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Brief paper

Robust port-Hamiltonian representations of passive systems☆

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ABSTRACT

We discuss robust representations of stable, passive systems in particular coordinate systems, focussing especially on *port-Hamiltonian* representations. Such representations are typically not unique and the degrees of freedom associated with nonuniqueness are related to the solution set of the Kalman–Yakubovich–Popov linear matrix inequality (LMI). In this paper we analyze robustness measures for different possible port-Hamiltonian representations and relate it to quality functions defined in terms of eigenvalues of the matrix solution of the LMI. In particular, we look at the analytic center of this LMI. Within this framework, we derive inequalities for the passivity radius of the given model representation.

1. Introduction

In this paper we consider realizations of *linear time-invariant (LTI) systems* that are variously characterized as positive real, passive, or port-Hamiltonian (pH). We restrict ourselves to systems of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \quad \text{with } x(0) = 0, \\ y &= Cx + Du, \end{aligned} \tag{1}$$

referred to by the tuple of matrices $\mathcal{M} := \{A, B, C, D\}$. Here $u : \mathbb{R} \to \mathbb{C}^m$, $x : \mathbb{R} \to \mathbb{C}^n$, and $y : \mathbb{R} \to \mathbb{C}^m$ are vector-valued functions denoting, respectively, the *input*, *state*, and *output* of the system. The coefficient matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times n}$, and $D \in \mathbb{C}^{m\times m}$ are constants. We assume that $\operatorname{rank} B = \operatorname{rank} C = m$ and that (1) is *minimal*, that is, the pair (A, B) is *controllable* (for all $s \in \mathbb{C}$, $\operatorname{rank}[\, sI - A \;\; B\,] = n$), and the pair (A, C) is *observable* ($(A^{\mathsf H}, C^{\mathsf H})$ is controllable). Here, I is the identity matrix, the (conjugate) transpose (transpose) of a vector or matrix V is denoted by $V^{\mathsf H}$ ($V^{\mathsf T}$). We denote the set of Hermitian matrices in $\mathbb{C}^{n\times n}$ by $\mathbb{H}_n$. Positive definiteness (semidefiniteness) of $A \in \mathbb{H}_n$ is denoted by $A > 0$ ($A \geq 0$). The set of all positive definite (semidefinite) matrices in $\mathbb{H}_n$ is denoted by $\mathbb{H}_n^{>}$ ($\mathbb{H}_n^{\geq}$). The real and imaginary parts of $Z \in \mathbb{C}^{n\times m}$ are written as $\mathfrak{Re}(Z)$ and $\mathfrak{Im}(Z)$, respectively.

Our focus is on pH system representations (see e.g. van der Schaft and Jeltsema (2014)) of *passive* and *positive-real* systems, see e.g. Willems (1971) and Willems (1972). These can be characterized via the solution set of the *Kalman–Yakubovich–Popov linear matrix inequality (KYP-LMI)*. We show, in particular, that the analytic center of certain barrier functions associated with the KYP-LMI leads to very robust pH realizations; we discuss robustness measures for such realizations; and we derive computable bounds for these measures.

2. Positive-realness, passivity, and pH systems

By applying the Laplace transform to (1) and eliminating the state, we obtain the associated *transfer function*,

$$\mathcal{T}(s) := D + C(sI_n - A)^{-1}B. \tag{2}$$

On the imaginary axis, $\imath\mathbb{R}$, $\mathcal{T}(\imath\omega)$ describes the *frequency response* of the system. Defining

$$\Phi(s) := \mathcal{T}^{\mathsf H}(-s) + \mathcal{T}(s), \tag{3}$$

the system is called *positive real* if $\Phi(\imath\omega) \in \mathbb{H}_m^{\geq}$ for all $\omega \in \mathbb{R}$. Note that usually it is assumed that the system is stable. If (1) is positive real, then it is known (Willems, 1971) that there exists $X \in \mathbb{H}_n$ such that the KYP-LMI:

$$W(X) := \begin{bmatrix} -XA - A^{\mathsf H}X & C^{\mathsf H} - XB \\ C - B^{\mathsf H}X & D + D^{\mathsf H} \end{bmatrix} \geq 0 \tag{4}$$

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holds. Define solution sets to the KYP-LMI (4) as

$$\mathbb{X} := \{X \in \mathbb{H}_n \mid W(X) \geq 0\}, \tag{5a}$$

$$\mathbb{X}^{>} := \{X \in \mathbb{H}_n \,|W(X) \geq 0,\ X > 0\} = \mathbb{H}_n^{>} \cap \mathbb{X}, \tag{5b}$$

$$\mathbb{X}^{\gg} := \{X \in \mathbb{H}_n \,|W(X) > 0,\ X > 0\}. \tag{5c}$$

We are interested especially in $\mathbb{X}^{>}$ and $\mathbb{X}^{\gg}$. Those solutions to the KYP-LMI (4) for which the rank of $W(X)$ is minimal will be referred to as *minimum rank solutions*. If $S_D := D + D^{\mathsf{H}}$ is nonsingular, the minimum rank solutions in $\mathbb{X}^{>}$ are those for which $\operatorname{rank} W(X) = \operatorname{rank} S_D = m$, which is the case if and only if the Schur complement of S_D in $W(X)$ vanishes. This Schur complement is associated with the *algebraic Riccati equation (ARE)*:

$$\begin{aligned}\mathsf{Ricc}(X) := &-XA - A^{\mathsf{H}}X \\ &- (C^{\mathsf{H}} - XB)S_D^{-1}(C - B^{\mathsf{H}}X) = 0.\end{aligned} \tag{6}$$

There are two extremal solutions of (6) in $\mathbb{X}$, X_- and X_+, see Willems (1971), so that all solutions X of (6) satisfy

$$0 \leq X_- \leq X \leq X_+. \tag{7}$$

$\mathbb{X}$ is bounded, but we may have $\mathbb{X} = \emptyset$ or $X_- = X_+$.

