of the p largest eigenvalues obtained by Ky Fan for Hermitian matrices. Some arguments of this proof appear in Fan's paper (see [Fan49]).

Proposition 3.2.7 *For every $1 \leq p \leq r$, $u \mapsto \sum_{j=1}^{p} \lambda_j(u)$ is the support function of $\mathcal{S}(\mathbf{1}_p)$.*

Proof
We fix an integer p between 1 and r. Let $u = \sum_{i=1}^{r} \lambda_i(u) d_i \in \mathcal{J}$ and $v = \sum_{i=1}^{r} \lambda_i(v) c_i \in \mathcal{S}(\mathbf{1}_p)$ be their respective complete spectral decomposition. In view of Remark 3.2.6, the eigenvalues of v should be between 0 and 1. We first notice that:

$$\mathrm{tr}(v d_j) = \sum_{i=1}^{r} \lambda_i(v) \mathrm{tr}(c_i d_j) \leq \sum_{i=1}^{r} \mathrm{tr}(c_i d_j) = \mathrm{tr}(d_j) = 1,$$

since c_i and d_j are in $\mathcal{K}_{\mathcal{J}}$ (and thus $\mathrm{tr}(c_i d_j) = \mathrm{tr}(Q_{c_i} d_j) \geq 0$).

Applying this inequality, we get:

$$\begin{aligned}
&\operatorname{tr}(uv)\\
&= \lambda_p(u)\sum_{j=1}^{r}\operatorname{tr}(vd_j) + \sum_{j=1}^{p}(\lambda_j(u)-\lambda_p(u))\operatorname{tr}(vd_j) + \sum_{j=p+1}^{r}(\lambda_j(u)-\lambda_p(u))\operatorname{tr}(vd_j)\\
&\leq \lambda_p(u)\operatorname{tr}(v) + \sum_{j=1}^{p}(\lambda_j(u)-\lambda_p(u))\operatorname{tr}(vd_j)\\
&\leq \lambda_p(u)\operatorname{tr}(v) + \sum_{j=1}^{p}\lambda_j(u) - p\lambda_p(u) = \sum_{j=1}^{p}\lambda_j(u).
\end{aligned}$$

Note that $v^* := \sum_{i=1}^{p} d_i$ lies in $\mathcal{S}(\mathbf{1}_p)$ and satisfies $\operatorname{tr}(uv^*) = \sum_{i=1}^{p}\lambda_i(u)$. We deduce that $\sum_{j=1}^{p}\lambda_j(u) = \max_{v\in\mathcal{S}(\mathbf{1}_p)}\operatorname{tr}(uv)$. ■

3.3 Subalgebras $\mathcal{J}_1(c)$ of a Jordan algebra $\mathcal{J}$

In this section, we describe several useful features of the Jordan subalgebras we want to use in our formulation of Wielandt's Theorem, that is, Jordan subalgebras of the form $\mathcal{J}_1(c)$. We especially focus on the possible relations between two subalgebras of the form $\mathcal{J}_1(c)$ and $\mathcal{J}_1(d)$, especially in the tricky situation where c and d do not operator commute.

We must emphasize the fact that *not all formally real Jordan subalgebras of $\mathcal{J}$ are of the form $\mathcal{J}_1(c)$*. Consider for instance the spin-factor Jordan algebra $\mathcal{S}$ of dimension $n > 3$. Its rank is equal to 2 and the only idempotent of rank 2 is the unit element e of this algebra. Suppose now that c and d are two distinct minimal idempotents of this algebra for which $cd \neq 0$. As shown in the proof of Lemma 2.10.10, the subspace $\mathbb{R}c + \mathbb{R}d + \mathbb{R}cd$ is a formally real Jordan subalgebra of rank 2 in $\mathcal{J}$. It is well-known that every formally real Jordan algebra has a unit element (see e.g Theorem 5 in [JvNW34]). The unit element of this subalgebra is thus e. However, its dimension equals 3, so it cannot be identical to $\mathcal{S}_1(e) = \mathcal{S}$, which has a dimension of n.

Lemma 3.3.1 *Let c_1 and c_2 be two idempotents of $\mathcal{J}$ such that $c_1c_2 = 0$. Then*

$$\mathcal{J}_1(c_1) = \mathcal{J}_1(c_1 + c_2) \cap \mathcal{J}_0(c_2).$$

Proof
The case $c_1 + c_2 = e$ is really easy to treat, because $\mathcal{J}_1(c_1 + c_2) = \mathcal{J}$ and $\mathcal{J}_1(c_1) = \mathcal{J}_0(e - c_1) = \mathcal{J}_0(c_2)$. Suppose now that $c_3 := e - c_1 - c_2 \neq 0$. Note that:

$$c_3^2 = e + c_1 + c_2 - 2c_1 - 2c_2 + 2c_1c_2 = e - c_1 - c_2 = c_3,$$

$$c_1c_3 = c_1 - c_1 - c_1c_2 = 0 \quad \text{and} \quad c_2c_3 = c_2 - c_1c_2 - c_2 = 0.$$

In other words, $\{c_1, c_2, c_3\}$ is a system of idempotents. Now, we set $\mathcal{J}_{ij} := Q_{c_i,c_j}\mathcal{J}$. Then $\mathcal{J}_1(c_1 + c_2) = \mathcal{J}_{11} \oplus \mathcal{J}_{12} \oplus \mathcal{J}_{22}$ and $\mathcal{J}_0(c_2) = \mathcal{J}_{11} \oplus \mathcal{J}_{13} \oplus \mathcal{J}_{33}$ in view of the second Pierce decomposition theorem. Finally,

$$\mathcal{J}_1(c_1 + c_2) \cap \mathcal{J}_0(c_2) = [\mathcal{J}_{11} \oplus \mathcal{J}_{12} \oplus \mathcal{J}_{22}] \cap [\mathcal{J}_{11} \oplus \mathcal{J}_{13} \oplus \mathcal{J}_{33}] = \mathcal{J}_{11} = \mathcal{J}_1(c_1).$$

Lemma 3.3.2 *Let c, d be two idempotents of $\mathcal{J}$. If $c \in \mathcal{J}_1(d)$, then $\mathcal{J}_1(c) = [\mathcal{J}_1(d)]_1(c) \subseteq \mathcal{J}_1(d)$.*

Proof
The inclusion $[\mathcal{J}_1(d)]_1(c) \subseteq \mathcal{J}_1(d)$ is trivial. As

$$[\mathcal{J}_1(d)]_1(c) = \{u \in \mathcal{J}_1(d) | cu = u\} \subseteq \{u \in \mathcal{J} | cu = u\} = \mathcal{J}_1(c),$$

it suffices to check the reverse inclusion. To this end, let us take an element $u \in \mathcal{J}_1(c)$. We need to prove that $du = u$. Note that $c \in \mathcal{J}_1(d)$ implies that $cd = c$ and that c and d operator commute in view of the first Pierce decomposition theorem. Thus:

$$c(du) = L(c)L(d)u = L(d)L(c)u = d(cu) = du.$$

Similarly, $u \in \mathcal{J}_1(c)$ implies that u and c operator commute. Hence:

$$c(du) = c(ud) = L(c)L(u)d = L(u)L(c)d = u(cd) = uc = u,$$

and the requested equality holds. ■

Lemma 3.3.3 *Let c, d be two idempotents of $\mathcal{J}$. The set $\mathcal{J}' := \mathcal{J}_1(c) \cap \mathcal{J}_1(d)$ is a formally real Jordan subalgebra of $\mathcal{J}$. Suppose that there exists an idempotent $c' \in \mathcal{J}'$; then $\mathcal{J}_1(c') \subseteq \mathcal{J}'$ and $\mathcal{J}_0(c') \supseteq \mathcal{J}_0(c) + \mathcal{J}_0(d)$.*

Proof
To prove the first claim, it is enough to check that the vector subspace $\mathcal{J}' := \mathcal{J}_1(c) \cap \mathcal{J}_1(d)$ is stable for the multiplication of $\mathcal{J}$. Let $u, v \in \mathcal{J}'$; since $u, v \in \mathcal{J}_1(c)$, we know that $uv \in \mathcal{J}_1(c)$ by the Pierce multiplication rules. Analogously, $uv \in \mathcal{J}_1(d)$, and $uv \in \mathcal{J}'$.

In view of Lemma 3.3.2, $c' \in \mathcal{J}' \subseteq \mathcal{J}_1(c)$ implies $\mathcal{J}_1(c') \subseteq \mathcal{J}_1(c)$. Similarly, $\mathcal{J}_1(c')$ is included in $\mathcal{J}_1(d)$. Thus $\mathcal{J}_1(c') \subseteq \mathcal{J}'$.

In order to show the second inclusion, we first prove the following auxiliary statement. Given a pair c_1, c_2 of idempotents of $\mathcal{J}$ satisfying $\mathcal{J}_1(c_1) \subseteq \mathcal{J}_1(c_2)$, we claim that $\mathcal{J}_0(c_1) \supseteq \mathcal{J}_0(c_2)$.

We can assume that $c_2 \neq e$, because otherwise $\mathcal{J}_0(c_2) = \{0\}$, and the claim trivially holds. Now, let $d_1 := c_1$, $d_2 := c_2 - c_1$, and $d_3 := e - c_2$. From $c_1 \in \mathcal{J}_1(c_2)$, i.e. $c_1 c_2 = c_1$, it is easily checked that $\{d_1, d_2, d_3\}$ is a system of idempotent. Let $\mathcal{J}_{ij} := Q_{d_i,d_j}\mathcal{J}$. Then, in view of the second Pierce decomposition theorem, we have:

$$\mathcal{J}_0(c_1) = \mathcal{J}_{22} \oplus \mathcal{J}_{23} \oplus \mathcal{J}_{33} \quad \text{and} \quad \mathcal{J}_0(c_2) = \mathcal{J}_{33},$$

and this settles the claim.

In the situation of the statement of our lemma, this simple result becomes $\mathcal{J}_0(c') \supseteq \mathcal{J}_0(c)$ and $\mathcal{J}_0(c') \supseteq \mathcal{J}_0(d)$. Since $\mathcal{J}_0(c')$ is a vector space, we get the desired inclusion. ■

Proposition 3.3.4 *Suppose that c and d are two idempotents for which $\mathrm{tr}(c) + \mathrm{tr}(d) > r$. Then the Jordan subalgebra $\mathcal{J}_1(c) \cap \mathcal{J}_1(d)$ contains a nonzero element. More precisely, it contains an idempotent of rank 1.*

Proof
Lemma 3.3.3 shows that $\mathcal{J}_1(c) \cap \mathcal{J}_1(d)$ is a formally real Jordan subalgebra of $\mathcal{J}$. Suppose that $\mathcal{J}_1(c) \cap \mathcal{J}_1(d) = \{0\}$, contrarily to our statement. Since:

$$\mathcal{J}_1(c) = \mathcal{J}_1(c) \cap \big(\mathcal{J}_1(d) \oplus \mathcal{J}_{1/2}(d) \oplus \mathcal{J}_0(d)\big) = \mathcal{J}_1(c) \cap \big(\mathcal{J}_{1/2}(d) \oplus \mathcal{J}_0(d)\big)\,,$$

the idempotent c can be decomposed into $c = c_{1/2}+c_0$, where $c_\alpha \in \mathcal{J}_\alpha(d)$ for $\alpha \in \{0, 1/2, 1\}$.

According to Lemma 2.7.21, we have $c_{1/2} = 0$ since $c_1 = 0$. Hence $c = c_0$, i.e. $c \in \mathcal{J}_0(d)$ and $\mathcal{J}_1(c) \subseteq \mathcal{J}_0(d)$. Thus

$$\mathcal{J} \supseteq \mathcal{J}_1(d) \oplus \mathcal{J}_0(d) \supseteq \mathcal{J}_1(d) \oplus \mathcal{J}_1(c).$$

By assumption, the rank $\mathrm{tr}(c) + \mathrm{tr}(d)$ of the algebra $\mathcal{J}_1(d) \oplus \mathcal{J}_1(c)$ is strictly greater than the rank r of $\mathcal{J}$. This contradiction proves that $\mathcal{J}_1(c) \cap \mathcal{J}_1(d)$ contains a nonzero element.

According to a classical result that dates back to the original paper of Jordan, von Neumann and Wigner (Theorem 5 of [JvNW34]), the algebra $\mathcal{J}'$ contains a unit element because it is formally real and satisfies Jordan's Axiom. This unit element is an idempotent of rank at least equal to 1 because $\mathcal{J}'$ is nonempty. ■

The next proposition is a key step in the proof of Courant-Fischer's Theorem and of Wielandt's Theorem for formally real Jordan algebras, as we will show in the next section.

Proposition 3.3.5 *Let c, d be two idempotents of $\mathcal{J}$ of trace k and l respectively, such that $k + l > r$. There exists an idempotent of trace $k + l - r$ in $\mathcal{J}_1(c) \cap \mathcal{J}_1(d)$.*

Proof
In view of Lemma 2.3.10 and of Proposition 2.7.19, we only need to prove the existence of an idempotent that has a trace *at least* equal to $k + l - r$ in $\mathcal{J}' := \mathcal{J}_1(c) \cap \mathcal{J}_1(d)$.

We already know that $\mathcal{J}'$ is a Jordan subalgebra in view of Lemma 3.3.3. Moreover, Proposition 3.3.4 shows that $\mathcal{J}'$ contains an idempotent of trace at least equal to 1, because $\mathcal{J}'$ is nonempty. We proceed below to improve our estimation of this trace.

A simple observation. Let $\{c_1, \ldots, c_r\}$ be a Jordan frame. From the relation

$$\sum_{j=1}^{r} \mathrm{tr}(c_j c) = \mathrm{tr}(c) = k,$$

we deduce that there are at most $r - k$ minimal idempotents c_j for which $\mathrm{tr}(c_j c) = 0$; similarly, there are at most $r - l$ minimal idempotents c_j for which $\mathrm{tr}(c_j d) = 0$. As a consequence of this, there exist at least $r - (r - k) - (r - l) = k + l - r$ indices j for which $\mathrm{tr}(c_j c) \neq 0$ and $\mathrm{tr}(c_j d) \neq 0$ *simultaneously* hold.

A contradiction argument. Let c' be the idempotent of maximal trace in $\mathcal{J}'$ and suppose that $t := \mathrm{tr}(c') < k+l-r$. We will construct an idempotent $\bar{c} \in \mathcal{J}'$ whose trace is strictly greater than t, leading us to a contradiction.

By definition, the idempotent c' is in $\mathcal{J}_1(c)$. Consequently, there exists a Jordan frame $\{f_1, \ldots, f_r\}$ such that $c' = f_1 + \cdots + f_t$ and $c = f_1 + \cdots + f_k$. In view of the above observation, we know that there is an integer j such that $t < j \leq k$ and $\mathrm{tr}(f_j d) \neq 0$ (observe that $\mathrm{tr}(f_j c)$ is null when $k < j \leq r$, and equals 1 otherwise).

This implies that $c - c' \neq 0$ and $d - c' \neq 0$. Of course, the element $c - c'$ is an idempotent, and so $d - c'$ is, because $c' \in \mathcal{J}_1(d)$.

Note that, in view of Lemma 3.3.1, we have $\mathcal{J}_1(c - c') \subseteq \hat{\mathcal{J}}$ and $\mathcal{J}_1(d - c') \subseteq \hat{\mathcal{J}}$. Moreover, we have $\mathrm{tr}(c - c') + \mathrm{tr}(d - c') = k + l - 2t > r - t$, which is the rank of $\hat{\mathcal{J}}$. Hence, we can apply Proposition 3.3.4 for the idempotents $c - c'$ and $d - c'$, considered in the algebra $\hat{\mathcal{J}} := \mathcal{J}_0(c)$ instead of $\mathcal{J}$.

Proposition 3.3.4 proves that there exists an idempotent c'' in $\mathcal{J}_1(c - c') \cap \mathcal{J}_1(d - c')$ of rank at least equal to 1 in $\hat{\mathcal{J}}$. Proposition 2.7.22 asserts that the rank of c'' is the same in $\hat{\mathcal{J}}$ and in $\mathcal{J}$. As $c'' \in \mathcal{J}_0(c')$, the element $\bar{c} := c' + c''$ is an idempotent and $\mathrm{tr}(\bar{c}) > t$. And since $(c - c')c'' = c''$ and $(d - c')c'' = c''$, we have $cc'' = dc'' = c''$ i.e. $c'' \in \mathcal{J}'$, hence $\bar{c} \in \mathcal{J}'$. This contradicts the maximality of c'. ■

The following remark insists on a useful aspect of the construction carried out in the proof of the previous proposition.

Remark 3.3.6 *The contradiction argument in the second part of the previous proof is based on the following fact. Suppose that there exists an idempotent $c' \in \mathcal{J}' := \mathcal{J}_1(c) \cap \mathcal{J}_1(d)$. Then, it is possible to find an idempotent $\bar{c} \in \mathcal{J}'$ of a trace at least as large as $r - \mathrm{tr}(c) - \mathrm{tr}(d)$ such that $\mathcal{J}_1(\bar{c})$ contains c'.* ■

This proposition allows us to reprove the following result of Hirzebruch. However, our technique applies to non-simple formally real Jordan algebras, in contrast with his approach.

Corollary 3.3.7 (Lemma 2.4 in [Hir70]) *Suppose that $\mathcal{J}$ is a simple Jordan algebra of rank r. Let $c_1, \ldots, c_{r-1}$ be $r - 1$ minimal idempotents of $\mathcal{J}$. There exists a minimal idempotent f for which $c_i f = 0$ for every $1 \leq i < r$.* ■

Proof

Let us apply the previous proposition to $c := e - c_1$ and $d := e - c_2$. It gives an idempotent f_{r-2} of rank $(r - 1) + (r - 1) - r = r - 2$ that belongs to $\mathcal{J}_0(c_1) \cap \mathcal{J}_0(c_2)$. In view of Lemma 3.3.2, $\mathcal{J}_1(f_{r-2}) \subseteq \mathcal{J}_0(c_1) \cap \mathcal{J}_0(c_2)$. Now, we apply again the previous proposition for $c := f_{r-2}$ and $d := e - c_3$. It gives an idempotent f_{r-3} of rank $(r-2)+(r-1)-r = r-3$ that belongs to $\mathcal{J}_1(f_{r-2}) \cap \mathcal{J}_0(c_3) \subseteq \mathcal{J}_0(c_1) \cap \mathcal{J}_0(c_2) \cap \mathcal{J}_0(c_3)$. Going on with this construction, we end up with an idempotent f_1 of rank 1, i.e. a minimal idempotent, that belongs to $\mathcal{J}_0(c_1) \cap \cdots \cap \mathcal{J}_0(c_{r-1})$. This is the idempotent we were looking for, as $c_i f_1 = 0$, because $f_1 \in \mathcal{J}_0(c_i)$. ■

As mentioned above, observe that our proof also applies for non-simple Jordan algebras.

3.4 Courant-Fischer's Theorem in Jordan algebras

Our proof of Wielandt's Theorem on Jordan algebras is loosely based on the original demonstration of Wielandt [Wie55]. His argument relies on a recurrence on the size of the Hermitian matrix. Here, we will undertake a recurrence on the rank r of the involved algebra $\mathcal{J}$. Our development requires an extension of the eigenvalue interlacing theorem (see Theorem 4.3.6 in [HJ96]), which is a consequence of the case where only *one* eigenvalue is involved in Wielandt's maximin characterization. This section focuses on this instance. As mentioned in the introduction, Wielandt's Theorem reduces in this case to the Courant-Fischer variational characterization of eigenvalues. In the framework of Hermitian matrices, this result can be stated as follows (see Theorem 4.2.11 in [HJ96]).

Let A be an $n \times n$ Hermitian matrix and let i be an integer between 1 and n. Then:

$$\lambda_i(A) = \max_{\dim(X)=i} \min_{\substack{h \in X \\ ||h||_2=1}} h^T A h = \min_{\dim(X)=n-i+1} \max_{\substack{h \in X \\ ||h||_2=1}} h^T A h. \tag{3.3}$$

■

As mentioned in the beginning of this chapter, the extension of this result to Jordan algebras requires to change the set on which the maximization is performed. To this end, we replace here the set of i-dimensional subspaces of $\mathbb{R}^n$ by the set:

$$\mathcal{E}_i := \{\mathcal{J}_1(c) | c \text{ is an idempotent of } \mathcal{J} \text{ of trace } i\}.$$

In the course of our development, we will need a slightly refined version of this definition. Suppose that $\mathcal{J}'$ is a subalgebra of $\mathcal{J}$. We denote:

$$\mathcal{E}_i(\mathcal{J}') := \{\mathcal{J}_1(c) | c \text{ is an idempotent of } \mathcal{J}' \text{ of trace } i\}.$$

Observe that, if $\mathcal{J}' = \mathcal{J}_1(d)$ and $\mathcal{J}_1(c) \in \mathcal{E}_i(\mathcal{J}')$, then $\mathcal{J}'_1(c) = \mathcal{J}_1(c)$ (see Lemma 3.3.2).

With this new object, the Courant-Fischer maximin relations can be extended as follows.

$$\lambda_i(u) = \max_{E \in \mathcal{E}_i} \min_{\substack{h \in E \\ ||h||=1}} \operatorname{tr}(Q_h u) = \min_{E \in \mathcal{E}_{r-i+1}} \max_{\substack{h \in E \\ ||h||=1}} \operatorname{tr}(Q_h u).$$

We have to mention that the subspaces of $\mathcal{E}_i$ do not necessarily have all the same dimension, especially when $\mathcal{J}$ is not a simple Jordan algebra. Note also that $\mathcal{E}_1$ does not contain all the straight lines of $\mathcal{J}$ passing through 0, but only the ones passing through a minimal idempotent. This is why our variational characterizations are deeply different from the Hermitian matrices version given in (3.3).

There already exists another version of the Courant-Fischer relations for *simple* formally real Jordan algebras, developed by Hirzebruch [Hir70]. We copy here his relations.

For every minimal idempotent, we define:

$$A_c := \{d | d \text{ is a minimal idempotent and } \operatorname{tr}(cd) = 0\}.$$

Then, the eigenvalue $\lambda_i(u)$ equals:

$$\min \left\{ \max \left\{ \operatorname{tr}(cu) | c \in A_{d_1} \cap \cdots \cap A_{d_{i-1}} \right\} | d_1, \ldots, d_{i-1} \text{ are min. idempotents of } \mathcal{J} \right\}.$$

However, our formulation appears to be better suited for a further extension to the full Wielandt's Theorem. Moreover, it allows us to prove easily several interesting corollaries, and especially interlacing relations of eigenvalues for formally real Jordan algebras.

Theorem 3.4.1 (Courant-Fischer's Theorem for Jordan algebras)
Let $u = \sum_{j=1}^{r} \lambda_j(u) c_j \in \mathcal{J}$ *and* $1 \leq i \leq r$. *We have:*

$$\lambda_i(u) = \max_{E \in \mathcal{E}_i} \min_{\substack{h \in E \\ ||h||=1}} \operatorname{tr}(Q_h u) = \min_{E \in \mathcal{E}_{r-i+1}} \max_{\substack{h \in E \\ ||h||=1}} \operatorname{tr}(Q_h u). \tag{3.4}$$

The subalgebra $\mathcal{J}_1(c_1 + \cdots + c_i) \in \mathcal{E}_i$ *achieves the maximum in the first formulation. The subalgebra* $\mathcal{J}_1(c_i + \cdots + c_r) \in \mathcal{E}_{r-i+1}$ *achieves the minimum in the second one.*

Proof

Let us fix a subalgebra E of $\mathcal{E}_i$. We denote:

$$H := \{h \in \mathcal{J} | \operatorname{tr}(Q_{c_j} h^2) = 0 \ \text{ for } \ 1 \leq j < i\}.$$

The inclusion $H \supseteq \mathcal{J}_0(c_1 + \cdots + c_{i-1}) = \mathcal{J}_1(c_i + \cdots + c_r)$ obviously holds, and the rank of the latter algebra is equal to $r - i + 1$. Since E is of rank i, Proposition 3.3.5 asserts us that $E \cap \mathcal{J}_0(c_1 + \cdots + c_{i-1})$ contains a nonzero element. Hence, the set $\{h \in E \cap \mathcal{J}_0(c_1 + \cdots + c_{i-1}) | \, ||h|| = 1\}$ is not empty, and a fortiori the set $\{h \in E \cap H | \, ||h|| = 1\}$ is not empty. Then, we can write:

$$\min_{\substack{h \in E \\ ||h||=1}} \operatorname{tr}(Q_h u) = \min_{\substack{h \in E \\ ||h||=1}} \sum_{j=1}^{r} \lambda_j(u) \operatorname{tr}(Q_h c_j) \leq \min_{\substack{h \in E \cap H \\ ||h||=1}} \sum_{j=1}^{r} \lambda_j(u) \operatorname{tr}(Q_h c_j).$$

Now, since $\operatorname{tr}(Q_h c_j) = \operatorname{tr}(Q_{c_j} h^2) = 0$ for every $h \in H$ and $1 \leq j < i$, while $\operatorname{tr}(Q_h c_j) \geq 0$ for $i \leq j \leq r$, we can proceed as follows:

$$\min_{\substack{h \in E \cap H \\ ||h||=1}} \sum_{j=1}^{r} \lambda_j(u) \operatorname{tr}(Q_h c_j) = \min_{\substack{h \in E \cap H \\ ||h||=1}} \sum_{j=i}^{r} \lambda_j(u) \operatorname{tr}(Q_h c_j) \leq \min_{\substack{h \in E \cap H \\ ||h||=1}} \lambda_i(u) \sum_{j=i}^{r} \operatorname{tr}(Q_h c_j).$$

This last minimum equals $\lambda_i(u)$ because, for every $h \in H$ of unitary norm, we have:

$$\sum_{j=i}^{r} \operatorname{tr}(Q_h c_j) = \sum_{j=1}^{r} \operatorname{tr}(Q_h c_j) = ||h||^2 = 1.$$

In order to prove the maximin relations, it remains to exhibit a subalgebra $E^* \in \mathcal{E}_i$ for which:

$$\lambda_i(u) \leq \min_{\substack{h \in E^* \\ ||h||=1}} \operatorname{tr}(Q_h u).$$

Let us take $E^* := \mathcal{J}_1(c_1 + \cdots + c_i)$. It is immediate from the first Pierce decomposition theorem that $Q_h c_j = 0$ for every $j > i$ and every $h \in E^*$. Thus $\sum_{j=1}^{i} \operatorname{tr}(Q_h c_j) = ||h||^2 = 1$ and:

$$\min_{\substack{h \in E^* \\ ||h||=1}} \operatorname{tr}(Q_h u) = \min_{\substack{h \in E^* \\ ||h||=1}} \sum_{j=1}^{r} \lambda_j(u) \operatorname{tr}(Q_h c_j) \geq \min_{\substack{h \in E^* \\ ||h||=1}} \lambda_i(u) \sum_{j=1}^{i} \operatorname{tr}(Q_h c_j) = \lambda_i(u).$$

In fact, this minimum is attained for $h := c_i$. This finishes the proof of the maximin relation.

It is well-known that the minimax Courant-Fischer relations follow directly from the maximin characterization. Indeed, we have:

$$\begin{aligned}
-\lambda_{r-i+1}(u) &= \lambda_i(-u) = \max_{E\in\mathcal{E}_i} \min_{\substack{h\in E\\ ||h||=1}} -\mathrm{tr}(Q_h u)\\
&= \min_{E\in\mathcal{E}_i} \left(- \min_{\substack{h\in E\\ ||h||=1}} \mathrm{tr}(Q_h u) \right) = -\max_{E\in\mathcal{E}_i} \min_{\substack{h\in E\\ ||h||=1}} \mathrm{tr}(Q_h u).
\end{aligned}$$

■

Courant-Fischer's Theorem has numerous important consequences. The first one we present here is a Jordan algebraic version of the interlacing relations between eigenvalues. We recall here the notational convention introduced in p. 79. When an element u belongs to a Jordan subalgebra $\mathcal{J}'$ of $\mathcal{J}$, one can consider its vector of eigenvalues in the algebra $\mathcal{J}'$ or in the algebra $\mathcal{J}$. Since these vectors can be different depending on the considered algebra (their size equals the rank of the corresponding algebra), we explicit this dependence by writing $\lambda(u;\mathcal{J}')$ for its ordered eigenvalue vector in $\mathcal{J}'$ and $\lambda(u;\mathcal{J})$ or simply $\lambda(u)$ for its eigenvalue vector in $\mathcal{J}$.

Corollary 3.4.2 *Let $u \in \mathcal{J}$ and c be an idempotent of $\mathcal{J}$ of trace k. For every $1 \leq i \leq k$, we have:*

$$\lambda_{r-k+i}(u;\mathcal{J}) \leq \lambda_i(Q_c u;\mathcal{J}_1(c)) \leq \lambda_i(u;\mathcal{J}). \tag{3.5}$$

Proof
Recall that $\mathcal{E}_i(\mathcal{J}_1(c)) \subseteq \mathcal{E}_i(\mathcal{J})$. According to Theorem 3.4.1,

$$\begin{aligned}
\lambda_i(Q_c u;\mathcal{J}_1(c)) &= \max_{E\in\mathcal{E}_i(\mathcal{J}_1(c))} \min_{\substack{h\in E\\ ||h||=1}} \mathrm{tr}(Q_h Q_c u) = \max_{E\in\mathcal{E}_i(\mathcal{J}_1(c))} \min_{\substack{h\in E\\ ||h||=1}} \mathrm{tr}(Q_h u)\\
&\leq \max_{E\in\mathcal{E}_i} \min_{\substack{h\in E\\ ||h||=1}} \mathrm{tr}(Q_h u) = \lambda_i(u;\mathcal{J}),
\end{aligned}$$

where Proposition 2.6.3 has been used to establish the second equality.

Using now the minimax version of Theorem 3.4.1 instead of the maximin one, we have:

$$\begin{aligned}
\lambda_{k-i+1}(Q_c u;\mathcal{J}_1(c)) &= \min_{E\in\mathcal{E}_i(\mathcal{J}_1(c))} \max_{\substack{h\in E\\ ||h||=1}} \mathrm{tr}(Q_h Q_c u) = \min_{E\in\mathcal{E}_i(\mathcal{J}_1(c))} \max_{\substack{h\in E\\ ||h||=1}} \mathrm{tr}(Q_h u)\\
&\geq \min_{E\in\mathcal{E}_i} \max_{\substack{h\in E\\ ||h||=1}} \mathrm{tr}(Q_h u) = \lambda_{r-i+1}(u;\mathcal{J}).
\end{aligned}$$

■

As an intriguing consequence of this corollary, one can characterize the number of nonzero eigenvalues of the elements in a Pierce subspace of the form $\mathcal{J}_{1/2}(c)$.

Corollary 3.4.3 *Suppose that c is an idempotent of $\mathcal{J}$ of trace k. Let $z \in \mathcal{J}_{1/2}(c)$. Then z has at most* $\min\{2k, 2r-2k\}$ *nonzero eigenvalues.*

Proof
Observe that $\mathcal{J}_{1/2}(c) = \mathcal{J}_{1/2}(e-c)$. We can then assume, without loss of generality, that $k \geq r/2$. Note that $Q_c z = 0$. Thus, using the interlacing relations (3.5), we have:

$$\lambda_{r-k+i}(z) \leq 0 \quad \text{and} \quad \lambda_i(z) \geq 0 \quad \text{for every } 1 \leq i \leq k.$$

If $r-k+1 \leq j \leq k$, the eigenvalue $\lambda_j(z)$ is null. Consequently, at least $k-(r-k+1)+1 = 2k-r$ eigenvalues are null, and at most $r-(2k-r) = 2r-2k$ are nonzero. ■

We can also deduce this useful result on the degeneracy of the projection of an element on off-diagonal Pierce decomposition.

Corollary 3.4.4 *Let d be an idempotent of trace p in $\mathcal{J}$, and let u be an element of $\mathcal{J}$. Suppose that $\sum_{i=1}^{p} \lambda_i(u) = \operatorname{tr}(du)$. Then $u \in \mathcal{J}_1(d) + \mathcal{J}_0(d)$.*
Proof
Let $u = u_1 + u_{1/2} + u_0$, with $u_\gamma \in \mathcal{J}_\gamma(d)$ for $\gamma \in \{0, 1/2, 1\}$. Note that:

$$\operatorname{tr}(du) = \operatorname{tr}(u_1) = \sum_{i=1}^{p} \lambda_i(Q_d u; \mathcal{J}_1(d)).$$

In view of the interlacing relations (3.5), we have $\lambda_i(Q_d u; \mathcal{J}_1(d)) \leq \lambda_i(u)$ for every $1 \leq i \leq p$. By the assumption $\sum_{i=1}^{p} \lambda_i(u) = \operatorname{tr}(du)$, we deduce that $\lambda_i(Q_d u; \mathcal{J}_1(d)) = \lambda_i(u)$. It follows that $\operatorname{tr}(u_0) = \operatorname{tr}(u) - \operatorname{tr}(u_1) = \sum_{i=p+1}^{r} \lambda_i(u)$. Moreover, we have:

$$\operatorname{tr}(u_0) = \sum_{i=1}^{r-p} \lambda_i(Q_{e-d} u; \mathcal{J}_0(d)).$$

The interlacing relations show that $\lambda_i(Q_{e-d} u; \mathcal{J}_0(d)) \geq \lambda_{p+i}(u)$ for $1 \leq i \leq r-p$, and, as above, all these inequalities are actually tight. Finally, from $||u_{1/2}||^2 = ||u||^2 - ||u_1||^2 - ||u_0||^2$, we can write:

$$||u_{1/2}||^2 = \sum_{i=1}^{r} \lambda_i^2(u) - \sum_{i=1}^{p} \lambda_i^2(Q_d u; \mathcal{J}_1(d)) - \sum_{i=1}^{r-p} \lambda_i^2(Q_{e-d} u; \mathcal{J}_0(d)) = 0,$$

so that $u_{1/2} = 0$ and $u \in \mathcal{J}_1(d) + \mathcal{J}_0(d)$. ■

Corollary 3.4.5 *Let $u \in \mathcal{K}_\mathcal{J}$ and c be an idempotent of $\mathcal{J}$. Then $\lambda_i(Q_c u) \leq \lambda_i(u)$ for $1 \leq i \leq r$.*
Proof
Since $Q_c u \in \mathcal{K}_\mathcal{J}$, we know that each eigenvalue $\lambda_i(Q_c u; \mathcal{J})$ is nonnegative. Let k be the trace of c. As $Q_c u \in \mathcal{J}_0(e-c)$, we know that $\lambda_i(Q_c u; \mathcal{J}) = 0$ for $i > k$. The desired inequality is then already shown for these indices $i > k$. Now, if $1 \leq i \leq k$, we know, by uniqueness of spectral decomposition in $\mathcal{J}$, that $\lambda_i(Q_c u; \mathcal{J}_1(c)) = \lambda_i(Q_c u; \mathcal{J})$. Applying the interlacing inequalities, we get $\lambda_i(Q_c u; \mathcal{J}) \leq \lambda_i(u; \mathcal{J})$ for all these indices i. ■

The following last consequence of Fischer's inequalities in Jordan algebras is rather technical. This statement has been written in order to be ready-to-use in the proof of Wielandt's Theorem. It relates the smallest eigenvalues of an element with the smallest eigenvalues of well-chosen projections of this element.

Proposition 3.4.6 *Let $u = \sum_{j=1}^{r} \lambda_j(u)c_j \in \mathcal{J}$ and let us fix an integer $1 \leq i \leq r$. We take $c := c_{r-i+1} + \cdots + c_r$, so that $\operatorname{tr}(c) = i$. Let $i \leq k \leq r$ and $\mathcal{J}_1(d) \in \mathcal{E}_k(\mathcal{J})$ be a subalgebra that contains $\mathcal{J}_1(c)$. Then*

$$\lambda_{r-i+1}(u; \mathcal{J}) = \lambda_{k-i+1}(Q_d u; \mathcal{J}_1(d)).$$

Proof

If $k = r$, then $d = e$, and there is nothing to prove.

If $k = i$, then $c = d$, and the identity to prove reduces to $\lambda_{r-i+1}(u; \mathcal{J}) = \lambda_1(Q_d u; \mathcal{J}_1(d))$. In fact, this is immediate if we write:

$$Q_d u = Q_c u = \sum_{j=r-i+1}^{r} \lambda_j(u; \mathcal{J})c_j.$$

Indeed, the latter identity can be interpreted as the spectral decomposition of $Q_d u$ in $\mathcal{J}_1(c) = \mathcal{J}_1(d)$, which then yields $\lambda_1(Q_d u; \mathcal{J}_1(d)) = \lambda_{r-i+1}(u; \mathcal{J})$.

We consider now the case where $i < k < r$. It turns out that $Q_{c,d-c}u = 0$. To see this, we use the Fundamental Identity (2.14), the operator commutativity of $e - c$ and d, and the relation $Q_d c = c$.

$$\begin{aligned} Q_d Q_{e-c} Q_d &= Q_{Q_d e-c} = Q_{d-c}, \\ Q_d Q_{e-c} Q_d &= Q_{e-c} Q_d^2 = Q_{e-c} Q_d. \end{aligned}$$

As $u = Q_c u + Q_{e-c} u$, we can thus write:

$$Q_d u = Q_d Q_c u + Q_d Q_{e-c} u = Q_c u + Q_{d-c} u.$$

We conclude that:

$$2Q_{c,d-c} u = Q_d u - Q_c u - Q_{d-c} u = 0,$$

i.e. that $Q_d u \in \mathcal{J}_1(c) \oplus \mathcal{J}_1(d - c)$. Let us have a closer look on the eigenvalues of $Q_c u$ and of $Q_{d-c}u$, which occur in the spectral decomposition of $Q_d u$ in $\mathcal{J}_1(d)$. We know by hypothesis that:

$$Q_c u = \sum_{j=r-i+1}^{r} \lambda_j(u; \mathcal{J})c_j.$$

In other words, the eigenvalues $\lambda_{r-i+1}(u; \mathcal{J}), \ldots, \lambda_r(u; \mathcal{J})$ are those of $Q_c u$ in $\mathcal{J}_1(c)$.

Let us now focus on the eigenvalues of $Q_{d-c}u$ in $\mathcal{J}_1(d - c)$. We prove below that they are all larger than those of $Q_c u$ in $\mathcal{J}_1(c)$. The smallest eigenvalue of $Q_{d-c}u$ in $\mathcal{J}_1(d - c)$ is, in view of Courant-Fischer's Theorem:

$$\begin{aligned} \lambda_{k-i}(Q_{d-c}u; \mathcal{J}_1(d - c)) &= \min_{\substack{h \in \mathcal{J}_1(d-c) \\ ||h||=1}} \operatorname{tr}(Q_h Q_{d-c} u) = \min_{\substack{h \in \mathcal{J}_1(d-c) \\ ||h||=1}} \operatorname{tr}(Q_h u) \\ &\geq \min_{\substack{h \in \mathcal{J}_1(e-c) \\ ||h||=1}} \operatorname{tr}(Q_h u) = \min_{\substack{h \in \mathcal{J}_1(e-c) \\ ||h||=1}} \operatorname{tr}(Q_h Q_{e-c} u) \\ &= \lambda_{r-i}(Q_{e-c}u; \mathcal{J}_1(e - c)) = \lambda_{r-i}(u; \mathcal{J}). \end{aligned}$$

Hence, the eigenvalue $\lambda_{r-i+1}(u; \mathcal{J})$, which is the largest eigenvalue of $Q_c u$ in $\mathcal{J}_1(c)$, is the $k - i + 1$st largest eigenvalue of $Q_d u$ in $\mathcal{J}_1(d)$. ■

3.5 Wielandt's Theorem in Jordan algebras

In the same vein as our extension (3.4) of the Courant-Fischer equations, we can reformulate Wielandt's Theorem (3.1) in the context of formally real Jordan algebras as follows.

