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A numerical method for deadbeat control of generalized state-space systems

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Abstract: In this paper we give a new numerical method for constructing a rank m correction BF to an $n \times n$ matrix A, such that the generalized eigenvalues of $\lambda E - (A + BF)$ are all at $\lambda = 0$. In the control literature, this problem is known as 'deadbeat control' of a generalized state-space system $Ex_{i+1} = Ax_i + Bu_i$, whereby the matrix F is the 'feedback matrix' to be constructed.

Keywords: Deadbeat control, Generalized state-space systems, Numerical methods.

1. Introduction

Consider the following generalized state-space system:

$$Ex_{i+1} = Ax_i + Bu_i, \quad i \geq 0 \text{ and } x_0 \text{ given,} \tag{1}$$

where E and A are $n \times n$ matrices and B is an $n \times m$ matrix. Substituting a control law $u_i = Fx_i$ in (1) we obtain the modified system

$$Ex_{i+1} = (A + BF)x_i, \quad i \geq 0 \text{ and } x_0 \text{ given.} \tag{2}$$

The problem of *deadbeat control* is now to find a feedback law F that will drive an arbitrary initial state x_0 to $x_k = 0$ after a *minimum* number of steps k. All subsequent x_i (for $i \geq k$) are then also zero, which explains the term 'deadbeat'. For standard state-space systems (i.e. $E = I_n$) this is known to be equivalent to finding a feedback F such that all eigenvalues of $A + BF$ lie at $\lambda = 0$, and $A + BF$ has Jordan chains of *minimal* length (see [1,6] for alternative definitions). Extending this *algebraic* definition to generalized state-space systems yields:

$$\begin{array}{l}\text{Let } E \text{ and } A \text{ be square } n \times n \text{ matrices and } B \text{ an } n \times m \text{ matrix.}\\ \text{Find an } m \times n \text{ matrix } F \text{ such that the pencil } \lambda E - (A + BF)\\ \text{has } n \text{ eigenvalues at } \lambda = 0 \text{ and Jordan chains of minimal length.}\end{array} \tag{3}$$

This we will call here the *generalized deadbeat problem* since it can be interpreted as the deadbeat problem of a generalized state-space system (see also [5]). We note here that if E is singular, (3) is not necessarily equivalent to the problem formulation we started from (take e.g. the simple 1×1 system with $E = 0$, $A = 1$, $B = 1$ and $F = -1$). The difference lies in the peculiarities one encounters with 'infinite modes' and 'singular pencils' occurring when E is singular. For systems with singular E its known that at most rank E eigenvalues can be placed by feedback and hence (3) can never be fulfilled. One could then yet consider the problem of placing as many eigenvalues as possible at $\lambda = 0$ using techniques such as those described in [7]. However, this does not imply that the original problem (1),(2) is then solved since x_{i+1} in (2) is not uniquely determined and can thus be chosen nonzero. Therefore it is assumed here that E is invertible (see also [5]). In this case the problem can be reduced to a deadbeat control problem for the standard state-space system

$$x_{i+1} = E^{-1}Ax_i + E^{-1}Bu_i, \quad i \geq 0 \text{ and } x_0 \text{ given}, \tag{4}$$

and it is easily seen that formulation (3) is then equivalent to the deadbeat requirement of an arbitrary state x_0 in (2). It should be emphasized that for reason of numerical stability one should certainly avoid the inversion of the matrix E, if possible. This paper precisely considers solving the generalized deadbeat control problem (3) from a numerical point of view, via an *implicit* algorithm which avoids the inversion of E. Nevertheless, our approach for solving problem (3) is strongly inspired from the method given by Van Dooren [12] for the standard deadbeat problem. Therefore, we adopt the notation used in [12] as much as possible.

2. Problem reformulation

When the system (1) is uncontrollable then the generalized deadbeat control problem (3) clearly has a solution if and only if the uncontrollable modes are already at $\lambda = 0$. If so, the problem can then be reduced to one involving only the controllable part of the system (see [12]). The system (1) may thus be assumed controllable without loss of generality, i.e. [8,9],

$$\operatorname{rank}[\lambda E - A \mid B] = n \quad \text{for all finite } \lambda \in \mathbf{C} \tag{5}$$

or equivalently (since E is assumed invertible),

$$\operatorname{rank}\left[\lambda I_n - E^{-1}A \mid E^{-1}B\right] = n \quad \text{for all finite } \lambda \in \mathbf{C}. \tag{6}$$