A system $\mathcal{M} := \{A, B, C, D\}$ is called *passive* if there exists a *storage function*, $\mathcal{H}(x) \geq 0$, such that for any $\mu, t_0 \in \mathbb{R}$ with $\mu > t_0$, the *dissipation inequality* holds:

$$\mathcal{H}(x(\mu)) - \mathcal{H}(x(t_0)) \leq \int_{t_0}^{\mu} \mathfrak{Re}\left(y(t)^{\mathsf{H}}u(t)\right)\, dt. \tag{8}$$

If for all $\mu > t_0$ the inequality in (8) is strict, then the system is *strictly passive*. It has been shown in Willems (1972) that if the system $\mathcal{M}$ is minimal, then the KYP-LMI (4) has a solution $X \in \mathbb{H}_n^{>}$ if and only if $\mathcal{M}$ is a passive system. If this is the case, then $\mathcal{H}(x) := \frac{1}{2}x^{\mathsf{H}}Xx$ defines a storage function associated with the supply rate $\mathfrak{Re}(y^{\mathsf{H}}u)$ satisfying (8). Furthermore, there exist extremal solutions $0 < X_- \leq X_+$ of (4) such that all solutions X of (4) satisfy $0 < X_- \leq X \leq X_+$. If $X \in \mathbb{X}^{>}$ exists, then the system $\mathcal{M}$ of (1) is Lyapunov stable and if $X \in \mathbb{X}^{\gg}$ exists, then it is asymptotically stable. Note, however, that for (asymptotic) stability of A it is sufficient if the $(1, 1)$ block of $W(X)$ is (positive definite) positive semidefinite. A minimal system $\mathcal{M}$ as in (1) is passive if and only if it is positive real and stable and it is strictly passive if and only if it is strictly positive real and asymptotically stable. In the latter case, $X_+ - X_- > 0$, see Willems (1971). But minimality is not necessary for passivity, e.g., the system $\dot{x} = -x, y = u$ is stable and passive but not minimal.

An LTI *port-Hamiltonian (pH) system* has the form

$$\begin{aligned}\dot{x} &= (J - R)Qx + (G - K)u, \\ y &= (G + K)^{\mathsf{H}}Qx + Du,\end{aligned} \tag{9}$$

with $Q = Q^{\mathsf{H}} > 0, J = -J^{\mathsf{H}}, \mathcal{W} := \begin{bmatrix} R & K \\ K^{\mathsf{H}} & S \end{bmatrix} \geq 0$, where $S = \frac{1}{2}(D + D^{\mathsf{H}}) = \frac{1}{2}S_D$, and $W(Q^{-1}) = \frac{1}{2}\mathcal{W}$.

PH systems are a major tool for energy-based modeling (van der Schaft & Jeltsema, 2014). With a storage function $\mathcal{H}(x) = \frac{1}{2}x^{\mathsf{H}}Qx$, the dissipation inequality (8) holds and so pH systems are always passive. Conversely, any minimal and passive system $\mathcal{M}$ may be represented as a pH system via the following construction. If $X = Q \in \mathbb{X}^{>}$ is a solution of (4) then one obtains a pH formulation with $J := \frac{1}{2}(AQ^{-1} - Q^{-1}A^{\mathsf{H}})$, $R := -\frac{1}{2}(AQ^{-1} + Q^{-1}A^{\mathsf{H}})$, $K := \frac{1}{2}\left(Q^{-1}C^{\mathsf{H}} - B\right)$, and $G := \frac{1}{2}\left(Q^{-1}C^{\mathsf{H}} + B\right)$.

The pH form seems to be a very robust representation of a passive system (Mehl, Mehrmann, & Sharma, 2016). Moreover, it has a variety of other advantages: it allows for structure preserving interconnection of systems; it encodes the physical properties directly in the coefficients, see van der Schaft and Jeltsema (2014); it allows for simple projective model reduction approaches that preserve structure (Gugercin, Polyuga, Beattie, & van der Schaft, 2012); and it simplifies optimization methods for computing stability and passivity radii (Gillis, Mehrmann, & Sharma, 2018; Gillis & Sharma, 2018; Overton & Van Dooren, 2005). For a detailed discussion of passivity, positive realness, and pH realizations, in particular in limiting cases, see Beattie, Mehrmann, and Van Dooren (2018).

3. The analytic center of the solution set $\mathbb{X}^{>}$

Solutions of the KYP-LMI (4) are usually obtained via optimization algorithms, see e.g. (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994; Nesterov & Nemirovski, 1994). A common approach involves introducing a *barrier function* $b : \mathbb{C}^{n\times n} \to \mathbb{R}$ that is defined and finite throughout the interior of the constraint set, becoming infinite as the boundary is approached. The minimum of the barrier function is called the *analytic center* of the constraint set (Genin, Nesterov, & Van Dooren, 1999).