Let $1 \leq k \leq r$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq r$. For every $u \in \mathcal{J}$, we have:

$$\sum_{j=1}^{k} \lambda_{i_j}(u) = \max_{\substack{E_j \in \mathcal{E}_{i_j} \\ E_1 \subseteq \cdots \subseteq E_k}} \min_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} \sum_{j=1}^{k} \mathrm{tr}(Q_{h_j} u) \tag{3.6}$$

and

$$\sum_{j=1}^{k} \lambda_{i_j}(u) = \min_{\substack{E_j \in \mathcal{E}_{r-i_j+1} \\ E_1 \supseteq \cdots \supseteq E_k}} \max_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} \sum_{j=1}^{k} \mathrm{tr}(Q_{h_j} u). \tag{3.7}$$

The condition "$h_i^T h_j = \delta_{ij}$ for $1 \leq i, j \leq k$" has been replaced here by the somewhat intriguing "$\lambda_1(h_1^2 + \cdots + h_k^2) \leq 1$". Observe that the conditions $||h_j|| = 1$ for $1 \leq j \leq k$ and $\lambda_1(h_1^2 + \cdots + h_k^2) \leq 1$ hold if one takes $\{h_1, \ldots, h_k\}$ as a set of orthogonal minimal idempotents.

The relation (3.7) is an immediate consequence of (3.6). As in the proof of Theorem 3.4.1, it suffices to replace u by $-u$ in (3.6); using $\lambda_i(-u) = -\lambda_{r-i+1}(u)$, we get:

$$\begin{aligned} -\sum_{j=1}^{k} \lambda_{r-i_j+1}(u) = \sum_{j=1}^{k} \lambda_{i_j}(-u) &= \max_{\substack{E_j \in \mathcal{E}_{i_j} \\ E_1 \subseteq \cdots \subseteq E_k}} \min_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} -\sum_{j=1}^{k} \mathrm{tr}(Q_{h_j} u) \\ &= -\min_{\substack{E_j \in \mathcal{E}_{i_j} \\ E_1 \subseteq \cdots \subseteq E_k}} \max_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} \sum_{j=1}^{k} \mathrm{tr}(Q_{h_j} u), \end{aligned}$$

which is (3.7).

Our task is now to prove (3.6). First, we need to ensure that the set on which we perform the minimization is not empty: given a sequence $E_1 \subseteq \cdots \subseteq E_k$ of Jordan subalgebras $E_j \in \mathcal{E}_{i_j}$, we need to check that there always exists a set of elements $\{h_1, \ldots, h_k\}$ with $h_j \in E_j$, $||h_j|| = 1$ and $\lambda_1(h_1^2 + \cdots + h_k^2) \leq 1$. We carry out this verification in the next lemma.

Lemma 3.5.1 *Given $1 \leq k \leq r$ and $1 \leq i_1 < \cdots < i_k \leq r$, we let $E_j \in \mathcal{E}_{i_j}$ be a sequence of subalgebras such that $E_1 \subseteq \cdots \subseteq E_k$. Then, there exists a Jordan frame $\{d_1, \ldots d_r\}$ such that $E_j = \mathcal{J}_1(d_1 + \cdots + d_{i_j})$. We can then take $h_j := d_{i_j}$ to satisfy the desired conditions.*

Proof

For every $1 \leq j \leq k$, we define $d^{(j)}$ as the idempotents for which $E_j = \mathcal{J}_1(d^{(j)})$; letting $d^{(0)} := 0$, we set $e_j := d^{(j)} - d^{(j-1)}$.

We first check that the elements of $S := \{e_1, \ldots, e_k\}$ are orthogonal idempotents. The element $e_1 = d^{(1)}$ is an idempotent by definition. Of course, $d^{(i)} \in \mathcal{J}_1(d^{(j)})$ when $i \leq j$, so that $d^{(i)}d^{(j)} = d^{(i)}$. This implies:

$$e_j^2 = (d^{(j)})^2 - 2d^{(j)}d^{(j-1)} + (d^{(j-1)})^2 = d^{(j)} - d^{(j-1)} = e_j$$

and

$$e_j e_i = (d^{(j)} - d^{(j-1)})(d^{(i)} - d^{(i-1)}) = d^{(i)} - d^{(i)} - d^{(i-1)} + d^{(i-1)} = 0$$

for $1 \leq i < j \leq k$.

If needed, that is if $i_k \neq r$, i.e. if $E_k \neq \mathcal{J}$, we can complete S into a *system* of idempotents by incorporating $e_{k+1} := e - d^{(k)}$ in it. Considering $u := -\sum_{j=1}^{k+1} je_j$, there exists a Jordan frame $\{d_1, \ldots, d_r\}$ such that $u = \sum_{i=1}^{r} \lambda_i(u)d_i$ by the Theorem of complete spectral decomposition. It remains to identify these two spectral decompositions to get $e_j = d_{i_j} + \cdots + d_{i_{j-1}+1}$ (with $i_0 := 0$), and $d^{(j)} = d_{i_j} + \cdots + d_1$. ■

Definition 3.5.2 *We call every Jordan frame given by the previous lemma a Jordan frame* compatible with $E_1, \ldots, E_k$. ■

The next lemma tackles the easy part of Wielandt's Theorem: proving that the partial sum of eigenvalues is smaller than the maximin expression.

Lemma 3.5.3 *Let $1 \leq k \leq r$, $1 \leq i_1 < \cdots < i_k \leq r$ and $u \in \mathcal{J}$. There exist subalgebras $E_1 \subseteq \cdots \subseteq E_k$ with $E_j \in \mathcal{E}_{i_j}$ such that, for every $h_j \in \mathcal{E}_j$ of norm 1, we have:*

$$\sum_{j=1}^{k} \lambda_{i_j}(u) \leq \sum_{j=1}^{k} \mathrm{tr}(Q_{h_j} u).$$

Proof
Let $u = \sum_{i=1}^{r} \lambda_i(u)c_i \in \mathcal{J}$, $d^{(j)} := c_1 + \cdots + c_{i_j}$, and $E_j := \mathcal{J}_1(d^{(j)})$. Since $\mathrm{tr}(d^{(j)}) = i_j$, the subalgebra E_j belongs to the set $\mathcal{E}_j$ for every j. Note also that $E_j \subseteq E_{j+1}$. Let h_j be an element of norm 1 in $\mathcal{E}_j$. By the choice of E_j and by the Courant-Fischer maximin relation given in Theorem 3.4.1, we can write, for every $1 \leq j \leq k$, that:

$$\lambda_{i_j}(u) = \min_{\substack{h'_j \in E_j \\ ||h'_j||=1}} \mathrm{tr}(Q_{h'_j} u) \leq \mathrm{tr}(Q_{h_j} u).$$

Adding these inequalities, we obtain the desired result. ■

Our task now is to prove the converse inequality. There is a particular case where this is not difficult: when $k = r$, this comes easily from Lemma 3.5.1, as shown by the following lemma.

Lemma 3.5.4 *Let u be an element of $\mathcal{J}$ and let $E_1 \subseteq \cdots \subseteq E_r$ be a sequence of subalgebras such that $E_j \in \mathcal{E}_j$. Then there exist $h_j \in E_j$ for $1 \leq j \leq r$ such that $||h_j|| = 1$, $\lambda_1(h_1^2 + \cdots + h_r^2) \leq 1$, and $\mathrm{tr}(u) = \sum_{j=1}^{r} \mathrm{tr}(Q_{h_j} u)$.*

Proof
Let $u \in \mathcal{J}$ and E_j be a sequence of subalgebras that satisfy the assumptions of the statement. Let $\{d_1, \ldots, d_r\}$ be a Jordan frame compatible with $E_1, \ldots, E_r$. (Observe that this Jordan frame is uniquely determined.) Taking $h_j := d_j$ for every j, we have:

$$\sum_{j=1}^{r} \operatorname{tr}(Q_{h_j} u) = \operatorname{tr}(Q_e u) = \operatorname{tr}(u) = \sum_{j=1}^{r} \lambda_j(Q_{h_j} u; \mathcal{J}).$$

■

Now, we are ready to give a proof of Wielandt's Theorem in Jordan algebras.

Theorem 3.5.5 (Wielandt's Theorem for Jordan algebras)
Let $1 \leq k \leq r$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq r$. For every $u \in \mathcal{J}$, we have:

$$\sum_{j=1}^{k} \lambda_{i_j}(u) = \max_{\substack{E_j \in \mathcal{E}_{i_j} \\ E_1 \subseteq \cdots \subseteq E_k}} \; \min_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} \; \sum_{j=1}^{k} \operatorname{tr}(Q_{h_j} u)$$

and

$$\sum_{j=1}^{k} \lambda_{i_j}(u) = \min_{\substack{E_j \in \mathcal{E}_{r-i_j+1} \\ E_1 \supseteq \cdots \supseteq E_k}} \; \max_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} \; \sum_{j=1}^{k} \operatorname{tr}(Q_{h_j} u).$$

Proof
We state below the assertion that remains to be checked.

In a formally real Jordan algebra $\mathcal{J}$ of rank r and for every element $u \in \mathcal{J}$, if $1 \leq k \leq r$ and $1 \leq i_1 < \cdots < i_k \leq r$, we have:

$$\sum_{j=1}^{k} \lambda_{i_j}(u; \mathcal{J}) \geq \max_{\substack{E_j \in \mathcal{E}_{i_j}(\mathcal{J}) \\ E_1 \subseteq \cdots \subseteq E_k}} \; \min_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2; \mathcal{J})\leq 1}} \; \sum_{j=1}^{k} \operatorname{tr}(Q_{h_j} u). \qquad (W_r)$$

The proof is carried by an induction *on the rank* r of the algebra $\mathcal{J}$.

The statement (W_1) is proved by applying Lemma 3.5.4 for $r := 1$.

We fix an integer $r > 1$, and we assume that (W_k) holds for every $1 \leq k < r$. We proceed below to prove the statement (W_r) for every formally real Jordan algebra of rank r.

Let $\mathcal{J}$ be a formally real Jordan algebra of rank r and let $u = \sum_{i=1}^{r} \lambda_i(u) c_i \in \mathcal{J}$. We choose an integer k between 1 and r, and a sequence of integers $1 \leq i_1 < \cdots < i_k \leq r$. We fix k Jordan subalgebras $E_j = \mathcal{J}_1(d^{(j)}) \in \mathcal{E}_{i_j}(\mathcal{J})$ such that $E_1 \subseteq \cdots \subseteq E_k$. We denote by $\{d_1, \ldots, d_r\}$ a Jordan frame compatible with $E_1, \ldots, E_k$.

Now, we need to find some elements $h_j \in E_j$ of norm 1, with:

$$\lambda_1(h_1^2 + \cdots + h_k^2; \mathcal{J}) \leq 1 \quad \text{and} \quad \sum_{j=1}^{k} \lambda_{i_j}(u; \mathcal{J}) \geq \sum_{j=1}^{k} \operatorname{tr}(Q_{h_j} u). \tag{3.8}$$

To this end, we distinguish two cases.

A. (Apply induction immediately) Suppose that $i_k \neq r$ and denote $d^{(k)}$ by d to simplify the notations. Then $E_k = \mathcal{J}_1(d)$ is a Jordan algebra of rank $i_k < r$ and $E_j \in \mathcal{E}_j(\mathcal{J}_1(d))$ because $d^{(j)} \in \mathcal{J}_1(d)$. By induction hypothesis, the statement (W_{i_k}) provides us with some elements $h_j \in E_j$ of norm 1 such that $\lambda_1(h_1^2 + \cdots + h_k^2; \mathcal{J}_1(d)) \leq 1$ and:

$$\sum_{j=1}^{k} \lambda_{i_j}(Q_d u; \mathcal{J}_1(d)) \geq \sum_{j=1}^{k} \mathrm{tr}(Q_{h_j} Q_d u). \tag{3.9}$$

In view of Proposition 2.6.3, we can write:

$$\sum_{j=1}^{k} \mathrm{tr}(Q_{h_j} Q_d u) = \sum_{j=1}^{k} \mathrm{tr}(Q_{h_j} u).$$

From Corollary 3.4.2, we also get:

$$\sum_{j=1}^{k} \lambda_{i_j}(Q_d u; \mathcal{J}_1(d)) \leq \sum_{j=1}^{k} \lambda_{i_j}(u; \mathcal{J}).$$

Moreover, since $h_1^2 + \cdots + h_k^2 \in \mathcal{J}_1(d)$ has nonnegative eigenvalues, its largest eigenvalue in $\mathcal{J}_1(d)$ equals its largest in $\mathcal{J}$. The first case is settled.

B. (Peel off and apply induction) Suppose now that $i_k = r$. The case $k = r$ has already been carried out in Lemma 3.5.4. If $k < r$, there exists an integer $1 \leq l < r$ such that $i_{l+1}, \ldots, i_k$ are consecutive and for which $i_l + 1 < i_{l+1}$. That is, $i_j = r - k + j$ for $l < j \leq r$ and $i_l < r - k + l$. We let:

$$\bar{c} := \sum_{j=r-k+l+1}^{r} c_j.$$

The rank of the subalgebra $\mathcal{J}_1(\bar{c})$ equals $k - l$, while the rank of the subalgebra E_l equals i_l. By Proposition 3.3.5, there exists an idempotent d' of trace at least 1 in $\mathcal{J}_0(\bar{c}) \cap \mathcal{J}_0(d^{(l)})$, because

$$(r - (k - l)) + (r - i_l) - r = r - k + l - i_l \geq 1.$$

In view of Lemma 3.3.3, the idempotent $d := e - d'$ satisfies $\mathcal{J}_1(d) \supseteq \mathcal{J}_1(\bar{c}) + \mathcal{J}_1(d^{(l)}) = \mathcal{J}_1(\bar{c}) + E_l$.

Let $\mathcal{J}' := \mathcal{J}_1(d)$. The trace of this algebra equals at most $r - 1$. We take $E'_1 := E_1$, $\ldots$, $E'_l := E_l$ and $u' := Q_d u$. Note that $E'_j \in \mathcal{E}_{i_j}(\mathcal{J}')$. For $l < j \leq r$, we can apply Proposition 3.3.5 to exhibit an idempotent $f^{(j)}$ of trace $i_j + \mathrm{tr}(d) - r \leq i_j - 1$ for which $\mathcal{J}_1(f^{(j)}) \subseteq E_j \cap \mathcal{J}'$. In view of Remark 3.3.6, the idempotents $f^{(j)}$ can be constructed such that $\mathcal{J}_1(f^{(j)}) \subseteq \mathcal{J}_1(f^{(j+1)})$ by applying the argument given in the proof of Proposition 3.3.5; for the same reason, we can require that $E_l = E_l \cap \mathcal{J}' \subseteq \mathcal{J}_1(f^{(l+1)})$. We then take $E'_j := \mathcal{J}_1(f^{(j)})$ for $l < j \leq r$, so that $E'_j \in \mathcal{E}_{i_j + \mathrm{tr}(d) - r}(\mathcal{J}')$ for these subalgebras. We apply now the statement $(W_{\mathrm{tr}(d)})$ for $\mathcal{J}'$, u' and $(E'_j)_{1\leq j\leq k}$. Accordingly, we get some elements $h_1 \in E'_1, \ldots, h_k \in E'_k$ such that $\lambda_1(h_1^2 + \cdots + h_k^2; \mathcal{J}') \leq 1$ and:

$$\sum_{j=1}^{l} \lambda_{i_j}(u'; \mathcal{J}') + \sum_{j=l+1}^{k} \lambda_{i_j+\mathrm{tr}(d)-r}(u'; \mathcal{J}') \geq \sum_{j=1}^{k} \mathrm{tr}(Q_{h_j} u'). \tag{3.10}$$

Let us check that these elements satisfy all the required properties. First, we have $h_j \in E'_j \subseteq E_j$ for every j. Second, since the element $h_1^2 + \cdots + h_k^2$ belongs to $\mathcal{K}_{\mathcal{J}}$, its largest eigenvalue in $\mathcal{J}'$ equals its largest eigenvalue in $\mathcal{J}$. Third, Proposition 2.6.3 shows that:

$$\sum_{i=1}^{k} \mathrm{tr}(Q_{h_j} u') = \sum_{i=1}^{k} \mathrm{tr}(Q_{h_j} u).$$

The interlacing relations of Corollary 3.4.2 prove that:

$$\sum_{j=1}^{l} \lambda_{i_j}(u) \geq \sum_{j=1}^{l} \lambda_{i_j}(u'; \mathcal{J}').$$

It remains to verify that:

$$\lambda_{i_j+\mathrm{tr}(d)-r}(u'; \mathcal{J}') = \lambda_{i_j}(u; \mathcal{J})$$

for every $l < j \leq k$. Actually, this comes from Proposition 3.4.6. Indeed, it suffices to replace in its statement

"i" by "$r - i_j + 1$", "c" by "$c_{i_j-1} + \cdots + c_r$" and "k" by "$\mathrm{tr}(d)$"

to get the desired inequality, because $\mathcal{J}_1(c) \subseteq \mathcal{J}_1(\bar{c}) \subseteq \mathcal{J}_1(d)$. ∎

The same proof can be carried out in order to show the following more general statement.

Theorem 3.5.6 *Let $1 \leq k \leq r$ be an integer, and let $-\infty \leq a \leq b \leq +\infty$. Suppose that $f : [a, b]^k \to \mathbb{R}$ is a function with the following properties:*

- ◇ *f is symmetric with respect to permutations of its arguments,*
- ◇ *f is increasing in every of its argument's components, and*
- ◇ *f is Schur-convex.*

Let $1 \leq i_1 < i_2 < \cdots < i_k \leq r$. For every $u \in \mathcal{J}$ with eigenvalues between a and b, we have:

$$f(\lambda_{i_1}(u), \ldots, \lambda_{i_k}(u)) = \max_{\substack{E_j \in \mathcal{E}_{i_j} \\ E_1 \subseteq \cdots \subseteq E_k}} \min_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} f(\mathrm{tr}(Q_{h_1} u), \ldots, \mathrm{tr}(Q_{h_k} u)) \tag{3.11}$$

and:

$$f(\lambda_{i_1}(u), \ldots, \lambda_{i_k}(u)) = \min_{\substack{E_j \in \mathcal{E}_{r-i_j+1} \\ E_1 \supseteq \cdots \supseteq E_k}} \max_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} f(\mathrm{tr}(Q_{h_1} u), \ldots, \mathrm{tr}(Q_{h_k} u)). \tag{3.12}$$

Remark 3.5.7 *The right-hand sides of (3.11) and of (3.12) are well-defined. Here is a verification. Observe that if the eigenvalues of an element u of $\mathcal{J}$ are in a range between a and b, then the number* $\mathrm{tr}(Q_h u)$ *lies also in the interval* $[a, b]$ *when* $||h|| = 1$. *Indeed, considering a complete spectral decomposition* $u = \sum_{i=1}^{r} \lambda_i(u)c_i$, *we have*

$$\mathrm{tr}(Q_h u) = \mathrm{tr}(h^2 u) = \sum_{i=1}^{r} \lambda_i(u)\mathrm{tr}(h^2 c_i).$$

The latter term indicates that $\mathrm{tr}(Q_h u)$ *is a convex combination of the eigenvalues of* u, *because the nonnegative coefficients* $\mathrm{tr}(h^2 c_i)$ *sum up to* $||h||^2 = 1$. *Henceforth, the number* $\mathrm{tr}(Q_h u)$ *ranges between* a *and* b. ■

Remark 3.5.8 *As an example of a function f that satisfies the three required properties, we can take* $f : \mathbb{R}^k \to \mathbb{R}$, $\gamma \mapsto f(\gamma) := \prod_{i=1}^{k} \gamma_i$. ■

Proof
The proof has the same structure than the proof of the previous theorem.

The maximin relation follows from the minimax relation. Consider the function: $g : [-b, -a]^k \to \mathbb{R}$, $\gamma \mapsto g(\gamma) := -f(-\gamma)$. Observe that this function is symmetric with respect to permutations of its arguments and is increasing in every component. It is also a Schur-convex function: as an immediate consequence of the definition of majorization, we have $\gamma \preceq \lambda$ if and only if $-\lambda \preceq -\gamma$; this implies $g(\lambda) = -f(-\lambda) \geq -f(-\gamma) = g(\gamma)$.

Assuming that the maximin statement (3.11) holds for every function that satisfies the three required properties, we can apply it to the function g above. We get:

$$\begin{aligned}
& -f(\lambda_{r-i_1+1}(-u), \ldots, \lambda_{r-i_k+1}(-u)) \\
= & -f(-\lambda_{i_1}(u), \ldots, -\lambda_{i_k}(u)) \\
= & \; g(\lambda_{i_1}(u), \ldots, \lambda_{i_k}(u)) \\
= & \max_{\substack{E_j \in \mathcal{E}_{i_j} \\ E_1 \subseteq \cdots \subseteq E_k}} \; \min_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} g(\mathrm{tr}(Q_{h_1}u), \ldots, \mathrm{tr}(Q_{h_k}u)) \\
= & \max_{\substack{E_j \in \mathcal{E}_{i_j} \\ E_1 \subseteq \cdots \subseteq E_k}} \; \min_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} -f(\mathrm{tr}(Q_{h_1}(-u)), \ldots, \mathrm{tr}(Q_{h_k}(-u))) \\
= & -\min_{\substack{E_j \in \mathcal{E}_{i_j} \\ E_1 \subseteq \cdots \subseteq E_k}} \; \max_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} f(\mathrm{tr}(Q_{h_1}(-u)), \ldots, \mathrm{tr}(Q_{h_k}(-u))),
\end{aligned}$$

which establishes (3.12). Let us now proceed to prove (3.11).

Generalization of Lemma 3.5.3: the easiest inequality. The extension we need takes the following form.

There exist subalgebras $E_1 \subseteq \cdots \subseteq E_k$ with $E_j \in \mathcal{E}_{i_j}$ such that, for every $h_j \in \mathcal{E}_j$ of norm 1, we have:

$$f(\lambda_{i_1}(u), \ldots, \lambda_{i_k}(u)) \leq f(\operatorname{tr}(Q_{h_1}u), \ldots, \operatorname{tr}(Q_{h_k}u)).$$

■

The beginning of its proof is a copy-paste of the demonstration of Lemma 3.5.3. The final argument comes from the fact that f is increasing in every of its argument's components.

Generalization of Lemma 3.5.4: the case $k = r$.

We assume here that $k = r$. Let $E_1 \subseteq \cdots \subseteq E_r$ be a sequence of subalgebras such that $E_j \in \mathcal{E}_j$. For every $1 \leq j \leq r$, there exists an element $h_j \in E_j$ of unit norm such that $\lambda_1(h_1^2 + \cdots + h_r^2) \leq 1$, and

$$f(\lambda_1(u), \ldots, \lambda_r(u)) \geq f(\operatorname{tr}(Q_{h_1}u), \ldots, \operatorname{tr}(Q_{h_r}u)).$$

■

Let $\{d_1, \ldots, d_r\}$ be a Jordan frame compatible with $E_1, \cdots, E_r$. We show below that $h_j := d_j$ is a satisfactory choice. Since f is Schur-convex, we only need to prove that the vector $\lambda(u)$ majorizes the vector $\gamma := (\operatorname{tr}(Q_{d_1}u), \ldots, \operatorname{tr}(Q_{d_r}u))^T$. Let us fix an integer $1 \leq p \leq r$, and a subset I of $\{d_1, \ldots, d_r\}$ of size p. With $d_I := \sum_{j \in I} d_j$, we have

$$\sum_{j \in I} \operatorname{tr}(Q_{d_j}u) = \sum_{j \in I} \operatorname{tr}(d_j u) = \operatorname{tr}(d_I u).$$

Observe that $d_I \in \mathcal{S}(\mathbf{1}_p)$. In view of Proposition 3.2.7, we obtain that $\operatorname{tr}(d_I u) \leq \sum_{i=1}^p \lambda_i(u)$. Thus $s_p(\gamma) \leq s_p(\lambda(u))$. Moreover, if $p = r$, we have

$$s_r(\gamma) = \sum_{j=1}^r \operatorname{tr}(d_j u) = \operatorname{tr}(u) = s_r(\lambda(u)).$$

Henceforth, we have $\lambda(u)$ majorizes γ in view of Lemma 3.2.5.

Following the proof of Theorem 3.5.5. We use here the notation and the objects defined in the proof of the aforementioned theorem. The problem (3.8) takes here the form: find some elements $h_j \in E_j$ of norm 1, with $\lambda_1(h_1^2 + \cdots + h_k^2; \mathcal{J}) \leq 1$ and:

$$f(\lambda_{i_1}(u; \mathcal{J}), \ldots, \lambda_{i_k}(u; \mathcal{J})) \geq f(\operatorname{tr}(Q_{h_1}u), \ldots, \operatorname{tr}(Q_{h_k}u)). \tag{3.13}$$

Case A. Again, the reader is referred to Case A of the proof of the Theorem 3.5.5 for the notation. The relation (3.9) becomes here:

$$f(\lambda_{i_1}(Q_d u; \mathcal{J}_1(d)), \ldots, \lambda_{i_k}(Q_d u; \mathcal{J}_1(d))) \geq f(\operatorname{tr}(Q_{h_1}Q_d u), \ldots, \operatorname{tr}(Q_{h_k}Q_d u)).$$

The interlacing relations (3.5) ensures that the arguments of f lie within the appropriate range. Note that $\lambda_{i_j}(Q_d u; \mathcal{J}_1(d)) \leq \lambda_{i_j}(u; \mathcal{J})$, so that the left-hand side is smaller than $f(\lambda_{i_1}(u; \mathcal{J}), \ldots, \lambda_{i_k}(u; \mathcal{J}))$ in view of the increasing property of f. From Proposition 2.6.3, we have $\operatorname{tr}(Q_{h_j}Q_d u) = \operatorname{tr}(Q_{h_j}u)$. Hence, the right-hand side equals $f(\operatorname{tr}(Q_{h_1}u), \ldots, \operatorname{tr}(Q_{h_k}u))$. The first case is settled.

Case B. In Case B of the proof of the Theorem 3.5.5, we have defined some elements $h_1, \ldots, h_k$, and an idempotent d. We reemploy these objects here, as well as the convenient notation we have introduced there.

Here, the relation (3.10) becomes:

$$f(\lambda_{i_1}(u';\mathcal{J}'), \ldots, \lambda_{i_l}(u';\mathcal{J}'), \lambda_{i_{l+1}+\mathrm{tr}(d)-r}(u';\mathcal{J}'), \ldots, \lambda_{i_k+\mathrm{tr}(d)-r}(u';\mathcal{J}'))$$
$$\geq f(\mathrm{tr}(Q_{h_1}u'), \ldots, \mathrm{tr}(Q_{h_k}u')). \tag{3.14}$$

It remains to check that this inequality implies the desired relation (3.13). Since we have $\mathrm{tr}(Q_{h_j}u') = \mathrm{tr}(Q_{h_j}u)$, the right-hand sides are equal. Now, we have shown in the proof of the Theorem that $\lambda_{i_j+\mathrm{tr}(d)-r}(u';\mathcal{J}') = \lambda_{i_j}(u;\mathcal{J})$ for $l < j \leq k$. Moreover, the interlacing relations show that $\lambda_{i_j}(u';\mathcal{J}') \leq \lambda_{i_j}(u;\mathcal{J})$ for $1 \leq j \leq l$. By the increasing property of f, we finally conclude that the left-hand side of the inequality (3.14) is smaller than the left-hand side of (3.13). Everything is now proved. ■

3.6 Applications of Wielandt's Theorem

We propose here two applications of Wielandt's Theorem. The first one is an extension of the Lidskii's inequalities to Jordan algebras. This extension has been mentioned as an open problem in [BGLS01] for the more general framework of hyperbolic polynomials (Open Problem 3.6). However, this question has recently been settled by Leonid Gurvits in the preprint [Gur04], as a direct consequence of the Lax Conjecture, proved in [LPR05].

Corollary 3.6.1 (Extension of Lidskii's inequalities) *Let us fix the integers $1 \leq k \leq r$ and $1 \leq i_1 < \cdots < i_k \leq r$. For every elements u and v of $\mathcal{J}$, we have:*

$$\sum_{j=1}^k \lambda_{i_j}(u) + \sum_{j=1}^k \lambda_j(v-u) \geq \sum_{j=1}^k \lambda_{i_j}(v) \geq \sum_{j=1}^k \lambda_{i_j}(u) + \sum_{j=1}^k \lambda_{r-j+1}(v-u).$$

Proof
By Proposition 3.2.7, we know that:

$$\sum_{j=1}^k \lambda_j(v-u) = \max_{h\in S(\mathbf{1}_k)} \mathrm{tr}(h(v-u)) \text{ and } \sum_{j=1}^k \lambda_{r-j+1}(v-u) = \min_{h\in S(\mathbf{1}_k)} \mathrm{tr}(h(v-u)),$$

where $S(\mathbf{1}_k) = \{u \in \mathcal{J} | \mathrm{tr}(u) = k,\ 0 \leq \lambda_i(u) \leq 1\}$.

Let $E_1^* \subseteq \cdots \subseteq E_k^*$ be the subalgebras for which the maximum is achieved in the minimax Wielandt relation for $\sum_{j=1}^k \lambda_{i_j}(u)$.

By Wielandt's Theorem, we can write:

$$\sum_{j=1}^{k} \lambda_{i_j}(v) = \min_{\substack{E_j \in \mathcal{E}_{r-i_j+1}(\mathcal{J}) \\ E_1 \supseteq \cdots \supseteq E_k}} \max_{\substack{h_j \in E_j \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} \sum_{j=1}^{k} \mathrm{tr}(Q_{h_j} v)$$

$$\leq \sum_{j=1}^{k} \lambda_{i_j}(u) + \max_{\substack{h_j \in E_j^* \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} \sum_{j=1}^{k} \mathrm{tr}(Q_{h_j}(v-u)).$$

Let us now proceed to check that for every $(h_j)_{1\leq j\leq k}$ such that $h_j \in E_j^*$, $||h_j|| = 1$, and $\lambda_1(h_1^2 + \cdots + h_k^2) \leq 1$, we have $h := \sum_{j=1}^{k} h_j^2 \in S(\mathbf{1}_k)$.

First, we can write $\mathrm{tr}(h) = \sum_{j=1}^{k} \mathrm{tr}(h_j^2) = \sum_{j=1}^{k} ||h_j||^2 = k$, and $0 \leq \lambda_i(h) \leq \lambda_1(h) \leq 1$. In view of Remark 3.2.6, it remains to check that for every $1 \leq p \leq r$, we have

$$\sum_{i=1}^{p} \lambda_i(u) \leq s_p(\mathbf{1}_k) = \min\{k, p\}.$$

This verification is immediate for $p \leq k$, because $\sum_{i=1}^{p} \lambda_i(u) \leq \sum_{i=1}^{p} 1 = p$, and for $p > k$, since $\sum_{i=1}^{p} \lambda_i(u) \leq \mathrm{tr}(h) = k$. Henceforth, we can write:

$$\sum_{j=1}^{k} \lambda_{i_j}(u) + \min_{\substack{h_j \in E_j^* \\ ||h_j||=1 \\ \lambda_1(h_1^2+\cdots+h_k^2)\leq 1}} \sum_{j=1}^{k} \mathrm{tr}(Q_{h_j}(v-u))$$

$$\leq \sum_{j=1}^{k} \lambda_{i_j}(u) + \min_{h \in S(\mathbf{1}_k)} \mathrm{tr}(h(v-u)) = \sum_{j=1}^{k} \lambda_{i_j}(u) + \sum_{j=1}^{k} \lambda_j(v-u).$$

The last equality comes from Ky Fan's relations (see Proposition 3.2.7). The other inequality can be easily proved using the identity $\lambda_{r-j+1}(u-v) = -\lambda_j(v-u)$. ∎

Definition 3.6.2 *A function $w : \mathbb{R}^r \mapsto \mathbb{R}$ is a* gauge function *if w is a norm for which $w(x) = w(|x|)$ for every $x \in \mathbb{R}^r$, where $|x| := (|x_1|, \cdots, |x_r|)^T$, and that is invariant with respect to permutations of the components of its argument (this invariance property is often called* symmetry*).* ∎

Recall that, for every $x \in \mathbb{R}^r$, the number $s_p(x)$ stands for the sum of its p largest components. A proof of the next lemma can be found in [SS90], Theorem II.3.17.

Lemma 3.6.3 *Let $w : \mathbb{R}^r \mapsto \mathbb{R}$ be a gauge function. We have for every $x, y \in \mathbb{R}^r_+$ that $w(x) \geq w(y)$ if $s_p(x) \geq s_p(y)$ for all $1 \leq p \leq r$.* ∎

Theorem 3.6.4 (Mirski's Theorem for Jordan algebras) *Let $w : \mathbb{R}^r \mapsto \mathbb{R}$ be a gauge function and let $W : \mathcal{J} \mapsto \mathbb{R}$, $u \mapsto W(u) := w(\lambda(u))$. For every $u, v \in \mathcal{J}$, we have*

$$W(v-u) \geq w(\lambda(v) - \lambda(u)).$$

Proof
In view of Lemma 3.6.3, it suffices to prove that for every $u, v \in \mathcal{J}$ and every $1 \leq p \leq r$, we have

$$s_p(|\lambda(v-u)|) \geq s_p(|\lambda(v) - \lambda(u)|).$$

So, let us fix such an integer p. Let $\bar{\mathcal{J}}$ be the Cartesian product $\mathcal{J} \oplus \mathcal{J}$; its unit element is (e, e). Moreover, the components of the eigenvalues vector $\lambda((u, v); \bar{\mathcal{J}})$ constitute an ordering of the components of the vector $(\lambda(u; \mathcal{J}), \lambda(v; \mathcal{J}))$.

Let $u, v \in \mathcal{J}$. We denote $\bar{u} := (u, -u)$ and $\bar{v} := (v, -v)$, so that $\bar{u}$ and $\bar{v}$ are two elements of $\bar{\mathcal{J}}$. Their respective eigenvalue vector consists in an ordering of the numbers

$$\{\lambda_1(u; \mathcal{J}), \ldots, \lambda_r(u; \mathcal{J}), -\lambda_1(u; \mathcal{J}), \ldots, -\lambda_r(u; \mathcal{J})\}$$

and

$$\{\lambda_1(v; \mathcal{J}), \ldots, \lambda_r(v; \mathcal{J}), -\lambda_1(v; \mathcal{J}), \ldots, -\lambda_r(v; \mathcal{J})\}$$

respectively. Similarly, the eigenvalues of $\bar{v} - \bar{u}$ are of the form $\pm\lambda_i(v - u; \mathcal{J})$. Hence $s_p(|\lambda(v - u; \mathcal{J})|) = \sum_{k=1}^{p} \lambda_k(\bar{v} - \bar{u}; \bar{\mathcal{J}})$.

We define now the indices $j_1, \ldots, j_p$ of the p greatest numbers of the sequence

$$|\lambda_1(v; \mathcal{J}) - \lambda_1(u; \mathcal{J})|, \ldots, |\lambda_r(v; \mathcal{J}) - \lambda_r(u; \mathcal{J})|.$$

Now, we put $i_k := j_k$ if $\lambda_{j_k}(v; \mathcal{J}) \geq \lambda_{j_k}(u; \mathcal{J})$, and $i_k := 2r - j_k$ otherwise. One can easily check that the selected sequence $i_1, \ldots, i_p$ satisfies:

$$|\lambda_{j_k}(v; \mathcal{J}) - \lambda_{j_k}(u; \mathcal{J})| = \lambda_{i_k}(\bar{v}; \bar{\mathcal{J}}) - \lambda_{i_k}(\bar{u}; \bar{\mathcal{J}})$$

and

$$s_p(|\lambda(v; \mathcal{J}) - \lambda(u; \mathcal{J})|) = \sum_{k=1}^{p} \lambda_{i_k}(\bar{v}; \bar{\mathcal{J}}) - \lambda_{i_k}(\bar{u}; \bar{\mathcal{J}}).$$

It suffices now to use Corollary 3.6.1 to get

$$\sum_{k=1}^{p} \lambda_{i_k}(\bar{u}; \bar{\mathcal{J}}) + \sum_{k=1}^{p} \lambda_k(\bar{v} - \bar{u}; \bar{\mathcal{J}}) \geq \sum_{k=1}^{p} \lambda_{i_k}(\bar{v}; \bar{\mathcal{J}}),$$

which is exactly what we planned to show. ■

As an application our version of Mirski's Theorem, we can obtain a sharp bound on the Lipschitz constant of spectral functions on formally real Jordan algebras.

Let $f : Q \to \mathbb{R} \cup \{+\infty\}$ be a function invariant with respect to permutations (i.e. a symmetric function). We define $K := \{v \in \mathcal{J} | \lambda(v) \in Q\}$ and $F : K \to \mathbb{R}$, $u \mapsto F(u) := f(\lambda(u))$. Let w be a gauge for $\mathbb{R}^r$ and W be the norm of $\mathcal{J}$ such that $u \mapsto W(u) := w(\lambda(u))$.

Theorem 3.6.5 *Suppose that there exists a constant $L > 0$ such that for every $\lambda, \gamma \in Q$ we have:*

$$|f(\lambda) - f(\gamma)| \leq Lw(\lambda - \gamma).$$

Then, for every $u, v \in K$, we have:

$$|F(v) - F(u)| \leq LW(v - u).$$

Proof
The proof is extremely easy. It suffices to write:

$$|F(v) - F(u)| = |f(\lambda(v)) - f(\lambda(u))| \leq Lw(\lambda(v) - \lambda(u)) \leq LW(v - u).$$

■

CHAPTER 4

Spectral functions on formally real Jordan algebras

IN THIS CHAPTER, we study several properties of the eigenvalue function of a formally real Jordan algebra, extending several known results in the framework of symmetric matrices. In particular, we give a concise form for the directional differential of a single eigenvalue.

More specifically, we focus on *spectral functions* F on formally real Jordan algebras, which are the composition of a symmetric real-valued function f with the eigenvalue function. In this context, we explore several properties of f that are transferred to F, in particular convexity, strong convexity, differentiability in the classical sense and subdifferentiability in the sense of Clarke.

4.1 Introduction

Formally real Jordan algebraic techniques are more and more used to generalize various results previously that have been obtained in the framework of symmetric matrices. These techniques apply now in such different fields as statistics (e.g. [MN98]), positivity theory [GST04] or operation research (e.g. [Fay97b]). Among other adaptations, these extensions are performed by replacing the eigenvalues of symmetric matrices with the more general eigenvalues defined in the context of formally real Jordan algebras.