Since E is invertible, the Jordan chians of $\lambda E - (A + BF)$ and those of $\lambda I - E^{-1}(A + BF)$ are equal and any feedback F for the pair $(E^{-1}A,\ E^{-1}B)$ is also one for the system $(\lambda E - A,\ B)$, and conversely. We can thus rely on the results of [12] to say that for any deadbeat feedback F the lengths of the chains are in fact equal to the controllability indices c_i, $i = 1, \ldots, m$, of $(E^{-1}A,\ E^{-1}B)$. To retrieve these indices c_i one could make use of the *staircase form* of $(E^{-1}A,\ E^{-1}B)$, which can be obtained under an orthogonal state transformation V [11]:

$$\left[\hat{B} \mid \hat{A}\right] = \left[V'E^{-1}B \mid V'E^{-1}AV\right] = \begin{array}{|c|ccccc|l}
\hat{B}_1 & \hat{A}_{1,1} & \hat{A}_{1,2} & \cdots & \cdots & \hat{A}_{1,k} & \}r_1 \\
0 & \hat{A}_{2,1} & \hat{A}_{2,2} & & & \vdots & \}r_2 \\
0 & 0 & \ddots & \ddots & & & \\
\vdots & \vdots & & \ddots & \ddots & \vdots & \\
0 & 0 & \cdots & 0 & \hat{A}_{k,k-1} & \hat{A}_{k,k} & \}r_k \\
\multicolumn{1}{c}{\underbrace{\qquad}_{m}} & \multicolumn{1}{c}{\underbrace{\qquad}_{r_1}} & \underbrace{\qquad}_{r_2} & & \underbrace{\qquad}_{r_{k-1}} & \multicolumn{1}{c}{\underbrace{\qquad}_{r_k}} &
\end{array}, \tag{7}$$

where

$$\begin{aligned} &\hat{B}_1 \text{ is } r_1 \times m \text{ and has full row rank } r_1, \\ &\hat{A}_{i,i-1} \text{ is } r_i \times r_{i-1} \text{ and has full row rank } r_i, \quad 2 \leq i \leq k. \end{aligned} \tag{8}$$

As shown in [12] these ranks r_i directly yield the controllability indices via the rule

$$\text{there are } r_j - r_{j+1} \text{ indices } c_i \text{ equal to } j. \tag{9}$$

In order to avoid the inversion of E, we use here a different approach. A similar form indeed exists for the generalized state-space system $(\lambda E - A,\ B)$ using orthogonal transformations W and V (see e.g. [8]):

$$\tilde{E} = WEV = \left|\begin{array}{cccc|l} \tilde{E}_{1,1} & \tilde{E}_{1,2} & \cdots & \tilde{E}_{1,k} & \}r_1 \\ 0 & \tilde{E}_{2,2} & & \vdots & \}r_2 \\ \vdots & & \ddots & \vdots & \\ 0 & \cdots & 0 & \tilde{E}_{k,k} & \}r_k \\ \underbrace{\qquad}_{r_1} & \underbrace{\qquad}_{r_2} & & \underbrace{\qquad}_{r_k} & \end{array}\right., \tag{10}$$

and

$$[\tilde{B} \mid \tilde{A}] = [WB \mid WAV] = \left|\begin{array}{c:cccccc|l} \tilde{B}_1 & \tilde{A}_{1,1} & \tilde{A}_{1,2} & \cdots & & \cdots & \tilde{A}_{1,k} & \}r_1 \\ 0 & \tilde{A}_{2,1} & \tilde{A}_{2,2} & & & & \vdots & \}r_2 \\ 0 & 0 & \ddots & \ddots & & & & \\ \vdots & \vdots & & \ddots & \ddots & & \vdots & \\ 0 & 0 & \cdots & 0 & \tilde{A}_{k,k-1} & & \tilde{A}_{k,k} & \}r_k \\ \underbrace{\qquad}_{m} & \underbrace{\qquad}_{r_1} & \underbrace{\qquad}_{r_2} & & \underbrace{\qquad}_{r_{k-1}} & & \underbrace{\qquad}_{r_k} & \end{array}\right., \tag{11}$$

where

$$\begin{aligned} &\tilde{B}_1 \text{ is } r_1 \times m \text{ and has full row rank } r_1, \\ &\tilde{E}_{i,i} \text{ is upper triangular and invertible}, \quad 1 \le i \le k, \\ &\tilde{A}_{i,i-1} \text{ is } r_i \times r_{i-1} \text{ and has full row rank } r_i, \quad 2 \le i \le k. \end{aligned} \tag{12}$$