For the solution of the KYP-LMI, if $\mathbb{X}^{>}$ is non-empty and bounded, then the *barrier function*

$$b(X) := -\log\det W(X),$$

is bounded from below and becomes infinite when $W(X)$ becomes singular. To characterize the analytic center of b, we study the *interior* of $\mathbb{X}^{>}$ given by

$$\begin{aligned}\operatorname{Int}\mathbb{X}^{>} := \Big\{&X \in \mathbb{X}^{>} \mid \text{there exists } \delta > 0 \text{ such that} \\ &X + \Delta_X \in \mathbb{X}^{>} \text{for all } \Delta_X \in \mathbb{H}_n \text{ with } \|\Delta_X\|_2 \leq \delta\Big\},\end{aligned}$$

where $\|\Delta_X\|_2$ is the spectral norm given by the maximal singular value of Δ_X. We compare $\operatorname{Int}\mathbb{X}^{>}$ with the open set $\mathbb{X}^{\gg} = \{X \in \mathbb{X}^{>} \mid W(X) > 0\}$. Since $b(X)$ is finite for all points in $\mathbb{X}^{\gg}$, there is an open neighborhood where it stays bounded, and thus $\mathbb{X}^{\gg} \subseteq \operatorname{Int}\mathbb{X}^{>}$. The converse inclusion is not necessarily true; a characterization when both sets are equal is given by the following theorem.

Theorem 1. *Consider the system $\mathcal{M}$ of (1) with $\operatorname{rank}(B) = m$. Then $\mathbb{X}^{\gg} \equiv \operatorname{Int}\mathbb{X}^{>}$.*

Proof. If $\mathbb{X}^{>} = \emptyset$ then $\mathbb{X}^{\gg} = \emptyset$ as well. Otherwise, pick an $X \in \operatorname{Int}\mathbb{X}^{>}$ and suppose that $W(X)$ is positive semidefinite and singular. Then there exists a nontrivial $[z_1^{\mathsf{T}}, z_2^{\mathsf{T}}]^{\mathsf{T}} \in \operatorname{Ker} W(X)$ and $\varepsilon > 0$ sufficiently small so that for all $\Delta X \in \mathbb{H}_n$ with $\|\Delta X\|_2 \leq \varepsilon$, we have $X + \Delta X \in \mathbb{X}^{>}$. Observe that for all such ΔX, we have $W(X + \Delta X) = W(X) + \Gamma(\Delta X) \geq 0$, where $\Gamma(\Delta X) = -\begin{bmatrix} \Delta XA + A^{\mathsf{H}}\Delta X & \Delta XB \\ B^{\mathsf{H}}\Delta X & 0 \end{bmatrix}$, and so

$$0 \leq \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^{\mathsf{H}} W(X + \Delta X) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^{\mathsf{H}} \Gamma(\Delta X) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{10}$$

If there was a choice for $\Delta X \in \mathbb{H}_n$ with $\|\Delta X\|_2 \leq \varepsilon$ producing strict inequality in (10), then we would arrive at a contradiction, since the choice $-\Delta X$ satisfies the same requirements yet violates the inequality. Thus, equality holds in (10) for all $\Delta X \in \mathbb{H}_n$, which in turn implies that

$$W(X + \Delta X) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \Gamma(\Delta X) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0.$$

This means that $B^{\mathsf{H}}\Delta X z_1 = 0$ for all $\Delta X \in \mathbb{H}_n$ with $\|\Delta X\|_2 \leq \epsilon$. If $z_1 = 0$, then we find that $\Delta XBz_2 = 0$ for all $\Delta X \in \mathbb{H}_n$ with $\|\Delta X\|_2 \leq \varepsilon$, which in turn means $Bz_2 = 0$. Since $\operatorname{rank}(B) = m$ this is a contradiction, and thus $W(X)$ is nonsingular after all, and hence positive definite. To eliminate the last remaining case, suppose that $z_1 \neq 0$. Choosing first $\Delta X = \varepsilon I$, we find that $z_1 \perp \operatorname{Ran}(B)$. Pick $0 \neq b \in \operatorname{Ran}(B)$ and define $\Delta X = \varepsilon\left(I - 2ww^{\mathsf{H}}\right)$ with $w = \frac{1}{\sqrt{2}}\left(\frac{z_1}{\|z_1\|} - \frac{b}{\|b\|}\right)$. Then $B^{\mathsf{H}}\Delta X z_1 = \varepsilon\frac{\|z_1\|}{\|b\|}B^{\mathsf{H}}b = 0$ which implies that $z_1 = 0$, and so, $z = 0$, $W(X) > 0$, and again the assertion holds. $\square$