The two following chapters lie within this scope. In the present chapter, we study the eigenvalues function on formally real Jordan algebras, and more specifically, *spectral functions on formally real Jordan algebras*. These functions can be build as follows. Consider a formally real Jordan algebra $\mathcal{J}$ of rank r. Given a symmetric function $f : \mathbb{R}^r \to \mathbb{R} \cup \{+\infty\}$, that is, a function invariant with respect to every component permutations of its argument, we let $F : \mathcal{J} \to \mathbb{R} \cup \{+\infty\}$, $u \mapsto F(u) := f(\lambda(u))$ to be *the spectral function generated by f*. We give in this chapter a collection of properties that f transmits to F. Some of them were known in the framework of symmetric matrices. For instance, differentiability properties (including subdifferentiability and conjugation relation) have been explored by Lewis and Sendov [Lew96a, LS02]. Further references are given in the text.

However, we must mention that there are properties that cannot be transferred from a symmetric function to the spectral function it generates. Adrian Lewis has shown in [Lew96b] that there are symmetric functions f that are directionally differentiable in $\lambda(u)$, while the spectral functions that they generate are not directionally differentiable in u.

The applications that motivate our work come mostly from convex optimization. Following [SA03], let us briefly recall how formally real Jordan algebras have turned out to be a powerful tool for investigation in the study of interior-point methods. In convex optimization, algorithms are often designed in a first stage to solve some class of linear problems efficiently. Then several attempts are made to generalize these algorithms to a broader class of instances. Formally real Jordan algebras, which unify linear, second-order and semidefinite programming, have proven to be a very efficient tool for performing such extensions. As noticed by Alizadeh and Schmieta in [AS00], these extensions are often done in a systematic way. Typically, an algorithm for linear programming is constructed via some symmetric functions (barrier functions, penalty functions and so on). In order to get the Jordan algebraic version of the algorithm, it essentially suffices to replace all these symmetric functions by the corresponding spectral function they generate. This is how Faybusovich could extend potential-reduction algorithms [Fay02]. Schmieta, Alizadeh and Muramatsu have also used formally real Jordan algebras in a similar way to design several primal-dual interior point algorithms with various neighborhoods [Mur02, SA03]. Rangarajan applied this construction to generalize his infeasible interior-point methods [Ran06].

Some recent results of Nesterov tend to show that interior-point methods are not always the best procedures to solve some very large scale linear problems [Nes05a]. Whereas the number of iterations of these methods is predictably low, each of them requires so much work than performing the very first one might already be out of reach. The new smoothing method of Nesterov has been designed to potentially avoid this problem, because, without affecting too severely the number of iterations, the iteration cost is much cheaper. This method can be implemented for solving efficiently some structured non-smooth linear optimization problems (see Section 4.1 of [Nes05a]). Can formally real Jordan algebras help to extend this implementation via the spectral function technique described in the previous paragraph ? In order to answer this question, we need to study how the *Lipschitz constant* of a symmetric function's gradient is transferred to the spectral function it generates, for various norms. Corollary 4.4.15 gives a partial result in this direction, since it focuses exclusively on Euclidean norm. For future research, it can also be interesting to use this approach to unify the techniques involving self-regular functions [PRT02] in the framework of formally real Jordan algebras, although they have already been studied for second-order and semidefinite programming separately in the given reference.

The chapter is organized as follows. In the two next subsections, we recall some definition and we define some notational conventions that we will use throughout all the chapter. An exposition of all the supplementary needed facts on Jordan algebras is provided in Section 4.2. We also introduce there the new concept of "similar joint decomposition", which plays an important role in describing the subdifferential of spectral functions. In Section 4.3, we review some properties that a symmetric domain transfers to the spectral domain it generates. Spectral functions on formally real Jordan algebra are studied in Section 4.4. First, we make sure that the known results on conjugate functions of spectral function of Hermitian matrices translate smoothly in the framework of formally real Jordan algebras. These observations allow us to carry out a differentiability analysis of partial sums of eigenvalues, from which we infer a formula for the directional derivative of a single eigenvalue. Differentiability of spectral functions is then discussed, and we close the section with several convexity results. In Section 4.5, we determine how the Clarke subdifferential of a symmetric function is linked with the Clarke subdifferential of the spectral function it generates.

4.1.1 Functions and differentials

The *domain* of a function $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is the set of points x in $\mathbb{R}^n$ where $f(x) < +\infty$; this set is denoted by $\operatorname{dom} f$. A function is called *proper* if its domain is nonempty. Provided that $\mathbb{R}^n$ is endowed with a scalar product $\langle\cdot,\cdot\rangle$, we define *the conjugate function of a proper function* f as follows:

$$f^* : \mathbb{R}^n \to \mathbb{R}, \quad s \mapsto f^*(s) := \sup_{x\in\operatorname{dom} f} \langle s, x\rangle - f(x) = \sup_{x\in\mathbb{R}^n} \langle s, x\rangle - f(x). \tag{4.1}$$

Throughout this chapter, the scalar product we use for $\mathbb{R}^r$ is, unless explicitly stated otherwise, the standard dot product: $\langle\gamma,\lambda\rangle := \sum_{i=1}^r \gamma_i\lambda_i$ for every $\gamma,\lambda \in \mathbb{R}^r$. The Euclidean norm it defines is denoted by $||\cdot||$.

Let $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a function whose domain has a nonempty interior. If x is a point of this interior and h an n-dimensional vector, we say that the function f is *differentiable in the direction h at the point x* if the limit:

$$\nabla^h_x f(x) := \lim_{t\downarrow 0} \frac{f(x+th) - f(x)}{t}$$

exists. If this limit exists for every direction h, and if the application $h \mapsto \nabla_x^h f(x)$ is linear, we say that f is *differentiable at the point* x. Finally, f is *differentiable*, or *differentiable in the classical sense*, if it is differentiable at every point of its domain. Some authors also qualify such a function as *smooth*.

The *differential* of f in x is the linear function $\nabla_x f(x)$, or $\nabla f(x)$, that maps every $h \in \mathbb{R}^n$ to $\nabla_x^h f(x)$. The Riesz representer of $\nabla f(x)$ with respect to the considered scalar product is written $f'(x)$. With a slight abuse of language, we also call this vector the differential of f in x.

It is well-known that addition, multiplication, and composition of functions preserve differentiability in the classical sense. It is also well-known that a convex and differentiable function reaches its minimum at every point of the interior of its domain where its differential vanishes. One of the common procedures in Convex Optimization is to build a new convex function from the maximization of given ones. However, this construction typically fails to preserve differentiability in the classical sense, and the previous criterion for minimality does not hold. Many generalizations of differentiability have been proposed to cope with such circumstances (see [Roc81], [RW98]). We consider two of them in this chapter.

Following Lewis [Lew96a], we define the *subdifferential* of a function f at a point x of its domain as:

$$\partial f(x) := \{s \in \mathbb{R}^r | f(x) + f^*(s) = \langle s, x\rangle\}.$$

In view of the definition (4.1) of the conjugate $f^*(s)$, if the supremum is attained at the point x^*, then $s \in \partial f(x^*)$. According to Theorem 23.5 in [Roc70], when f is convex and proper, $g \in \partial f(x)$ if and only if $f(y) \geq f(x) + \langle g, y - x\rangle$ for each $y \in \mathbb{R}^n$. Hence, a point x of the domain of f is a minimum if and only if 0 belongs to $\partial f(x)$. Moreover, the function f is differentiable at x in the classical sense if and only $\partial f(x)$ contains exactly one element (Theorem 25.1 in [Roc70]).

However, the subdifferential does not behave well for non-convex functions f, because $\partial f(x)$ might be empty. Francis Clarke [Cla75] has proposed a different viewpoint for solving this issue. His concept is well-defined for every locally Lipschitz continuous function. It turned out to be extremely fertile – Google finds about 10600 pages with the words "Clarke subdifferential".

Definition 4.1.1 *Let E and F be two vector spaces endowed with the norm $||\cdot||_E$ and $||\cdot||_F$ respectively. Let f be a function from a set $A \subseteq E$ with a nonempty interior to F. We say that f is* locally Lipschitz continuous *if, for every $x \in \operatorname{int} A$, there exist a neighborhood U of x in A and a constant $L_a > 0$ for which:*

$$y \in U \quad \Rightarrow \quad ||f(y) - f(x)||_F \leq L_a||y - x||_E.$$

■

Consider a function $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ that is locally Lipschitz continuous. From Rademacher's Theorem, quoted as Theorem 4.5.2 below, we know that this function is differentiable in the classical sense on a dense part $D(f)$ of its domain. The *Clarke subdifferential* of f

at $x \in \operatorname{dom} f$ is:

$$\partial_C f(x) := \operatorname{conv}\{v \in \mathbb{R}^n | \text{there exists } (x_m)_{m\geq 0} \subseteq D(f) \text{ such that } f'(x_m) \to v\}. \quad (4.2)$$

There exist several equivalent definitions of this set (we refer the reader to [Roc81] for further details), but this form is the most suitable for computing the Clarke subdifferential of locally Lipschitz continuous spectral functions on formally real Jordan algebras. The set between braces in (4.2) is called the *Bouligand subdifferential* of f at x, and is denoted by $\partial_B f(x)$. If f is a proper convex function, its Clarke subdifferential coincide with its subdifferential.

4.1.2 Symmetric functions

Complying with the notation introduced in Chapter 3, we write $\mathcal{P}$ for the set of all permutations of r-dimensional vectors considered as $r \times r$ 0-1 matrices. We denote by σ_P the permutation on indices $\{1, 2, \ldots, r\}$ that a matrix $P \in \mathcal{P}$ defines. We label each element of $\mathcal{P}$ with an index, so that $\mathcal{P} = \{P_i,\ 1 \leq i \leq r!\}$. A subset of $\mathbb{R}^r$ is said to be *symmetric* if it remains unchanged under every permutation of $\mathcal{P}$.

Definition 4.1.2 *A real-valued function defined on a symmetric set $Q \in \mathbb{R}^r$ is a* symmetric function *if for every permutation $P \in \mathcal{P}$ and each $\gamma \in Q$, we have $f(P\gamma) = f(\gamma)$.* ■

For the ease of reference, we recall below a classical statement involving the set $\mathcal{P}$. Its proof can be found in [HJ96], Theorem 8.7.1. We denote here the all-one r-dimensional vector by $\mathbf{1}$.

Definition 4.1.3 *A matrix $A \in \mathbb{R}^{r\times r}$ is* doubly stochastic *if $A\mathbf{1} = \mathbf{1}$, if $A^T\mathbf{1} = \mathbf{1}$, and if all its coefficients are nonnegative.* ■

Theorem 4.1.4 (Birkhoff's Theorem) *The convex hull of $\mathcal{P}$ is the set of doubly stochastic matrices.* ■

4.2 Further results on Jordan algebras

We assume throughout this chapter that $\mathcal{J}$ is a formally real Jordan algebra of dimension $N < +\infty$ and of rank r, as defined in Chapter 2. We comply with all the notation we introduced there. For instance, we denote the eigenvalues function of an element u that belongs to a subalgebra $\mathcal{J}'$ of $\mathcal{J}$ by $\lambda(u; \mathcal{J}')$ or by $\lambda(u; \mathcal{J})$, depending on the algebra where we consider u (see on p. 79). We often abbreviate the writing $\lambda(u; \mathcal{J})$ into $\lambda(u)$.

The following new concept will help us to describe the subdifferential of some spectral functions.

Definition 4.2.1 *Let $u, v \in \mathcal{J}$. If there exists a Jordan frame $\{c_1, \ldots, c_r\}$ (possibly not unique) such that $u = \sum_{i=1}^r \lambda_i(u)c_i$ and $v = \sum_{i=1}^r \lambda_i(v)c_i$, we say that u and v have a* similar joint decomposition. ■

It is important to underline the fact that, according to our numbering convention, we have $\lambda_1(u) \geq \cdots \geq \lambda_r(u)$ and $\lambda_1(v) \geq \cdots \geq \lambda_r(v)$. So, "similar joint decomposition" is *not* a synonym of "operator commutativity", where the ordering of eigenvalues is not taken into account. The following proposition gives an alternative description of similar joint decomposition, which is instructive to compare with Proposition 2.7.29. This characterization is sometimes easier to manipulate than the existence statement of the definition.

Proposition 4.2.2 *Let us fix two elements u and v of $\mathcal{J}$. Using the unique subspace spectral decomposition theorem, we can decompose u as $u = \sum_{j=1}^s \xi_j e_j$, where the real numbers ξ_j are distinct and ordered decreasingly, and the elements e_j are idempotent. We denote by $\mathcal{J}_{jj}$ the subalgebra $\mathcal{J}_1(e_j)$.*

The elements u and v have a similar joint decomposition if and only if:

***a.** for all $1 \leq j \leq s$, there exists an element $v_j \in \mathcal{J}_{jj}$ such that $v = \sum_{j=1}^s v_j$, and*

***b.** the smallest eigenvalue of v_j on $\mathcal{J}_{jj}$ is greater than or equal to the largest eigenvalue of v_{j+1} on $\mathcal{J}_{j+1,j+1}$ for each $1 \leq j < s$.*

Proof
We first show the "if" part.

According to the hypothesis ***a***, we assume that $v \in \bigoplus_{j=1}^s \mathcal{J}_{jj}$, so that $v = \sum_{j=1}^s v_j$ for some $v_j \in \mathcal{J}_{jj}$. We know from the first Pierce decomposition theorem that $\mathcal{J}_{jj} = \mathcal{J}_1(e_j)$ is a Jordan subalgebra of $\mathcal{J}$. It is also formally real, as a restriction of the formally real algebra $\mathcal{J}$. Hence, we can apply the complete spectral decomposition theorem *in this subalgebra* to decompose v_j into $v_j = \sum_{i=1}^{\mathrm{tr}(e_j)} \lambda_{ji} c_{ji}$. This theorem assures us that the idempotents c_{ji} are minimal in their respective subalgebras. In view of Proposition 2.7.22, they are also minimal in the full algebra $\mathcal{J}$. Thus, the set

$$\{c_{11}, \ldots, c_{1,\mathrm{tr}(e_1)}, c_{21}, \ldots, c_{s,\mathrm{tr}(e_s)}\} \tag{4.3}$$

is a Jordan frame.

In view of the requirement ***b***, we further assume that the smallest eigenvalue of v_j on $\mathcal{J}_{jj}$ (i.e. $\lambda_{j,\mathrm{tr}(e_j)}$) is greater than or equal to the largest eigenvalue of v_{j+1} on $\mathcal{J}_{j+1,j+1}$ (i.e. $\lambda_{j+1,1}$) for every $1 \leq j < s$. In other words, we have $\lambda_{11} \geq \cdots \geq \lambda_{1,\mathrm{tr}(e_1)} \geq \lambda_{21} \geq \cdots \geq \lambda_{s,\mathrm{tr}(e_s)}$. Since

$$u = \sum_{j=1}^s \xi_j e_j = \sum_{j=1}^s \xi_j \sum_{i=1}^{\mathrm{tr}(e_j)} c_{ji} \quad \text{and} \quad v = \sum_{j=1}^s \sum_{i=1}^{\mathrm{tr}(e_j)} \lambda_{ji} c_{ji},$$

we can use the Jordan frame (4.3) to show that u and v have indeed a similar joint decomposition.

Now, we turn to the "only if" part of the statement. We assume that u and v have a similar joint decomposition: there exists a Jordan frame $\{c_1, \ldots, c_r\}$ such that $u = \sum_{i=1}^r \lambda_i(u) c_i$ and $v = \sum_{i=1}^r \lambda_i(v) c_i$. We define the integers s, $k_1, \ldots, k_s$ such that $k_s := r$ and:

$$\lambda_1(u) = \cdots = \lambda_{k_1}(u) > \lambda_{k_1+1}(u) = \cdots = \lambda_{k_2}(u) > \cdots \lambda_{k_s}(u).$$

We let $M_j := \{k_{j-1}+1, \ldots, k_j\}$ (with $k_0 = 0$), $e_j := \sum_{i \in M_j} c_i$ and $\mathcal{J}_{jj} := \mathcal{J}_1(e_j)$. The decomposition $u = \sum_{i=1}^{s} \lambda_{k_j}(u) e_j$ is the unique decomposition of u provided by Theorem 2.7.13.

It suffices now to let $v_j := \sum_{i \in M_j} \lambda_i(v) c_i$, which belongs to $\mathcal{J}_{jj}$; the eigenvalues of v_j in $\mathcal{J}_{jj}$ are thus $\lambda_{k_{j-1}+1}(v), \ldots, \lambda_{k_j}(v)$, and the required condition ***b*** is also satisfied. ■

In order to compute the subdifferential of some spectral functions, we need an extension to formally real Jordan algebras of the *von Neumann inequality* (4.4), and, even more importantly, we have to determine when the equality occurs.

Similar joint decomposition allows us to propose a compact description of the equality case. Adrian Lewis [Lew96a] has obtained a corresponding result when $\mathcal{J}$ is the algebra of Hermitian matrices. An alternative description of the equality case has already been provided in [LKF03], although it only covers the case where $\mathcal{J}$ is a simple Jordan algebra (more details are given in Remark 4.2.6). As our argumentation uses a rather different technique, we include here a proof.

We need the following simple technical lemma.

Lemma 4.2.3 *Let α, β and γ be three s-dimensional vectors such that:*

- ◇ $\sum_{j=1}^{p} \beta_j \geq \sum_{j=1}^{p} \alpha_j$ *for every* $1 \leq p \leq s$,
- ◇ $\sum_{j=1}^{s} \beta_j = \sum_{j=1}^{s} \alpha_j$, *and*
- ◇ $\gamma_1 > \ldots > \gamma_r$.

If $\gamma^T \alpha = \gamma^T \beta$, then $\alpha = \beta$.

Proof
We have:

$$\begin{aligned} 0 = \gamma^T(\beta - \alpha) \;=\; & (\gamma_1 - \gamma_2)(\beta_1 - \alpha_1) + (\gamma_2 - \gamma_3)(\beta_1 + \beta_2 - \alpha_1 - \alpha_2) + \cdots \\ & + (\gamma_{r-1} - \gamma_r)(\beta_1 + \cdots + \beta_{r-1} - \alpha_1 - \cdots - \alpha_{r-1}) \\ & + \gamma_r(\beta_1 + \cdots + \beta_r - \alpha_1 - \cdots - \alpha_r). \end{aligned}$$

The last term is null by assumption, and the factors $\beta_1 + \cdots + \beta_p - \alpha_1 - \cdots - \alpha_p$ are nonnegative. Since $\gamma_p - \gamma_{p+1} > 0$, we have $\beta_1 + \cdots + \beta_p = \alpha_1 + \cdots + \alpha_p$ for every $1 \leq p < r$. Henceforth $\alpha = \beta$. ■

Theorem 4.2.4 *Let $u, v \in \mathcal{J}$. We have:*

$$\sum_{i=1}^{r} \lambda_i(u) \lambda_i(v) \geq \operatorname{tr}(uv). \tag{4.4}$$

The equality holds if and only if u and v have a similar joint decomposition.

Proof
Let $u = \sum_{i=1}^{r} \lambda_i(u) c_i$ be a complete spectral decomposition of u, and let $v = \sum_{i \leq j} v_{ij}$ be the second Pierce decomposition of v with respect to the Jordan frame $\{c_1, \ldots, c_r\}$, so

that $v_{ij} \in \mathcal{J}_{ij} := Q_{c_i,c_j}\mathcal{J} = \mathcal{J}_{1/2}(c_i) \cap \mathcal{J}_{1/2}(c_j)$. Note that, for each pair i, j of different numbers, we have:

$$uv_{ij} \in (\mathcal{J}_1(c_i) + \mathcal{J}_0(c_i)) \circ \mathcal{J}_{1/2}(c_i) \subseteq \mathcal{J}_{1/2}(c_i),$$

since $v \in \bigoplus_{k=1}^r \mathcal{J}_1(c_k) \subseteq \mathcal{J}_1(c_i) + \mathcal{J}_0(c_i)$. Thus $\mathrm{tr}(uv_{ij}) = 0$ in view of item 7 of Theorem 2.6.1.

Let $v = \sum_{i=1}^r \lambda_i(v)c_i'$ be the complete spectral decomposition of v. We successively get:

$$\begin{aligned}\mathrm{tr}(uv) &= \sum_{1\leq i\leq j\leq r} \mathrm{tr}(uv_{ij}) = \sum_{i=1}^r \mathrm{tr}(uv_{ii}) = \sum_{i=1}^r \lambda_i(u)\mathrm{tr}(c_i v) \\ &= \sum_{i=1}^r \sum_{j=1}^r \lambda_i(u)\mathrm{tr}(c_i c_j')\lambda_j(v) = \lambda(u)^T B\lambda(v),\end{aligned}$$

where B is the $r \times r$ matrix with coefficients $B_{ij} := \mathrm{tr}(c_i c_j')$.

First, note that $B_{ij} \geq 0$ since $\mathrm{tr}(c_i c_j') = \mathrm{tr}(Q_{c_i} c_j') \geq 0$ as $c_j' \in \mathcal{K}_{\mathcal{J}}$ (see item 7 of the first Pierce decomposition theorem).

Second, observe that the sum of elements in every row or column of B is equal to 1 since the idempotents c_i and c_j' are minimal. In other words, B is doubly stochastic. Hence, by Birkhoff's Theorem 4.1.4, $B \in \mathrm{conv}(\mathcal{P})$. Thus:

$$\lambda(u)^T B\lambda(v) \leq \max_{P\in\mathrm{conv}(\mathcal{P})} \lambda(u)^T P\lambda(v) = \max_{P\in\mathcal{P}} \lambda(u)^T P\lambda(v) = \lambda(u)^T\lambda(v). \tag{4.5}$$

The last equality holds because $\lambda(u)$ and $\lambda(v)$ are in $\mathbb{R}^r_\downarrow$ (see the rearrangement inequality). The second-to-last equality is a well-known fact in convex analysis (see Corollary 11.5.1 from [Roc70] for instance).

Now, we determine the equality conditions. The "if" part is trivial. In order to prove the "only if" part, let us define the integers s, $k_1, \ldots, k_s$ such that $k_s := r$ and:

$$\lambda_1(u) = \cdots = \lambda_{k_1}(u) > \lambda_{k_1+1}(u) = \cdots = \lambda_{k_2}(u) > \cdots \lambda_{k_s}(u).$$

Set $M_j := \{k_{j-1}+1, \ldots, k_j\}$ (with $k_0 = 0$) and $e_j := \sum_{i\in M_j} c_i$. We assume that the element v of $\mathcal{J}$ satisfies $\mathrm{tr}(uv) = \sum_{i=1}^r \lambda_i(u)\lambda_i(v)$. We have $\mathrm{tr}(uv) = \sum_{j=1}^s \lambda_{k_j}(u)\mathrm{tr}(e_j v)$. Let us denote by α the s-dimensional vector with components $\alpha_j := \mathrm{tr}(e_j v)$, and by β the s-dimensional vector with components $\beta_j := \sum_{i\in M_j} \lambda_i(v)$. Observe that:

$$\sum_{j=1}^s \beta_j = \sum_{j=1}^s \sum_{i\in M_j} \lambda_i(v) = \mathrm{tr}(v) = \sum_{j=1}^s \mathrm{tr}(e_j v) = \sum_{j=1}^s \alpha_j;$$

moreover, for $1 \leq p \leq r$, we have

$$\sum_{j=1}^p \beta_j = \sum_{i=1}^{k_p} \lambda_i(v) \geq \mathrm{tr}\left(\sum_{j=1}^p e_j v\right) = \sum_{j=1}^p \alpha_j,$$

in view of Ky Fan's inequalities of Proposition 3.2.7. In view of Lemma 4.2.3, we obtain $\alpha = \beta$, that is:

$$\sum_{i=1}^{k_p} \lambda_i(v) = \sum_{j=1}^{p} \mathrm{tr}(e_j v).$$

Applying Corollary 3.4.4 successively for $d := e_1$, for $d := e_1 + e_2$, ..., we deduce that:

$$v \in \mathcal{J}_1(e_1) + \mathcal{J}_1(e_2) + \cdots + \mathcal{J}_1(e_s).$$

Moreover, the eigenvalues of v on $\mathcal{J}_1(e_j)$ are $\{\lambda_i(v) | i \in M_j\}$. Thus the smallest eigenvalue of $Q_{e_j} v$ on $\mathcal{J}_1(e_j)$ is larger than the largest eigenvalue of $Q_{e_{j+1}} v$ on $\mathcal{J}_1(e_{j+1})$. In view of Proposition 4.2.2, we conclude that u and v have a similar joint decomposition. ■

It is possible to reprove a particular case of Mirski's inequality (see Theorem 3.6.4) from this theorem, where the gauge function we choose is the Euclidean norm. But now, we can easily describe the equality case.

Corollary 4.2.5 *For every $u, v \in \mathcal{J}$, we have $||\lambda(u) - \lambda(v)|| \leq ||u - v||_{\mathcal{J}}$. The equality holds if and only if u and v have a similar joint spectral decomposition.*

Proof
We have:

$$\begin{aligned} ||\lambda(u) - \lambda(v)||^2 &= ||\lambda(u)||^2 - 2\sum_{i=1}^{r} \lambda_i(u)\lambda_i(v) + ||\lambda(v)||^2 \\ &\leq ||\lambda(u)||^2 - 2\mathrm{tr}(uv) + ||\lambda(v)||^2 = ||u - v||_{\mathcal{J}}^2. \end{aligned}$$

The equality case follows immediately for the previous theorem. ■

Remark 4.2.6 *Lim, Kim, and Faybusovich have described the equality case of the von Neumann inequality as follows in Theorem 2 of [LKF03].*

> *Suppose that $\mathcal{J}$ is a formally real simple Jordan algebra of finite dimension. Let K be the connected component of the identity in $\mathcal{A}(\mathcal{J})$, the set of automorphisms of $\mathcal{J}$. We fix a Jordan frame $\{c_1, \ldots, c_r\}$, and we define the operator $\gamma : \mathcal{J} \to \mathcal{J}$, $u \mapsto \gamma(u) := \sum_{i=1}^{r} \lambda_i(u) c_i$. Then $\mathrm{tr}(uv) = \lambda(u)^T \lambda(v)$ for two elements u, v of $\mathcal{J}$ if and only if there exists an automorphism $k \in K$ such that $k(u) = \gamma(u)$ and $k(v) = \gamma(v)$.* ■

We illustrate here the fact that this viewpoint cannot be easily extended to non-simple formally real Jordan algebras.

Let us consider a formally real Jordan algebra $\mathcal{J}$ made of two copies of the $n \times n$ symmetric matrix Jordan algebra, say $\mathcal{J} = \mathcal{J}^a \oplus \mathcal{J}^b$. We assume that $n > 1$. The connected component of the identity in the set of automorphisms of $\mathcal{J}$, denoted here as K, is made of all the applications of the form

$$k := \begin{pmatrix} k^a & 0 \\ 0 & k^b \end{pmatrix},$$

where k^i is in the connected component of the identity in the set of automorphisms of $\mathcal{J}^i$. We fix a Jordan frame $\{c_1^a, \ldots, c_n^a, c_1^b, \ldots, c_n^b\}$ of $\mathcal{J}$, with $c_j^i \in \mathcal{J}^i$. Let $u = (u^a, u^b)$ and $v = (v^a, v^b)$ be two elements of $\mathcal{J}$ for which there is an automorphism $k \in K$ such that:

$$k(u) = k^a(u^a) + k^b(u^b) = \sum_{i=1}^r \lambda_i(u^a; \mathcal{J}^a)c_i^a + \sum_{i=1}^r \lambda_i(u^b; \mathcal{J}^b)c_i^b$$

and

$$k(v) = k^a(v^a) + k^b(v^b) = \sum_{i=1}^r \lambda_i(v^a; \mathcal{J}^a)c_i^a + \sum_{i=1}^r \lambda_i(v^b; \mathcal{J}^b)c_i^b.$$

At first glance, one could think that this kind of link between u and v, as it seems to naturally extend Kim, Lim and Faybusovich's characterization, is enough to have the equality case in von Neumann inequality. Unfortunately, this is not the case, as

$$\mathrm{tr}(uv) = \mathrm{tr}(u^a v^a) + \mathrm{tr}(u^b v^b) = \sum_{i=1}^r \lambda_i(u^a; \mathcal{J}^a)\lambda_i(v^a; \mathcal{J}^a) + \sum_{i=1}^r \lambda_i(u^b; \mathcal{J}^b)\lambda_i(v^b; \mathcal{J}^b)$$

is not necessarily equal to

$$\sum_{i=1}^{2r} \lambda_i(u; \mathcal{J})\lambda_i(v; \mathcal{J}),$$

due to the fact that the corresponding eigenvalues $\lambda_i(u; \mathcal{J})$ and $\lambda_i(v; \mathcal{J})$ might not come from the same simple subalgebra $\mathcal{J}^a$ or $\mathcal{J}^b$.

In fact, similar joint decomposition ensures that nothing wrong happens with the numbering of the eigenvalues $\lambda_i(u^a; \mathcal{J}^a)$, $\lambda_i(u^b; \mathcal{J}^b)$, $\lambda_i(v^a; \mathcal{J}^a)$ and $\lambda_i(v^b; \mathcal{J}^b)$ with respect to the numbering of the eigenvalues $\lambda_i(u; \mathcal{J})$ and $\lambda_i(v; \mathcal{J})$. ■

4.3 Properties of spectral domains

Before analyzing more closely the spectral functions, we concentrate in this section on several simple properties that are transmitted form a symmetric set $Q \subseteq \mathbb{R}^r$ to the subset K of elements of $\mathcal{J}$ whose eigenvalue vector lies in Q.

Remember from Chapter 3 that the set

$$\mathcal{SC}(\lambda) := \mathrm{conv}\{P\lambda | P \text{ is a permutation matrix}\}$$

can be described for every $\lambda \in \mathbb{R}^r$ (see Lemma 3.2.5) as follows:

$$\gamma \in \mathcal{SC}(\lambda) \;\Leftrightarrow\; s_p(\gamma) \le s_p(\lambda) \quad \text{for all } 1 \le p \le r \quad \text{and} \quad s_r(\gamma) = s_r(\lambda),$$

where the function $s_p : \mathbb{R}^r \to \mathbb{R}$ maps every vector $\lambda \in \mathbb{R}^r$ to the sum of its p largest components. This simple characterization and Fan's inequalities (see Proposition 3.2.7) are the only needed tools to show how the convexity of a set Q can be transmitted to the set $K := \{u \in \mathcal{J} | \lambda(u) \in Q\}$. An anonymous referee mentioned that this result can also be derived in the framework of Hermitian matrices by applying Corollary 2.7 of [Lew96a] to the characteristic function of the set Q.

Theorem 4.3.1 *Let $Q \subseteq \mathbb{R}^r$ be a symmetric set and let $K := \{v \in \mathcal{J} | \lambda(v) \in Q\}$.*

1. *If Q is convex, then K is convex.*
2. *If Q is closed, then K is closed.*
3. *If Q is open, then K is open.*
4. *If Q is bounded, then K is bounded.*

Proof
Suppose that Q is convex and fix $v_0, v_1 \in K$ and $\alpha \in [0,1]$. Using the characterization given in Lemma 3.2.5, we can prove that:

$$\lambda(\alpha v_1 + (1-\alpha)v_2) \in \mathcal{SC}(\alpha\lambda(v_1) + (1-\alpha)\lambda(v_2)) \tag{4.6}$$

as follows. We denote $v_\alpha := \alpha v_1 + (1-\alpha)v_0$ and $\lambda_\alpha := \alpha\lambda(v_1) + (1-\alpha)\lambda(v_0)$; observe that λ_α is an ordered vector. We first have:

$$\begin{aligned} s_r(\lambda_\alpha) &= \alpha s_r(\lambda(v_1)) + (1-\alpha)s_r(\lambda(v_0)) \\ &= \alpha \operatorname{tr}(v_1) + (1-\alpha)\operatorname{tr}(v_0) = \operatorname{tr}(v_\alpha) = s_r(\lambda(v_\alpha)). \end{aligned}$$

Second, as shown in Proposition 3.2.7, the function $S_p(v) = s_p(\lambda(v))$ is convex (it is even a support function). This allows us to write, since the components of λ_α are ordered decreasingly:

$$s_p(\lambda_\alpha) = s_p(\alpha\lambda(v_1) + (1-\alpha)\lambda(v_0)) = \alpha S_p(v_1) + (1-\alpha)S_p(v_0) \geq S_p(v_\alpha) = s_p(\lambda(v_\alpha)),$$

and (4.6) is shown. Now, $\lambda_\alpha \in Q$ because Q is convex. The symmetry of Q implies $\mathcal{SC}(\lambda_\alpha) \subseteq Q$. From (4.6), we have $\lambda(v_\alpha) \in Q$ i.e. $v_\alpha \in K$.

Items 2 and 3 are immediate consequences of the continuity of the eigenvalue functions.

Item 4 is easy to prove as well. It suffices to apply Corollary 4.2.5 with $v := 0$ and to observe that the equality holds in that case. ■

The compactness of a set $Q \subseteq \mathbb{R}^r$ is therefore transferred to the set $K \subseteq \mathcal{J}$ it generates. This fact is used in the following proposition, which will allow us to prove some continuity results in Jordan algebras.

Proposition 4.3.2 *Suppose that we have an element $u \in \mathcal{J}$ and a sequence $(u_m)_{m\geq 0}$ of $\mathcal{J}$ that converges to u. We denote the complete spectral decompositions of these elements by $u_m = \sum_{i=1}^r \lambda_i(u_m)c_{i,m}$ and $u = \sum_{i=1}^r \lambda_i(u)c_i$ respectively. We define the numbers $s, k_1, \ldots, k_s$ so that $k_s := r$ and:*

$$\lambda_1(u) = \cdots = \lambda_{k_1}(u) > \lambda_{k_1+1}(u) = \cdots = \lambda_{k_2}(u) > \cdots \lambda_{k_s}(u).$$

We set $M_j := \{k_{j-1}+1, \ldots, k_j\}$, $e_j := \sum_{i\in M_j} c_i$, and $e_{j,m} := \sum_{i\in M_j} c_{i,m}$.
For every $1 \leq j \leq s$, the sequences $(e_{j,m})_{m\geq 0}$ converge respectively to e_j as m goes to infinity.

Proof
We can successively write:

$$0 = \lim_{m\to\infty} u_m - u = \lim_{m\to\infty} \sum_{i=1}^{r} \lambda_i(u_m)c_{i,m} - \lambda_i(u)c_{i,m} + \lambda_i(u)c_{i,m} - \lambda_i(u)c_i$$
$$= \lim_{m\to\infty} \sum_{i=1}^{r} \lambda_i(u)\big(c_{i,m} - c_i\big) = \lim_{m\to\infty} \sum_{j=1}^{s} \lambda_{k_j}(u)\big(e_{j,m} - e_j\big);$$

we have used the continuity of eigenvalues for the second equality.

Now, let $m_0, m_1, m_2, \ldots$ be an increasing sequence of integers such that $\big(e_{j,m_k}\big)_{k\geq 0}$ converges for every $1 \leq j \leq s$. This sequence is known to exists, as the idempotents $e_{j,m}$ all lie in the set $\{v \in \mathcal{K}_{\mathcal{J}} | \mathrm{tr}(v) \leq r\}$, which is compact in view of the previous theorem.

Let f_j be the respective limits of these subsequences; obviously $\{f_1, \ldots, f_s\}$ is a system of idempotents. The equality above shows that $u = \sum_{j=1}^{s} \lambda_{k_j}(u) f_j$. By the first spectral decomposition theorem, we then have $e_j = f_j$.

We have proved that every converging subsequence of $\big(e_{j,m}\big)_{m\geq 0}$ must converge to e_j. Since the sequences are all in a compact set, we obtain the result[1]. ■

This proposition will be refined in Lemma 5.4.4, Lemma 5.4.5, and Lemma 5.4.6.

We conclude this section with a density result that will be useful in the computation of the Clarke differential of spectral functions.

Proposition 4.3.3 *Given an open symmetric set $Q \subseteq \mathbb{R}^r$ and a dense subset D of Q, we let $K := \{u \in \mathcal{J} | \lambda(u) \in Q\}$ and $U := \{u \in \mathcal{J} | \lambda(u) \in D\}$. The set U is a dense subset of K.*

Proof
We first prove that $D \cap \mathbb{R}^r_\downarrow$ is dense in $Q \cap \mathbb{R}^r_\downarrow$. Suppose that it is not the case. Then, there exists a vector $x \in Q \cap \mathbb{R}^r_\downarrow$ and a real number $\epsilon > 0$ for which the set $B(x,\epsilon) \cap D \cap \mathbb{R}^r_\downarrow$ is empty. Without loss of generality, we can assume that ϵ is small enough for the ball $B := B(x,\epsilon)$ to be entirely contained in Q. Since $\mathbb{R}^r_\downarrow = \mathrm{adh}(\mathrm{int}\, \mathbb{R}^r_\downarrow)$, there exist a vector $y \in B$ and a real number $\epsilon' > 0$ for which $B(y,\epsilon')$ is included at the same time in $\mathbb{R}^r_\downarrow$ and in B. By density of D in Q, there exists a vector $w \in B(y,\epsilon') \cap D$. As $w \in B(y,\epsilon') \subseteq B(x,\epsilon) \cap \mathbb{R}^r_\downarrow$, we have reached a contradiction.

The statement is now easy to prove. Let $u = \sum_{i=1}^{r} \lambda_i(u)c_i \in K$ and $\epsilon > 0$. According to what we showed above, there exists a vector $\mu \in D \cap \mathbb{R}^r_\downarrow$ for which $||\lambda(u) - \mu|| < \epsilon$. Letting $v := \sum_{i=1}^{r} \mu_i c_i$, which is in U, we have $||u - v||_{\mathcal{J}} = ||\lambda(u) - \lambda(v)|| = ||\lambda(u) - \mu|| < \epsilon$. Thus $v \in U \cap B(u,\epsilon)$, and U is dense in K. ■

[1] This argument is very standard in the theory of metric spaces. We give here a five-lines proof. Let $(a_n)_{n\geq 0}$ be a sequence on the compact K, every subsequence of which converge to the same point a. Let $\epsilon > 0$ and consider the open ball B centered in a and of radius ϵ. We need to find an integer m' such that, for every $m \geq m'$, we have $||a_m - a|| \leq \epsilon$. Suppose that this m' does not exist. Then there is an infinite number of points $(b_n)_{n\geq 0}$ of our sequence in the compact $K' := K \backslash B$. Thus, there exists a converging subsequence of $(b_n)_{n\geq 0}$ in K'. But the point a is not in K', and we have reached a contradiction.

Observe that it is absolutely necessary to ensure that the vector μ has ordered components in the proof of the previous proposition. Otherwise, the equality $||\lambda(u) - \lambda(v)|| = ||\lambda(u) - \mu||$ would not be valid.

4.4 Inherited properties of spectral functions

Given a symmetric function $f : \mathbb{R}^r \to \mathbb{R} \cup \{+\infty\}$, we let $F : \mathcal{J} \to \mathbb{R} \cup \{+\infty\}$, $u \mapsto F(u) := f(\lambda(u))$ to be the spectral function generated by f. We give in this chapter a preliminary collection of properties that f transmits to F. Some of them were known in the framework of symmetric matrices. For instance, differentiability properties (including subdifferentiability and conjugation relation) have been explored by Lewis and Sendov [Lew96a, LS02]. Further references are given in the text.