It is easily seen from (10), (11) that $(\tilde{E}^{-1}\tilde{A},\ \tilde{E}^{-1}\tilde{B}) = (V'E^{-1}AV,\ V'E^{-1}B) = (\hat{A},\ \hat{B})$ is in fact the staircase form (7) and that the rank properties (12) are equivalent to (8). With (10)–(12) we thus *implicitly* computed the transformation V reducing $(E^{-1}A,\ E^{-1}B)$ to staircase form without ever computing E^{-1}.

For the construction of a deadbeat feedback we now rely on the connections between these two forms. We start from the algorithmic solution given in [12] for state-space systems in the form (7) to derive an *implicit* algorithm for generalized state-space systems in the form (10), (11). This relation e.g. implies that the constructed feedback F is also the minimum norm solution to the problem (see Remark 1). For the construction of the deadbeat feedback F of $(E^{-1}A,\ E^{-1}B)$ as explained in [12] one is looking for an additional orthogonal transformation $\tilde{U}$ such that

$$\tilde{U}'(\hat{A} + \hat{B}F)\tilde{U} = \left|\begin{array}{ccccc|l} 0 & \bar{A}_{1,2} & \cdots & \cdots & \bar{A}_{1,k} & \}r_1 \\ & 0 & & & & \\ & & \ddots & & \vdots & \\ & & & 0 & \bar{A}_{k-1,k} & \}r_{k-1} \\ & & & & 0 & \}r_k \\ \underbrace{\qquad}_{r_1} & \underbrace{\qquad}_{r_2} & & & \underbrace{\qquad}_{r_k} & \end{array}\right., \tag{13}$$

which clearly is a nilpotent matrix. Moreover it can be shown [10,12] that the Jordan chains at $\lambda = 0$ of

(13) are indeed equal to c_i as given in (9). In the generalized state-space case we are now looking for additional orthogonal transformations $\tilde{Q}$ and $\tilde{U}$ and a feedback matrix F such that

$$\tilde{Q}\tilde{E}\tilde{U}=\left|\begin{array}{cccc} E_{1,1} & E_{1,2} & \cdots & E_{1,k} \\ 0 & E_{2,2} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & E_{k,k} \end{array}\right| \begin{array}{l} \}r_1 \\ \}r_2 \\ \\ \}r_k \end{array}, \qquad \begin{array}{cccc} \underbrace{\quad}_{r_1} & \underbrace{\quad}_{r_2} & & \underbrace{\quad}_{r_k} \end{array} \tag{14}$$

and

$$\tilde{Q}(\tilde{A}+\tilde{B}F)\tilde{U}=\left|\begin{array}{cccc} 0 & A_{1,2} & \cdots & A_{1,k} \\ & 0 & & \vdots \\ & & \ddots & A_{k-1,k} \\ & & & 0 \end{array}\right| \begin{array}{l} \}r_1 \\ \\ \}r_{k-1} \\ \}r_k \end{array}. \qquad \begin{array}{cccc} \underbrace{\quad}_{r_1} & \underbrace{\quad}_{r_2} & & \underbrace{\quad}_{r_k} \end{array} \tag{15}$$

Again, it is easily seen from (14), (15) that $\tilde{U}'\tilde{E}^{-1}(\tilde{A}+\tilde{B}F)\tilde{U}$ has the form (13) and that in fact we implicitly computed the transformation $\tilde{U}$ of (13) without passing via $(\hat{A}, \hat{B})=(\tilde{E}^{-1}\tilde{A}, \tilde{E}^{-1}\tilde{B})$. This implicit algorithm is now given in the next section.