We now use an optimization algorithm to compute the analytic center of $\mathbb{X}^{>}$ as a candidate for a 'good' solution to the LMI (4) yielding a robust representation. For this we assume that $\mathbb{X}^{\gg} \neq \emptyset$. The *gradient* of the barrier function $b(X)$ with respect to W is given by

$$\partial b(X)/\partial W = -W(X)^{-1}.$$

For $X, Y \in \mathbb{H}_n$, define the *Frobenius inner product* $\langle X, Y\rangle := \operatorname{trace}\big(\mathfrak{Re}(Y)^{\mathsf{T}}\mathfrak{Re}(X) + \mathfrak{Im}(Y)^{\mathsf{T}}\mathfrak{Im}(X)\big)$. It follows from (Nesterov & Nemirovski, 1994) that $X \in \mathbb{C}^{n\times n}$ is an extremal point of $b(X)$ if and only if

$$\langle \partial b(X)/\partial W, \Delta W(X)[\Delta_X]\rangle = 0 \quad \text{for all } \Delta_X \in \mathbb{H}_n,$$

where $\Delta W(X)[\Delta_X]$ is the incremental step in the direction Δ_X given by

$$\Delta W(X)[\Delta_X] = -\begin{bmatrix} A^{\mathsf{H}}\Delta_X + \Delta_X A & \Delta_X B \\ B^{\mathsf{H}}\Delta_X & 0 \end{bmatrix}.$$

For an extremal point, it is then necessary that

$$\left\langle W(X)^{-1}, \begin{bmatrix} A^{\mathsf{H}}\Delta_X + \Delta_X A & \Delta_X B \\ B^{\mathsf{H}}\Delta_X & 0 \end{bmatrix}\right\rangle = 0 \tag{11}$$

for all $\Delta_X \in \mathbb{H}_n$. Defining $F := S_D^{-1}(C - B^{\mathsf{H}}X)$, $P := -A^{\mathsf{H}}X - XA - F^{\mathsf{H}}S_D F$, and $A_F := A - BF$, and using

$$W(X) = \begin{bmatrix} I & F^{\mathsf{H}} \\ 0 & I \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & S_D \end{bmatrix}\begin{bmatrix} I & 0 \\ F & I \end{bmatrix} \tag{12}$$

one obtains (see also Genin et al., 1999) the equivalent condition

$$\begin{aligned}&\left\langle \begin{bmatrix} P^{-1} & 0 \\ 0 & S_D^{-1} \end{bmatrix}, \begin{bmatrix} A_F^{\mathsf{H}}\Delta_X + \Delta_X A_F & \Delta_X B \\ B^{\mathsf{H}}\Delta_X & 0 \end{bmatrix}\right\rangle \\ &\quad = \left\langle P^{-1}, A_F^{\mathsf{H}}\Delta_X + \Delta_X A_F\right\rangle = 0 \quad \forall \Delta_X \in \mathbb{H}_n.\end{aligned}$$

This implies that (11) holds if and only if P is invertible and

$$A_F^{\mathsf{H}}P + PA_F = 0. \tag{13}$$

Note that P is the evaluation of the Riccati operator (6) at X, and that A_F is the corresponding closed loop matrix. It has been shown in Beattie et al. (2018) that the closed loop matrix A_F of the analytic center has its spectrum on the imaginary axis and thus is also 'central' in a certain sense. Since $P \in \mathbb{H}_n^{>}$, we can rewrite the equations defining the analytic center of $\mathbb{X}^{>}$ as the solutions $X \in \mathbb{H}_n$, $P \in \mathbb{H}_n^{>}$, $F \in \mathbb{C}^{m,n}$ of the system

$$\begin{aligned} S_D F &= C - B^{\mathsf{H}}X, \\ P &= -A^{\mathsf{H}}X - XA - F^{\mathsf{H}}S_D F, \\ 0 &= P(A - BF) + (A^{\mathsf{H}} - F^{\mathsf{H}}B^{\mathsf{H}})P = PA_F + A_F^{\mathsf{H}}P. \end{aligned} \tag{14}$$

which can be solved iteratively, using a starting value X_0 to compute P_0, F_0 and then solving consecutively for X_i, P_i and F_i with $i = 1, 2, \ldots$

Remark 1. For a given P the solution X of (14) can be obtained via the computation of an *extended Lagrangian invariant subspace*, see Benner, Losse, Mehrmann, and Voigt (2015), satisfying

$$\begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -X \\ I_n \\ -F \end{bmatrix} Z = \begin{bmatrix} 0 & A & B \\ A^{\mathsf{H}} & -P & C^{\mathsf{H}} \\ B^{\mathsf{H}} & C & S_D \end{bmatrix}\begin{bmatrix} -X \\ I_n \\ -F \end{bmatrix}.$$

Computing this subspace for $P = \mathsf{Ricc}(X) = 0$ allows one to compute solutions X_+ and X_- of (6), which can be used to construct a starting point for further optimization.

4. The passivity radius

Our goal to achieve robust pH representations of a passive system can be realized in different ways. A natural way to obtain a robust representation is to maximize the *passivity radius* $\rho_{\mathcal{M}}$, which is the smallest perturbation (in an appropriate norm) to the coefficients of a model $\mathcal{M}$ that makes the system non-passive. Using the computational methods to determine $\rho_{\mathcal{M}}$ introduced in Overton and Van Dooren (2005), we can optimize $\rho_{\mathcal{M}}$ over all solutions of (4), and then determine the pH representation (9), so that the system is automatically passive (but not necessarily strictly passive). Alternatively, for $X \in \operatorname{Int}\mathbb{X}^{>}$, we can determine the smallest (in Frobenius norm) perturbation $\Delta_{\mathcal{M}}$ of the model $\mathcal{M}$ that leads to a loss of positive definiteness of $W(X)$, because then we are on the boundary of the set of passive systems. This is a very suitable approach for perturbation analysis, since for fixed X the matrix $W(X)$ is linear in the unknowns A, B, C, D and when we perturb the coefficients, then we preserve strict passivity as long as