4.4.1 The conjugate and the subdifferential of a spectral function

According to its definition, it is obvious that the conjugate of a convex function f is tightly linked with the subdifferential of f. These interactions are described in Section 23 of [Roc70], and especially in Theorems 23.4 and 23.5. In particular, they allow us to deduce easily from considerations on the conjugate a precise description of the subdifferential of a spectral function in the context of formally real Jordan algebras.

Most of the results presented in this subsection were previously known in the framework of Hermitian matrices, and our contribution consists in checking if they translate smoothly for Jordan algebras.

We recall below that the conjugate function of a symmetric function is itself symmetric.

Lemma 4.4.1 *Let Q be a symmetric set of $\mathbb{R}^r$ and let $f : Q \to \mathbb{R}$ be a symmetric function. The conjugate of f with respect to the dot scalar product on $\mathbb{R}^r$ is a symmetric function too.*

Proof

Let $s \in \mathbb{R}^r$ and let $P \in \mathcal{P}$. We have :

$$\begin{aligned} f^*(Ps) &= \sup_{x\in Q}\langle Ps, x\rangle - f(x) = \sup_{x\in Q}\langle s, P^T x\rangle - f(x) \\ &= \sup_{x\in Q}\langle s, P^T x\rangle - f(P^T x) = \sup_{x\in Q}\langle s, x\rangle - f(x) = f^*(s), \end{aligned}$$

by symmetry of Q (note that $f^*(Ps)$ may be equal to $\pm\infty$). ∎

From this lemma, we can consider the spectral function generated by f^*. The next theorem shows that this is exactly F^*. Its (short) proof follows the demonstration of Theorem 2.6 in [Lew96a], where the same result was obtained in the framework of Hermitian matrices. Corollary 4.4.3 is the Jordan algebraic version of Theorem 3.2 of [Lew96a].

Theorem 4.4.2 *Let Q be a nonempty symmetric set of $\mathbb{R}^r$, let $f : Q \to \mathbb{R}$ be a symmetric function and let F be the spectral function generated by f. Then F^* is the spectral function generated by f^*.*

Proof
Let $s \in \mathcal{J}$ be such that $f^*(\lambda(s)) < +\infty$. Denoting $K := \{u \in \mathcal{J} | \lambda(u) \in Q\}$, we successively have:

$$\begin{aligned} F^*(s) &= \sup_{x \in K} [\operatorname{tr}(xs) - F(x)] = \sup_{x \in K} [\operatorname{tr}(xs) - f(\lambda(x))] \\ &= \sup_{\lambda \in Q} [\langle \lambda(s), \lambda \rangle - f(\lambda)] = f^*(\lambda(s)). \end{aligned}$$

Theorem 4.2.4 justifies the second to last equality. ■

As a straightforward corollary, we can establish how the subdifferential of a spectral function is linked to the subdifferential of the function from which it has been generated.

Corollary 4.4.3 *Using the same notation as in the previous theorem, we have:*

$$\partial F(x) = \{s \in \mathcal{J} | \lambda(s) \in \partial f(\lambda(x)),\ s \textit{ and } x \textit{ have a similar joint decomposition}\}.$$

Proof
We have for all $x, s \in \mathcal{J}$:

$$F^*(s) + F(x) = f^*(\lambda(s)) + f(\lambda(x)) \geq \sum_{i=1}^{r} \lambda_i(x)\lambda_i(s) \geq \operatorname{tr}(xs).$$

An element $s \in \mathcal{J}$ belongs to $\partial F(x)$ if and only if $F^*(s) + F(x) = \operatorname{tr}(xs)$. The upper bound of the first inequality is reached if and only if $\lambda(s) \in \partial f(\lambda(x))$; by Theorem 4.2.4, the second inequality turns to an equality if and only if x and s have a similar joint decomposition. ■

4.4.2 Directional derivative of eigenvalue functions

In this subsection, we apply the results derived above to compute the subdifferential of the function S_p, that is, the sum of the p largest eigenvalues. Then, we deduce an expression for the directional derivative of λ_i. Related results in the framework of symmetric matrices can be found in [OW93], Theorem 3.5, Theorem 3.9 and corollaries. It turns out that the present analysis, culminating in Theorem 4.4.8 below, settles an open question in [SS04]. Moreover, the results of this subsection play an important role in the computation of the Hessian of a spectral function and in related problems.

In the next lemma, we determine the differential of the support function of $\mathcal{SC}(\mathbf{1}_p)$ for every $1 \leq p \leq r$. We assume throughout this subsection that for every $\mu \in \mathbb{R}^r_\downarrow$, the number μ_0 is strictly greater than μ_1, and μ_{r+1} is strictly lower than μ_r.

Lemma 4.4.4 *Let $1 \leq p \leq r$, and f be the support function of $\mathcal{SC}(\mathbf{1}_p)$. We fix a vector $\mu \in \mathbb{R}^r_\downarrow$. We define the integers $l_p \geq 1$ and $u_p \geq 0$ such that:*

$$\mu_{p-l_p} > \mu_{p-l_p+1} = \cdots = \mu_p = \cdots = \mu_{p+u_p} > \mu_{p+u_p+1}.$$

Then

$$\partial f(\mu) = \left\{(\mathbf{1}_{p-l_p}; B\mathbf{1}_{l_p}; 0) | B \textit{ is a } (l_p + u_p) \times (l_p + u_p) \textit{ doubly stochastic matrix}\right\}.$$

Proof
According to Theorem 23.5 of [Roc70], we have

$$\partial f(\mu) = \arg\max\{\langle \gamma, \mu\rangle | \gamma \in \mathcal{SC}(\mathbf{1}_p)\}.$$

Observe that, by an elementary application of Lemma 3.2.5, the relation $\gamma \in \mathcal{SC}(\mathbf{1}_p)$ can be equivalently rewritten as $0 \leq \gamma_i \leq 1$ for every i and $s_r(\gamma) = p$. The above optimization problem can then be reformulated as the following continuous knapsack problem:

$$\begin{array}{rll} \partial f(\mu) = \arg\max & \langle \gamma, \mu \rangle & \\ \text{s.t.} & \sum_{i=1}^r \gamma_i = p & \\ & 0 \leq \gamma_i \leq 1, & i = 1, \ldots, r. \end{array}$$

According to the standard greedy approach for this knapsack problem (see for instance Section 2.6 in [Wol98]), all the optimal solutions γ^* to this problem satisfy:

$$\gamma_1^* = \cdots = \gamma_{p-l_p}^* = 1 \quad \text{and} \quad \gamma_{p+u_p+1}^* = \cdots = \gamma_r^* = 0.$$

Hence, we are left with the conditions

$$\sum_{i=p-l_p+1}^{p+u_p} \gamma_i^* = l_p, \quad \text{and} \quad 0 \leq \gamma_i^* \leq 1 \text{ for } p - l_p < i \leq p + u_p. \tag{4.7}$$

Observe that every γ^* that complies with these conditions satisfies $\langle \gamma^*, \mu\rangle = \langle \mathbf{1}_p, \mu \rangle = f(\mu)$. Hence, they describe the subdifferential of f at μ.

For notational convenience, the $(l_p + u_p)$-dimensional vector consisting of components $p - l_p + 1$ to $p + u_p$ of γ^* is denoted by γ^*_{mid}. The condition (4.7) on coefficients of γ^*_{mid} can be equivalently formulated as $\gamma^*_{\text{mid}} \in \mathcal{SC}(\mathbf{1}'_{l_p})$, where the vector $\mathbf{1}'_{l_p}$ is $(l_p + u_p)$-dimensional.

In view of Birkhoff's Theorem, we finally get the desired form. ■

As the reader may guess, the possible multiplicity of the eigenvalues of u should be carefully treated in the computation of the subdifferential of $S_p(u)$. Keeping this point in mind, let us introduce a few notational conventions.

For each $u = \sum_{i=1}^r \lambda_i(u) c_i \in \mathcal{J}$ and each $1 \leq p \leq r$, we define the integers $l_p(u) \geq 1$ and $u_p(u) \geq 0$ such that they satisfy:

$$\begin{aligned} \lambda_1(u) \geq \cdots \geq \lambda_{p-l_p(u)}(u) &> \lambda_{p-l_p(u)+1}(u) = \cdots = \lambda_p(u) = \cdots = \lambda_{p+u_p(u)}(u) \\ &> \lambda_{p+u_p(u)+1}(u) \geq \cdots \geq \lambda_r(u). \end{aligned}$$

If we represent on a line the indices of the eigenvalues of u that are equal to $\lambda_p(u)$, we obtain a segment in $\mathbb{N}$. Starting from p and going to the left, one can go as far as $l_p(u) - 1$ on this segment; going to the right, the largest distance one can move is $u_p(u)$. The full length of the segment is $l_p(u) + u_p(u) - 1$, and the multiplicity of the eigenvalue $\lambda_p(u)$ is $l_p(u) + u_p(u)$.

Moreover, we denote

$$\mathsf{f}'_p(u) := c_{p-l_p(u)+1} + \ldots + c_{p+u_p(u)};$$

we use a sans-serif typeface for this idempotent to avoid a possible confusion with a component of the differential of a function f. In fact, $\mathsf{f}'_p(u)$ is the idempotent given by the unique eigenspaces spectral decomposition theorem (see Theorem 2.7.13) for the root $\xi_j = \lambda_p(u)$. Consequently, $\mathsf{f}'_p(u)$ is uniquely defined, whatever may be the Jordan frame we have chosen for the complete spectral decomposition of u.

We also write:

$$\mathsf{f}_p(u) := c_1 + c_2 + \cdots + c_{p-l_p(u)} \quad \text{and} \quad \mathsf{f}''_p(u) := c_{p+u_p(u)+1} + \cdots + c_{r-1} + c_r,$$

so that $e = \mathsf{f}_p(u) + \mathsf{f}'_p(u) + \mathsf{f}''_p(u)$. Observe that these elements are uniquely defined and that their pairwise products are all null.

Proposition 4.4.5 *Let $u \in \mathcal{J}$. We have:*

$$\begin{aligned}\partial S_p(u) \;=\; \{v \in \mathcal{S}(\mathbf{1}_p) \mid\ & v = \mathsf{f}_p(u) + v',\ v' \in \mathcal{J}_1(\mathsf{f}'_p(u)),\\ & \operatorname{tr}(v') = l_p(u),\ 1 \geq \lambda_i(v') \geq 0\}.\end{aligned}$$

Proof
Let us fix an element u in $\mathcal{J}$. Observe that, in view of Fan's inequalities (see Proposition 3.2.7), the function S_p is the support function of the set $\mathcal{S}(\mathbf{1}_p)$, which can be in turn constructed from $\mathcal{SC}(\mathbf{1}_p)$ by the usual Jordan eigenvalues lifting.

Applying Corollary 4.4.3, we have:

$$\partial S_p(u) = \{v \in \mathcal{J} | \lambda(v) \in \partial f(\lambda(u)),\ u \text{ and } v \text{ have a similar joint decomposition}\},$$

where the function f is the support function of $\mathcal{SC}(\mathbf{1}_p)$. From Lemma 4.4.4, we know that $\gamma \in \partial f(\lambda(u))$ if and only if:

- $\diamond$ $\gamma_i = 1$ for $1 \leq i \leq p - l_p(u)$;
- $\diamond$ $0 \leq \gamma_i \leq 1$ for $p - l_p(u) + 1 \leq i \leq p + u_p(u)$, and the sum of these components equals $l_p(u)$;
- $\diamond$ and $\gamma_i = 0$ for $p + u_p(u) + 1 \leq i \leq r$.

In view of Proposition 4.2.2 on similar joint decomposition, we deduce that:

$$v \in \partial S_p(u) \quad \Leftrightarrow \quad v = \mathsf{f}_p(u) + v',$$

where $v' \in \mathcal{J}_1(\mathsf{f}'_p(u))$ is an element whose eigenvalues are between 0 and 1 and whose trace is equal to $l_p(u)$. ■

The following two corollaries are direct consequences of this explicit description of $\partial S_p(u)$. The first one has been obtained independently in [SS04], Proposition 4.

Corollary 4.4.6 *Let $u = \sum_{i=1}^r \lambda_i(u) c_i \in \mathcal{J}$ and $1 \leq p \leq r$. If p is the ending rank of a group of equal eigenvalues of u, i.e. if $u_p(u) = 0$, then S_p is differentiable at u and $\partial S_p(u) = \{\mathsf{f}_p(u) + \mathsf{f}'_p(u)\} = \{\sum_{i=1}^p c_i\}$.*

Proof
Let $v \in \partial S_p(u)$. By Proposition 4.4.5, we can write $v = \mathsf{f}_p(u) + v'$, where $v' \in \mathcal{J}_1(\mathsf{f}'_p(u))$. Since $\mathcal{J}_1(\mathsf{f}'_p(u))$ is a subalgebra of $\mathcal{J}$ of rank $l_p(u) + u_p(u) = l_p(u)$, since the eigenvalues of v' are between 0 and 1, and since $\mathrm{tr}(v') = l_p(u)$, all the eigenvalues of v' are equal to 1 in $\mathcal{J}_1(\mathsf{f}'_p(u))$. Thus v' is the unit element of $\mathcal{J}_1(\mathsf{f}'_p(u))$, i.e. $v' = \mathsf{f}'_p(u)$. Hence $\partial S_p(u) = \{\mathsf{f}_p(u) + \mathsf{f}'_p(u)\} = \{\sum_{i=1}^p c_i\}$. This subdifferential contains only one element, and it suffices to apply Theorem 25.1 in [Roc70] to conclude that S_p is differentiable at u. ■

In the next corollary, we adopt the notation $S_p(u; \mathcal{J}') := \sum_{i=1}^p \lambda_i(u; \mathcal{J}')$ for every u in a subalgebra $\mathcal{J}'$ of $\mathcal{J}$ and every $1 \leq p \leq \mathrm{rank}(\mathcal{J}')$. As recalled at the beginning of Section 4.2, $\lambda_i(u; \mathcal{J}')$ is the ith eigenvalue of u in the subalgebra $\mathcal{J}'$.

In the special case where $u \in \mathcal{J}' := \mathcal{J}_1(c)$ for an idempotent c, one can easily reconstruct $\lambda(u; \mathcal{J})$ from $\lambda(u; \mathcal{J}')$: it suffices to enlarge this vector by adding enough zero components (see Theorem 2.6.1 and Proposition 2.7.22). [2]

In particular, the function $S_p(\cdot; \mathcal{J}')$ is equal to S_p on every $u \in \mathcal{J}'$ for which $\lambda_p(u; \mathcal{J}) \geq 0$. Moreover, the trace of $\mathcal{J}'$ is equal to the restriction of the trace of $\mathcal{J}$ to $\mathcal{J}'$.

Corollary 4.4.7 *Let $u, h \in \mathcal{J}$, $1 \leq p \leq r$, and $\mathcal{J}' := \mathcal{J}_1(\mathsf{f}'_p(u))$. Then:*

$$\nabla_u^h S_p(u) = \mathrm{tr}(\mathsf{f}_p(u)h) + S_{l_p(u)}(Q_{\mathsf{f}'_p(u)}h; \mathcal{J}').$$

Observe that $Q_{\mathsf{f}'_p(u)}h$ is the orthogonal projection of h on $\mathcal{J}'$.

Proof
By convexity of S_p, and in view of Theorem 23.4 in [Roc70], we can write:

$$\nabla_u^h S_p(u) = \sup_{v \in \partial S_p(u)} \mathrm{tr}(vh).$$

Thus, with $l_p := l_p(u)$, $\mathsf{f}_p := \mathsf{f}_p(u)$ and $\mathsf{f}'_p := \mathsf{f}'_p(u)$, we successively have:

$$\begin{aligned}
\nabla_u^h S_p(u) &= \sup\{\mathrm{tr}(vh) | v \in \partial S_p(u)\} \\
&= \mathrm{tr}(\mathsf{f}_p h) + \sup\{\mathrm{tr}(v'h) \,|\, v' \in \mathcal{J}',\ 0 \leq \lambda_i(v') \leq 1\, \forall i,\ \mathrm{tr}(v') = l_p\} \\
&= \mathrm{tr}(\mathsf{f}_p h) + \sup\{\mathrm{tr}(v' Q_{\mathsf{f}'_p} h) | v' \in \mathcal{J}',\ 0 \leq \lambda_i(v'; \mathcal{J}') \leq 1\, \forall i,\ \mathrm{tr}(v') = l_p\} \\
&= \mathrm{tr}(\mathsf{f}_p h) + S_{l_p}(Q_{\mathsf{f}'_p} h; \mathcal{J}').
\end{aligned}$$

The second equality comes from Proposition 4.4.5. The third one follows from the fact that the eigenvalues of v' in $\mathcal{J}$ and in $\mathcal{J}'$ are identical, except for the multiplicity of 0. We have also applied the identity $\mathrm{tr}[(Q_c x)y] = \mathrm{tr}[(Q_c(Q_c x))y] = \mathrm{tr}[(Q_c x)(Q_c y)]$, which holds for every idempotent c in view of the fact that Q_c is self-adjoint. The fourth one is an application of Proposition 3.2.7 in the subalgebra $\mathcal{J}'$. ■

This corollary confirms that S_p is in general not differentiable because the expression of $\nabla_u^h S_p(u)$ is not linear in h. Here, we have a linear part $[\mathrm{tr}(\mathsf{f}_p(u)h)]$ and a convex part $[S_{l_p(u)}(Q_{\mathsf{f}'_p(u)}h; \mathcal{J}')]$.

[2] Not all the Jordan subalgebras of $\mathcal{J}$ are of the form $\mathcal{J}_1(c)$ for an idempotent c of $\mathcal{J}$. Consider indeed the subalgebra $\mathbb{R}c + \mathbb{R}d + \mathbb{R}cd$ suggests (see the proof of Lemma 2.10.10), where c and d are minimal idempotents of $\mathcal{J}$ such that $tr(cd) \in]0, 1[$.

We have now everything we need to compute the directional derivative of an eigenvalue. This answers the first open question given in the conclusion of the preprint [SS04].

Theorem 4.4.8 *Let $u, h \in \mathcal{J}$, and $1 \leq p \leq r$. We write $\mathcal{J}'$ for $\mathcal{J}_1(\mathsf{f}'_p(u))$. The directional derivative $\nabla_u^h \lambda_p(u)$ exists and equals:*

$$\nabla_u^h \lambda_p(u) = \lambda_{l_p(u)}(Q_{\mathsf{f}'_p(u)} h; \mathcal{J}').$$

Proof
Actually, this is a direct application of the previous corollary. For simplicity, we write again $l_p := l_p(u)$, $\mathsf{f}_p := \mathsf{f}_p(u)$ and $\mathsf{f}'_p := \mathsf{f}'_p(u)$.
Suppose first that $p = 1$. Since $l_p = 1$ and $\mathsf{f}_p = 0$, we have:

$$\nabla_u^h \lambda_1(u) = \nabla_u^h S_1(u) = S_1(Q_{\mathsf{f}'_p} h; \mathcal{J}') = \lambda_1(Q_{\mathsf{f}'_p} h; \mathcal{J}').$$

Now, if $p > 1$, we have $\lambda_p(u) = S_p(u) - S_{p-1}(u)$. Let us consider the case where $l_p > 1$. Since $\mathsf{f}_p = \mathsf{f}_{p-1}$, $l_{p-1} = l_p - 1$ and $\mathsf{f}'_p = \mathsf{f}'_{p-1}$, we have:

$$\begin{aligned}
\nabla_u^h \lambda_p(u) &= \nabla_u^h S_p(u) - \nabla_u^h S_{p-1}(u) \\
&= \operatorname{tr}(\mathsf{f}_p h) + S_{l_p}(Q_{\mathsf{f}'_p} h; \mathcal{J}') - \operatorname{tr}(\mathsf{f}_{p-1} h) - S_{l_{p-1}}(Q_{\mathsf{f}'_{p-1}} h; \mathcal{J}_1(\mathsf{f}'_{p-1})) \\
&= \lambda_{l_p}(Q_{\mathsf{f}'_p} h; \mathcal{J}').
\end{aligned}$$

It remains to analyze the situation where $p > 1$ and $l_p = 1$. In this case, $\lambda_{p-1}(u) > \lambda_p(u)$ and $u_{p-1} = 0$; using now Corollary 4.4.6, we get:

$$\begin{aligned}
\nabla_u^h \lambda_p(u) &= \nabla_u^h S_p(u) - \nabla_u^h S_{p-1}(u) \\
&= \operatorname{tr}(\mathsf{f}_p h) + S_1(Q_{\mathsf{f}'_p} h; \mathcal{J}') - \operatorname{tr}(\mathsf{f}_p h) = \lambda_1(Q_{\mathsf{f}'_p} h; \mathcal{J}').
\end{aligned}$$

■

4.4.3 First derivatives of spectral functions

We show in this subsection how to differentiate spectral functions on Jordan algebras. For Hermitian matrices, this problem has been solved by Adrian Lewis in [Lew96b], Theorem 1.1. Our proof loosely follows his argument. Our result has been obtained, independently of our work, in the preprint [SS04], Theorem 21.

We first start by an observation concerning the symmetry of the differential of a symmetric function.

Remark 4.4.9 *Let $Q \subseteq \mathbb{R}^r$ be an open symmetric set and let $f : Q \to \mathbb{R}$ be a function that is symmetric with respect to permutations. Suppose that f is differentiable at $\lambda \in Q$ and that $\lambda_i = \lambda_j$. Then $f'_i(\lambda) = f'_j(\lambda)$.*

Indeed, f is differentiable at $P\lambda$ for every $P \in \mathcal{P}$. For every direction $h \in \mathbb{R}^r$, we can write by symmetry of the function f:

$$\begin{aligned}
\nabla_{P\lambda}^h f(P\lambda) &= \lim_{t \to 0} \frac{f(P\lambda + th) - f(P\lambda)}{t} \\
&= \lim_{t \to 0} \frac{f(P(\lambda + tP^T h)) - f(P\lambda)}{t} = \lim_{t \to 0} \frac{f(\lambda + tP^T h) - f(\lambda)}{t} = \nabla_\lambda^{P^T h} f(\lambda).
\end{aligned}$$

Equivalently, $\nabla f(P\lambda) = \nabla f(\lambda)P^T$ *or* $f'(P\lambda) = Pf'(\lambda)$*. In particular, if* P *is the permutation that only exchanges the components* i *and* j*, we have* $P\lambda = \lambda$*, and* $f'(\lambda) = P^T f'(\lambda)$*. Thus* $f_i'(\lambda) = [P^T f'(\lambda)]_i = f_j'(\lambda)$*.* ■

Theorem 4.4.10 *Let* $Q \subseteq \mathbb{R}^r$ *be an open symmetric set, and let* $f : Q \to \mathbb{R}$ *be a symmetric function. We define* $K := \{v \in \mathcal{J} | \lambda(v) \in Q\}$ *and* $F : K \to \mathbb{R}$*,* $v \mapsto F(v) := f(\lambda(v))$*. Let* $u = \sum_{i=1}^r \lambda_i(u)c_i \in K$*. If the function* f *is differentiable at* $\lambda(u)$*, then the function* F *is differentiable at* u *and:*

$$F'(u) = \sum_{i=1}^{r} f_i'(\lambda(u))c_i. \tag{4.8}$$

Proof
Observe first that the formula (4.8) is independent of the particular spectral decomposition of u we have taken, due to the symmetry of f (see Remark 4.4.9).

Let $\epsilon > 0$ and define the integers $s, k_1, \ldots, k_s$ such that $k_s := r$ and:

$$\lambda_1(u) = \cdots = \lambda_{k_1}(u) > \lambda_{k_1+1}(u) = \cdots = \lambda_{k_2}(u) > \cdots \lambda_{k_s}(u).$$

We also take $M_j := \{k_{j-1}+1, \ldots, k_j\}$ (with $k_0 := 0$) and $e_j := \sum_{i \in M_j} c_i$. By differentiability of f in $\lambda(u)$, there exists an open and bounded neighborhood Λ of $\lambda(u)$ such that:

$$|f(\gamma) - f(\lambda(u)) - f'(\lambda(u))^T(\gamma - \lambda(u))| \leq \epsilon ||\gamma - \lambda(u)||$$

for every $\gamma \in \Lambda$. Let $V := \{v \in \mathcal{J} | \lambda(v) \in \Lambda\}$; according to Theorem 4.3.1, this is an open and bounded neighborhood of u. We can further assume that:

$$\left| f_{k_j}'(\lambda(u))[S_{k_j}(u+h) - S_{k_j}(u) - \nabla_u^h S_{k_j}(u)] \right| \leq \epsilon ||h||_{\mathcal{J}}$$

for every $h \in V - u$ and each $1 \leq j \leq s$ by possibly considering for Λ a smaller neighborhood of $\lambda(u)$. Using the directional derivative formula for S_p from Corollary 4.4.6, we can deduce from these inequalities:

$$\begin{aligned}
&\left| f_{k_j}'(\lambda(u)) \left[\sum_{i \in M_j} \Big(\lambda_i(u+h) - \lambda_i(u) - \mathrm{tr}(c_i h) \Big) \right] \right| \\
\leq\ & \left| f_{k_j}'(\lambda(u)) \left[S_{k_j}(u+h) - S_{k_j}(u) - \nabla_u^h S_{k_j}(u) \right] \right| \\
&\quad + \left| f_{k_j}'(\lambda(u)) \left[S_{k_{j-1}}(u+h) - S_{k_{j-1}}(u) - \nabla_u^h S_{k_{j-1}}(u) \right] \right| \\
\leq\ & 2\epsilon ||h||_{\mathcal{J}}
\end{aligned}$$

for all $h \in V - u$. Now, we can write:

$$
\begin{aligned}
&|f'(\lambda(u))^T(\lambda(u+h) - \lambda(u)) - \sum_{j=1}^{s} f'_{k_j}(\lambda(u))\mathrm{tr}(e_j h))| \\
&= \left| \sum_{j=1}^{s} f'_{k_j}(\lambda(u)) \left[\sum_{i \in M_j} \Big(\lambda_i(u+h) - \lambda_i(u) - \mathrm{tr}(c_i h) \Big) \right] \right| \\
&\leq \sum_{j=1}^{s} \left| f'_{k_j}(\lambda(u)) \left[\sum_{i \in M_j} \Big(\lambda_i(u+h) - \lambda_i(u) - \mathrm{tr}(c_i h) \Big) \right] \right| \\
&\leq \sum_{j=1}^{s} 2\epsilon ||h||_{\mathcal{J}} = 2s\epsilon ||h||_{\mathcal{J}} \qquad (4.9)
\end{aligned}
$$

The Lipschitz property for eigenvalues showed in Corollary 4.2.5 allows us to write for all $h \in V - u$:

$$
|f(\lambda(u+h)) - f(\lambda(u)) - f'(\lambda(u))^T(\lambda(u+h) - \lambda(u))| \leq \epsilon ||\lambda(u+h) - \lambda(u)|| \leq \epsilon ||h||_{\mathcal{J}}.
$$

In view of (4.9), we then get:

$$
|F(u+h) - F(u) - \sum_{j=1}^{s} f'_{k_j}(\lambda(u))\mathrm{tr}(e_j h)| \leq \epsilon(1+2s)||h||_{\mathcal{J}}.
$$

Since V is open, $h/||h||_{\mathcal{J}}$ can be arbitrarily chosen on the unit sphere of $\mathcal{J}$, and

$$
F'(u) = \sum_{j=1}^{s} f'_{k_j}(\lambda(u)) e_j = \sum_{i=1}^{r} f'_i(\lambda(u)) c_i.
$$

■

Observe that the previous theorem requires no assumption on the convexity of the function F in the previous theorem.

Corollary 4.4.11 *If the function f is continuously differentiable in Q, the spectral function F generated by f is continuously differentiable in K.*

Proof
Let $u \in K$ and $(u_m)_{m \geq 0}$ be a sequence of K that converges to u. We denote the respective complete spectral decompositions of these elements by $u = \sum_{i=1}^{r} \lambda_i(u) c_i$ and $u_m = \sum_{i=1}^{r} \lambda_i(u_m) c_{i,m}$. The continuity of eigenvalues and of f' implies that $\lim_{m \to \infty} f'(\lambda(u_m)) = f'(\lambda(u))$. It remains now to use Proposition 4.3.2 to get:

$$
\lim_{m \to \infty} F'(u_m) = \sum_{j=1}^{s} f'_{k_j}(\lambda(u)) \lim_{m \to \infty} \sum_{i \in M_j} c_{i,m} = \sum_{j=1}^{s} f'_{k_j}(\lambda(u)) e_j = F'(u).
$$

■

4.4.4 Convex properties of spectral functions

This subsection discusses how convex properties of symmetric functions can be transferred to the corresponding spectral function. The first item of the following Theorem has been obtained by Adrian Lewis in [Lew96a], Corollary 2.7, for convex lower semicontinuous spectral function on Hermitian matrices. While Lewis' proof relies on some relationships between conjugate functions, we use here a more elementary argument based on the description of the permutahedron. It is interesting to note that, in view of our proof, the convexity of spectral functions generated by convex functions follows directly from the convexity of the functions S_p and from Birkhoff's Theorem.

Definition 4.4.12 *Let $Q \subseteq \mathbb{R}^n$ be a convex set. A function $f : Q \to \mathbb{R}$ is strongly convex with parameter σ with respect to the norm $||\cdot||$ if and only if for every $x, y \in Q$ and every $\alpha \in [0,1]$, we can write:*

$$\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y) \geq \frac{\sigma}{2}\alpha(1-\alpha)||x-y||^2.$$

If the function f is differentiable on Q, this requirement is equivalent to

$$f(y) - f(x) - \langle f'(x), y-x\rangle \geq \frac{\sigma}{2}||y-x||^2 \quad \textit{for all } x, y \in Q.$$

■

Theorem 4.4.13 *Let $Q \subseteq \mathbb{R}^r$ be a symmetric set and $K := \{v \in \mathcal{J} | \lambda(v) \in Q\}$, let $f : Q \to \mathbb{R}$ be a symmetric function. Let $F : K \to \mathbb{R}$, $v \mapsto F(v) := f(\lambda(v))$.*

- ◇ *If f is convex, F is convex.*
- ◇ *If f is quasi-convex, F is quasi-convex.*
- ◇ *If f is twice differentiable and strongly convex with parameter σ for the Euclidean norm, F is strongly convex with parameter σ for the norm $||\cdot||_{\mathcal{J}}$.*

Proof
Let $v_0, v_1 \in K$ and $\alpha \in [0,1]$; denote $v_\alpha := (1-\alpha)v_0 + \alpha v_1$ and $\lambda_\alpha := (1-\alpha)\lambda(v_0) + \alpha\lambda(v_1)$. Given that K is convex, the element v_α belongs to K. Further, we know from Lemma 3.2.5, and more specifically from (4.6), that $\lambda(v_\alpha)$ belongs to $\mathcal{SC}(\lambda_\alpha)$. Let us now take an arbitrary element μ of the permutahedron $\mathcal{SC}(\lambda_\alpha)$. We can write this vector as $\mu = \sum_{j=1}^{r!} \alpha_j P_j \lambda_\alpha$, where the nonnegative numbers α_j sum up to 1, and where the matrices P_j are all in $\mathcal{P}$.

Suppose first that f is convex. Using convexity and symmetry of f, we get $f(\mu) \leq \sum_{j=1}^{r!} \alpha_j f(P_j \lambda_\alpha) = \sum_{j=1}^{r!} \alpha_j f(\lambda_\alpha) = f(\lambda_\alpha)$. Hence:

$$\begin{aligned} F(v_\alpha) = f(\lambda(v_\alpha)) \leq f(\lambda_\alpha) &\leq (1-\alpha)f(\lambda(v_0)) + \alpha f(\lambda(v_1)) \\ &= (1-\alpha)F(v_0) + \alpha F(v_1), \end{aligned} \tag{4.10}$$

and F is convex as well.

Next, if f is quasi-convex, we get $f(\mu) \leq \max_{1\leq j\leq r!}\{f(P_j\lambda_\alpha)\} = f(\lambda_\alpha)$ by symmetry of f. Hence, as in (4.10), we can write:

$$F(v_\alpha) = f(\lambda(v_\alpha)) \leq f(\lambda_\alpha) \leq \max\{f(\lambda(v_0)), f(\lambda(v_1))\} = \max\{F(v_0), F(v_1)\},$$

and F is quasi-convex.

If f is strongly convex with parameter σ, it is easy to show that $f''(\lambda) - \sigma I_r$ is positive semidefinite (see Theorem 2.1.11 of [Nes03]); equivalently, $g(\lambda) := f(\lambda) - \sigma||\lambda||^2/2$ is convex, where $||\cdot||$ is the Euclidean norm of $\mathbb{R}^r$. Let $F(v) := f(\lambda(v))$ and $G(v) := g(\lambda(v))$ for every $v \in K$. Of course, we have $F(v) = G(v) + \sigma\mathrm{tr}(v^2)/2$; we also know that G is convex by the first item. Observe that:

$$\begin{aligned}
& \mathrm{tr}(v_\alpha^2) - (1-\alpha)\mathrm{tr}(v_0^2) - \alpha\mathrm{tr}(v_1^2) \\
= \; & (1-\alpha)^2\mathrm{tr}(v_0^2) + 2\alpha(1-\alpha)\mathrm{tr}(v_0v_1) + \alpha^2\mathrm{tr}(v_1^2) - (1-\alpha)\mathrm{tr}(v_0^2) - \alpha\mathrm{tr}(v_1^2) \\
= \; & -\alpha(1-\alpha)\mathrm{tr}(v_0^2) + 2\alpha(1-\alpha)\mathrm{tr}(v_0v_1) - \alpha(1-\alpha)\mathrm{tr}(v_1^2) \\
= \; & -\alpha(1-\alpha)\mathrm{tr}(v_0 - v_1)^2.
\end{aligned} \tag{4.11}$$

Now, by convexity of G, we can write:

$$G(v_\alpha) \leq (1-\alpha)G(v_0) + \alpha G(v_1),$$

or:

$$F(v_\alpha) - \sigma\mathrm{tr}(v_\alpha^2)/2 \leq (1-\alpha)F(v_0) + \alpha F(v_1) - \sigma[(1-\alpha)\mathrm{tr}(v_0^2) + \alpha\mathrm{tr}(v_1^2)]/2.$$

The identity on traces (4.11) entails the following inequality:

$$F(v_\alpha) \leq (1-\alpha)F(v_0) + \alpha F(v_1) - \sigma[\alpha(1-\alpha)\mathrm{tr}(v_0 - v_1)^2]/2,$$

which is equivalent to the strong convexity of F with the parameter σ. ■

As an application of this theorem, one can check that the *condition number* $\mathrm{cond}(u) := \lambda_1(u)/\lambda_r(u)$ for $u \in \mathrm{int}\,\mathcal{K}_\mathcal{J}$ is a quasi-convex function, due to the fact that

$$f(x) = \max_{1\leq i\leq r} x_i / \min_{1\leq i\leq r} x_i$$

is a symmetric quasi-convex function on the positive orthant.

Lipschitz continuity of the gradient of a function is one of the most frequently used properties in the development of a number of optimization algorithms, as well as in the evaluation of their performances. In order to see how this smoothness property can be transmitted from a symmetric function to the spectral function it generates, we first recall a classical result in convex analysis.

Lemma 4.4.14 *Let A be a nonempty convex subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a differentiable convex function. Let $\langle\cdot,\cdot\rangle$ be a scalar product on $\mathbb{R}^n$, $||\cdot||$ the Euclidean norm it generates, and f^* the conjugate function of f constructed with it. Then, the following equivalence holds:*

$$f(y) - f(x) - \langle f'(x), y - x\rangle \leq \frac{L}{2}||y - x||^2 \quad \forall\, x, y \in \mathrm{dom}\, f \tag{4.12}$$

(i.e. if f has a Lipschitz continuous gradient with parameter L) if and only if:

$$f^*(y) - f^*(x) - \langle f^{*\prime}(x), y - x\rangle \geq \frac{1}{2L}||y - x||^2 \quad \forall\, x, y \in \operatorname{dom} f^* \tag{4.13}$$

(i.e. f^* is strictly convex with parameter $1/L$).

The "only if" part is proved in [HUL93], Theorem X.4.2.2. The "if" part is a straightforward adaptation of their proof. ■

Corollary 4.4.15 *Let $Q \subseteq \mathbb{R}^r$ be a nonempty symmetric set and $K := \{v \in \mathcal{J} | \lambda(v) \in Q\}$, let $f : Q \to \mathbb{R}$ be a convex, symmetric and differentiable function with a closed epigraph. We denote by F the spectral function generated by f. If there exists a constant $L > 0$ such that, for every $\lambda_1, \lambda_2 \in Q$, we can write:*

$$||f'(\lambda_1) - f'(\lambda_2)|| \leq L||\lambda_1 - \lambda_2||, \tag{4.14}$$

where $||\cdot||$ stands for the Euclidean norm of $\mathbb{R}^r$, then

$$||F'(v_1) - F'(v_2)||_{\mathcal{J}} \leq L||v_1 - v_2||_{\mathcal{J}}.$$

Proof
First of all, F is convex and differentiable because f is. It can be easily shown that (4.14) is equivalent to (4.12). By the previous lemma, the conjugate f^* of f is strongly convex with parameter $1/L$ for the Euclidean norm. The spectral function generated by f^* is F^* by Theorem 4.4.2. The third item of Theorem 4.4.13 shows that F^* is strongly convex with a parameter equal to $1/L$. Since f has a closed epigraph, $\operatorname{epi} F$ is also closed by continuity of eigenvalues; hence the conjugate of F^* is F (See [Roc70], Corollary 12.2.1). Applying again the previous lemma, we deduce that F has a Lipschitz continuous gradient with parameter L. ■

4.5 Clarke subdifferentiability

We give in this section a closed form for the Clarke subdifferential of a spectral function F on a formally real Jordan algebra. Interestingly, this Clarke subdifferential can be characterized by means of the Clarke subdifferential of the symmetric function f that generates F. We first compute the Bouligand subdifferential of F before devising a formula for $\partial_C F(u)$. As the following proposition states, local Lipschitz continuity is transferred from f to F. We omit its trivial proof, based on Corollary 4.2.5.

Proposition 4.5.1 *Let $f : Q \subseteq \mathbb{R}^r \to \mathbb{R}$ be a symmetric function on a symmetric set Q. If f is locally Lipschitz continuous, then the spectral function generated by f is also locally Lipschitz continuous.* ■

The reader can find a proof of Rademacher's Theorem [Rad19] in [EG92], Section 3.1.