$$\begin{aligned} W_{\Delta_{\mathcal{M}}}(X) &:= \begin{bmatrix} 0 & (C+\Delta_C)^{\mathsf{H}} \\ (C+\Delta_C) & (D+\Delta_D)+(D+\Delta_D)^{\mathsf{H}} \end{bmatrix} \\ &\quad - \begin{bmatrix} (A+\Delta_A)^{\mathsf{H}}X + X(A+\Delta_A) & X(B+\Delta_B) \\ (B+\Delta_B)^{\mathsf{H}}X & 0 \end{bmatrix} > 0. \end{aligned}$$

Hence, for a given $X \in \operatorname{Int}\mathbb{X}^{>}$, we can determine the smallest perturbation $\Delta_{\mathcal{M}}$ to $\mathcal{M}$ that makes $\det(W_{\Delta_{\mathcal{M}}}(X)) = 0$, which defines the *X-passivity radius*

$$\rho_{\mathcal{M}}(X) := \inf_{\Delta_{\mathcal{M}}\in\mathbb{C}^{n+m,n+m}} \left\{\|\Delta_{\mathcal{M}}\|_F \mid \det W_{\Delta_{\mathcal{M}}}(X) = 0\right\}.$$

If for any given $X \in \operatorname{Int}\mathbb{X}^{>}$, $\|\Delta_{\mathcal{M}}\|_F < \rho_{\mathcal{M}}(X)$, then all systems $\mathcal{M} + \Delta_{\mathcal{M}}$ with $\|\Delta_{\mathcal{M}}\|_F < \rho_{\mathcal{M}}(X)$ are strictly passive. Therefore $\rho_{\mathcal{M}} \geq \sup_{\operatorname{Int}\mathbb{X}^{>}} \rho_{\mathcal{M}}(X)$. Equality follows, since there exists a perturbation $\Delta_{\mathcal{M}}$ of norm $\rho_{\mathcal{M}}$ for which there does not exist a point $X \in \operatorname{Int}\mathbb{X}^{>}$ with $W_{\Delta_{\mathcal{M}}}(X) > 0$. This thus yields the following definition.

Definition 1. The passivity radius of $\mathcal{M}$ is given by

$$\begin{aligned} \rho_{\mathcal{M}} &= \sup_{X\in\operatorname{Int}\mathbb{X}^{>}} \inf_{\Delta_{\mathcal{M}}\in\mathbb{C}^{n+m,n+m}} \{\|\Delta_{\mathcal{M}}\|_F \mid \det W_{\Delta_{\mathcal{M}}}(X) = 0\} \\ &= \sup_{X\in\operatorname{Int}\mathbb{X}^{>}} \rho_{\mathcal{M}}(X). \end{aligned}$$

To compute $\rho_{\mathcal{M}}(X)$ via an optimization problem, setting

$$\hat{W} := W(X),\ \hat{X} := \begin{bmatrix} X & 0 \\ 0 & I_m \end{bmatrix},\ \Delta_T := \begin{bmatrix} -\Delta_A & -\Delta_B \\ \Delta_C & \Delta_D \end{bmatrix}, \tag{15}$$

we can express $W_{\Delta_{\mathcal{M}}}(X) > 0$ as the LMI

$$W_{\Delta_{\mathcal{M}}} = \hat{W} + \hat{X}\Delta_T + \Delta_T^{\mathsf{H}}\hat{X} > 0 \tag{16}$$

as long as the system is still passive. To violate this condition, determine the smallest (in Frobenius norm) Δ_T such that $\det W_{\Delta_{\mathcal{M}}} = 0$. Multiplying $\hat{W}^{-\frac{1}{2}}$ on both sides of (16) yields

$$\begin{aligned} &\det\left(I_{n+m} + \hat{W}^{-\frac{1}{2}}\hat{X}\Delta_T\hat{W}^{-\frac{1}{2}} + \hat{W}^{-\frac{1}{2}}\Delta_T^{\mathsf{H}}\hat{X}\hat{W}^{-\frac{1}{2}}\right) \\ &= \det\left(I_{n+m} + \begin{bmatrix} \hat{W}^{-\frac{1}{2}}\hat{X} & \hat{W}^{-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} 0 & \Delta_T \\ \Delta_T^{\mathsf{H}} & 0 \end{bmatrix}\begin{bmatrix} \hat{X}\hat{W}^{-\frac{1}{2}} \\ \hat{W}^{-\frac{1}{2}} \end{bmatrix}\right) \\ &= \det\left(I_{2(n+m)} + \begin{bmatrix} 0 & \Delta_T \\ \Delta_T^{\mathsf{H}} & 0 \end{bmatrix}\begin{bmatrix} \hat{X}\hat{W}^{-\frac{1}{2}} \\ \hat{W}^{-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} \hat{W}^{-\frac{1}{2}}\hat{X} & \hat{W}^{-\frac{1}{2}} \end{bmatrix}\right) \\ &= 0. \end{aligned} \tag{17}$$

The minimal perturbation Δ_T for which this is the case was described in Overton and Van Dooren (2005) using the following theorem, which we have slightly modified in order to take into account the positive semi-definiteness of the considered matrix.

Theorem 2. *Consider the matrices $\hat{X}, \hat{W}$ in (15) and the pointwise positive semidefinite matrix function*

$$M(\gamma) := \begin{bmatrix} \gamma \hat{X}\hat{W}^{-\frac{1}{2}} \\ \hat{W}^{-\frac{1}{2}}/\gamma \end{bmatrix} \begin{bmatrix} \gamma \hat{W}^{-\frac{1}{2}}\hat{X} & \hat{W}^{-\frac{1}{2}}/\gamma \end{bmatrix} \tag{18}$$

in the real parameter γ. Then the largest eigenvalue $\underline{\lambda}_{\max}$ of $M(\gamma)$ is a unimodal function of γ (i.e. it is first monotonically decreasing and then monotonically increasing in γ). At the minimizing value $\underline{\gamma}$, we have

$$M(\underline{\gamma})z = \underline{\lambda}_{\max} z, \quad z := \begin{bmatrix} u \\ v \end{bmatrix},$$

where $\|u\|_2^2 = \|v\|_2^2 = \frac{1}{2}$. The minimum norm perturbation Δ_T is of rank 1 and is given by $\Delta_T = 2uv^{\mathsf{H}}/\underline{\lambda}_{\max}$. It has norm $1/\underline{\lambda}_{\max}$ both in spectral and Frobenius norm.

Proof. The proof for a modified formulation was given in Overton and Van Dooren (2005) and can be easily adapted for this case, see Beattie et al. (2018). $\square$

A bound for $\underline{\lambda}_{\max}$ in Theorem 2 is as follows.

Corollary 1. *Consider the matrices $\hat{X}, \hat{W}$ in (15) and the matrix function $M(\gamma)$ as in (18). The largest eigenvalue $\underline{\lambda}_{\max}$ of $M(\gamma)$ is also the largest eigenvalue of*

$$\gamma^2 \hat{W}^{-\frac{1}{2}}\hat{X}^2\hat{W}^{-\frac{1}{2}} + \hat{W}^{-1}/\gamma^2.$$

A simple upper bound for $\underline{\lambda}_{\max}$ is given by $\underline{\lambda}_{\max} \le \frac{2}{\alpha\beta}$ where $\alpha^2 := \lambda_{\min}(\hat{W})$ and $\beta^2 = \lambda_{\min}(\hat{X}^{-1}\hat{W}\hat{X}^{-1})$. The corresponding lower bound for $\|\Delta_T\|_F$ then becomes

$$\rho_{\mathcal{M}}(X) = \min_{\gamma} \|\Delta_T\|_F \ge \alpha\beta/2.$$

Proof. Clearly $\|\hat{W}^{-1}\|_2 \le \frac{1}{\alpha^2}$ and $\|\hat{W}^{-\frac{1}{2}}\hat{X}^2\hat{W}^{-\frac{1}{2}}\|_2 \le \frac{1}{\beta^2}$. So if we choose $\gamma^2 = \frac{\beta}{\alpha}$ then

$$\min_{\gamma} \|\gamma^2 \hat{W}^{-\frac{1}{2}}\hat{X}^2\hat{W}^{-\frac{1}{2}} + \hat{W}^{-1}/\gamma^2\|$$
$$\le \|(\beta/\alpha)\hat{W}^{-\frac{1}{2}}\hat{X}^2\hat{W}^{-\frac{1}{2}} + (\alpha/\beta)\hat{W}^{-1}\| \le \frac{2}{\alpha\beta}. \qquad \square$$

To construct a perturbation $\Delta_T = \epsilon(\alpha\beta)vu^{\mathsf{H}}$ of norm $|\epsilon|(\alpha\beta)$ which makes $W_{\Delta_{\mathcal{M}}}$ singular and thus gives an upper bound for $\rho_M(X)$, let u, v and w be vectors of norm 1, satisfying $\hat{W}^{-\frac{1}{2}}u = u/\alpha$, $\hat{W}^{-\frac{1}{2}}\hat{X}v = w/\beta$, $\Delta_T = \epsilon(\alpha\beta)vu^{\mathsf{H}}$, and $\epsilon u^{\mathsf{H}}w = -|\epsilon u^{\mathsf{H}}w|$, i.e., u, v and w are singular vectors to the largest singular values $1/\alpha$ of $\hat{W}^{-\frac{1}{2}}$ and $1/\beta$ of $\hat{W}^{-\frac{1}{2}}\hat{X}$. Inserting these in (17), we get

$$\det\left(I_{n+m} + \begin{bmatrix} \hat{W}^{-\frac{1}{2}}\hat{X} & \hat{W}^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 0 & \Delta_T \\ \Delta_T^{\mathsf{H}} & 0 \end{bmatrix} \begin{bmatrix} \hat{X}\hat{W}^{-\frac{1}{2}} \\ \hat{W}^{-\frac{1}{2}} \end{bmatrix}\right)$$
$$= \det\left(I_{n+m} + \begin{bmatrix} w & u \end{bmatrix} \begin{bmatrix} 0 & \epsilon \\ \bar{\epsilon} & 0 \end{bmatrix} \begin{bmatrix} w^{\mathsf{H}} \\ u^{\mathsf{H}} \end{bmatrix}\right)$$
$$= \det\left(I_2 + \begin{bmatrix} 0 & \epsilon \\ \bar{\epsilon} & 0 \end{bmatrix} \begin{bmatrix} w^{\mathsf{H}} \\ u^{\mathsf{H}} \end{bmatrix} \begin{bmatrix} w & u \end{bmatrix}\right),$$