Theorem 4.5.2 (Rademacher's Theorem) *Let G and H be two Euclidean vector spaces and let $\phi : G \to H$ be a locally Lipschitz continuous function. The function ϕ is non-differentiable only in a negligible part of G (in the Lebesgue sense).* ■

Consequently, if $\phi : G \to \mathbb{R}$ is a locally Lipschitz continuous function, the set D_ϕ of points in G where ϕ is differentiable is a dense subset of G. The *Bouligand differential* of ϕ in a point x of its domain is defined as follows:

$$\partial_B\phi(x) := \{V \in G \mid \text{there exists } (x_m)_{m\geq 0} \subseteq D_\phi \text{ such that } \lim_{m\to\infty} x_m = x \text{ and } V = \lim_{k\to\infty} \phi'(x_m)\}.$$

In view of Proposition 4.5.1, we can similarly define the Bouligand differential of the spectral function F generated by a locally Lipschitz continuous symmetric function f. We denote by K the domain of F and by D_f the set of points where f is differentiable. In view of Theorem 4.4.10, we know that the spectral function F generated by f is differentiable in $D_F := \{u \in \mathcal{J} | \lambda(u) \in D_f\}$. It is not difficult to show that F cannot be differentiable at a point u outside D_F, because otherwise f would be differentiable at $\lambda(u) \notin D_f$. Moreover, D_F is dense in K in view of Proposition 4.3.3. Hence,

$$\partial_B F(u) = \{V \in \mathcal{J} \mid \text{there exists } (u_m)_{m\geq 0} \subseteq D_F \text{ such that } \lim_{m\to\infty} u_m = u \text{ and } V = \lim_{k\to\infty} F'(u_m)\}.$$

We observe that the symmetry of a function f induces a precise block structure for the elements of its Bouligand subdifferential. This structure is similar to the one of the differential of symmetric functions as expressed in Remark 4.4.9.

Remark 4.5.3 *Let f be a symmetric locally Lipschitz function on a symmetric domain $Q \subseteq \mathbb{R}^r$. We denote by D_f the subset of Q where f is differentiable. Note that D_f is symmetric in view of Remark 4.4.9. We consider a point λ in Q, a vector g of $\partial_B f(\lambda)$, and a permutation matrix $P \in \mathcal{P}$. By definition, there exists a sequence $(\lambda_m)_{m\geq 0}$ of D_f converging to λ for which $f'(\lambda_m)$ converges to g as m goes to infinity. Now, consider the vectors $\mu_m := P\lambda_m$ for every $m \geq 0$. By symmetry, the sequence $(\mu_m)_{m\geq 0}$ belongs to D_f. It converges to $P\lambda$, and $f'(\mu_m)$ tends to Pg as m goes to infinity. We conclude that $\partial_B f(P\lambda) \supseteq P\partial_B f(\lambda)$. Since P is invertible, we analogously have $\partial_B f(\lambda) \supseteq P^{-1}\partial_B f(P\lambda)$, and:*

$$P\partial_B f(\lambda) \supseteq \partial_B f(P\lambda) \supseteq P\partial_B f(\lambda),$$

implying $P\partial_B f(\lambda) = \partial_B f(P\lambda)$.

If we assume that $\lambda_i = \lambda_j$ and that the permutation P consists in swapping the ith component with the jth, then the above equality simplifies into $P\partial_B f(\lambda) = \partial_B f(\lambda)$. ■

Theorem 4.5.4 (Bouligand subdifferential) *Let $f : Q \subseteq \mathbb{R}^r \to \mathbb{R}$ be a symmetric function on an open symmetric set Q. We assume that f is locally Lipschitz continuous. Let $u \in K := \{v \in \mathcal{J} | \lambda(v) \in Q\}$. Then $g \in \partial_B F(u)$ if and only if there exists a Jordan frame $\{c_1, \ldots, c_r\}$ and a vector $\gamma \in \partial_B f(\lambda(u))$ for which $u = \sum_{i=1}^r \lambda_i(u) c_i$ and $g = \sum_{i=1}^r \gamma_i c_i$.*

Proof
We let D_f be the set of points in Q where f is differentiable and $D_F := \{v \in \mathcal{J} | \lambda(v) \in D_f\}$. Of course, F is only differentiable on D_F. We denote by $\{c_1, \ldots, c_r\}$ a Jordan frame for which $u = \sum_{i=1}^r \lambda_i(u) c_i$.

We consider first the "if" part. Let us take an element $g \in \mathcal{J}$ such that $g = \sum_{i=1}^r \gamma_i c_i$ and $\gamma \in \partial_B f(\lambda(u))$. Then, there exists a sequence $(\mu_m)_{m \geq 0}$ in $D_f \cap Q$ that converges to $\lambda(u)$ and for which $f'(\mu_m)$ converges to γ. We denote by $\mu_{i,m}$ the ith component of the vector μ_m and take $u_m := \sum_{i=1}^r \mu_{i,m} c_i$ for every $m \geq 0$. Then $u_m \in D_F$ for every $m \geq 0$, because $\lambda(u_m)$, which is a permutation of the components of μ_m belongs to the symmetric set D_f. Moreover, u_m tends to u as m goes to infinity. Finally, we have by Theorem 4.4.10 that $F'(u_m) = \sum_{i=1}^r f_i'(\mu_m) c_i$, which converges to g, and g is in $\partial_B F(u)$. (Observe that it is not necessary to ensure that each vector μ_m has ordered components.)

Now, we turn our attention to the "only if" part of the statement. Let $g = \sum_{i=1}^r \lambda_i(g) d_i$ be an element of $\partial_B F(u)$. There exists a sequence $(u_m)_{m \geq 0}$ in D_F that converges to u and for which $F'(u_m) \to g$. We denote $u_m = \sum_{i=1}^r \lambda_i(u_m) c_{i,m}$, and we introduce the integers s, $k_1, \ldots, k_s$ such that $k_s := r$ and:

$$\lambda_1(u) = \cdots = \lambda_{k_1}(u) > \lambda_{k_1+1}(u) = \ldots = \lambda_{k_2}(u) > \cdots \lambda_{k_s}(u).$$

We also take $M_j := \{k_{j-1} + 1, \ldots, k_j\}$ (with $k_0 = 0$), $e_j := \sum_{i \in M_j} c_i$, and $e_{j,m} := \sum_{i \in M_j} c_{i,m}$ for every $m \geq 0$. We finally set P_m to be a permutation that orders the components of $f'(\lambda(u_m))$, that is, for which $P_m f'(\lambda(u_m)) \in \mathbb{R}^r_\downarrow$. Since f is a symmetric function, we have $P_m f'(\lambda(u_m)) = f'(P_m \lambda(u_m)) \in \mathbb{R}^r_\downarrow$ in view of Remark 4.4.9. These permutations can also be represented by mappings σ_m from $\{1, \ldots, r\}$ into itself, so that if P_m transforms the ith component of its argument into the jth one, then $\sigma_m(i) = j$. By continuity of eigenvalues, and since the following limit holds when m goes to infinity:

$$F'(u_m) = \sum_{i=1}^r f_i'(P_m \lambda(u_m)) c_{\sigma_m(i),m} \to \sum_{i=1}^r \lambda_i(g) d_i, \tag{4.15}$$

we already have:

$$f_i'(P_m \lambda(u_m)) \to \lambda_i(g)$$

as m goes to infinity.

Now comes the tricky part. The analysis of the limit (4.15) with Proposition 4.3.2 can teach us something more. For the sake of notational simplicity, let us temporarily write $d_{i,m} := c_{\sigma_m(i),m}$. We introduce the integers t, $l_1, \ldots, l_t$ such that $l_s := t$ and:

$$\lambda_1(g) = \cdots = \lambda_{l_1}(g) > \lambda_{l_1+1}(g) = \ldots = \lambda_{l_2}(g) > \cdots \lambda_{l_t}(g).$$

Similarly as above, we define the sets $N_j := \{l_{j-1}+1, \ldots, l_j\}$ (with $l_0 = 0$) and the idempotents $f_j := \sum_{i \in N_j} d_i$ for $1 \leq j \leq t$. Proposition 4.3.2 ensures us that $\sum_{i \in N_j} d_{i,m} \to f_j$ for every j; in other words, the set $\{1, 2, \ldots, r\}$ can be partitioned into t subsets $M_1', \ldots, M_t'$ for which $\sum_{i \in M_j'} c_{i,m} \to f_j$.

Summarized, we simultaneously have:

$$\lim_{m \to \infty} \sum_{i \in M_j} c_{i,m} = e_j \quad \text{and} \quad \lim_{m \to \infty} \sum_{i \in M_k'} c_{i,m} = f_k$$

for $1 \leq j \leq s$ and $1 \leq k \leq t$.

For every possible values of j and k, we claim that:

$$\lim_{m\to\infty} \sum_{i\in M_j\cap M'_k} c_{i,m} = Q_{f_k} e_j.$$

Observe that some sets $M_j \cap M'_k$ may be empty, in which case our claim reduces to $Q_{f_k} e_j = 0$.

In order to prove our claim, we define $f_{k,m} := \sum_{i\in M'_k} c_{i,m}$. We have $\sum_{i\in M_j\cap M'_k} c_{i,m} = \sum_{i\in M_j} Q_{f_{k,m}} c_{i,m}$ and $Q_{f_{k,m}} \to Q_{f_k}$. Hence, we can write:

$$\lim_{m\to\infty} \sum_{i\in M_j\cap M'_k} c_{i,m} = \lim_{m\to\infty} \sum_{i\in M_j} Q_{f_{k,m}} c_{i,m} = \lim_{m\to\infty} \sum_{i\in M_j} Q_{f_k} c_{i,m} = Q_{f_k} e_j.$$

Actually, an analogous reasoning leads us to the identity $Q_{f_k} e_j = Q_{e_j} f_k$.

Now, we put:

$$\mathcal{I} := \{Q_{f_k} e_j | 1 \leq j \leq s, 1 \leq k \leq t\}.$$

This set constitutes a system of idempotents (that may contain null elements), as a consequence of the fact that it represents the limit of the systems of idempotents

$$\{Q_{f_{k,m}} e_{j,m} = \sum_{i\in M_j\cap M'_k} c_{i,m} | 1 \leq j \leq s, 1 \leq k \leq t\}.$$

Since $\mathcal{I}$ is a system of idempotents (some of them might be null), each of its elements can be decomposed into the sum of minimal idempotents that belong to a well-chosen Jordan frame $\{c'_1, \ldots, c'_r\}$. Up to a renumbering of these minimal idempotents, we can assume that $\sum_{i\in M_j} c'_i = e_j$, so that $u = \sum_{i=1}^r \lambda_i(u) c'_i$. Of course, the interesting feature of this Jordan frame is that we can use it to write the spectral decomposition of g:

$$g = \sum_{j=1}^t \lambda_{l_j}(g) f_j = \sum_{j=1}^t \sum_{i\in M'_j} \lambda_{l_j}(g) c'_i = \sum_{i=1}^r \gamma_i c'_i.$$

Now,

$$F'(u_m) = \sum_{i=1}^r f'_i(\lambda(u_m)) c_{i,m} = \sum_{j=1}^s f'_{k_j}(\lambda(u_m)) e_{j,m}$$

has the same limit as

$$\sum_{j=1}^s f'_{k_j}(\lambda(u_m)) e_j = \sum_{i=1}^r f'_i(\lambda(u_m)) c'_i$$

when m goes to infinity. By assumption, this limit equals $g = \sum_{i=1}^r \gamma_i c'_i$. We deduce that $f'(\lambda(u_m)) \to \gamma$. Thus γ belongs to $\partial_B f(\lambda(u))$. ∎

As already mentioned, the Clarke subdifferential of a locally Lipschitz continuous function is defined as the convex hull of its Bouligand subdifferential. It remains to use the previous theorem to compute the Clarke subdifferential of a spectral function.

Theorem 4.5.5 (Clarke subdifferential) *Let $f : Q \subseteq \mathbb{R}^r \to \mathbb{R}$ be a symmetric function on an open symmetric set Q. We assume that f is locally Lipschitz continuous. Let $K := \{u \in \mathcal{J} | \lambda(u) \in Q\}$ and fix an element $u \in K$. Then $g \in \partial_C F(u)$ if and only if there exists a Jordan frame $\{c_1, \ldots, c_r\}$ and a vector $\gamma \in \partial_C f(\lambda(u))$ for which $u = \sum_{i=1}^r \lambda_i(u) c_i$ and $g = \sum_{i=1}^r \gamma_i c_i$.* ■

Proof
Following our usual notation, we define the integers $s, k_1, \ldots, k_s$ such that $k_s := r$ and:

$$\lambda_1(u) = \cdots = \lambda_{k_1}(u) > \lambda_{k_1+1}(u) = \ldots = \lambda_{k_2}(u) > \cdots \lambda_{k_s}(u).$$

We also take $M_j := \{k_{j-1}+1, \ldots, k_j\}$ (with $k_0 = 0$), and $e_j := \sum_{i \in M_j} c_i$. We denote by $\pi_j : \mathbb{R}^r \to \mathbb{R}^{|M_j|}$ the projector:

$$\lambda = (\lambda_1, \ldots, \lambda_r)^T \mapsto \pi_j(\lambda) := (\lambda_{k_{j-1}+1}, \ldots, \lambda_{k_j})^T.$$

We resolve to prove the following equivalent form (in view of Corollary 2.7.30) of the statement:

$$g \in \partial_C F(u) \text{ if and only if } g = \sum_{j=1}^{s} g_j, \quad \text{where for every } j, \text{ we have } g_j \in \mathcal{J}_1(e_j)$$
$$\text{and } \lambda(g_j; \mathcal{J}_1(e_j)) \in \pi_j[\partial_C f(\lambda(u))]. \tag{4.16}$$

By Theorem 4.5.4, we know that g belongs to the Bouligand subdifferential $\partial_B F(u)$ if and only if there exist a Jordan frame $\{c_1, \ldots, c_r\}$ and a vector $\gamma \in \partial_B f(\lambda(u))$ such that $u = \sum_{i=1}^r \lambda(u)_i c_i$ and $g = \sum_{i=1}^r \gamma_i c_i$. Writing $g_j := Q_{e_j} g \in \mathcal{J}_1(e_j)$, we observe that $\lambda(g_j; \mathcal{J}_1(e_j)) = P_j \pi_j(\gamma)$ for an appropriate permutation P_j of $|M_j|$-dimensional vectors. In view of Corollary 2.7.30 and of Remark 4.5.3, this characterization of the Bouligand subdifferential can be restated as follows:

$$g \in \partial_B F(u) \text{ if and only if } g = \sum_{j=1}^{s} g_j, \quad \text{where for every } j, \text{ we have } g_j \in \mathcal{J}_1(e_j)$$
$$\text{and } \lambda(g_j; \mathcal{J}_1(e_j)) \in \pi_j[\partial_B f(\lambda(u))].$$

Observe that, by linearity of the projector π_j, we have:

$$\text{conv}[\pi_j[\partial_B f(\lambda(u))]] = \pi_j[\text{conv}[\partial_B f(\lambda(u))]] = \pi_j[\partial_C f(\lambda(u))]. \tag{4.17}$$

We first check the "if" part of the statement (4.16). Assume that $g = \sum_{j=1}^s g_j$, where $g_j := Q_{e_j} g$, and that $\lambda_j := \lambda(g_j; \mathcal{J}_1(e_j))$ is in the projection $\pi_j[\partial_C f(\lambda(u))]$ of the Clarke subdifferential. We denote by $g_j = \sum_{i=1}^{|M_j|} \lambda_{ij} c_{ij}$ a complete spectral decomposition of g_j in the subalgebra $\mathcal{J}_1(e_j)$.

In view of (4.17), we can represent λ_j as:

$$\lambda_j = \sum_{\alpha \in A_j} t_{j,\alpha} \lambda_{j,\alpha}, \text{ where } \lambda_{j,\alpha} = (\lambda_{1,j,\alpha}, \ldots, \lambda_{|M_j|,j,\alpha})^T \in \pi_j[\partial_B f(\lambda(u))],$$

and where the positive coefficients $t_{j,\alpha}$ satisfy $\sum_{\alpha\in A_j} t_{j,\alpha} = 1$. Let

$$g_{j,\alpha} := \sum_{i=1}^{|M_j|} \lambda_{i,j,\alpha} c_{ij},$$

where $\lambda_{1,j,\alpha}, \ldots, \lambda_{|M_j|,j,\alpha}$ are the decreasingly ordered components of the vector $\lambda_{j,\alpha}$. Observe that:

$$g = \sum_{\alpha_1\in A_1} \cdots \sum_{\alpha_s\in A_s} t_{1,\alpha_1} \ldots t_{s,\alpha_s}(g_{1,\alpha_1} + \cdots + g_{s,\alpha_s}),$$

and that $g_{1,\alpha_1} + \cdots + g_{s,\alpha_s}$ belongs to $\partial_B F(u)$ for every $\alpha_1 \in A_1$, ..., $\alpha_s \in A_s$ in view of Theorem 4.5.4. Thus $g \in \mathrm{conv}[\partial_B F(u)] = \partial_C F(u)$.

In order to prove the "only if part", suppose that $g \in \partial_C F(u)$. There exists a set of positive real numbers $(t_\alpha)_{\alpha\in A}$ that sums up to 1, and a set $\{g_\alpha | \alpha \in A\} \subseteq \partial_B F(u)$ such that $g = \sum_{\alpha\in A} t_\alpha g_\alpha$. We denote by $g_{j,\alpha}$ the projection of g_α on $\mathcal{J}_1(e_j)$, so that $g_j := \sum_{\alpha\in A} t_\alpha g_{j,\alpha}$ is the projection of g on $\mathcal{J}_1(e_j)$.

In view of relation (4.6) on the eigenvalues of a convex combination of elements, we have:

$$\lambda(g_j; \mathcal{J}_1(e_j)) \in \mathcal{SC}\left[\sum_{\alpha\in A} t_\alpha \lambda(g_{j,\alpha}; \mathcal{J}_1(e_j))\right].$$

Since the eigenvalue vector $\lambda(g_{j,\alpha}; \mathcal{J}_1(e_j))$ is in $\pi_j[\partial_B f(\lambda(u))]$, which is a symmetric set in view of Remark 4.5.3, we have by (4.17):

$$\begin{aligned}\lambda(g_j; \mathcal{J}_1(e_j)) \;\in\; & \mathcal{SC}\left[\sum_{\alpha\in A} t_\alpha \lambda(g_{j,\alpha}; \mathcal{J}_1(e_j))\right] = \sum_{\alpha\in A} t_\alpha \mathcal{SC}\left[\lambda(g_{j,\alpha}; \mathcal{J}_1(e_j))\right] \\ & \subseteq \mathrm{conv}[\pi_j[\partial_B f(\lambda(u))]] = \pi_j[\partial_C f(\lambda(u))].\end{aligned}$$

The first equality follows directly from the definition of $\mathcal{SC}$. ■

CHAPTER 5

Spectral mappings on formally real Jordan algebras

WE INTRODUCE IN THIS CHAPTER the notion of *spectral mapping on formally real Jordan algebras*, a particular case of which is the gradient of a spectral function on formally real Jordan algebras. We perform a differentiability analysis of the sequence of Jordan frames corresponding to a converging sequence of elements. This analysis enables us to derive a closed form formula for the Jacobian of a spectral mapping. As a byproduct, we obtain a close expression for the Hessian of a twice differentiable spectral function on formally real Jordan algebras. This results settles an open question proposed by H. Sendov.

We apply our formula to study two smoothing strategies to solve the symmetric cone complementarity problem, namely, the Chen-Mangasarian and the Fischer-Burmeister smoothing schemes. We demonstrate that these strategies are well-posed, and we provide some indication concerning their convergence.

5.1 Introduction

The previous chapter was dedicated to the study of *spectral functions on Jordan algebras*. In this chapter, we consider a related construction, namely *spectral mapping on Jordan algebras*, which are also built by means of the spectral decomposition theorem for Jordan algebras. Let $Q \subseteq \mathbb{R}^r$ be a symmetric set. We are given a function $g : Q \to \mathbb{R}^r$ that is symmetric in the following sense: for every permutation matrix P and every $\lambda \in Q$, we have $g(P\lambda) = Pg(\lambda)$. Examples of such functions include the gradient mapping of a symmetric function, and projection operators on convex symmetric sets of $\mathbb{R}^r$. From the function g, we build a function G in the following way. Let u be an element of $\mathcal{J}$ that have its eigenvalues vector in Q. Suppose that $u = \sum_{i=1}^r \lambda_i(u)c_i$, using the complete spectral decomposition theorem for formally real Jordan algebras (see Theorem 2.7.25); we set $G(u) := \sum_{i=1}^r g_i(\lambda(u))c_i$. Our aim is to study how the differentiability of the function g transfers to the function G and we give a concise formula for the Jacobian. This provides an answer to an open question given in the PhD thesis of Sendov ([Sen00], Chapter 8, question 12).

We also show how our results can be applied to deal with complementarity problem defined on the cone of squares of a formally real Jordan algebra. More specifically, we demonstrate how the Chen-Mangasarian and the Fischer-Burmeister smoothing algorithms (see [CM95] and [Kan96] respectively) can be analyzed in the Jordan algebraic framework with the help of a spectral mapping. In [FLT01], Fukushima, Luo and Tseng have already considered an extension to second-order cone programming.

Our formula for the differential of spectral mappings can serve as a starting point for the analysis of more general notions of differentiability, such as Bouligand or Clarke subdifferentials. A first result in this direction has been obtained in [MS05], related to the projection operator on the cone of positive semidefinite matrices.

This chapter is organized as follows. Sections 5.2 to 5.6 cover the computation of the Jacobian of a spectral mapping, while its application to symmetric complementarity problems is presented in Section 5.7.

Our differentiability analysis is inspired by the work of Lewis and Sendov [LS02], who computed the Hessian of spectral functions on symmetric matrices. However, we need to solve some extra technical difficulties due to the more general context we deal with. In Section 5.2 and 5.3, we introduce the notation and the objects we deal with in the computation of the spectral mapping. Section 5.4 is devoted to a careful differentiability analysis of Jordan frames. The formula for the Jacobian is derived in 5.5, and the continuity of the Jacobian is studied in Section 5.6. In Section 5.7, we apply this formula to show that the Chen-Mangasarian (Subsection 5.7.1) and the Fischer-Burmeister (Subsection 5.7.2) strategies designed for solving the complementarity problem on symmetric cones are well-defined.

Results similar to our Corollary 5.5.3 were found independently in [SS04]. However, our technique allows us to treat the more general situation of Theorem 5.5.1. Our formula for directional differential of the eigenvalue functions (see Theorem 4.4.8) is an essential ingredient of our proof, and actually solves the first open question stated in the conclusion of [SS04].

5.2 Defining the problem

We start by introducing some notational conventions and some objects that we will keep throughout the whole chapter.

Definition 5.2.1 *A function $g : Q \to \mathbb{R}^r$ is called a* symmetric mapping *if, for every $r \times r$ permutation matrix P and each $\gamma \in Q$, we have $g(P\gamma) = Pg(\gamma)$.* ■

Definition 5.2.2 *Let $Q \subseteq \mathbb{R}^r$ be a symmetric set and $g : Q \to \mathbb{R}^r$ be a symmetric mapping. The* spectral mapping generated by *g is the function G whose domain is $K := \{v \in \mathcal{J} | \lambda(v) \in Q\}$ and such that $G(v) := \sum_{i=1}^r g_i(\lambda(v))c_i$ for every $v \in K$, where $v = \sum_{i=1}^r \lambda_i(v)c_i$ is a complete spectral decomposition of v.* ■

Example 5.2.1 *In view of Remark 4.4.9, the gradient of a symmetric function f is a symmetric mapping. Hence, the gradient of the spectral function generated by f is a spectral mapping.* ■

Example 5.2.2 *Consider a convex symmetric set $Q \subseteq \mathbb{R}^r$. We write $||\cdot||$ for a norm of $\mathbb{R}^r$ that is invariant with respect to permutations of the components of its argument, as, for instance, gauge functions. The projector on Q, defined as:*

$$\begin{aligned} \pi_Q : \mathbb{R}^r &\to Q \\ x &\mapsto \pi_Q(x) := \arg\min_{y\in Q} ||x - y||, \end{aligned}$$

is a symmetric mapping. Indeed, we have for every $r \times r$ permutation matrix P:

$$\pi_Q(Px) = \arg\min_{y\in Q} ||Px - y|| = \arg\min_{y\in Q} ||x - P^T y|| = P\pi_Q(x),$$

since the set Q is symmetric. ■

From Theorem 2.7.13, and from the required symmetry property of g, one can easily deduce that the definition of $G(v)$ does not depend on the particular complete spectral decomposition of v we have taken.

We fix once forever an open symmetric set $Q \subseteq \mathbb{R}^r$ and a symmetric mapping $g : Q \to \mathbb{R}^r$. We build the set $K := \{v \in \mathcal{J} | \lambda(v) \in Q\}$ and we take $G : K \to \mathcal{J}$ for the spectral mapping generated by g. Let us fix an element u in K and one of its complete spectral decomposition $u = \sum_{i=1}^r \lambda_i(u)c_i$. We set the integers $s, k_1, \ldots, k_s$ such that $k_s := r$ and:

$$\lambda_1(u) = \cdots = \lambda_{k_1}(u) > \lambda_{k_1+1}(u) = \cdots = \lambda_{k_2}(u) > \cdots \lambda_{k_s}(u).$$

We let $M_\alpha := \{k_{\alpha-1} + 1, \ldots, k_\alpha\}$ (with $k_0 = 0$) and $e_\alpha := \sum_{i\in M_\alpha} c_i$; the idempotents e_α are uniquely determined, because $u = \sum_{\alpha=1}^r \lambda_{k_\alpha}(u)e_\alpha$ is the unique eigenspaces spectral decomposition of u given by Theorem 2.7.13.

The corresponding Pierce subspaces are written $\mathcal{J}_{\alpha\beta} := Q_{e_\alpha,e_\beta}\mathcal{J}$. Of course, they are not necessarily generated by minimal idempotents.

Suppose that g is continuous on an open neighborhood Λ of $\lambda(u)$ and differentiable at $\lambda(u)$. Proving the continuity of G on $V := \{v \in \mathcal{J} | \lambda(v) \in \Lambda\}$ requires only a straightforward adaptation of Corollary 4.4.11. The main focus of this chapter is to check whether G is differentiable at u, and, if it is the case, to provide a formula for its differential.

The symmetry of g implies that its Jacobian matrix has the very specific block structure we describe below, echoing Lemma 2.1 of [LS02].

Remark 5.2.3 *Let P be a $r \times r$ permutation matrix and let σ_P be the corresponding mapping of $\{1, \ldots, r\}$ to itself. Abbreviating $\lambda(u)$ to λ, we can write for every $h \in \mathbb{R}^r$, in view of the symmetry of g:*

$$\begin{aligned} \nabla g(\lambda) Ph &= \lim_{\epsilon \to 0} \frac{g(\lambda + \epsilon Ph) - g(\lambda)}{\epsilon} = \lim_{\epsilon \to 0} \frac{g(PP^T\lambda + \epsilon Ph) - g(PP^T\lambda)}{\epsilon} \\ &= P \lim_{\epsilon \to 0} \frac{g(P^T\lambda + \epsilon h) - g(P^T\lambda)}{\epsilon} = P\nabla g(P^T\lambda)h. \end{aligned}$$

When $P^T\lambda = \lambda$, we have for every indices i, j:

$$g'_{ij}(\lambda) = g'_{\sigma_P(i),\sigma_P(j)}(\lambda).$$

Suppose that $i = j$; the previous relation implies that $g'_{ii}(\lambda) = g'_{\sigma_P(i),\sigma_P(i)}(\lambda)$ when i and $\sigma_P(i)$ are in the same set M_α. Now, if $i \neq j$, we get $g'_{ij}(\lambda) = g'_{\sigma_P(i),\sigma_P(j)}(\lambda)$ when i and $\sigma_P(i)$ lie in the same set M_α, and j and $\sigma_P(j)$ also belong to the same set M_β.

Summarized,

$$g'(\lambda) = B(\lambda) + \mathrm{diag}(b(\lambda)),$$

where $B_{ij}(\lambda) = B_{kl}(\lambda)$ and $b_i(\lambda) = b_k(\lambda)$ if i, k are in the same M_α, and j, l are in the same M_β. If $|M_\alpha| = 1$, we agree to set $b_{k_\alpha}(\lambda) := g'_{k_\alpha k_\alpha}(\lambda)$ and $B_{k_\alpha k_\alpha}(\lambda) := 0$. Note that $B(\lambda)$ is not necessarily a symmetric matrix. ■

Our task is to prove the existence of a linear operator $\Delta : \mathcal{J} \to \mathcal{J}$ that satisfies the following differentiability statement.

> For every $\epsilon > 0$ and each sequence $(h_m)_{m \geq 0}$ that converges to zero, there exists an integer m' large enough to satisfy:
>
> $$\text{for every } m \geq m', \quad ||G(u + h_m) - G(u) - \Delta(h_m)|| \leq \epsilon ||h_m||. \tag{5.1}$$

If this statement is true, the operator Δ is uniquely defined. It will then be the Jacobian $\nabla G(u)$ of G at u.

Thus, we need to evaluate the ratio $[G(u + h_m) - G(u)]/||h_m||$ when m goes to infinity.

5.3 Fixing a converging sequence

Let us fix a sequence $(h_m)_{m \geq 0} \subset \mathcal{J} \backslash \{0\}$ that converges to zero. We assume beforehand that it satisfies the following three properties.

- ⋄ The point $u + h_m$ belongs to V for all $m \geq 0$.
- ⋄ The limit $\lim_{m\to\infty} h_m/||h_m||$ exists. We denote it by h.
- ⋄ Fixing a spectral decomposition $u + h_m = \sum_{i=1}^r \lambda_i(u + h_m)c_{i,m}$ for each m, the limit $d_i := \lim_{m\to\infty} c_{i,m}$ exists for every i. It is readily seen that $\{d_1, \ldots, d_r\}$ is a Jordan frame. (This observation has already been used in the proof of Theorem 2.7.25)

These three properties are not very restrictive in essence. Indeed, every sequence of $\mathcal{J}$ that converges to zero has a subsequence that fulfils each of them by compactness of a converging sequence.

We denote $e_{\alpha,m} := \sum_{i\in M_\alpha} c_{i,m}$. We have proved in Proposition 4.3.2, that $e_{\alpha,m}$ tends to e_α as m goes to infinity.

In order to understand the link between the idempotents d_i and e_α, we observe that:

$$\sum_{i=1}^r \lambda_i(u)c_i = u = \lim_{m\to\infty} u + h_m = \lim_{m\to\infty} \sum_{i=1}^r \lambda_i(u + h_m)c_{i,m} = \sum_{i=1}^r \lambda_i(u)d_i.$$

By the complete spectral decomposition theorem, we have $\sum_{i\in M_\alpha} d_i = e_\alpha$. Hence $d_i \in \mathcal{J}_1(e_\alpha)$ for all $i \in M_\alpha$.

We summarize below the limiting behavior of the sequences introduced above. For every $1 \leq i \leq r$ and $1 \leq \alpha \leq s$, we have:

$$h_m \to 0, \quad \frac{h_m}{||h_m||} \to h, \quad \lambda(u + h_m) \to \lambda(u), \quad c_{i,m} \to d_i, \quad e_{\alpha,m} \to e_\alpha; \tag{5.2}$$

moreover,

$$d_i \text{ operator commutes with } e_\alpha, \quad \text{and} \quad e_\alpha = \sum_{j\in M_\alpha} c_j = \sum_{j\in M_\alpha} d_j. \tag{5.3}$$

5.4 Limiting behavior of a sequence of Jordan frames

We want to evaluate $[G(u + h_m) - G(u)]/||h_m||$ when m goes to infinity. The existence of the directional derivative of eigenvalue functions established in Theorem 4.4.8 and the differentiability of g in $\lambda(u)$ allows us to reformulate this fraction as:

$$\frac{G(u + h_m) - G(u)}{||h_m||} \tag{5.4}$$

$$= \sum_{i=1}^r \frac{g_i(\lambda(u + h_m))c_{i,m} - g_i(\lambda(u))c_i}{||h_m||}$$

$$= \sum_{i=1}^r \frac{g_i\left(\lambda(u) + \nabla_u^{h_m}\lambda(u) + o(||h_m||)\right)c_{i,m} - g_i(\lambda(u))c_i}{||h_m||}$$

$$= \sum_{i=1}^r g_i(\lambda(u))\frac{c_{i,m} - c_i}{||h_m||} + \sum_{i=1}^r\sum_{j=1}^r g'_{ij}(\lambda(u))\frac{\nabla_u^{h_m}\lambda_j(u)c_{i,m}}{||h_m||} + o(1), \tag{5.5}$$

where, in accordance with the classical convention, $o(\cdot)$ denotes an asymptotically negligible quantity with respect to its argument.

It is not *a priori* obvious that the right-hand side converges to a quantity independent of the specific Jordan frame $\{c_1, \ldots, c_r\}$ we have taken to describe u. Therefore, we need to check carefully that this is indeed the case. Also, this limit should not change if we take another converging sequence $(h'_m)_{m\geq 0}$ for which the ratio $h'_m/||h'_m||$ tends to h as well.

To avoid a tedious notation overfilled with indecipherable indices of exponents, we write $\lambda'_i(u, h_m) := \nabla_u^{h_m}\lambda_i(u)$. By Theorem 4.4.8, we know that:

$$\lambda'_i(u, h_m) = \lambda_{l_i(u)}(Q_{\mathsf{f}'_i(u)}h_m; \mathcal{J}_1(\mathsf{f}'_i(u))),$$

where we have used the notational conventions of Subsection 4.4.2.

We divide both sides by $||h_m||$ and let m goes to infinity. Since the eigenvalues are continuous, this limit exists and is equal to $\lambda_{l_i(u)}(Q_{\mathsf{f}'_i(u)}h; \mathcal{J}_1(\mathsf{f}'_i(u))) = \lambda'_i(u, h)$.

The following observation indicates that $\lambda'_i(u, h)$ is invariant with respect to uniform shifts of u.

Remark 5.4.1 *For every direction $\bar{h} \in \mathcal{J}$ and every real number t, we have $\lambda'_i(u+te, \bar{h}) = \lambda'_i(u, \bar{h})$. Indeed, $\lambda(u + te) = \lambda(u) + t\mathbf{1}$; hence $l_i(u + te) = l_i(u)$ and $\mathsf{f}'_i(u + te) = \mathsf{f}'_i(u)$, implying that:*

$$\begin{aligned}\lambda'_i(u + te, \bar{h}) &= \lambda_{l_i(u+te)}(Q_{\mathsf{f}'_i(u+te)}\bar{h}; \mathcal{J}_1(\mathsf{f}'_i(u + te))) \\ &= \lambda_{l_i(u)}(Q_{\mathsf{f}'_i(u)}\bar{h}; \mathcal{J}_1(\mathsf{f}'_i(u))) = \lambda'_i(u, \bar{h}).\end{aligned}$$

■

We finally set

$$z_m(u) := \sum_{i=1}^{r} \lambda'_i(u, h_m)c_{i,m}, \quad \text{and} \quad z(u) := \sum_{i=1}^{r} \lambda'_i(u, h)d_i,$$

which is of course the limit of $z_m(u)/||h_m||$ when m goes to infinity. The element $z_m(u)$ is linear in h_m and operator commutes with $u + h_m$; it can be interpreted as the coefficient of degree 1 in $||h_m||$ in an expansion of $u + h_m$ on the Jordan frame $\{c_{1,m}, \ldots, c_{r,m}\}$.

The differential ratio (5.5) takes now the following form:

$$\begin{aligned}&\frac{G(u + h_m) - G(u)}{||h_m||} \\ = &\sum_{\alpha=1}^{s} g_{k_\alpha}(\lambda(u))\frac{e_{\alpha,m} - e_\alpha}{||h_m||} + \sum_{i=1}^{r}\sum_{j=1}^{r} g'_{ij}(\lambda(u))\frac{\lambda'_j(u, h_m)c_{i,m}}{||h_m||} + o(1) \\ \to &\lim_{m\to\infty}\sum_{\alpha=1}^{s} g_{k_\alpha}(\lambda(u))\frac{e_{\alpha,m} - e_\alpha}{||h_m||} + \sum_{i=1}^{r}\sum_{j=1}^{r} g'_{ij}(\lambda(u))\lambda'_j(u, h)d_i \end{aligned} \tag{5.6}$$

The recalcitrant part of this expression lies in its first term. In order to handle with it, we need to understand the first order behavior of a converging sequence of idempotents. In the following lemmas, we carry out an asymptotic analysis of ratios of the type $c_{i,m}/||h_m||$. We found it simpler to perform this analysis separately over each Pierce subspaces $\mathcal{J}_{\alpha\beta} = Q_{e_\alpha,e_\beta}\mathcal{J}$.

As the following technical lemma suggests, the elements $z_m(u)$ and $z(u)$ are closely linked to the ratios we want to describe.

Lemma 5.4.2 *We have:*

1. $h - z(u) = \lim_{m\to\infty} \sum_{\alpha=1}^{s} \lambda_{k_\alpha}(u) \frac{e_{\alpha,m} - e_\alpha}{||h_m||}$.

2. $\mathrm{tr}[Q_{e_\alpha}(h - z(u))] = 0$ *for every* $1 \leq \alpha \leq s$.

Proof

1. In view of the directional differentiability of eigenvalues, we can write:

$$\begin{aligned} u + h_m &= \sum_{i=1}^{r} \lambda_i(u + h_m)c_{i,m} = \sum_{i=1}^{r} \lambda_i(u)c_{i,m} + \sum_{i=1}^{r} \lambda_i'(u, h_m)c_{i,m} + o(h_m) \\ &= \sum_{i=1}^{r} \lambda_i(u)c_{i,m} + z_m(u) + o(h_m). \end{aligned}$$

Thus,

$$\lim_{m\to\infty} \frac{h_m - z_m(u)}{||h_m||} = \lim_{m\to\infty} \frac{\sum_{i=1}^{r} \lambda_i(u)c_{i,m} - u}{||h_m||} = \lim_{m\to\infty} \sum_{i=1}^{r} \lambda_i(u) \frac{c_{i,m} - c_i}{||h_m||}.$$

The first term equals $h - z(u)$.

2. Notice that:

$$\begin{aligned} \mathrm{tr}\,(Q_{e_\alpha} z(u)) &= \sum_{\beta=1}^{s} \sum_{i\in M_\beta} \lambda_i'(u, h)\mathrm{tr}(Q_{e_\alpha} d_i) = \sum_{i\in M_\alpha} \lambda_i'(u, h)\mathrm{tr}(Q_{e_\alpha} d_i) \\ &= \sum_{i\in M_\alpha} \lambda_{l_i(u)}(Q_{\mathsf{f}_i'(u)} h; \mathcal{J}_1(\mathsf{f}'(u))) \end{aligned}$$

as $Q_{e_\alpha} d_i = d_i$ when $i \in M_\alpha$ and $Q_{e_\alpha} d_i = 0$ otherwise (remember the relations (5.3)). In this summation, the numbers $l_i(u)$ go from 1 to $|M_\alpha|$. Moreover, $\mathsf{f}_i'(u) = e_\alpha$ for every $i \in M_\alpha$. Hence all the eigenvalues in the right-hand side stem from the same subalgebra $\mathcal{J}_1(e_\alpha)$, and:

$$\mathrm{tr}\,(Q_{e_\alpha} z(u)) = \sum_{i=1}^{|M_\alpha|} \lambda_i(Q_{e_\alpha} h; \mathcal{J}_1(e_\alpha)) = \mathrm{tr}(Q_{e_\alpha} h), \tag{5.7}$$

because the eigenvalues of $Q_{e_\alpha} h$ in $\mathcal{J}_1(e_\alpha)$ are identical to those of $Q_{e_\alpha} h$ in $\mathcal{J}$, except for the multiplicity of the eigenvalue 0. ■

The following observation continues the analysis of projections of $z(u)$ on Pierce subspaces $\mathcal{J}_{\alpha\beta}$ that we have started in the second item of the previous lemma.

Remark 5.4.3 *Let $1 \leq \alpha \neq \beta \leq s$ be two integers. We have:*

$$Q_{e_\alpha,e_\beta} z(u) = \sum_{i=1}^{r} \lambda_i'(u,h) Q_{e_\alpha,e_\beta} d_i = 0,$$

since d_i operator commutes with every idempotent e_j, in view of (5.3). ■

Lemma 5.4.4 *For every $1 \leq \alpha \leq s$, we have:*

$$\lim_{m\to\infty} Q_{c_j,c_{j'}} \frac{c_{i,m}}{||h_m||} = 0 \quad \textit{when } j,j' \in M_\alpha \textit{ and } i \notin M_\alpha, \tag{5.8}$$

$$\lim_{m\to\infty} Q_{c_{j,m},c_{j',m}} \frac{c_i}{||h_m||} = 0 \quad \textit{when } j,j' \in M_\alpha \textit{ and } i \notin M_\alpha \tag{5.9}$$

and

$$\lim_{m\to\infty} \frac{Q_{e_\alpha} e_{\alpha,m} - e_\alpha}{||h_m||} = 0. \tag{5.10}$$

Proof
For simplicity, we set $\lambda_{k_{s+1}}$ to $-\infty$.

We start with the following simple observation: for every $1 \leq i \leq r$, every $1 \leq \alpha \leq s$ and each subset I of M_α, we have:

$$\mathrm{tr}(Q_{e_\alpha} c_{i,m}) = \mathrm{tr}(Q_c c_{i,m}) + \mathrm{tr}(Q_{e_\alpha - c} c_{i,m}) \quad \text{for} \quad c = \sum_{j\in I} c_j.$$

(This is an immediate application of item 7 of Theorem 2.6.1).

Now, we prove (5.8) and (5.9) for $\alpha := 1$. Let $I \subseteq M_1$ and $c = \sum_{j\in I} c_j$. For every real number t, we can write:

$$\begin{aligned} 0 &= \mathrm{tr}[Q_{e_1}(h - z(u))] = \mathrm{tr}[Q_{e_1}(h - z(u+te))] \\ &= (\lambda_1(u) + t) \lim_{m\to\infty} \frac{\mathrm{tr}(Q_{e_1} e_{1,m}) - |M_1|}{||h_m||} \\ &\quad + \sum_{j=2}^{s} \sum_{i\in M_j} (\lambda_{k_j}(u) + t) \lim_{m\to\infty} \left(\frac{\mathrm{tr}(Q_c c_{i,m})}{||h_m||} + \frac{\mathrm{tr}(Q_{e_1 - c} c_{i,m})}{||h_m||} \right). \end{aligned} \tag{5.11}$$

The first equality comes from Lemma 5.4.2, item 2. The second one is justified by Remark 5.4.1. The third one relies on Lemma 5.4.2, item 1, while our preliminary observation justifies the decomposition in the parentheses. Now, note that:

$$|M_1| = \mathrm{tr}(Q_{e_1} e) = \mathrm{tr}(Q_{e_1} e_{1,m}) + \mathrm{tr}(Q_{e_1}(e - e_{1,m})) \geq \mathrm{tr}(Q_{e_1} e_{1,m})$$

and that $\mathrm{tr}(Q_c c_{i,m})$, $\mathrm{tr}(Q_{e-c} c_{i,m})$ are nonnegative. Taking $t := -(\lambda_1(u) + \lambda_{k_2}(u))/2$, all the terms in (5.11) are negative or null. Consequently, they are all null in the limit. When

$i \notin M_1$, the ratio $\mathrm{tr}(Q_c c_{i,m})/||h_m||$ tends to 0 as m goes to infinity. Proposition 2.8.10 implies that $Q_c c_{i,m}/||h_m||$ also goes to 0 at the limit. The relation (5.8) for $\alpha := 1$ follows trivially for $I := \{j, j'\}$, $I := \{j\}$ and $I := \{j'\}$.

Now, since

$$\lim_{m\to\infty} \mathrm{tr}(Q_{e_{1,m}}(h - z(u + te))) = 0$$

and since $\mathrm{tr}(Q_c c') = \mathrm{tr}(Q_{c'} c)$ for every pair of idempotents c and c', one can prove the case $\alpha := 1$ of relations (5.9) by the same argument.

Suppose now that (5.8) and (5.9) are proved for $\alpha \in \{1, \dots, l-1\}$ and let us show these relations for $\alpha := l$. Let $I \subseteq M_l$ and $c = \sum_{j\in I} c_j$. We have now:

$$\begin{aligned} 0 &= \mathrm{tr}(Q_{e_l}(h - z(u + te))) \\ &= \sum_{j=1}^{l-1} (\lambda_{k_j}(u) + t) \lim_{m\to\infty} \frac{\mathrm{tr}(Q_{e_l} e_{j,m})}{||h_m||} + (\lambda_{k_l}(u) + t) \lim_{m\to\infty} \frac{\mathrm{tr}(Q_{e_l} e_{l,m}) - |M_l|}{||h_m||} \qquad (5.12) \\ &\quad + \sum_{j=l+1}^{s} \sum_{i\in M_j} (\lambda_{k_j}(u) + t) \lim_{m\to\infty} \left(\frac{\mathrm{tr}(Q_c c_{i,m})}{||h_m||} + \frac{\mathrm{tr}(Q_{e_l - c} c_{i,m})}{||h_m||} \right). \end{aligned}$$

By the recurrence hypothesis and by the relation $\mathrm{tr}(Q_{e_l} e_{j,m}) = \mathrm{tr}(Q_{e_{j,m}} e_l)$, the first sum is equal to zero. Now, if we take $-\lambda_{k_l} < t < -\lambda_{k_{l+1}}$, all the remaining terms in (5.12) are negative or null. Applying the same argument as before, we deduce from it the relations (5.8) for $\alpha = l$. The corresponding equations (5.9) are proved by a similar argument.

It remains to show (5.10). For this, just observe that:

$$0 = \lim_{m\to\infty} \frac{Q_{e_\alpha} e - e_\alpha}{||h_m||} = \sum_{l\neq\alpha} \lim_{m\to\infty} \frac{Q_{e_\alpha} e_{l,m}}{||h_m||} + \lim_{m\to\infty} \frac{Q_{e_\alpha} e_{\alpha,m} - e_\alpha}{||h_m||}.$$

In view of (5.8), the first limit is equal to 0. The remaining term is thus equal to zero, and everything is shown. ■

If we combine this lemma with item 2 of Lemma 5.4.2, we obtain:

$$Q_{e_\alpha}(h - z(u)) = \lim_{m\to\infty} \lambda_{k_\alpha}(u) \frac{Q_{e_\alpha} e_{\alpha,m} - e_\alpha}{||h_m||} + \lim_{m\to\infty} \sum_{\beta\neq\alpha} \lambda_{k_\beta}(u) \frac{Q_{e_\alpha} e_{\beta,m}}{||h_m||} = 0 \qquad (5.13)$$

for all $1 \leq \alpha \leq s$. This relation, together with Remark 5.4.3, proves that $z(u)$ does not depend on the particular sequence $(h_m)_{m\geq 0}$ we have chosen, but only on h. In fact, we can now easily formulate $z(u)$ as a function of h:

$$z(u) = \sum_{\alpha=1}^{s} Q_{e_\alpha} z(u) + \sum_{\beta\neq\alpha} Q_{e_\alpha, e_\beta} z(u) = \sum_{\alpha=1}^{s} Q_{e_\alpha} z(u) = \sum_{\alpha=1}^{s} Q_{e_\alpha} h.$$

As expected, $z(u)$ varies linearly with respect to h.

We can further exploit the previous lemma to compute other projections of the limit $\lim_{m\to\infty} c_{i,m}/||h_m||$.

Lemma 5.4.5 *Suppose that $i \in M_\alpha$, $j \in M_\beta$ and $k \in M_\gamma$, and that M_α, M_β and M_γ are three* different *sets. Then*

$$\lim_{m\to\infty} \frac{Q_{c_i,c_j} c_{k,m}}{||h_m||} = 0. \tag{5.14}$$

Moreover,

$$\lim_{m\to\infty} \frac{Q_{c_i,c_j}\,(e_{\alpha,m} + e_{\beta,m})}{||h_m||} = 0. \tag{5.15}$$

Proof
We know by the previous lemma that:

$$\lim_{m\to\infty} \frac{Q_{c_i} c_{k,m}}{||h_m||} = 0 \quad \text{and} \quad \lim_{m\to\infty} \frac{Q_{c_j} c_{k,m}}{||h_m||} = 0.$$

Hence

$$\lim_{m\to\infty} \frac{\operatorname{tr}(Q_{c_i} c_{k,m})}{||h_m||} = 0 \quad \text{and} \quad \lim_{m\to\infty} \frac{\operatorname{tr}(Q_{c_j} c_{k,m})}{||h_m||} = 0.$$

Item 7 of Theorem 2.6.1 allows us to add these two equalities as follows:

$$\lim_{m\to\infty} \frac{\operatorname{tr}(Q_{c_i+c_j} c_{k,m})}{||h_m||} = 0,$$

and, by Proposition 2.8.10, we have:

$$0 = \lim_{m\to\infty} \frac{Q_{c_i+c_j} c_{k,m}}{||h_m||} = \lim_{m\to\infty} \frac{Q_{c_i} c_{k,m}}{||h_m||} + \lim_{m\to\infty} \frac{Q_{c_j} c_{k,m}}{||h_m||} + 2 \lim_{m\to\infty} \frac{Q_{c_i,c_j} c_{k,m}}{||h_m||}.$$

It remains to note that the two first terms are null to show (5.14).

The computation of limit (5.15) results as a straightforward consequence of (5.14). To see this, it suffices to write:

$$0 = \lim_{m\to\infty} \frac{Q_{c_i,c_j} e}{||h_m||} = \lim_{m\to\infty} \frac{Q_{c_i,c_j}\,(e_{\alpha,m} + e_{\beta,m})}{||h_m||} + \sum_{l \notin M_\alpha, M_\beta} \lim_{m\to\infty} \frac{Q_{c_i,c_j} c_{l,m}}{||h_m||},$$

and to notice that the second term is null. ■

The next lemma covers the only situation we have not yet considered in the two previous demonstrations.

Lemma 5.4.6 *Let $i \in M_\alpha$ and $j \in M_\beta$, with $\alpha \neq \beta$. Then:*

$$\lim_{m\to\infty} \left[\lambda_{k_\alpha}(u) \frac{Q_{c_i,c_j} e_{\alpha,m}}{||h_m||} + \lambda_{k_\beta}(u) \frac{Q_{c_i,c_j} e_{\beta,m}}{||h_m||} \right] = Q_{c_i,c_j} h. \tag{5.16}$$

For every $A, B \in \mathbb{R}$, we have:

$$\lim_{m\to\infty} \left[A \frac{Q_{c_i,c_j} e_{\alpha,m}}{||h_m||} + B \frac{Q_{c_i,c_j} e_{\beta,m}}{||h_m||} \right] = \frac{A - B}{\lambda_{k_\alpha}(u) - \lambda_{k_\beta}(u)} Q_{c_i,c_j} h. \tag{5.17}$$

Proof
By Lemma 5.4.2, we have:

$$Q_{c_i,c_j}(h - z(u)) = \lim_{m\to\infty}\left[\lambda_{k_\alpha}(u)\frac{Q_{c_i,c_j}e_{\alpha,m}}{||h_m||} + \lambda_{k_\beta}(u)\frac{Q_{c_i,c_j}e_{\beta,m}}{||h_m||}\right] + \sum_{\gamma\neq\alpha,\beta}\lambda_{k_\gamma}(u)\lim_{m\to\infty}\frac{Q_{c_i,c_j}e_{\gamma,m}}{||h_m||}.$$

In view of Lemma 5.4.5, the second limit equals zero. Remark 5.4.1 shows that $Q_{c_i,c_j}z(u) = 0$ and the limit (5.16) is proved.

For the sake of simplicity, we set:

$$x := \lambda_{k_\alpha}(u), \quad y := \lambda_{k_\beta}(u),$$

$$a_m := Q_{c_i,c_j}e_{\alpha,m}/||h_m|| \quad \text{and} \quad b_m := Q_{c_i,c_j}e_{\beta,m}/||h_m||.$$

We know that $a_m + b_m$ goes to 0 when m tends to infinity in view of equation (5.15). Elementary manipulations yield:

$$(Bx - Ay)(a_m + b_m) = (x - y)(Aa_m + Bb_m) + (B - A)(xa_m + yb_m).$$

Using (5.16), we note that the last term tends to $(B - A)Q_{c_i,c_j}h$ as m goes to ∞. We are thus left with:

$$\lim_{m\to\infty} Aa_m + Bb_m = \frac{A - B}{x - y}Q_{c_i,c_j}h.$$

■

At this point, we have all the necessary instruments for computing the Jacobian matrix of a spectral function.

5.5 Jacobian of spectral mapping

We specify now the operator Δ, our candidate for the Jacobian of G in u. As mentioned earlier, it is convenient to describe its behavior on each Pierce subspace $\mathcal{J}_{\alpha\beta} := Q_{e_\alpha,e_\beta}\mathcal{J}$ separately.

◇ For all $1 \leq \alpha \leq s$, we set:

$$Q_{e_\alpha}\Delta(h) := b_{k_\alpha}(\lambda(u))Q_{e_\alpha}h + \sum_{\beta=1}^{s} B_{k_\alpha k_\beta}(\lambda(u))\mathrm{tr}(Q_{e_\beta}h)e_\alpha,$$

where the functions $B_{k_\alpha k_\beta}$, b_{k_α} were defined in Remark 5.2.3. They are constructed from the coefficients of $g'(\lambda(u))$.

◇ For all $\alpha, \beta \in \{1, \ldots, s\}$ with $\alpha \neq \beta$, we set:

$$Q_{e_\alpha,e_\beta}\Delta(h) := \frac{g_{k_\alpha}(\lambda(u)) - g_{k_\beta}(\lambda(u))}{\lambda_{k_\alpha}(u) - \lambda_{k_\beta}(u)}Q_{e_\alpha,e_\beta}h.$$

Of course, this description is free from any ambiguity that could be caused by the particular choice of the Jordan frame $\{c_1, \ldots, c_r\}$ in the spectral decomposition of u. Note also that $\Delta(h)$ is linear in h.

The following theorem constitutes the main result of this section. Its proof loosely follows Lewis and Sendov's demonstration of the corresponding statement for Hessians of spectral functions in the framework of symmetric matrices (see Theorem 3.2 in [LS02]).

Theorem 5.5.1 *The spectral mapping G is differentiable at u, and $\nabla G(u)h = \Delta(h)$.*

Proof
Let $(v_m)_{m\geq 0} \subseteq \mathcal{J}$ be a sequence converging to 0. Suppose that, contrary to the statement, the fraction

$$\frac{G(u+v_m) - G(u) - \Delta(v_m)}{||v_m||} \tag{5.18}$$

does not tend to 0 as m goes to ∞. In other words, suppose that there exists a real number $\epsilon > 0$ that is strictly smaller than the Jordan norm of each of the ratios (5.18). There exists a subsequence $(h_m)_{m\geq 0}$ of our sequence $(v_m)_{m\geq 0}$ that satisfies the three hypotheses we stated on p. 160. Let

$$\Omega_m := \frac{G(u+h_m) - G(u) - \Delta(h_m)}{||h_m||};$$

we check below that Ω_m tends to zero as m goes to ∞, yielding a contradiction.

In view of the development (5.5), we expand this differential ratio as follows:

$$\Omega_m = \sum_{\alpha=1}^{s} g_{k_\alpha}(\lambda(u)) \frac{e_{\alpha,m} - e_\alpha}{||h_m||} + \sum_{i,j=1}^{r} g'_{ij}(\lambda(u)) \frac{\lambda'_j(u, h_m)c_{i,m}}{||h_m||} - \Delta\left(\frac{h_m}{||h_m||}\right) + o(1).$$

Part 1: the subspaces $\mathcal{J}_1(e_\alpha)$

We fix an integer $1 \leq \alpha \leq s$. The projection of Ω_m on $\mathcal{J}_1(e_\alpha)$ is:

$$\begin{aligned}
Q_{e_\alpha}\Omega_m \;=\; & g_{k_\alpha}(\lambda(u)) \frac{Q_{e_\alpha} e_{\alpha,m} - e_\alpha}{||h_m||} + \sum_{\beta\neq\alpha} g_{k_\beta}(\lambda(u)) \frac{Q_{e_\alpha} e_{\beta,m}}{||h_m||} \\
& + \sum_{i\in M_\alpha} \sum_{j=1}^{r} g'_{ij}(\lambda(u)) \frac{\lambda'_j(u, h_m) Q_{e_\alpha} c_{i,m}}{||h_m||} \\
& + \sum_{i\notin M_\alpha} \sum_{j=1}^{r} g'_{ij}(\lambda(u)) \frac{\lambda'_j(u, h_m) Q_{e_\alpha} c_{i,m}}{||h_m||} \\
& - Q_{e_\alpha}\Delta\left(\frac{h_m}{||h_m||}\right) + o(1).
\end{aligned}$$

The two first terms of the right-hand side, as well as the fourth term, go to 0 as m goes to infinity by (5.10) and (5.8). Exploiting the structure of $g'(\lambda(u))$ given in Remark 5.2.3, we derive:

$$\begin{aligned}\lim_{m\to\infty} Q_{e_\alpha}\Omega_m &= \sum_{j=1}^{r} B_{k_\alpha j}(\lambda(u)) \lim_{m\to\infty} \frac{\lambda_j'(u,h_m)Q_{e_\alpha}e_{\alpha,m}}{||h_m||} \\ &\quad + \sum_{i\in M_\alpha} b_i(\lambda(u)) \lim_{m\to\infty} \frac{\lambda_i'(u,h_m)Q_{e_\alpha}c_{i,m}}{||h_m||} - Q_{e_\alpha}\Delta(h)\end{aligned}$$

The first limit of the right-hand side goes to $\lambda_j'(u,h)e_\alpha$ in view of Proposition 4.3.2 and of the fact that $\lambda_j'(u,h_m)/||h_m||$ tends to $\lambda_j'(u,h)$ as m goes to infinity. The second limit is simply:

$$\lim_{m\to\infty} Q_{e_\alpha}Q_{c_{i,m}}\frac{z_m(u)}{||h_m||} = Q_{e_\alpha}Q_{d_i}z(u) = Q_{d_i}Q_{e_\alpha}z(u).$$

Thus, we get:

$$\begin{aligned}\lim_{m\to\infty} Q_{e_\alpha}\Omega_m &= \sum_{j=1}^{r} B_{k_\alpha j}(\lambda(u))\lambda_j'(u,h)e_\alpha + \sum_{i\in M_\alpha} b_i(\lambda(u))Q_{d_i}Q_{e_\alpha}z(u) - Q_{e_\alpha}\Delta(h) \\ &= \sum_{j=1}^{r} B_{k_\alpha j}(\lambda(u))\lambda_j'(u,h)e_\alpha + b_{k_\alpha}(\lambda(u))Q_{e_\alpha}h - Q_{e_\alpha}\Delta(h).\end{aligned}$$

The last equality holds because the considered components of $b_i(\lambda(u))$ are identical and in view of the identity (5.13).

Finally, the first term can successively be reformulated as:

$$\begin{aligned}\sum_{j=1}^{r} B_{k_\alpha j}(\lambda(u))\lambda_j'(u,h)e_\alpha &= \sum_{\beta=1}^{s} B_{k_\alpha k_\beta}(\lambda(u)) \sum_{i\in M_\beta} \lambda_i'(u,h)e_\alpha \\ &= \sum_{\beta=1}^{s} B_{k_\alpha k_\beta}(\lambda(u))\mathrm{tr}(Q_{e_\beta}h)e_\alpha,\end{aligned}$$

where the last inequality results from the same argument as in (5.7), based on the links between eigenvalues of an element $v \in \mathcal{J}_1(c)$ on $\mathcal{J}$ and on $\mathcal{J}_1(c)$. The limit of $Q_{e_\alpha}\Omega_m$ is then exactly equal to 0 by definition of Δ.

Part 2: the subspaces $Q_{e_\alpha,e_\beta}\mathcal{J}$

We focus now on the projections of Ω_m on the other Pierce subspaces. Let $1 \leq \alpha \neq \beta \leq s$. We have now:

$$\begin{aligned} Q_{e_\alpha,e_\beta}\Omega_m &= g_{k_\alpha}(\lambda(u))\frac{Q_{e_\alpha,e_\beta}e_{\alpha,m}}{||h_m||} + g_{k_\beta}(\lambda(u))\frac{Q_{e_\alpha,e_\beta}e_{\beta,m}}{||h_m||} \\ &+ \sum_{\gamma\neq\alpha,\beta} g_{k_\gamma}(\lambda(u)) \sum_{i\in M_\gamma} \frac{Q_{e_\alpha,e_\beta}c_{i,m}}{||h_m||} \\ &+ \sum_{i,j=1}^{r} g'_{ij}(\lambda(u))\frac{\lambda'_j(u,h_m)}{||h_m||}Q_{e_\alpha,e_\beta}c_{i,m} \\ &- Q_{e_\alpha,e_\beta}\Delta\left(\frac{h_m}{||h_m||}\right) + o(1). \end{aligned}$$

According to (5.14), the third term goes to 0 as m tends to infinity. Also, in view of the operator commutativity of d_i with every e_γ, we have:

$$\lim_{m\to\infty} \frac{\lambda'_j(u,h_m)}{||h_m||}Q_{e_\alpha,e_\beta}c_{i,m} = \lambda'_j(u,h)Q_{e_\alpha,e_\beta}d_i = 0,$$

and the fourth terms also tends to zero. We are left with:

$$\begin{aligned} \lim_{m\to\infty} Q_{e_\alpha,e_\beta}\Omega_m &= \lim_{m\to\infty}\left[g_{k_\alpha}(\lambda(u))\frac{Q_{e_\alpha,e_\beta}e_{\alpha,m}}{||h_m||} + g_{k_\beta}(\lambda(u))\frac{Q_{e_\alpha,e_\beta}e_{\beta,m}}{||h_m||}\right] - Q_{e_\alpha,e_\beta}\Delta(h) \\ &= \frac{g_{k_\alpha}(\lambda(u)) - g_{k_\beta}(\lambda(u))}{\lambda_{k_\alpha} - \lambda_{k_\beta}} Q_{e_\alpha,e_\beta}h - Q_{e_\alpha,e_\beta}\Delta(h) = 0. \end{aligned}$$

For the second equality, we have used (5.17) for $A := g_{k_\alpha}(\lambda(u))$ and $B := g_{k_\beta}(\lambda(u))$. ■

In this formula, the nonsymmetric part lies in the second term and exclusively originates in the (possible) nonsymmetry of $\nabla g(u)$; that is, the way we have built G from g does not change the symmetry of the Jacobian matrix. The third term of the above expression points out the Jordan frame differentiation.

Remark 5.5.2 *As an immediate consequence of the formula (5.19), we can observe that if the matrix B is null, if the vector b is positive, and if the numbers*

$$\frac{g_{k_\alpha}(\lambda(u)) - g_{k_\beta}(\lambda(u))}{\lambda_{k_\alpha}(u) - \lambda_{k_\beta}(u)}$$

are all positive, then the operator $\nabla G(u)$ is positive definite. ■

Theorem 5.5.1 allows us to obtain very easily a formula for the Hessian of a spectral function. This formula has been found independently, using a different technique, in the preprint [SS04]. As mentioned earlier, Lewis and Sendov proved it in the framework of symmetric matrices [LS02].

Corollary 5.5.3 *Let f be a spectral function that is twice differentiable at a point λ of its domain and continuously differentiable on an open neighborhood Λ of λ. The spectral function F generated by f is twice differentiable at each point u whose eigenvalues equals λ.*

Jacobian of the spectral mapping G

Let $u = \sum_{i=1}^{r} \lambda_i(u)c_i = \sum_{\alpha=1}^{s} \lambda_{k_\alpha}(u)e_\alpha \in \operatorname{dom} G$. If $g'(\lambda(u)) = \operatorname{diag}(b) + B$, we have for every $h \in \mathcal{J}$:

- $\diamond$ $Q_{e_\alpha} \nabla G(u)h = b_{k_\alpha} Q_{e_\alpha} h + \sum_{\beta=1}^{s} B_{k_\alpha k_\beta} \operatorname{tr}(Q_{e_\beta} h) e_\alpha$.
- $\diamond$ $Q_{e_\alpha, e_\beta} \nabla G(u)h = \dfrac{g_{k_\alpha}(\lambda(u)) - g_{k_\beta}(\lambda(u))}{\lambda_{k_\alpha}(u) - \lambda_{k_\beta}(u)} Q_{e_\alpha, e_\beta} h$.

Let $v, w \in \mathcal{J}$ and $v_{\alpha\beta} := Q_{e_\alpha, e_\beta} v$, $w_{\alpha\beta} := Q_{e_\alpha, e_\beta} w$. Then:

$$\begin{aligned}\langle \nabla G(u)v, w \rangle &= \sum_{\alpha=1}^{s} b_{k_\alpha} \operatorname{tr}(v_{\alpha\alpha} w_{\alpha\alpha}) + \sum_{\alpha,\beta=1}^{s} B_{k_\alpha k_\beta} \operatorname{tr}(v_{\beta\beta}) \operatorname{tr}(w_{\alpha\alpha}) \\ &\quad + 2 \sum_{\alpha \neq \beta} \frac{g_{k_\alpha}(\lambda(u)) - g_{k_\beta}(\lambda(u))}{\lambda_{k_\alpha}(u) - \lambda_{k_\beta}(u)} \operatorname{tr}(v_{\alpha\beta} w_{\alpha\beta}). \qquad (5.19)\end{aligned}$$

Proof
Let $V := \{v \in \mathcal{J} | \lambda(v) \in \Lambda\}$. We know from Theorem 4.4.10 that $G := F'$ is continuous on V. Observe that $g := f'$ is a symmetric mapping in view of Remark 4.4.9 and that G is the spectral mapping it generates. It suffices now to apply Theorem 5.5.1 to obtain the final result. In the framed formula above, we only have to replace g by f' and to set $b(\lambda(u))$ and $B(\lambda(u))$ so that $f''(\lambda(u)) = \operatorname{diag}(b(\lambda(u))) + B(\lambda(u))$ following the same rules as in Remark 5.2.3. Observe that the matrix $B(\lambda(u))$ is symmetric in this case. ■

5.6 Continuous differentiability of spectral mappings

Using our formula for the Jacobian matrix of a spectral mapping, we verify here that, if g is continuously differentiable, then G is also continuously differentiable. The structure of our proof essentially follows [LS02], Lemma 4.1 and Theorem 4.2. As some adaptations to the Jordan algebraic framework are necessary at several places, we include here its proof.

With respect to the previous sections, we add the extra hypothesis that the symmetric mapping g is continuously differentiable on the set Λ. The following theorem shows that G is continuously differentiable on the set $V := \{v \in \mathcal{J} | \lambda(v) \in \Lambda\}$.

Theorem 5.6.1 *Suppose that $(u_m)_{m \geq 0} \subseteq V$ is a sequence converging to u. Then*

$$\lim_{m \to \infty} \nabla G(u_m)h = \nabla G(u)h;$$

that is, G is continuously differentiable at u.
Proof
Note that G is differentiable on V by the theorem on the Jacobian of a spectral mapping.

On a first step, we prove the theorem for sequences $(u_m)_{m\geq 0}$ that satisfy the two following conditions.

- ◇ Each element u_m is regular, that is, all the eigenvalues of u_m are distinct.
- ◇ Let $u_m = \sum_{i=1}^r \lambda_i(u_m)c_{i,m}$ be the spectral decomposition of u_m. In view of the first hypothesis on u_m, both Theorem 2.7.13 and Theorem 2.7.25 entail the same decomposition. *We assume that* $\lim_{m\to\infty} c_{i,m}$ *exists and equals* c_i *for every* i.

Of course, we have by assumption the following limiting behaviors:

$$\begin{aligned} \lambda(u_m) &\to \lambda(u), \\ f''(\lambda(u_m)) = B(\lambda(u_m)) + \mathrm{diag}(b(\lambda(u_m))) &\to f''(\lambda(u)) = B(\lambda(u)) + \mathrm{diag}(b(\lambda(u))). \end{aligned} \tag{5.20}$$

This last limit can subsequently be rewritten as:

$$B_{ij}(\lambda(u_m)) \to B_{ij}(\lambda(u)) \quad \text{ for } i \neq j,$$

$$b_i(\lambda(u_m)) \to b_i(\lambda(u)) \quad \text{ if } i \in M_\alpha \text{ and } |M_\alpha| = 1.$$

We need to check that every projection $Q_{c_i,c_j}\nabla G(u_m)h$ effectively converges to the corresponding projection $Q_{c_i,c_j}\nabla G(u)h$. Equivalently, since the sequence of operators $Q_{c_{i,m},c_{j,m}}$ tends to Q_{c_i,c_j}, we need to verify that:

$$\lim_{m\to\infty} Q_{c_{i,m},c_{j,m}}\nabla G(u_m)h = Q_{c_i,c_j}\nabla G(u)h \quad \text{ for every } i, j.$$

Let us fix two integers $1 \leq \alpha, \beta \leq s$. We also consider two integers $i \in M_\alpha$ and $j \in M_\beta$. We distinguish three cases.

The case $i = j$. Since $Q_{c_i}Q_{e_\alpha} = Q_{c_i}$, we have in view of the Jacobian's formula:

$$\begin{aligned} Q_{c_i}\nabla G(u)h &= b_i(\lambda(u))Q_{c_i}h + \sum_{\gamma=1}^s B_{k_\gamma i}(\lambda(u))\mathrm{tr}(Q_{e_\gamma}h)c_i \\ &= b_i(\lambda(u))Q_{c_i}h + \sum_{l=1}^r B_{li}(\lambda(u))\mathrm{tr}(Q_{c_l}h)c_i. \end{aligned}$$

Suppose that $i \in M_\alpha$. Considering the cases where M_α contains one element or more than one separately, it is now obvious from (5.20) that

$$Q_{c_{i,m}}\nabla G(u_m)h = b_i(\lambda(u_m))Q_{c_{i,m}}h + \sum_{l=1}^r B_{li}(\lambda(u_m))\mathrm{tr}(Q_{c_{lm}}h)c_{i,m}$$

goes to $Q_{c_i}\nabla G(u)h$ as m tends to infinity.

The case $i \neq j$ and $\alpha = \beta$. Now, we have $Q_{c_i,c_j}Q_{e_\alpha} = Q_{c_i,c_j}$ and $Q_{c_i,c_j}e_\alpha = 0$. Hence,

$$Q_{c_i,c_j}\nabla G(u)h = b_{k_\alpha}(\lambda(u))Q_{c_i,c_j}h = [g'_{ii}(\lambda(u)) - g'_{ij}(\lambda(u))]Q_{c_i,c_j}h \tag{5.21}$$

by the definition of b given in Remark 5.2.3. Also, we have:

$$Q_{c_{i,m},c_{j,m}} \nabla G(u_m)h = \frac{g_j(\lambda(u_m)) - g_i(\lambda(u_m))}{\lambda_j(u_m) - \lambda_i(u_m)} Q_{c_{i,m},c_{j,m}} h. \tag{5.22}$$

Now, let P denote the $r \times r$ permutation matrix that exchanges the ith component with the jth, and let $\hat{v}_k$ be the vector of $\mathbb{R}^r$ that has its kth component equal to 1, all the others being null. For all $\mu \in \mathbb{R}^r$, we have by definition of P:

$$P\mu = \mu + (\mu_j - \mu_i)(\hat{v}_i - \hat{v}_j) \quad \text{and} \quad g_j(\mu) = [P^T g(P\mu)]_j = g_i(P\mu).$$

Hence, taking $\alpha_m := \lambda_j(u_m) - \lambda_i(u_m)$, we obtain:

$$\begin{aligned}
&\lim_{m\to\infty} \frac{g_j(\lambda(u_m)) - g_i(\lambda(u_m))}{\lambda_j(u_m) - \lambda_i(u_m)} \\
&= \lim_{m\to\infty} \frac{g_i(\lambda(u_m) + \alpha_m(\hat{v}_i - \hat{v}_j)) - g_i(\lambda(u_m))}{\alpha_m} \\
&= \lim_{m\to\infty} \sum_{k=1}^{r} \nabla g_i(\lambda(u_m))(\hat{v}_i - \hat{v}_j) = g'_{ii}(\lambda(u)) - g'_{ij}(\lambda(u)),
\end{aligned}$$

and (5.22) tends to (5.21).

The case $\alpha \neq \beta$. The component of $\nabla G(u)h$ we deal with here has the following form:

$$Q_{c_i,c_j} \nabla G(u)h = \frac{g_j(\lambda(u)) - g_i(\lambda(u))}{\lambda_j(u) - \lambda_i(u)} Q_{c_i,c_j} h.$$

In view of (5.20), it is just obvious that this is the limit point of:

$$Q_{c_{im},c_{jm}} \nabla G(u_m)h = \frac{g_j(\lambda(u_m)) - g_i(\lambda(u_m))}{\lambda_j(u_m) - \lambda_i(u_m)} Q_{c_{im},c_{jm}} h.$$

Consequently, the theorem is proved for the particular subsequences $(u_m)_{m\geq 0}$ we have chosen at first. A simple compactness argument allows us to drop the second assumption on the convergence of the sequences $(c_{i,m})_{m\geq 0}$. It remains to discard the regularity assumption: assume now that the elements of sequence $(u_m)_{m\geq 0} \subseteq V$ may have multiple eigenvalues. Since the set of regular elements is dense in $\mathcal{J}$ (see Proposition 2.7.24), there exists for every $m \geq 0$ a sequence $(u_{ml})_{l\geq 0}$ of regular elements that converges to u_m. From the first part of the proof, we know that $\lim_{l\to\infty} \nabla G(u_{ml})h = \nabla G(u_m)h$. Hence there exists a $l_m \in \mathbb{N}$ such that for every $l \geq l_m$

$$||u_{ml} - u_m|| \leq \frac{1}{m} \quad \text{and} \quad ||\nabla G(u_{ml})h - \nabla G(u_m)h|| \leq \frac{1}{m}.$$

Let $u'_m := u_{ml_m}$ and take an $\epsilon > 0$. Since $(u'_m)_{m\geq 0}$ is a sequence of regular elements that converges to u, there exists a $m' \geq 2/\epsilon$ such that, for all $m \geq m'$, we have $||\nabla G(u'_m)h - \nabla G(u)h|| \leq \epsilon/2$. If $m \geq m'$, we can thus write:

$$\begin{aligned}
||\nabla G(u_m)h - \nabla G(u)h|| &\leq ||\nabla G(u_m)h - \nabla G(u'_m)h|| + ||\nabla G(u'_m)h - \nabla G(u)h|| \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

and $\nabla G(u_m)h$ converges to $\nabla G(u)h$. ■

Needless to say, this theorem allows us to show that if a symmetric function is twice continuously differentiable, the spectral function it generates is also twice differentiable.

5.7 Application: complementarity problems in Jordan algebras

Fukushima, Luo and Tseng have studied a smoothing technique for solving the second-order complementarity problem in [FLT01]. This section shows how their analysis can be extended to deal with complementarity problems defined on symmetric cones. Some of the proofs in the first subsection are minor updates of the work of Fukushima, Luo and Tseng (see also [CQT03], where similar results have been obtained for the semidefinite cone complementarity problem). However, our argumentation in the second subsection is rather different from theirs, as it relies on a purely Jordan algebraic machinery.

Complementarity problems constitute a natural generalization of the convex conic optimization problem. In order to introduce them, we define the following notation. Let $\langle\cdot,\cdot\rangle$ be a scalar product on $\mathbb{R}^N$, and let $\mathcal{K}$ be a regular cone, that is, a cone closed, convex, pointed, and with a nonempty interior. We denote its dual by:

$$\mathcal{K}^* := \{y \in \mathbb{R}^N | \langle y, x\rangle \geq 0 \text{ for every } x \in \mathcal{K}\}.$$

We also consider a continuously differentiable mapping $\Phi : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^k \to \mathbb{R}^N \times \mathbb{R}^k$. The *general complementarity problem* consists in finding, if it exists, a point $(x, y, z) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^k$ for which:

$$x \in \mathcal{K}, \quad y \in \mathcal{K}^*, \quad \langle y, x\rangle = 0, \quad \text{and} \quad \Phi(x, y, z) = 0.$$

Example 5.7.1 (Primal-dual optimization problem) *Let us recall the general conic optimization problem. We denote by $\langle\cdot,\cdot\rangle_N$ and by $\langle\cdot,\cdot\rangle_k$ the respective scalar products of $\mathbb{R}^N$ and $\mathbb{R}^k$. We identify $\mathbb{R}^N$ and $\mathbb{R}^k$ with their corresponding dual. Let $A : \mathbb{R}^N \to \mathbb{R}^k$ be a surjective linear operator, and let A^* be its adjoint: for every $x \in \mathbb{R}^N$ and every $y \in \mathbb{R}^k$, we have $\langle Ax, y\rangle_k = \langle x, A^*y\rangle_N$. We fix a vector $b \in \mathbb{R}^k$ and a vector $c \in \mathbb{R}^N$. Finally, let $\mathcal{K} \subseteq \mathbb{R}^N$ be a regular cone, and let $\mathcal{K}^*$ be its dual.*

The general conic optimization problem can be stated as follows:

$$\begin{aligned} \min\ & \langle c, x\rangle_N \\ \text{s.t. } & Ax = b \\ & x \in \mathcal{K}. \end{aligned}$$

Closely related to this problem comes its dual:

$$\begin{aligned} \max\ & \langle b, y\rangle_k \\ \text{s.t. } & A^*y + s = c \\ & s \in \mathcal{K}^*, y \in \mathbb{R}^k. \end{aligned}$$

We recall that this primal-dual pair of problems is called strictly feasible *if there exists a point* $(x, s, y) \in \operatorname{int} \mathcal{K} \times \operatorname{int} \mathcal{K}^* \times \mathbb{R}^k$ *that satisfies all the linear constraints. It is well known (see for instance in [ET76]) that, if these problems are strictly feasible, they can be reformulated as the following complementarity problem: find a point* $(x^*, s^*, y^*) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^k$ *such that:*

$$x \in \mathcal{K}, \quad s \in \mathcal{K}^*, \quad \langle x, s \rangle_N = 0, \quad \textit{and} \quad \Phi(x, s, y) = 0,$$

where

$$\Phi(x, s, y) := \begin{pmatrix} Ax - b \\ A^* y + s - c \end{pmatrix}.$$

This point (x^*, s^*, y^*) *represents a solution to the primal-dual optimization problem.* ■

In the smoothing approach of [FLT01], the condition

$$x \in \mathcal{K}, \quad y \in \mathcal{K}^*, \quad \langle x, y \rangle = 0$$

is approximated by the following one:

$$\phi_\mu(x, y) = 0 \quad \text{for } \mu > 0,$$

where $(\phi_\mu)_{\mu > 0}$ is a class of continuously differentiable functions, parameterized by the positive scalar μ, such that the pointwise limit function $\phi_0(x, y) := \lim_{\mu \downarrow 0} \phi_\mu(x, y)$ exists and satisfies:

$$\phi_0(x, y) = 0 \quad \text{if and only if} \quad x \in \mathcal{K}, \quad y \in \mathcal{K}^*, \quad \langle x, y \rangle = 0. \tag{5.23}$$

The algorithm that solves the generalized complementarity problem starts from an initial $\mu > 0$, and finds at each iteration an approximate solution $(\hat{x}_\mu, \hat{y}_\mu, \hat{z}_\mu)$ of the smoothed system:

$$\phi_\mu(x, y) = 0, \quad \Phi(x, y, z) = 0. \tag{5.24}$$

This is typically achieved by a few steps of a Newton algorithm. The scalar μ is then decreased to μ_-, and the process is repeated. The approximate solution $(\hat{x}_\mu, \hat{y}_\mu, \hat{z}_\mu)$ can be used as a starting guess for solving the smoothed system parameterized by μ_-.