which we can make 0 by choosing ϵ such that $\epsilon u^{\mathsf{H}}w$ is real and negative and satisfies $1 = |\epsilon u^{\mathsf{H}}w| + |\epsilon|$. Since $0 \le |u^{\mathsf{H}}w| \le 1$, we have that $\frac{1}{2} \le |\epsilon| \le 1$ and thus we have

$$\alpha\beta/2 \le \rho_{\mathcal{M}}(X) \le |\epsilon|\alpha\beta. \tag{19}$$

Moreover, if u and w are linearly dependent, then this interval shrinks to a point and the estimate is exact. We have the following corollary.

Corollary 2. *If for a system $\mathcal{M}$ we have $X = I_n \in \operatorname{Int}\mathbb{X}^{>}$ then the representation is pH, i.e., $\mathcal{M} := \{J - R, G - K, G^{\mathsf{H}} + K^{\mathsf{H}}, D\}$ and*

$$\rho_{\mathcal{M}}(I) = \lambda_{\min}(W(I)).$$

Moreover, if $X = I_n$ is the analytic center of $\operatorname{Int}\mathbb{X}^{>}$, then $\rho_{\mathcal{M}}(I)$ equals the passivity radius $\rho_{\mathcal{M}}$ of $\mathcal{M}$.

Proof. This follows directly from (19), since then $\alpha = \beta$ and we can choose $u = w$. $\square$

Remark 2. In a pH representation, the conditions $\hat{W} \ge \alpha^2 I_{n+m}$ and $\hat{X}^{-1}\hat{W}\hat{X}^{-1} \ge \beta^2 I_{n+m}$ yield the necessary (but not sufficient) condition for passivity that

$$\begin{bmatrix} \hat{W} & \alpha\beta I_{n+m} \\ \alpha\beta I_{n+m} & \hat{X}^{-1}\hat{W}\hat{X}^{-1} \end{bmatrix} \ge 0,$$

With $\hat{T} := \hat{X}^{\frac{1}{2}}$ this is equivalent to $\hat{T}^{-1}\hat{W}\hat{T}^{-1} \ge \alpha\beta I_{n+m}$. Then

$$\xi := \lambda_{\min}(\hat{T}^{-1}\hat{W}\hat{T}^{-1}) \ge \alpha\beta,$$

which suggests that pH representations are likely to give a good passivity margin. In order to compute the optimal product $\xi = \alpha\beta$, we could maximize ξ under the constraint $\hat{W} - \xi\hat{X} > 0$.

Our previous discussions suggest that if we want a state-space representation that has a maximal passivity radius, we should not maximize $\det(W(X))$, but instead

$$\det\left(\begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & I_m \end{bmatrix} W(X) \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & I_m \end{bmatrix}\right)$$
$$= \det(W(X)\begin{bmatrix} X^{-1} & 0 \\ 0 & I_m \end{bmatrix}) := \det \tilde{W}(X) \tag{20}$$

under the constraint $X > 0$ so that $T = X^{\frac{1}{2}}$ exists. The gradient and the incremental step for the associated barrier function $\tilde{b}(X) := -\log\det \tilde{W}(X)$ in this case are then given by

$$\partial\tilde{b}(X/\partial\tilde{W}) = -\tilde{W}(X)^{-\mathsf{H}} = -W(X)^{-1}\begin{bmatrix} X & 0 \\ 0 & I_m \end{bmatrix},$$

$$\Delta\tilde{W}(X)[\Delta_X] = \begin{bmatrix} XAX^{-1}\Delta_X - \Delta_X A & -\Delta_X B \\ -CX^{-1}\Delta_X & 0 \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ 0 & I_m \end{bmatrix},$$

and the necessary optimality condition is that

$$\left\langle \tilde{W}(X)^{-1}, \begin{bmatrix} XAX^{-1}\Delta_X - \Delta_X A & -\Delta_X B \\ -CX^{-1}\Delta_X & 0 \end{bmatrix} \right\rangle = 0 \tag{21}$$

for all $\Delta_X \in \mathbb{H}_n$. Proceeding as before, with the same P and F it then follows that (21) holds if and only if P is invertible and

$$\left\langle \Delta_X, P^{-1}(XAX^{-1} + F^{\mathsf{H}}CX^{-1}) - (A - BF)P^{-1} \right\rangle = 0$$

for all $\Delta_X \in \mathbb{H}_n$. With $T = X^{\frac{1}{2}}$, $P_T = T^{-1}PT^{-1}$, and $F_T = FT^{-1}$, setting $\{A_T, B_T, C_T, D_T\} := \{TAT^{-1}, TB, CT^{-1}, D\}$ we get equivalently

$$P_T[(A_T^{\mathsf{H}} - A_T) + (C_T^{\mathsf{H}} + B_T)F_T]$$
$$+[(A_T^{\mathsf{H}} - A_T) + (B_T + C_T^{\mathsf{H}})F_T]^{\mathsf{H}}P_T = 0.$$

Using a pH representation $\mathcal{M}_T = \{A_T, B_T, C_T, D_T\} = \{J - R, G - K, (G + K)^{\mathsf{H}}, D\}$, then at the analytic center of $\tilde{b}$ we have $X_T = I$, $SF_T = K^{\mathsf{H}}$, $P_T = R - F_T^{\mathsf{H}}SF_T$, and $0 = P_T(J - GF_T) + (J - GF_T)^{\mathsf{H}}P_T$ which implies that the passivity radius is given by $\lambda_{\min}(2\mathcal{W})$. On the other hand, since we have optimized $\det(\tilde{W}(X))$ which has the same determinant as (20), although we cannot prove this, we can expect to have a nearly optimal passivity margin.