Various classes of functions ϕ_μ are proposed in the literature when $\mathcal{K}$ is the positive orthant or when $\mathcal{K}$ is the second-order cone. We generalize here two classes to the Jordan algebraic cone of squares $\mathcal{K}_{\mathcal{J}}$, namely, the Chen-Mangasarian smoothing functions introduced in [CM95], and the Fischer-Burmeister smoothing functions. We check that these two classes satisfy property (5.23). We also make sure that, under suitable regularity assumptions on the function Φ, they yield an invertible Jacobian for the function

$$\begin{pmatrix} \phi_\mu(x, y) \\ \Phi(x, y, z) \end{pmatrix}.$$

This last requirement ensures that Newton's algorithm is well defined for the nonlinear equations that have to be solved. Finally, we show to what extent the approximate solution of a smoothed problem with a parameter μ can serve as a good initial guess for the solution of the same smoothed problem with a smaller parameter μ_-. From this result, it is also possible to see how the exact solutions to the problems parameterized by μ converge to the actual solution of the complementarity problem.

5.7.1 Chen-Mangasarian smoothing functions

Definition 5.7.1 *Let $\hat{g} : \mathbb{R} \to \mathbb{R}_+$ be a continuously differentiable convex function that satisfies:*

- $\diamond$ $\lim_{t\to-\infty} \hat{g}(t) = 0$*;*
- $\diamond$ $\lim_{t\to+\infty} \hat{g}(t) - t = 0$*;*
- $\diamond$ $0 < g'(t) < 1$ *for every* $t \in \mathbb{R}$*.*

Let $g_{\mathcal{CM}} : \mathbb{R}^r \to \mathbb{R}$, $\lambda \mapsto g(\lambda) := (\hat{g}(\lambda_1), \ldots, \hat{g}(\lambda_r))^T$. We denote by $G_{\mathcal{CM}}$ the spectral mapping generated from $g_{\mathcal{CM}}$. In view of Theorem 5.6.1, the function $G_{\mathcal{CM}}$ is continuously differentiable. The Chen-Mangasarian smoothing function induced by $\hat{g}$ *is:*

$$\begin{aligned} \phi_{\mathcal{CM},\mu} : \mathcal{J} \times \mathcal{J} &\to \mathcal{J} \\ (u, v) &\mapsto \phi_{\mathcal{CM},\mu}(u, v) := u - \mu G_{\mathcal{CM}}\left(\frac{u - v}{\mu}\right). \end{aligned}$$

■

Note that $\phi_{\mathcal{CM},\mu}$ is continuously differentiable by Theorem 5.6.1. For the sake of notational simplicity, we drop the subscript $_{\mathcal{CM}}$ in this subsection.

Remark 5.7.2 *Observe that*

$$\lim_{\mu\downarrow 0} \mu\hat{g}(t/\mu) = \max\{t, 0\}.$$

Indeed, if $t > 0$, we have $\lim_{\mu\downarrow 0} \hat{g}(t/\mu) - t/\mu = 0$ in view of the second assumption on $\hat{g}$. If $t < 0$, we have $\lim_{\mu\downarrow 0} \hat{g}(t/\mu) = 0$ by the first assumption on $\hat{g}$. ■

Remark 5.7.3 *We can easily check that $\hat{g}(t) > \max\{0, t\}$ for every $t \in \mathbb{R}$. First $\hat{g}$ is strictly increasing, and $\lim_{t\to-\infty} \hat{g}(t) = 0$, implying that $\hat{g}(t) > 0$ for every $t \in \mathbb{R}$.*

Second, consider the function $\psi(t) := \hat{g}(t) - t$. This function converges to 0 when t tends to $+\infty$ and is strictly decreasing, as $\psi' = \hat{g}' - 1 < 0$. Thus this function has no root, and since $\psi(0) = \hat{g}(0) > 0$, we deduce that $\psi(t) > 0$ for every $t \in \mathbb{R}$. ■

Lemma 5.7.4 *Let u be an element of $\mathcal{J}$. The element of $\mathcal{K}_{\mathcal{J}}$ which is the closest to $u = \sum_{i=1}^r \lambda_i(u)c_i$ with respect to the Jordan norm is $u_+ := \sum_{i=1}^r \max\{\lambda_i(u), 0\}c_i$.*

Proof
Corollary 4.2.5 states that $||u - v||_{\mathcal{J}} \geq ||\lambda(u) - \lambda(v)||$, where the equality holds if and only if u and v have a similar joint decomposition. Hence

$$\min_{v\in\mathcal{K}_{\mathcal{J}}} ||u - v||^2_{\mathcal{J}} = \min_{\lambda\in\mathbb{R}^n_+} ||\lambda(u) - \lambda||^2,$$

whose solution λ^* is easily seen to have as components $\lambda^*_i = \max\{\lambda_i(u), 0\}$. Observe also that $u = u_+ - (-u)_+$. ■

Proposition 5.7.5 *For every function $\hat{g}$ that satisfies the assumptions in Definition 5.7.1, the Chen-Mangasarian smoothing function induced by $\hat{g}$ complies with the complementarity property (5.23).*

Proof
Let u and v be two elements of $\mathcal{J}$, and let $u - v = \sum_{i=1}^r \gamma_i c_i$ be the complete spectral decomposition of $u - v$. In view of Remark 5.7.2, we can write:

$$\begin{aligned}\phi_0(u,v) &= \lim_{\mu\downarrow 0}\phi_\mu(u,v) = \lim_{\mu\downarrow 0} u - \mu G((u-v)/\mu) \\ &= u - \sum_{i=1}^r \lim_{\mu\downarrow 0}\mu\hat{g}(\gamma_i/\mu)c_i = u - \sum_{i=1}^r \max\{\gamma_i, 0\}c_i = u - (u-v)_+,\end{aligned}$$

where the last equality comes from Lemma 5.7.4. It suffices now to apply Proposition 6 in [GST04], which states that the condition $\phi_0(u,v) = u - (u-v)_+ = 0$ is equivalent to $u, v \in \mathcal{K}_\mathcal{J}$ and $\operatorname{tr}(uv) = 0$. ■

Proposition 5.7.6 *If the linear application $\nabla_z\Phi(u,v,z)$ has a full column rank, and if for every $u, v \in \mathcal{K}_\mathcal{J}$ and $z \in \mathbb{R}^k$, we have:*

$$\nabla\Phi(u,v,z)\begin{pmatrix} h_u \\ h_v \\ h_z\end{pmatrix} = 0 \quad \textit{implies that} \quad \operatorname{tr}(h_u h_v) = 0, \tag{5.25}$$

then the Jacobian

$$J_\mu(u,v,z) = \begin{pmatrix} \nabla_u\phi_\mu(u,v) & \nabla_v\phi_\mu(u,v) & 0 \\ \nabla_u\Phi_\mu(u,v,z) & \nabla_v\Phi_\mu(u,v,z) & \nabla_z\Phi_\mu(u,v,z)\end{pmatrix}$$

is invertible.

Proof
Observe that $\nabla_u\phi_\mu(u,v) = I_N - \nabla G((u-v)/\mu)$ and $\nabla_v\phi_\mu(u,v) = \nabla G((u-v)/\mu)$. Let us check that these two matrices are positive definite.

Let $(u-v)/\mu = \sum_{j=1}^s \xi_j e_j$ be the unique eigenspaces spectral decomposition of $(u-v)/\mu$. By our formula for the Jacobian of spectral mappings, we have:

$$M := \nabla G\left(\frac{u-v}{\mu}\right) = \sum_{j=1}^s \hat{g}'(\xi_j)Q_{e_j} + \sum_{j\neq k}\frac{\hat{g}(\xi_j) - \hat{g}(\xi_k)}{\xi_j - \xi_k}Q_{e_j,e_k}. \tag{5.26}$$

By assumption, we know that $\hat{g}'(t) > 0$ for every $t \in \mathbb{R}$. Thus every scalar coefficient of (5.26) is positive, and, in view of Remark 5.5.2, the operator M is positive definite. Furthermore, the operator $I_N - M$ can be written as follows:

$$I_N - \nabla G\left(\frac{u-v}{\mu}\right) = \sum_{j=1}^s(1 - \hat{g}'(\xi_j))Q_{e_j} + \sum_{j\neq k}\left(1 - \frac{\hat{g}(\xi_j) - \hat{g}(\xi_k)}{\xi_j - \xi_k}\right)Q_{e_j,e_k}.$$

Since $\hat{g}'(t) < 1$ by hypothesis, we conclude that $I_N - M$ is positive definite as well. This implies that $M^{-1/2}(I_N - M)M^{-1/2} = M^{-1} - I_N$ is also positive definite.

Now, suppose that:

$$J_\mu(u,v,z)\begin{pmatrix} h_u \\ h_v \\ h_z \end{pmatrix} = 0.$$

In view of the assumption on $\Phi(u,v,z)$, we know that $h_z = 0$ and that $\mathrm{tr}(h_u h_v) = 0$. On the other hand, since M is invertible

$$(I_N - M)h_u - Mh_v = 0, \qquad \text{or} \qquad (M^{-1} - I_N)h_u - h_v = 0.$$

Thus $\langle h_u, (M^{-1} - I_N)h_u\rangle = \mathrm{tr}(h_u h_v) = 0$, and $h_u = 0$ since $M^{-1} - I_N$ is positive definite. It follows that $h_v = 0$, and $J_\mu(u,v,z)$ is invertible. ■

Remark 5.7.7 *Observe that the function $\Phi(x,s,y)$ of the primal-dual conic problem stated in Example 5.7.1 complies with the assumptions of the previous proposition. Indeed, as $\nabla_y\Phi(x,s,y) = A^*$ is injective, it has a full column rank. Moreover, the condition (5.25) becomes*

$$Ah_x = 0, \quad h_s + A^* h_y = 0,$$

form which we deduce $\langle h_s, h_x\rangle_N = 0$. ■

It remains to analyze the convergence of the successive smoothings of the complementarity problem.

Proposition 5.7.8 *For all elements u and v of $\mathcal{J}$, and every parameters $\mu > \nu > 0$, we have*

$$\phi_\nu(u,v) - \phi_\mu(u,v) \in \mathrm{int}\,\mathcal{K}_{\mathcal{J}}$$

and

$$\hat{g}(0)(\mu - \nu)e - [\phi_\nu(u,v) - \phi_\mu(u,v)] \in \mathcal{K}_{\mathcal{J}}.$$

In other words, for every $1 \le i \le r$, we have

$$\hat{g}(0)(\mu - \nu) \ge \lambda_i\big(\phi_\nu(u,v) - \phi_\mu(u,v)\big) > 0.$$

Proof

These inclusion results are equivalent to the following inequalities:

$$\hat{g}(0)(\mu - \nu) \ge \mu\hat{g}\left(\frac{\lambda_i(u-v)}{\mu}\right) - \nu\hat{g}\left(\frac{\lambda_i(u-v)}{\nu}\right) > 0.$$

The first one follows from the convexity of $\hat{g}$:

$$\frac{\mu-\nu}{\mu}\hat{g}(0) + \frac{\nu}{\mu}\hat{g}\left(\frac{\lambda_i(u-v)}{\nu}\right) \ge \hat{g}\left(\frac{\lambda_i(u-v)}{\mu}\right).$$

In order to prove the second inequality, we proceed by checking that for every real number t, the function $\psi : \mathbb{R}_{++} \to \mathbb{R}$, $\alpha \mapsto \psi(\alpha) := \hat{g}(\alpha t)/\alpha$ is strictly decreasing. Observe that the derivative of this function is $\psi'(\alpha) = (\alpha t\hat{g}'(\alpha t) - \hat{g}(\alpha t))/\alpha^2$. Now, we have $\hat{g}(x) > \max\{0, x\}$ for every $x \in \mathbb{R}$ in view of Remark 5.7.3. Further, $0 < \hat{g}'(x) < 1$ implies that $\max\{0, x\} \ge x\hat{g}'(x)$. Henceforth, the derivative ψ' is negative on its domain. ■

5.7.2 Fischer-Burmeister smoothing functions

Definition 5.7.9 *The* Fischer-Burmeister *class of smoothing functions is defined as follows:*

$$\begin{aligned} \phi_{\mathcal{FB},\mu} : \mathcal{J} \times \mathcal{J} &\to \mathcal{J} \\ (u,v) &\mapsto \phi_{\mathcal{FB},\mu}(u,v) := u + v - (u^2 + v^2 + 2\mu^2 e)^{1/2}. \end{aligned}$$

For the sake of notational simplicity, we drop the subscript $_{\mathcal{FB}}$ in this subsection.

Lemma 5.7.10 *For every u and v of $\mathcal{J}$, the element $(u^2+v^2)^{1/2} - u$ belongs to $\mathcal{K}_{\mathcal{J}}$.*

Proof
First, we prove this statement for elements u of $\operatorname{int}\mathcal{K}_{\mathcal{J}}$. Recall that, in view of Remark 2.8.9, the solution x to the equation $wx = z$ is unique and belongs to $\mathcal{K}_{\mathcal{J}}$ if $w \in \operatorname{int}\mathcal{K}_{\mathcal{J}}$ and $z \in \mathcal{K}_{\mathcal{J}}$. Note that $w := (u^2+v^2)^{1/2} + u$ belongs to $\operatorname{int}\mathcal{K}_{\mathcal{J}}$ since u lies in $\operatorname{int}\mathcal{K}_{\mathcal{J}}$. Taking $z := v^2$ and $x := (u^2+v^2)^{1/2} - u$, we conclude that $wx = z$. Thus x belongs to $\mathcal{K}_{\mathcal{J}}$.

A simple argument based on the continuity of the smallest eigenvalue allows us to extend this result to every $u \in \mathcal{K}_{\mathcal{J}}$.

For every $w = \sum_{i=1}^r \lambda_i(w)c_i$ of $\mathcal{J}$, we denote the element $\sum_{i=1}^r |\lambda_i(w)|c_i$ by $|w|$; it belongs obviously to $\mathcal{K}_{\mathcal{J}}$. According to the notation of the previous subsection, we have $|w| = w_+ + (-w)_+$. Observe that $|w| - w = 2(-w)_+ \in \mathcal{K}_{\mathcal{J}}$. Thus the element

$$(u^2+v^2)^{1/2} - u = \Big((u^2+v^2)^{1/2} - |u|\Big) + (|u| - u)$$

belongs to $\mathcal{K}_{\mathcal{J}}$ by convexity of the cone of squares. ■

Proposition 5.7.11 *The Fischer-Burmeister class of smoothing functions satisfies the property (5.23).*

Proof
Since the square root function is continuous on $[0,+\infty[$, we have $\phi_0(u,v) = \lim_{\mu\downarrow 0}\phi_\mu(u,v) = u+v-(u^2+v^2)^{1/2}$. Hence, the condition $\phi_0(u,v) = 0$ implies that $u^2+2uv+v^2 = u^2+v^2$, or $uv = 0$. In view of Lemma 5.7.10, the element $v = (u^2+v^2)^{1/2} - u$ belongs to $\mathcal{K}_{\mathcal{J}}$. Similarly, u is in $\mathcal{K}_{\mathcal{J}}$ as well.

Conversely, if the elements u and v satisfy $uv = 0$, then $(u+v)^2 = u^2+v^2$. If they are also in $\mathcal{K}_{\mathcal{J}}$, then $u+v \in \mathcal{K}_{\mathcal{J}}$. Taking the square root of each side, we get $u+v = (u^2+v^2)^{1/2}$, or $\phi_0(u,v) = 0$. ■

Let A and B be two self-adjoint linear operators. In the next lemma, we adopt the standard notation $A \succcurlyeq B$ when the operator $A - B$ is positive semidefinite. If $A - B$ is positive definite, we write $A \succ B$.

Lemma 5.7.12 *For every elements u and v in $\mathcal{J}$, and every real number $\mu > 0$, we have $L\left((u^2+v^2+2\mu^2 e)^{1/2}\right) \succ L(u)$.*

Proof
In view of Lemma 5.7.10, the element $(x^2+y^2)^{1/2}-x$ belongs to the cone of squares for every $x,y\in\mathcal{J}$. According to Proposition 2.7.31, the operator $L\left((x^2+y^2)^{1/2}\right)-L(x)$ is positive semidefinite. Now, let $x:=(u^2+2\mu^2e)^{1/2}$ and $y:=v$. We have:

$$L\left((x^2+y^2)^{1/2}\right)\succcurlyeq L(x)=L\left((u^2+2\mu^2e)^{1/2}\right)\succ L(|u|)\succcurlyeq L(u).$$

The second to last relation comes from the fact that $(u^2+2\mu^2e)^{1/2}-|u|$ belongs to the interior of $\mathcal{K}_\mathcal{J}$, as its eigenvalues are of the form $\sqrt{\lambda_i(|u|)^2+2\mu^2}-\lambda_i(|u|)$, which is positive when $\mu>0$. ■

Lemma 5.7.13 *Let $f:\mathbb{R}^r_+\to\mathbb{R}^r$, $\lambda\mapsto g(\lambda):=(\sqrt{\lambda_1},\ldots,\sqrt{\lambda_r})^T$, and let $G:\mathcal{K}_\mathcal{J}$ be the spectral mapping generated by g. Then*

$$\nabla G(u)=\frac{1}{2}L(u^{1/2})^{-1}$$

for every invertible element u of $\mathcal{K}_\mathcal{J}$.

Proof
This proof is an easy application of our formula for the Jacobian of spectral mappings, and of Proposition 2.7.31 on the spectral decomposition of the multiplication operator. Let $u=\sum_{i=1}^s\xi_je_j$ be the unique subspaces spectral decomposition of $u\in\operatorname{int}\mathcal{K}_\mathcal{J}$. We have:

$$\begin{aligned}
\nabla G(u) &= \sum_{j=1}^s\frac{1}{2\sqrt{\xi_j}}Q_{e_j}+\sum_{j\neq k}\frac{\sqrt{\xi_j}-\sqrt{\xi_k}}{\xi_j-\xi_k}Q_{e_j,e_k}\\
&= \frac{1}{2}\left(\sum_{j=1}^s\frac{1}{\sqrt{\xi_j}}Q_{e_j}+\sum_{j\neq k}\frac{2}{\sqrt{\xi_j}+\sqrt{\xi_k}}Q_{e_j,e_k}\right)\\
&= \frac{1}{2}L(u^{1/2})^{-1}.
\end{aligned}$$

This result can also be obtained with the help of the formula for $x^{1/2}$ given in the proof of Proposition 2.7.6, and using the algebraic calculus developed in Section 2.4. ■

We are now ready to prove a statement similar to Proposition 5.7.6 on the invertibility of the Jacobian of the system (5.24).

Proposition 5.7.14 *If the linear application $\nabla_z\Phi(u,v,z)$ has a full column rank, and if for every $u,v\in\mathcal{K}_\mathcal{J}$ and $z\in\mathbb{R}^k$, we have:*

$$\nabla\Phi(u,v,z)\begin{pmatrix}h_u\\h_v\\h_z\end{pmatrix}=0\quad\textit{implies that}\quad\operatorname{tr}(h_uh_v)=0,$$

then the Jacobian

$$J_\mu(u,v,z)=\begin{pmatrix}\nabla_u\phi_\mu(u,v) & \nabla_v\phi_\mu(u,v) & 0\\ \nabla_u\Phi_\mu(u,v,z) & \nabla_v\Phi_\mu(u,v,z) & \nabla_z\Phi_\mu(u,v,z)\end{pmatrix}$$

is invertible.

Proof
Using Proposition 2.4.2, it is easy to see that

$$\nabla_u[u^2 + v^2 + 2\mu^2 e] = 2L(u).$$

With Lemma 5.7.13, we can easily compute that:

$$\nabla_u \phi_\mu(u, v) = I_N - L(u)L\left((u^2 + v^2 + 2\mu^2 e)^{1/2}\right)^{-1}.$$

In view of Lemma 5.7.12, the operator $\nabla_u \phi_\mu(u, v)$ is positive definite. By symmetry, the operator $\nabla_v \phi_\mu(u, v)$ is positive definite as well. We can now repeat the final argument of the proof of Proposition 5.7.6 to conclude. ∎

In the following proposition, we show that the Fischer-Burmeister smoothing yields an algorithm that theoretically behaves similarly to the Chen-Mangasarian procedure.

Proposition 5.7.15 *For every elements u and v of $\mathcal{J}$, and every parameters $\mu > \nu > 0$, we have*

$$\phi_\mu(u, v) - \phi_\nu(u, v) \in \operatorname{int} \mathcal{K}_\mathcal{J}$$

and

$$\sqrt{2}(\mu - \nu)e - [\phi_\mu(u, v) - \phi_\nu(u, v)] \in \mathcal{K}_\mathcal{J}.$$

Proof
The two inclusions amounts to proving the following inequalities:

$$\sqrt{2}(\mu - \nu) \geq \sqrt{\lambda_i(u^2 + v^2) + 2\mu^2} - \sqrt{\lambda_i(u^2 + v^2) + 2\nu^2} > 0.$$

It is not difficult to prove that the function $f(t) := \sqrt{t + 2\mu^2} - \sqrt{t + 2\nu^2}$ decreases on $\mathbb{R}_+$. Its maximum is reached in $t = 0$, and equals $\sqrt{2}(\mu - \nu)$. This yields the first inequality. The second inequality is trivial. ∎

CHAPTER 6

Smoothing techniques in formally real Jordan algebras

BENEFITING FROM OUR ANALYSIS of spectral functions on formally real Jordan algebras, we extend the powerful smoothing techniques of Yu. Nesterov to the framework of formally real Jordan algebras. This study allows us to design a new scheme for minimizing the largest eigenvalue of an affine function on a formally real Jordan algebra. We prove that its complexity is in the order of $\mathcal{O}(1/\epsilon)$, where ϵ is the absolute tolerance on the value of the objective.

Particularizing our result, we propose a new algorithm designed to minimize a sum of Euclidean norms and we perform a complete analysis of its complexity. Further numerical experiments show that smoothing techniques are numerically stable and competitive with respect to interior-point methods. We finally propose an heuristic that relies on our smoothing technique, and appears to be efficient for very large-scale sum-of-norms problems.

6.1 Introduction

Some recent results of Nesterov [Nes05a] tend to show that, in spite of their popularity, interior-point methods are not always the best procedures to solve some very large scale optimization problems. Whereas the number of iterations of these methods is *predictably* low, each of them requires so many computations that performing the very first one of them may already be out of reach.

In order to bypass this problem, Nesterov has essentially managed to combine the cheap iteration cost of subgradient methods and the efficiency of structural optimization in a powerful generic method for solving some structured non-smooth optimization problems. This method is generic in the sense that, given a class of problems to be solved, an appropriate *prox-function* has to be specified by the practitioner. The efficiency of the method relies heavily on the choice of this prox-function (more details are given in Section 6.2). In [Nes05a], Nesterov shows how a class of piecewise linear optimization problem can be solved *provably* fast. He extended his result to a class of non-smooth problems involving symmetric matrices [Nes05b]. A natural question arises in this context: can formally real Jordan algebras help to further extend this method ? We give a positive answer in this chapter, and we particularize our study to the sum-of-norms problem. Our solution is mainly based on the results on spectral functions and spectral mappings shown in the previous chapters.

We are indebted to Donald Goldfarb for the observation that our technique can immediately be applied to solve the more general problem of minimizing a sum of Euclidean norms and a linear function. Many thanks to him.

The chapter is organized as follows. In Section 6.2, we briefly recall how the smoothing techniques of Nesterov work. Section 6.4 contains the main result of the chapter, namely, the inequality (6.5). This inequality allows us to estimate the complexity of smoothing techniques applied to the maximal eigenvalue optimization problem in Jordan algebras. We specify the obtained algorithm in Section 6.5 in order to solve the sum-of-norms problem, obtaining, up to our knowledge, the first theoretical complexity result for this problem. We have implemented our method in Matlab, and we compare its numerical behavior with the best available interior-point scheme for the sum-of-norms problem in Section 6.6. (It is interesting to note that, up to our knowledge, no theoretical analysis can explain the particularly good efficiency of this interior-point method; the available complexity analysis turns out to be very pessimistic with respect to its practical behavior). It appears that our algorithm is competitive with respect to this interior-point method for very large instances if the required accuracy is not too small. We also describe a heuristic procedure that performs surprisingly well for high-dimensional instances.

6.2 Smoothing techniques in non-smooth convex optimization

In this section, we briefly recall the historical context of the present study. A smoother account can be found in the introductory chapter of this thesis.

The general problem of Convex Optimization can be formulated as follows. Given a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a nonempty convex set $Q \subseteq \mathbb{R}^n$, find the minimal value f^* that f takes on Q, and, if possible, find a point of Q where this value is attained. On a finite-arithmetic computer, this goal is typically unreachable, and we content ourselves with an approximation of this minimal value: given an absolute tolerance $\epsilon > 0$, the problem consists in finding a point $\hat{x}$ in Q such that $f(\hat{x}) - f^* < \epsilon$.

The first methods proposed and studied for solving convex optimization problems were the *subgradient schemes* (our brief exposition in Section 1.3 can be completed by [Sho85], or in Chapter 2 and 3 of [Nes03]). Following the terminology of [NY83], these algorithms are *black-box methods*. This means that the only information on the objective function that they can get, given a point of its domain, is the value of the objective and of one of its subgradients at that point. In other words, they only have access to local information. It has been proved, by a resisting oracle technique, that these methods cannot have a better complexity than $\Theta(1/\epsilon^2)$ in terms of number of iterations of the scheme [NY83]. Now, suppose that the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is *smooth*, more precisely that it is differentiable and that its gradient is Lipschitz continuous:

$$||\nabla f(x) - \nabla f(y)||_* \leq L||x - y|| \quad \text{for every } x, y \in \operatorname{dom} f$$

where $||\cdot||$ is a norm of $\mathbb{R}^n$, $||\cdot||_*$ is the associated dual norm, and the positive constant L the gradient Lipschitz continuity corresponding to the norm $||\cdot||$. In this case, the complexity analysis of subgradient schemes – we can actually call them *gradient schemes* – shows that an approximate solution can be found in no more than $\mathcal{O}(\sqrt{L/\epsilon})$ iterations (see Chapter 3 of [Nes03]).

Later on appeared efficient interior-point methods for Convex Programming. The most decisive breakthrough in the field has been achieved in [NN93]. In contrast with subgradient schemes, these methods do not content themselves with local information on the problem. They explicitly exploit its structure. The construction of the *self-concordant barrier* needed in the algorithm mimics the mathematical description of the specific problem to be solved. These methods have a complexity in the order of $\mathcal{O}(\sqrt{\nu}\ln(\nu/\epsilon))$ iterations, where ν is a structural parameter of the problem, usually a multiple of its dimension or of the number of constraints (see Section 1.5).

Subgradient schemes for non-smooth problems may seem completely dwarfed by interior-point methods. But the complexity of an iteration required by an interior-point method is much larger than the cost of a subgradient scheme iteration: indeed, interior-point methods typically require the resolution of a (sometimes sparse) linear system of equations at each step, while subgradient methods only need vector manipulations (addition, computation of scalar products, projections on simple sets, ...). Hence, very large-scale problems might be out of reach for interior-point methods because the very first iteration is already prohibitively expensive.

The smoothing method of Nesterov [Nes05a] has been designed to potentially solve this issue, because, without affecting too severely the number of iterations, their cost is much cheaper for every generated point. It can be applied to optimization problems with the following very specific *structure* and performs at each iteration a cheap gradient-like step. We are given Q_1 and Q_2 two bounded convex sets, contained in the Euclidean vector spaces

E_1 and E_2 respectively. The objective function, to be minimized over Q_1, is supposed to have the following form:

$$f(x) = \hat{f}(x) + \max_{u \in Q_2}[\langle Ax, u\rangle - \hat{\phi}(u)], \tag{6.1}$$

where $\hat{f}$ and $\hat{\phi}$ are smooth convex functions, and A is a linear operator from E_1 to E_2^*. We assume that an evaluation of f is not too expensive, i.e. that the maximization of $\langle Ax, u\rangle - \hat{\phi}(u)$ over Q_2 can be performed very efficiently, or even that a closed form of the solution is available.

The idea is to replace the non-smooth objective function f by a smooth approximation of it via a *prox-function* d_2 of Q_2, that is, a twice continuously differentiable function $d_2 : Q_2 \to \mathbb{R}$ whose minimal value is 0 and is attained in the relative interior of Q_2. We also require for a prox-function d_2 of Q_2 to be strongly convex on Q_2, i.e. that:

$$\text{for every} \quad u \in Q_2 \quad \text{and} \quad h \in E_2, \quad \langle d_2''(u)h, h\rangle \geq \sigma_2 ||h||_{E_2}^2$$

for some norm $||\cdot||_{E_2}$ of E_2 and some *strong convexity constant* $\sigma_2 > 0$. For each parameter $\mu > 0$, we define the function:

$$f_\mu(x) := \hat{f}(x) + \max_{u \in Q_2}\langle Ax, u\rangle - \hat{\phi}(u) - \mu d_2(u).$$

This family of functions approaches f from below as μ goes to 0, and each of them has a Lipschitz continuous gradient. We choose a norm $||\cdot||_{E_1}$ of E_1 and we define:

$$||A||_{E_1,E_2} := \max\{\langle Ax, u\rangle : ||x||_{E_1} \leq 1, ||u||_{E_2} \leq 1\}.$$

It can be proved (see Theorem 1 in [Nes05a]) that the Lipschitz constant of f_μ' equals $L_\mu := ||A||_{E_1,E_2}^2/(\mu\sigma_2)$. Therefore, we can apply a low-cost gradient-like scheme in order to minimize it.

This gradient-like scheme requires a prox-function d_1 of Q_1, whose strong convexity constant for the norm $||\cdot||_{E_1}$ is denoted by σ_1 and its minimizer by x_0. The scheme updates at each step the three sequences of points $(x_k)_{k\geq 0}$, $(y_k)_{k\geq 0}$, and $(z_k)_{k\geq 0}$. Letting $D_1 := \max_{x\in Q_1} d_1(x)$ and $D_2 := \max_{x\in Q_2} d_2(x)$, we put $\mu := \epsilon/(2D_2)$.

Algorithm 6.2.1 *For $k \geq 0$:*

1. *Compute* $f_\mu'(x_k)$.
2. *Find* $y_k := \arg\min_{y\in Q_1}\left\{\langle f_\mu'(x_k), y - x_k\rangle + \frac{L_\mu}{2}||y - x_k||_{E_1}^2\right\}$.
3. *Find* $z_k := \arg\min_{y\in Q_1}\left\{\frac{L_\mu}{\sigma_1}d_1(y) + \sum_{i=1}^k \frac{i+1}{2}(\langle f_\mu'(x_i), y\rangle)\right\}$.
4. *Set* $x_{k+1} := \frac{k+1}{k+3}y_k + \frac{2}{k+3}z_k$. ∎

Theorem 6.2.1 (Theorem 3 in [Nes05a]) *For the sequence $(y_k)_{k\geq 0}$ generated by the algorithm, we have that $f(y_N) - f^* \leq \epsilon$ as soon as:*

$$N+1 \geq 4||A||_{E_1,E_2}\sqrt{\frac{D_1 D_2}{\sigma_1\sigma_2}}\cdot\frac{1}{\epsilon}+\sqrt{\frac{4\hat{L}D_1}{\sigma_1\epsilon}},$$

where $\hat{L}$ is the gradient Lipschitz constant of $\hat{f}$ corresponding to the norm $||\cdot||_{E_1}$. ■

Observe that this complexity result concerns the actual non-smooth problem, and not its smoothed approximation. Nesterov's method is then in $\mathcal{O}(1/\epsilon)$, which is the best known complexity so far for this class of non-smooth problems.

In this chapter, we only consider problems where $\hat{f} \equiv 0$, or where $\hat{f}$ is a linear function. In both cases, the Lipschitz constant $\hat{L}$ equals zero.

6.3 Smoothing techniques for piecewise linear optimization problems

It turns out that there is a particular instantiation of the general smoothing technique that is particularly efficient, that is, for which the parameters $||A||_{E_1,E_2}$, D_1, and D_2 are not too large, while σ_1 and σ_2 are not too small.

Definition 6.3.1 *The n-dimensional* simplex *is the set:*

$$\Delta_n := \{x \in \mathbb{R}^n | x_1 + \cdots + x_n = 1,\ x_1, \ldots, x_n \geq 0\}.$$

■

Definition 6.3.2 *The function*

$$\begin{aligned} f : \Delta_n &\to \mathbb{R} \\ x &\mapsto d(x) := \sum_{i=1}^n x_i \ln(x_i) + \ln(n) \end{aligned}$$

is called the entropy function *of dimension n.* ■

The following lemma summarizes the main properties of this function. A proof is included for the sake of completeness.

Lemma 6.3.3 *The entropy function is infinitely differentiable in the interior of its domain. It is strongly convex on its domain: we have*

$$\langle f''(x)h, h\rangle \geq ||h||_1^2,$$

where the norm $||\cdot||_1$ is the 1-norm, i.e. $||x||_1 := \sum_{i=1}^n |x_i|$. The entropy function attains its minimum in $x^* := (1/n, \ldots, 1/n)^T$ and is bounded from above in Δ_n by $\ln(n)$. The conjugate function of f is:

$$f^*(s) = \ln(\exp(s_1) + \cdots + \exp(s_n)) - \ln(n).$$

Proof
The infinite differentiability of f in $\operatorname{int}\Delta_n$ is obvious. Furthermore, we can easily derive:

$$[f'(x)]_i = \ln(x_i) + 1, \quad \text{and} \quad [f''(x)]_{ij} = \frac{\delta_{ij}}{x_i}.$$

Hence, since $x \in \Delta_n$, Cauchy-Schwartz inequality yields:

$$\langle f''(x)h, h\rangle = \sum_{i=1}^n \frac{h_i^2}{x_i} = \left(\sum_{i=1}^n \frac{h_i^2}{x_i}\right)\left(\sum_{i=1}^n x_i\right) \geq \left(\sum_{i=1}^n |h_i|\right)^2 = ||h||_1^2.$$

The minimum of f on Δ_n is unique because this function is strongly convex. By its symmetry, the function f attains its minimum in $x^* := (1/n, \ldots, 1/n)^T$, and $f(x^*) = 0$. Since $\ln(t) \leq 0$ when $0 < t \leq 1$, we have $f(x) \leq \ln(n)$ on Δ_n, with equality holding on the vertices of this domain. In order to compute the conjugate of f, we must solve the problem:

$$\max_{x\in\Delta_n} \sum_{i=1}^n s_i x_i - \sum_{i=1}^n x_i \ln(x_i) - \ln(n).$$

In view of Karush-Kuhn-Tucker Theorem (reproduced in Theorem 1.5.5), the stationary points (x^*, μ^*, y^*) of its Lagrangian

$$\mathcal{L}(x, \mu, y) = \sum_{i=1}^n s_i x_i - \sum_{i=1}^n x_i \ln(x_i) - \ln(n) + \mu\left(1 - \sum_{i=1}^n x_i\right) + \sum_{i=1}^n y_i x_i$$

have to satisfy, since $y_i^* x_i^* = 0$:

$$x_i^* \frac{\partial\mathcal{L}(x^*, \mu^*, y^*)}{\partial x_i} = x_i^* s_i - x_i^* \ln(x_i^*) - x_i^*(\mu^* + 1) = 0$$

Summing on the indices i, we obtain $f^*(s) + \ln(n) = \mu^* + 1$. Next, from the condition $\partial\mathcal{L}(x^*, \mu^*, y^*)/\partial x_i = 0$, we deduce that $x_i^* \neq 0$, so that $y^* = 0$, and $\mu^* = \ln(\sum_{i=1}^n \exp(s_i)) - 1$. Observe also that

$$x_i^* = \frac{\exp(s_i)}{\sum_{i=1}^n \exp(s_i)}. \tag{6.2}$$

■

We specify now the non-smooth problem we are interested in. Let $E_1 := \mathbb{R}^m$ and $E_2 := \mathbb{R}^n$, endowed with their respective dot products as scalar products. We set $\hat{\phi}(u) := b^T u$ for an n-dimensional vector b. We also take Δ_m and Δ_n for the sets Q_1 and Q_2 respectively. The prox-functions d_1 and d_2 that we consider are the entropy functions of corresponding dimensions. In view of the previous lemma, their strong convexity constant equals 1 for the norm $||\cdot||_1$. (Observe that this norm has the smallest ball that contains unit vectors.) Moreover, we have $D_1 = \ln(m)$ and $D_2 = \ln(n)$. Finally, we derive:

$$||A||_{E_1,E_2} = \max_{||u||_1=||x||_1=1} u^T Ax = \max_{||x||_1=1} \max_i |[Ax]_i| = \max_{ij} |A_{ij}|.$$

Observe that this norm is typically much smaller than the Frobenius norm or than a spectral norm. The resulting complexity of the Algorithm 6.2.1 on the problem

$$\min_{x \in \Delta_m} \max_{u \in \Delta_n} u^T(Ax - b) \tag{6.3}$$

is then bounded from above by:

$$\frac{4}{\epsilon} \max_{ij} |A_{ij}| \sqrt{\ln(m)\ln(n)}.$$

The rest of the chapter is devoted to the resolution of the spectral problem generated by (6.3), namely

$$\min_{x \in \Delta_m} \max_{u \in \Delta_{\mathcal{J}}} \langle u, (Ax - b) \rangle_{\mathcal{J}}, \tag{6.4}$$

where A is a linear operator from $\mathbb{R}^m$ to a formally real Jordan algebra $\mathcal{J}$; the element b belongs to $\mathcal{J}$, and

$$\Delta_{\mathcal{J}} = \{u \in \mathcal{K}_{\mathcal{J}} | \operatorname{tr}(u) = 1\}.$$

The natural prox-function for this set seems to be the spectral function generated by the entropy function. We need to check whether this spectral function satisfies properties that are similar to those of Lemma 6.3.3 in order to establish the complexity of the subsequent smoothing algorithm.

6.4 An upper bound on the Hessian of the power function

In this section, we generalize to formally real Jordan algebras an inequality obtained recently by Nesterov [Nes05b] in the framework of symmetric matrices.

For every nonnegative integer k and every real r-dimensional vector λ, we let:

$$p_k(\lambda) := \lambda_1^k + \cdots + \lambda_r^k.$$

The spectral function generated by p_k is denoted by P_k:

$$\begin{aligned} P_k : \mathcal{J} &\to \mathbb{R} \\ u &\mapsto P_k(u) := \operatorname{tr}(u^k). \end{aligned}$$

The main result of this section is the following inequality.

For every integer $k \geq 2$, for every element $u = \sum_{i=1}^r \lambda_i(u) c_i$ of $\mathcal{J}$, and for every direction h of $\mathcal{J}$, we have:

$$\langle P_k''(u)h, h \rangle \leq k(k-1) \langle |u|^{k-2} h, h \rangle, \tag{6.5}$$

where $|u| := \sum_{i=1}^r |\lambda_i(u)| c_i$. ■

This inequality is the key for extending smoothing techniques in the framework of Jordan algebras, and for determining a complexity bound of the obtained scheme.

Lemma 6.4.1 *Let p and q be two nonnegative integers. For every $u \in \mathcal{J}$, the operator $L(|u|^{p+q}) - Q_{u^p,u^q}$ is positive semidefinite. In other words, for every $h \in \mathcal{J}$, we have:*

$$\langle |u|^{p+q} h, h\rangle \geq \langle Q_{u^p,u^q} h, h\rangle.$$

Proof
Let us fix an element $u \in \mathcal{J}$, and let us consider one of its complete spectral decomposition $u = \sum_{i=1}^r \lambda_i(u) c_i$. For the sake of notational simplicity, we write λ for $\lambda(u)$. From Proposition 2.7.31 and Corollary 2.7.32, we know that $L(|u|^{p+q})$ and Q_{u^p,u^q} have identical eigenspaces, which are direct sums of the subspaces $\mathcal{J}_{ij} := Q_{c_i,c_j}\mathcal{J}$. The eigenvalues corresponding to the eigenspace $\mathcal{J}_{ij}$ are respectively $(|\lambda_i|^{p+q} + |\lambda_j|^{p+q})/2$ for $L(|u|^{p+q})$, and $(\lambda_j^p\lambda_i^q + \lambda_i^p\lambda_j^q)/2$ for Q_{u^p,u^q}. Observe that:

$$(|\lambda_i|^p - |\lambda_j|^p)(|\lambda_i|^q - |\lambda_j|^q) \geq 0,$$

so that:

$$|\lambda_i|^{p+q} + |\lambda_j|^{p+q} \geq |\lambda_j|^p|\lambda_i|^q + |\lambda_i|^p|\lambda_j|^q \geq \lambda_j^p\lambda_i^q + \lambda_i^p\lambda_j^q.$$

In other words, the eigenvalues of $L(|u|^{p+q}) - Q_{u^p,u^q}$ are nonnegative. ■

Proposition 6.4.2 *For every u and h of $\mathcal{J}$, the inequality (6.5):*

$$\langle P_k''(u)h, h\rangle \leq k(k-1)\langle |u|^{k-2}h, h\rangle$$

holds true for all $k \geq 2$.

Proof
Since the Hessian is continuous in view of Theorem 5.6.1, it suffices to show the inequality for regular elements u, because they form a dense set in $\mathcal{J}$ (see Proposition 2.7.24). Let us fix a regular element $u = \sum_{i=1}^r \lambda_i(u)c_i$ of $\mathcal{J}$, and let us compute $\langle P_k''(u)h, h\rangle$ using the formula (5.19) for the Hessian.

We easily get:

$$[p_k'(\lambda)]_i = k\lambda_i^{k-1} \quad \text{and} \quad [p_k''(\lambda)]_{ij} = \delta_{ij}k(k-1)\lambda_i^{k-2},$$

where δ_{ij} is the Kronecker symbol. Let h be an element of $\mathcal{J}$, and let $h_{ij} := Q_{c_i,c_j}h$, so that $h = \sum_{i,j=1}^r h_{ij}$. Therefore, the second Pierce decomposition of h with respect to the Jordan frame $\{c_1, \ldots, c_r\}$ results in:

$$h = \sum_{i=1}^r h_{ii} + 2\sum_{i<j} h_{ij}.$$

By regularity of u, we have:

$$\begin{aligned}
\langle P_k''(u)h, h\rangle &= \sum_{i=1}^{r} k(k-1)\lambda_i^{k-2}\mathrm{tr}(h_{ii}^2) + 2\sum_{i\neq j} k\frac{\lambda_i^{k-1}-\lambda_j^{k-1}}{\lambda_i-\lambda_j}\mathrm{tr}(h_{ij}^2)\\
&= k\left(\sum_{i=1}^{r}(k-1)\lambda_i^{k-2}\mathrm{tr}(h_{ii}^2) + 2\sum_{i\neq j}\sum_{l=0}^{k-2}\lambda_i^l\lambda_j^{k-l-2}\mathrm{tr}(h_{ij}^2)\right)\\
&= k\left(\sum_{i=1}^{r}(k-1)\lambda_i^{k-2}\mathrm{tr}(h_{ii}^2) + 2\sum_{i\neq j}\sum_{l=0}^{k-2}\frac{\lambda_i^l\lambda_j^{k-l-2}+\lambda_j^l\lambda_i^{k-l-2}}{2}\mathrm{tr}(h_{ij}^2)\right).
\end{aligned}$$

Observe now that, for every nonnegative integers p and q, we can write:

$$\langle Q_{u^p,u^q}h, h\rangle = \sum_{i,j=1}^{r}\frac{\lambda_i^p\lambda_j^q+\lambda_j^p\lambda_i^q}{2}\mathrm{tr}(h_{ij}h) = \sum_{i=1}^{r}\lambda_i^{p+q}\mathrm{tr}(h_{ii}^2) + \sum_{i\neq j}\frac{\lambda_i^p\lambda_j^q+\lambda_j^p\lambda_i^q}{2}\mathrm{tr}(h_{ij}^2).$$

With this relation, we can continue as follows:

$$\begin{aligned}
\langle P_k''(u)h, h\rangle &= k\left(\sum_{i=1}^{r}(k-1)\lambda_i^{k-2}\mathrm{tr}(h_{ii}^2) + \sum_{l=0}^{k-2}\left(\langle Q_{u^l,u^{k-l-2}}h, h\rangle - \sum_{i=1}^{r}\lambda_i^{k-2}\mathrm{tr}(h_{ii}^2)\right)\right)\\
&= k\sum_{l=0}^{k-2}\langle Q_{u^l,u^{k-l-2}}h, h\rangle \leq k\sum_{l=0}^{k-2}\langle L(|u|^{k-2})h, h\rangle = k(k-1)\langle |u|^{k-2}h, h\rangle,
\end{aligned}$$

where the inequality comes from Lemma 6.4.1. ■

The following corollaries are simple but very useful consequences of the previous proposition. Their proofs follow closely those in [Nes05b].

Corollary 6.4.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that has a power series expansion*

$$f(t) = \sum_{k\geq 0} a_k t^k$$

such that all the coefficients a_k are nonnegative. Let us denote the domain of f by I, and the set containing all the elements of $\mathcal{J}$ that have their eigenvalues in I by K. We define $F : K \to \mathbb{R}$, $u \mapsto F(u) := \sum_{i=1}^{r} f(\lambda_i(u))$. For every $u \in K$ and all $h \in \mathcal{J}$, we have:

$$\langle F''(u)h, h\rangle \leq \sum_{i=1}^{r} f''(\lambda_i(|u|))\lambda_i(h)^2.$$

Proof
By Proposition 6.4.2, we can write:

$$\langle F''(u)h, h\rangle = \sum_{k\geq 2} a_k\langle P_k''(u)h, h\rangle \leq \sum_{k\geq 2} k(k-1)a_k\mathrm{tr}(|u|^{k-2}h^2)$$

The von Neumann inequality (4.4) gives us $\mathrm{tr}(|u|^{k-2}h^2) \leq \sum_{i=1}^r \lambda_i(|u|)^{k-2}\lambda_i(h^2)$, from which we obtain:

$$\langle F''(u)h, h\rangle \leq \sum_{k\geq 2}\sum_{i=1}^r k(k-1)a_k\lambda_i(|u|)^{k-2}\lambda_i(h^2).$$

Now, since $f''(t) = \sum_{k\geq 2} k(k-1)a_k t^{k-2}$, we conclude that:

$$\langle F''(u)h, h\rangle \leq \sum_{i=1}^r f''(\lambda_i(|u|))\lambda_i(h^2).$$

■

Corollary 6.4.4 *Consider the function $F : \mathcal{J} \to \mathbb{R}$, $u \mapsto F(u) := \sum_{i=1}^r \exp(\lambda_i(u))$, and the function $E(u) := \ln F(u)$. Then*

$$\langle E''(u)h, h\rangle \leq \lambda_1(h^2)$$

for every u and h of $\mathcal{J}$.

Proof
A straightforward computation gives us:

$$\langle E''(u)h, h\rangle = \frac{\langle F''(u)h, h\rangle}{F(u)} - \frac{\langle F'(u), h\rangle^2}{F(u)^2} \leq \frac{\langle F''(u)h, h\rangle}{F(u)}.$$

Suppose preliminarily that $u \in \mathcal{K}_{\mathcal{J}}$, so that $u = |u|$. It is well-known that the coefficients of the power-series expansion of exp are positive. Therefore, using the previous corollary, we can continue as follows:

$$\langle E''(u)h, h\rangle \leq \frac{\langle F''(u)h, h\rangle}{F(u)} \leq \frac{\sum_{i=1}^r \exp(\lambda_i(|u|))\lambda_i(h^2)}{\sum_{i=1}^r \exp(\lambda_i(u))} \leq \lambda_1(h^2).$$

Now, observe that the element $u - Te$ is always in the cone of squares when T is smaller than $\lambda_r(u)$. Note also that $E(u - Te) = E(u) - T$, thus $E''(u - Te) = E''(u)$. Hence, the above inequality holds even for elements u that are not in $\mathcal{K}_{\mathcal{J}}$. ■

Corollary 6.4.5 *Let $\Delta_{\mathcal{J}} := \{v \in \mathcal{K}_{\mathcal{J}} | \mathrm{tr}(v) = 1\}$ be the Jordan algebraic extension of the standard simplex. The function $d : \Delta_{\mathcal{J}} \to \mathbb{R}$, $v \mapsto d(v) := \sum_{i=1}^r \lambda_i(v)\ln\lambda_i(v)$ satisfies, for all $h \in \mathcal{J}$ and all $u \in \Delta_{\mathcal{J}}$, the following inequality:*

$$\langle d''(u)h, h\rangle \geq ||h||_1^2,$$

where $||h||_1 := \sum_{i=1}^r |\lambda_i(h)|$ is the norm generated by the 1-norm in $\mathbb{R}^r$.

Proof
Let $\eta(\lambda) := \ln\sum_{i=1}^r \exp(\lambda_i)$ for every $\lambda \in \mathbb{R}^r$. The conjugate function of η is $\delta(\lambda) := \sum_{i=1}^r \lambda_i \ln\lambda_i$ on the standard r-dimensional simplex Δ_r.

In view of Lemma 6.3.3, the function d is then the conjugate of the spectral function E defined in the previous corollary. It is well-known (see Theorem 4.2.2 in [HUL93]) that strong convexity and Lipschitz continuity of the gradient are dual notions. In other words, suppose that the function $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is twice differentiable; then:

$$\langle f''(x)h, h\rangle \leq L||h||^2 \quad \forall\, x \in \operatorname{dom} f \text{ and } h \in \mathcal{J}$$

if and only if:

$$\langle f^{*\prime\prime}(x)h, h\rangle \geq \frac{1}{L}||h||_*^2 \quad \forall\, x \in \operatorname{dom} f^* \text{ and } h \in \mathcal{J},$$

where $||\cdot||_*$ is the dual norm of $||\cdot||$. As the dual norm of $||h||_\infty := \sqrt{\lambda_1(h^2)}$ is the norm $||h||_1 = \sum_{i=1}^r |\lambda_i(h)|$ (see also Theorem 4.4.2), we get that $\langle d''(u)h, h\rangle \geq ||h||_1^2$. ■

We have now everything we need to describe and analyze a smoothing algorithm for Jordan algebras. Let us consider the function:

$$f(x) := \max_{u \in \Delta_{\mathcal{J}}} \langle Ax, u\rangle - \langle b, u\rangle,$$

which maps $\mathbb{R}^m$ to $\mathbb{R}$. Analogously to the above corollary, the set $\Delta_{\mathcal{J}}$ represents the Jordan algebraic extension of the standard simplex. The linear application A maps $\mathbb{R}^m$ to $\mathcal{J}$, and the element b belongs to $\mathcal{J}$. The scalar product should be understood as the Jordan scalar product. In view of Proposition 3.2.7, the function f is exactly equal to $\lambda_1(Ax - b)$.

Using the prox-function $d_2(u) := \sum_{i=1}^r (\lambda_i(u) \ln \lambda_i(u)) + \ln r$ for the Jordan algebraic simplex $\Delta_{\mathcal{J}}$, we obtain for every $\mu > 0$ that:

$$f_\mu(x) := \max_{u \in \Delta_{\mathcal{J}}} \langle Ax, u\rangle - \langle b, u\rangle - \mu d_2(u) = \mu d_2^*\big((Ax - b)/\mu\big),$$

or:

$$f_\mu(x) = \mu \ln\left(\sum_{i=1}^r \exp\big(\lambda_i(Ax - b)/\mu\big)\right) - \mu \ln(r).$$

The above corollary ensures that the strong convexity constant σ_2 related to this smoothing equals 1 for the best possible norm (i.e. with the smallest unit ball), namely $||h||_{E_2} := \sum_{i=1}^r |\lambda_i(h)|$.

The complexity of the resulting smoothing algorithm, adapted for Algorithm 6.2.1 is thus:

$$4||A||_{E_1,E_2}\sqrt{\frac{D_1 D_2}{\sigma_1}} \cdot \frac{1}{\epsilon}.$$

It turns out that this algorithm can be slightly modified to solve the more general problem:

$$\min_{x \in \Delta_m} \max_{u \in \Delta_{\mathcal{J}}} \langle u, (Ax - b)\rangle + \langle c, x\rangle,$$

where c is an m-dimensional vector. The resulting complexity of this modified algorithm remains the same as the complexity for the original problem (6.4), because the linear function $\hat{f}(x) := \langle c, x\rangle$ of x has a strong convexity constant of zero.

6.5 Sum-of-norms problem

The *sum-of-norms problem* can be formulated as follows. Given p real matrices $\{A_1, \dots, A_p\}$ of dimension $m \times n$ and p real m-dimensional vectors $\{b_1, \dots, b_p\}$, we need to minimize the function

$$f(x) := \sum_{j=1}^{p} ||A_j x - b_j||$$

over $Q_1 := \{x \in \mathbb{R}^n : ||x|| \leq R\}$, where $||\cdot||$ stands for the standard Euclidean norm of $\mathbb{R}^n$ or of $\mathbb{R}^m$. We also consider the problem of minimizing the function

$$f^{(c)}(x) := \sum_{j=1}^{p} ||A_j x - b_j|| + c^T x,$$

where c is an n-dimensional vector.

In this section, we apply the machinery of smoothing techniques to solve these problems. We start by considering the minimization of f, then we indicate how the problem involving the function $f^{(c)}$ can be treated.

We define the following elements for all $1 \leq j \leq p$:

$$\bar{A}_j = \begin{pmatrix} 0 \\ A_j \end{pmatrix} \quad \text{and} \quad \bar{b}_j = \begin{pmatrix} 0 \\ b_j \end{pmatrix}.$$

We also introduce the function:

$$\begin{aligned} \bar{f} : \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto \bar{f}(x) := \sum_{j=1}^{p} \lambda_1(\bar{A}_j x - \bar{b}_j), \end{aligned}$$

where λ_1 is the largest eigenvalue of its argument in the formally real Jordan algebra $\mathcal{S}_m$. Observe that minimizing $\bar{f}$ over Q_1 is completely equivalent to the sum-of-norms problem.

Since λ_1 is the support function of the Jordan algebraic version of the standard simplex:

$$\Delta := \{\bar{u} \in \mathcal{S}_m | \lambda_1(\bar{u}) + \lambda_2(\bar{u}) = 1, \lambda_2(\bar{u}) \geq 0\} = \left\{ \bar{u} = \begin{pmatrix} 1/2 \\ u \end{pmatrix} \in \mathcal{J} : ||u|| \leq 1/2 \right\},$$

we can rewrite the function $\bar{f}$ as follows:

$$\bar{f}(\bar{x}) = \sum_{j=1}^{p} \lambda_1(\bar{A}_j x - \bar{b}_j) = \sum_{j=1}^{p} \max_{\bar{u}_j \in \Delta} \langle \bar{u}_j, \bar{A}_j x - \bar{b}_j \rangle_{\mathcal{J}}.$$

Now, we define

$$A := \begin{pmatrix} \bar{A}_1 \\ \vdots \\ \bar{A}_p \end{pmatrix} \quad \text{and} \quad b := \begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_p \end{pmatrix}.$$

Our expression of $\bar{f}$ then becomes $\bar{f}(x) = \max_{\bar{u}\in\Delta^p}\langle Ax, \bar{u}\rangle_{\mathcal{J}^p} - \langle b, \bar{u}\rangle_{\mathcal{J}^p}$, and our problem enters into the class of those for which smoothing techniques are applicable.

In the notation of Section 6.2, we let $||\cdot||_{E_1}$ be the *Euclidean norm*, and we take as prox-function for Q_1 the function:

$$d_1(x) := \frac{||x||^2_{E_1}}{2}.$$

It is now easy to show that the constant σ_1 equals 1 and that $D_1 = \max\{d_1(x)|x \in Q_1\} = R^2/2$.

The space E_2 will be $\mathcal{J}^p$. For the set $Q_2 := \Delta^p$, we propose the following prox-function:

$$\begin{aligned} d_2(\bar{u}) &:= \sum_{j=1}^p ||\bar{A}_j^*|| \cdot [\lambda_1(\bar{u}_j)\ln(\lambda_1(\bar{u}_j)) + \lambda_2(\bar{u}_j)\ln(\lambda_2(\bar{u}_j)) + \ln 2] \\ &= \sum_{j=1}^p ||\bar{A}_j^*|| \cdot \left[\frac{1}{2}\ln\left(1/4 - ||u_j||^2\right) + ||u_j||\ln\left(\frac{1/2 + ||u_j||}{1/2 - ||u_j||}\right) + \ln 2\right], \end{aligned}$$

and the following norm:

$$||\bar{u}||_{E_2} := \sqrt{\sum_{j=1}^p ||\bar{A}_j^*|| \cdot ||\bar{u}_j||_1^2}.$$

We have used the notation $||\cdot||_1$ to denote the spectral norm generated by the 1-norm on $\mathbb{R}^2$. The number $||\bar{A}_j^*||$ denotes here the maximum value that $\langle \bar{A}_j^* u_j, x\rangle$ can take when $||u_j||_1 \leq 1$ and $||x||_{E_1} \leq 1$. A straightforward computation shows that this maximum equals the maximal singular value of A_j, that is, $||\bar{A}_j^*|| = \sqrt{\lambda_{\max}(A_j^* A_j)}$.

We know from Corollary 6.4.5 that, for every $\bar{h}_1, \ldots, \bar{h}_p \in \mathcal{J}$, the following inequality holds:

$$\sum_{j=1}^p ||\bar{A}_j^*||\langle[\lambda_1(\bar{u}_j)\ln(\lambda_1(\bar{u}_j)) + \lambda_2(\bar{u}_j)\ln(\lambda_2(\bar{u}_j))]''\bar{h}_j, \bar{h}_j\rangle \geq \sum_{j=1}^p ||\bar{A}_j^*|| \cdot ||\bar{h}_j||_1^2 = ||\bar{h}||^2_{E_2},$$

where

$$\bar{h} := \begin{pmatrix} \bar{h}_1 \\ \vdots \\ \bar{h}_p \end{pmatrix}.$$

Hence, we can take $\sigma_2 := 1$. Now, $D_2 = \max\{d_2(\bar{u})|\bar{u} \in Q_2\} = \sum_{j=1}^r ||\bar{A}_j^*|| \ln 2$. It remains to compute the quantity $||A||_{E_1,E_2}$:

$$\begin{aligned}
||A||_{E_1,E_2} &= \max\left\{\langle Ax, \bar{u}\rangle_{\mathcal{J}^p} : ||x||_{E_1} \le 1, \sum_{j=1}^{p} ||\bar{A}_j^*|| \cdot ||\bar{u}_j||_1^2 \le 1\right\} \\
&= \max\left\{\sum_{j=1}^{p}\langle \bar{A}_j x, \bar{u}_j\rangle_{\mathcal{J}} : ||x||_{E_1} \le 1, \sum_{j=1}^{p} ||\bar{A}_j^*|| \cdot ||\bar{u}_j||_1^2 \le 1\right\} \\
&\le \max\left\{\sum_{j=1}^{p} ||\bar{A}_j^*|| \cdot ||\bar{u}_j||_1 \cdot ||x||_{E_1} : ||x||_{E_1} \le 1, \sum_{j=1}^{p} ||\bar{A}_j^*|| \cdot ||\bar{u}_j||_1^2 \le 1\right\} \\
&\le \max\left\{\sum_{j=1}^{p} ||\bar{A}_j^*|| \cdot ||\bar{u}_j||_1 : \sum_{j=1}^{p} ||\bar{A}_j^*|| \cdot ||\bar{u}_j||_1 \le \sqrt{\sum_{j=1}^{p} ||\bar{A}_j^*||}\right\} = \sqrt{\sum_{j=1}^{p} ||\bar{A}_j^*||}.
\end{aligned}$$

The last inequality comes from the Cauchy-Schwarz relation:

$$\left(\sum_{j=1}^{p} ||\bar{A}_j^*|| \cdot ||\bar{u}_j||_1\right)^2 \le \left(\sum_{j=1}^{p} ||\bar{A}_j^*||\right)\left(\sum_{j=1}^{p} ||\bar{A}_j^*|| \cdot ||\bar{u}_j||_1^2\right).$$

Letting $M := \sum_{j=1}^{p} ||\bar{A}_j^*||$, we can now conclude that the Algorithm 6.2.1 has the following rate of convergence:

$$\bar{f}(\bar{y}_N) - \bar{f}^* \le \frac{4||A||}{N+1}\sqrt{\frac{D_1 D_2}{\sigma_1\sigma_2}} = \frac{4\sqrt{\ln 2}MR}{N+1} = \mathcal{O}\left(\frac{RM}{N}\right).$$

If the matrices A_j are scaling matrices, that is, matrices of the form $A_j := m_j I_n$, Nesterov has shown that the same order of convergence holds with the following smoothed version of f:

$$f_\mu(x) := \sum_{i=1}^{p} m_j \psi_\mu(||x - c_j||),$$
$$\text{with } \psi_\mu(t) = \begin{cases} t^2/2\mu & \text{if } 0 \le t \le \mu, \\ t - \mu/2 & \text{if } \mu \le t. \end{cases}$$

Remark 6.5.1 *Observe that the problem:*

$$\min_{||x|| \le R} \sum_{j=1}^{p} |\langle a_j, x\rangle - b_j|$$

is a particular case of the problem we have considered above, obtained with $m := 1$*. In this case, the constant* M *is the sum of Euclidean norms of the vectors* a_j*.* ■

Now, the smoothing of the objective $\sum_{j=1}^{p} ||A_j x - b_j||$ is easily computed as:

$$f_\mu(x) = \sum_{j=1}^{p} \mu||\bar{A}_j^*|| \ln\cosh 2\left(\frac{||A_j x - b_j||}{\mu||\bar{A}_j^*||}\right).$$

Its gradient equals:

$$\nabla_x f_\mu(x) = \sum_{j=1}^{p} \mu ||\bar{A}_j^*|| \tanh\left(\frac{||A_j x - b_j||}{\mu ||\bar{A}_j^*||}\right) \nabla_x ||A_j x - b_j||,$$

where

$$\nabla_x^h ||A_j x - b_j|| = \frac{(A_j x - b_j)^T A_j h}{||A_j x - b_j||}.$$

In the Algorithm 6.2.1, the sequences $(y_k)_{k\geq 1}$ and $(z_k)_{k\geq 1}$ are the solution of a quadratic optimization problem, and can be written explicitly. In sum, we obtain the following algorithm.

Algorithm 6.5.1 *For* $k \geq 0$*:*

1. *Compute* $\nabla_x f_\mu(x_k) = \sum_{j=1}^{p} \mu ||\bar{A}_j^*|| \tanh\left(\frac{||A_j x_k - b_j||}{\mu ||\bar{A}_j^*||}\right) \nabla_x ||A_j x_k - b_j||$.
2. *Find* $y_k := \arg\min_{y \in Q_1} \left\{\langle f'_\mu(x_k), y - x_k\rangle + L_\mu ||y - x_k||^2/2\right\}$.
3. *Find* $z_k := \arg\min_{y \in Q_1} \left\{L_\mu ||y||^2/2 + \sum_{i=1}^{k} \frac{i+1}{2}(\langle f'_\mu(x_i), y\rangle)\right\}$.
4. *Let* $x_{k+1} := \frac{k+1}{k+3} y_k + \frac{2}{k+3} z_k$. ■

The iteration cost is in $\mathcal{O}(mnp)$. It compares favorably with the iteration cost of the interior-point approach proposed by G. Xue and Y. Ye in [XY97], which is in $\mathcal{O}(m^3 + pm^2 n)$. The number of iterations of their theoretical interior-point method is bounded by:

$$\mathcal{O}\left(\sqrt{p}\left(\log\left(\frac{\max_{1\leq j\leq p} ||b_j||}{\epsilon}\right) + \log p\right)\right),$$

where $\epsilon > 0$ is the absolute tolerance on objective's value. However, their actual implementation differs significantly from their theoretical algorithm, and its practical number of iterations seems to be much lower than the above estimate. Indeed, their complexity analysis is based on a short-step potential-reduction algorithm, while their implementation uses a long-step dual one. We refer the reader to the next section for an experimental comparison of their implementation and our smoothing method.

Consider now the problem of minimizing the function:

$$f^{(c)} = \sum_{j=1}^{p} ||A_j x - b_j|| + c^T x.$$

In the general non-smooth model (6.1), we let $\hat{f}(x) := c^T x$. In the smoothing Algorithm 6.5.1, only the computation of $f'_\mu(x_k)$ need to be modified to:

$$f'_\mu(x_k) = c + \sum_{j=1}^{p} \mu ||\bar{A}_j^*|| \tanh\left(\frac{||A_j x_k - b_j||}{\mu ||\bar{A}_j^*||}\right) \frac{\nabla_x A_j x_k}{||A_j x_k - b_j||}.$$

The resulting complexity remains the same.

6.6 Computational experiments

In this section, several test instances for the sum-of-norms problem are solved numerically using the smoothing scheme described in the previous section. The aim of this study is threefold. First, we want to check empirically that our method is numerically stable. Second, we would like to compare the worst-case complexity estimated in the previous section with the practical behavior of our algorithm. And third, we confront our scheme with the efficient potential-reduction approach developed by G. Xue and Y. Ye in [XY97].

The implementation of our scheme follows closely Algorithm 6.5.1. In order to check the stopping criterion, we use the following proposition, which combines results from Section 2 and Theorem 3 of [Nes05a].

Proposition 6.6.1 *We use the same notation and objects as in Section 6.2. Let us introduce the function:*

$$\begin{aligned} \phi : Q_2 &\to \mathbb{R} \\ u &\mapsto \phi(u) := -\hat{\phi}(u) + \min_{x \in Q_1} \langle Ax, u \rangle. \end{aligned}$$

Then $\phi(u) \leq f(x)$ for every $x \in Q_1$ and $u \in Q_2$.

We denote by $u(x)$ the optimal solution of the problem:

$$\max_{u \in Q_2} \langle Ax, u \rangle - \hat{\phi}(u) - \mu d_2(u),$$

so that $f_\mu(x) = \hat{f}(x) + \langle Ax, u(x) \rangle - \hat{\phi}(u(x)) - \mu d_2(u(x))$. Writing

$$u_k := \sum_{i=0}^{k} \frac{2(i+1)}{(k+1)(k+2)} u(x_i),$$

we have:

$$0 \leq f(y_k) - \phi(u_k) \leq \mu D_2 + \frac{4||A||_{E_1,E_2} D_1}{\mu \sigma_1 \sigma_2 (k+1)^2} + \frac{4\hat{L} D_1}{\sigma_1 (k+1)^2}.$$

■

In our implementation, we periodically compute $f(y_k)$ and $\phi(u_k)$, and we check if their difference is smaller than the fixed tolerance ϵ. In all our experiments, this test is run every ten iterations.

We have also developed a simple heuristic derived from our algorithm, which seems to perform very well with respect to the theoretical worst-case complexity of our algorithm for problems of very large dimension. Here is a brief description. Suppose that the desired tolerance on the objective value is of the form $\epsilon = d^M$ for a positive integer M. We run the smoothing algorithm until the duality gap $f(y_k) - \phi(u_k)$ is smaller than d, and obtain a first approximation $\hat{x}_1$. Then, starting from the approximation $\hat{x}_1$, we rerun the smoothing algorithm until the duality gap becomes lower than d^2, and we repeat the whole procedure, until the ϵ tolerance is reached. In a sense, we delete sometimes all accumulated information on the problem. Moreover, in this heuristic strategy, we do not change the prox-function. Hence, the starting point of a reinitialization is not necessarily the minimum of the prox-function.

The set of instances we consider here are *continuous location problems*, which represents a particular case of sum-of-norms problems where the matrices A_j are of the form $m_j I_n$, with $m_j \geq 0$.

Example 6.6.1 *The* social distance *between a point x and an m inhabitants city that is located in $c \in \mathbb{R}^2$ is defined as $m||x-c||$, where $||\cdot||$ is the standard Euclidean norm. Suppose now that there are p cities, located in $c_1, \ldots, c_p$, and with population $m_1, \ldots, m_p$ respectively. We want to find the optimal place to build a service center, that is, we want to find the location x^* that minimize its total social distance with respect to all the cities. This problem can be formulated as:*

$$\textit{Find} \quad x^* = \arg\min_{x \in Q} \sum_{j=1}^{p} m_j ||x - c_j||,$$

where $Q \subset \mathbb{R}^2$ is an Euclidean ball sufficiently large to contain every city c_j. Note that the predicted complexity takes the form

$$\mathcal{O}\left(\frac{R\left(\sum_{i=1}^{p} m_j\right)}{\epsilon}\right),$$

where R is the radius of Q. It seems reasonable to consider an approximate solution of this problem with respect to a relative accuracy $\epsilon/(\sum_{i=1}^{p} m_j)$ instead of the absolute precision ϵ. ■

We have generated uniformly distributed locations of the p cities on the hypercube $[0,1]^n$. Their populations are uniformly distributed on $[0,1]$.

Table 6.1 indicates the computational time needed to solve the continuous location problem up to a relative accuracy of ϵ, and the ratio ρ between the actual and the predicted number N of iterations for the pure smoothing method. The next column (CPU TIME SM) indicates the CPU time required by the pure smoothing technique. The third to last (CPU TIME HSM) mentions the CPU time needed for our heuristic based on the smoothing algorithm, for $d := 0.1$. The two last columns (CPU TIME DSA and n. iter.) displays the CPU time and the number of iterations of Xue and Ye's Long Steps Dual Scaling Algorithm [XY97] that we have implemented in MATLAB, exactly as described in their paper.

All computations were performed on an Pentium4 2.8GHz processor, with 512Mb of Random Access Memory (RAM), using version 5.1.0.421 of MATLAB.

As it was clear from the complexity analysis, interior-point schemes outperform smoothing techniques when the desired accuracy is high. However, for the continuous location problem, it is often useless to strive for such a precision (what is the use to know where to build a service center between cities or houses to within 0.01 millimeter ?). Due to their very cheap iteration cost, our methods, and especially our heuristic in very high dimension, are faster than the interior-point method. Note that when $p = 50$ and $n = 2000$, the

p	n	ϵ	N	ρ	CPU TIME SM.	CPU TIME HSM.	CPU TIME DSA.	n. iter.
50	2	0.01	467	0.193	0.062”	0.047”	0.094”	7
50	2	0.001	4676	0.113	0.344”	0.125”	0.109”	8
50	2	0.0001	45882	0.064	1.812”	0.281”	0.125”	10
50	2	0.00001	452603	0.084	23.02”	0.985”	0.156”	12
500	2	0.001	4697	0.023	0.453”	0.390”	1.125”	9
5000	2	0.001	4709	0.021	4.125”	2.562”	12.09”	9
50000	2	0.001	4709	0.019	69.61”	121.9”	237.5”	10
50	20	0.001	14839	0.200	2.532”	0.109”	0.219”	9
50	200	0.001	47049	0.161	21.16”	1.032”	3.125”	9
50	2000	0.001	148843	0.160	628.5”	16.89”	494.2”	10

Table 6.1: Computational results for the continuous location problem

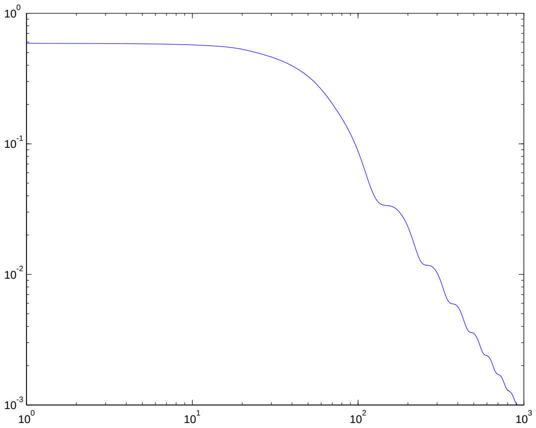

Figure 6.1: The duality gap of the smoothing algorithm decreases super-linearly after a transitory phase.

method of Xue and Ye requires only 10 iterations to achieve the desired accuracy. During each of them, the resolution of a dense system of linear equations of size 2000 is needed. This clearly reveals the major drawback of interior-point methods.

It is interesting to note that, after a short transitory phase, the duality gap $f(y_k)-\phi(u_k)$ seems to decrease *quadratically fast* rather that linearly fast as asserted by the theoretical arguments displayed earlier. Figure 6.1 has been generated for a problem involving 50 cities, in two dimensions. As indicated in Table 6.1, the actual number of iterations that smoothing techniques require represents indeed a small fraction of the theoretical complexity.

CHAPTER 7

Conclusions and perspectives

We review here the main results presented in this work and discuss some directions for future research.

The idea of combining two fields of mathematics that seems so different as Jordan Algebras and Convex Optimization appeared a decade ago. From the viewpoint of a pure algebraist, Jordan algebras has now become an exhausted field, because many of its most important open questions have been solved. From the viewpoint of an applied mathematician, Jordan algebras are now a fantastic tool of investigation. First, it is based on a completely mature mathematical theory. And second, it represents the common link between many practical problems, and it allows us to treat them in an elegant unified way without loosing too much of their structure. However, the theory of Jordan algebras is still quite new to optimizers. The second chapter of this thesis aims at providing a self-contained introduction to this field, which is also as complete as possible.

In the third chapter, we have extended some variational characterizations of eigenvalues that already existed in the framework of symmetric matrices. Our first aim was to prove Mirski's inequalities in the general framework of formally real Jordan algebras. However, in the course of our reasoning, we have obtained several very useful features on eigenvalues. Courant-Fischer's characterization and its corollary on interlacing relations plays a decisive role in the next chapters. These results might be the key to prove more interesting inequalities between eigenvalues. Also, the generalization of Wielandt's Theorem (see Theorem 3.5.6) might entail other useful relations, in particular between partial products of eigenvalues of different elements.

The main goal of our graduate work was to investigate on how the smoothing techniques of Nesterov can be used in more general situations than the few that were known so far. The crucial technical tool needed to make these methods work is a prox-function adapted to the problem. Advantageous prox-functions are not easy to find, and an automatic procedure to create them would be very useful. Our idea works as follows: given a good symmetric prox-function f for a non-smooth linear problem P, we can extend this problem P using the Jordan algebraic machinery, thus creating a nonlinear non-smooth problem. The most natural prox-function for this problem would be the spectral function F generated by f. However, we cannot hope a priori that the interesting properties are transmitted from f to F. Consequently, it is not guaranteed that the corresponding smoothing algorithm will have the nice properties of its linear counterpart.

This issue would be solved if the following conjecture is answered positively.

Conjecture 7.1 *Let $f : \mathbb{R}^r \to \mathbb{R} \cup \{+\infty\}$ be a continuously differentiable symmetric function. We assume that its domain is convex and has a nonempty interior. Let $||\cdot||$ be a norm. Suppose that there exists a constant $\sigma > 0$ the function f satisfies for every pair of points x, y of its domain the following inequality:*

$$f(y) \geq f(x) + \langle f'(x), y - x\rangle + \frac{\sigma}{2}||y - x||^2.$$

Consider a Jordan algebra $\mathcal{J}$ of rank r. We denote by F the spectral function generated by f, and by $|||\cdot|||$ the spectral norm generated by $||\cdot||$. Do we have for every u, v i the domain of F that:

$$F(v) \geq F(u) + \langle F'(u), v - u\rangle_{\mathcal{J}} + \frac{\sigma}{2}|||v - u|||^2,$$

with the same *constant σ ?* ■

The three next chapters offers partial positive answers to this open question.

In the fourth chapter, we start our study of spectral functions on Jordan algebras. We essentially investigate on how the properties of a symmetric function are transmitted to the spectral function it generates. We found that many features of the symmetric function are inherited by the correspondent spectral function. However, it is known that not all the properties follow this pattern, e.g. directional differentiation. Nevertheless, we have shown that the following properties are smoothly transmitted: Convexity, Quasi-Convexity, Strong Convexity (with the same strong convexity constant if the norm of interest is the Euclidean one), Local Lipschitz Continuity, Subdifferentiability, Differentiability and Continuous Differentiability, Bouligand Subdifferentiability, and Clarke Subdifferentiability. We also have closed form expressions for these four latter ones.

The fifth chapter focuses on a generalization of twice differentiability of spectral functions, that is, differentiability of spectral mappings. Although we do not need such a generality to solve Conjecture 7.1 for the very specific case of interest investigated in Chapter 6, our setting proves to be useful in the analysis of the extension to Jordan algebras of some classical methods for solving complementarity problems. We have generalized to the much larger class of problems defined on Jordan algebras the results that were obtained in [FLT01] for the second-order cone.

A natural possibility of extension of these results concerns the computation of higher-order derivatives of spectral mapping. Ultimately, these investigations might lead to an answer to Levent Tunçel's conjecture: given a symmetric ν-self-concordant function f of a set Q, is the spectral function generated by f a ν-self-concordant function of its domain as well ?

Another research direction concerns the computation of Bouligand and Clarke subdifferential for spectral mappings. These object would allow us to define and study generalized Newton methods for spectral mappings on Jordan algebras (see [SS03]).

The sixth chapter is devoted to the generalization of a smoothing technique of Nesterov to the Jordan algebraic framework. We have proved its competitive theoretical complexity, and we have tested its practical efficiency for very large continuous location problems. We have defined a heuristic based on smoothing techniques that outperforms the best existing interior-point methods for this problem when the dimension of the problem is large. It would be very interesting to understand theoretically why this heuristic works so well. We have also noticed that a duality gap measure seems to decrease *quadratically* instead of linearly, as asserted by the theory. A next challenge is to attempt to understanding this interesting feature.

Smoothing techniques are thus efficient in practice, and a promising research direction would be to discover more efficient prox-functions, in order to enlarge the range of application of these methods.

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