To illustrate our construction, we present a numerical example, for an analytic solution in the case $m = n = 1$ see Beattie et al. (2018). As a test case we look at a random numerical model $\{A, B, C, D\}$ in pH form of state dimension $n = 6$ and input/output dimension $m = 3$ via $\mathcal{W} := MM^{\mathsf{H}}$, where M is a random $(n+m) \times (n+m)$ matrix generated in MATLAB. From this we then identified the model $A := -R/2, B := -C^{\mathsf{H}} := -K/2$ and $D := S/2$, so that $X_0 = I_n$ satisfies the LMI positivity constraint for the model $\mathcal{M} := \{A, B, C, D\}$. We then used a Newton iteration to compute the analytic center X_c of the LMI $W(X) > 0$ with barrier function $b(X) := -\ln\det W(X)$. We determined the quantities $\alpha^2 := \lambda_{\min}(\hat{W}), \beta^2 := \lambda_{\min}(\hat{X}^{-1}\hat{W}\hat{X}^{-1})$, and $\xi := \lambda_{\min}(\hat{X}^{-\frac{1}{2}}\hat{W}\hat{X}^{-\frac{1}{2}})$. The constructed matrix $\hat{X}^{-\frac{1}{2}}\hat{W}\hat{X}^{-\frac{1}{2}}$ contains the parameters of the pH realization at X_c. The results are given in the table:

α^2	β^2	ξ	$\alpha\beta$
0.002366	0.001065	0.002381	0.001587

Note that ξ at the analytic center is a reasonable approximation of the passivity radius estimate $\alpha\beta$.

5. Conclusion

In this paper we have introduced the concept of the analytic center for a barrier function derived from the KYP LMI for passive systems. We have shown that its analytic center tends to optimize the passivity radius of the model corresponding to the KYP LMI. Moreover, we present a modified LMI which combined with a pH representation, yields a nearly optimal passivity margin.

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References

Beattie, C., Mehrmann, V., & Van Dooren, P. (2018). Robust port-Hamiltonian representations of passive systems. Inst. f. Mathematik, TU Berlin. Preprint 02-2018 http://arxiv:1801.05018.

Benner, P., Losse, P., Mehrmann, V., & Voigt, M. (2015). Numerical linear algebra methods for linear differential–algebraic equations. In A. Ilchmann, & T. Reis (Eds.), *Differ.-Algebr. Equ. Forum. Surveys in Differential-Algebraic Equations III* (pp. 117–175). Cham, CH: Springer-Verlag.

Boyd, S., El Ghaoui, L., Feron, E., & Balakrishnan, V. (1994). *Linear Matrix Inequalities in Systems and Control Theory*. Philadelphia: SIAM.

Genin, Y., Nesterov, Y., & Van Dooren, P. (1999). The analytic center of lmi's and riccati equations. In *Proceedings of the European Control Conference*.

Gillis, N., Mehrmann, V., & Sharma, P. (2018). Computing nearest stable matrix pairs. *Numerical Linear Algebra with Applications*. http://dx.doi.org/10.1002/nla.2153.

Gillis, N., & Sharma, P. (2018). Finding the nearest positive-real system. *SIAM Journal on Numerical Analysis*, *56*(2), 1022–1047.

Gugercin, S., Polyuga, R. V., Beattie, C., & van der Schaft, A. J. (2012). Structure-preserving tangential interpolation for model reduction of port-hamiltonian systems. *Automatica*, *48*, 1963–1974.

Mehl, C., Mehrmann, V., & Sharma, P. (2016). Stability radii for linear hamiltonian systems with dissipation under structure-preserving perturbations. *SIAM Journal of Mathematical Analysis*, *37*(4), 1625–1654.

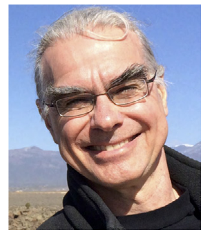
Nesterov, Yu., & Nemirovski, A. (1994). *Interior Point Polynomial Methods in Convex Programming*. Phildadelphia: SIAM.

Overton, M., & Van Dooren, P. (2005). On computing the complex passivity radius. In *Proceedings CDC-ECC 2005* (pp. 7960–7964).

van der Schaft, A. J., & Jeltsema, D. (2014). Port-hamiltonian systems theory: An introductory overview. *Foundations and Trends in Systems and Control*, *1*(2–3), 173–378.

Willems, J. C. (1971). Least squares stationary optimal control and the algebraic riccati equation. *IEEE Transactions on Automatic Control*, *AC-16*(6), 621–634.

Willems, J. C. (1972). Dissipative dynamical systems –part I: General theory. *Archive for Rational Mechanics and Analysis*, *45*, 321–351.



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