3. An algorithm for generalized deadbeat control

The construction of $\tilde{Q}$ and $\tilde{U}$ is performed by a recursive algorithm consisting of k steps where k is defined in (10), (11). The starting situation for the algorithm is given in (10), (11). We show that at the end of each step i $(1 \le i \le k)$ we have

$$Q_i(A+BF_i)U_i=\left|\begin{array}{c|c} A_d^i & X \\ \hline 0 & A_s^i \end{array}\right| \begin{array}{l} \}d_i \\ \}n-d_i \end{array}, \qquad Q_iEU_i=\left|\begin{array}{c|c} E_d^i & X \\ \hline 0 & E_s^i \end{array}\right| \begin{array}{l} \}d_i \\ \}n-d_i \end{array},$$

$$Q_iB=\left|\begin{array}{c} B_d^i \\ \hline B_s^i \end{array}\right| \begin{array}{l} \}d_i \\ \}n-d_i \end{array}, \tag{16}$$

where Q_i and U_i are orthogonal and $d_i=\Sigma_{j=1}^{i} r_j$. The subsystem $(\lambda E_d^i - A_d^i,\ B_d^i)$ in (16) is already in the form (14), (15), i.e. E_d^i is an invertible upper triangular matrix and

$$\left[B_d^i \mid A_d^i\right]=\left|\begin{array}{c|cccc} B_1^i & 0 & A_{1,2}^i & \cdots & A_{1,i}^i \\ & & 0 & & \vdots \\ \vdots & & & \ddots & A_{i-1,i}^i \\ B_i^i & & & & 0 \end{array}\right| \begin{array}{l} \}r_1 \\ \\ \}r_{i-1} \\ \}r_i \end{array}, \tag{17}$$

and the subsystem $(\lambda E_s^i - A_s^i,\ B_s^i)$ is still in staircase form, i.e. E_s^i is an invertible upper triangular matrix and

$$[B_s^i \,|\, A_s^i] = \left|\begin{array}{c|cccc|l} B_{i+1}^i & A_{i+1,i+1}^i & \cdots & \cdots & A_{i+1,k}^i & \}\,r_{i+1} \\ 0 & A_{i+2,i+1}^i & & & A_{i+2,k}^i & \}\,r_{i+2} \\ \vdots & & \ddots & & \vdots & \\ 0 & & & A_{k,k-1}^i & A_{k,k}^i & \}\,r_k \end{array}\right., \tag{18}$$

$$E_s^i = \left|\begin{array}{cccc|l} E_{i+1,i+1}^i & \cdots & \cdots & E_{i+1,k}^i & \}\,r_{i+1} \\ & \ddots & & E_{i+2,k} & \}\,r_{i+2} \\ & & \ddots & \vdots & \\ & & & E_{k,k}^i & \}\,r_k \end{array}\right. \tag{19}$$

where the blocks B_{i+1}^i and $A_{j,j-1}^i$ $(j = i+2,\dots,k)$ have full row rank and the blocks $E_{j,j}^i$ $(j = i+1,\dots,k)$ are invertible. Clearly, the situation at the beginning of step 1 (i.e. at the end of step 0) is indeed given by (16)–(19).

We now describe the i-th step of the algorithm. To this end, we assume we have the situation in (16)–(19) with i replaced by $i-1$.

Next, we construct an orthogonal matrix U_s^i that transforms A_s^{i-1} to upper triangular form R^i, i.e.,

$$A_s^{i-1}U_s^i = R^i = \left|\begin{array}{ccc|l} R_{i,i}^i & \cdots & R_{i,k}^i & \}\,r_i \\ & \ddots & \vdots & \\ & & R_{k,k}^i & \}\,r_k \\ \hline \underbrace{\phantom{R_{i,i}^i}}_{r_i} & & \underbrace{\phantom{R_{k,k}^i}}_{r_k} & \end{array}\right.. \tag{20}$$

Using the special rank properties of A_s^{i-1} it can easily be shown (see [12]) that the matrices $R_{j,j}^i$ with $j > i$ (but not necessarily $R_{i,i}^i$) are invertible. Before proceeding with the descriptions of the transformations we mention the following properties of matrix $(U_s^i)'$ to be used later.

Rewriting (20) as $A_s^{i-1} = R^i(U_s^i)'$ we find that the matrix $(U_s^i)'$ has the same upper block Hessenberg form as A_s^{i-1}, i.e.,

$$(U_s^i)' = \left|\begin{array}{cccc|l} U_{i,i} & \cdots & U_{i,k-1} & U_{i,k} & \}\,r_i \\ U_{i+1,i} & & & & \}\,r_{i+1} \\ & \ddots & & \vdots & \\ & & U_{k,k-1} & U_{k,k} & \}\,r_k \\ \hline \underbrace{\phantom{U_{i,i}}}_{r_i} & & \underbrace{\phantom{U_{k,k-1}}}_{r_{k-1}} & \underbrace{\phantom{U_{k,k}}}_{r_k} & \end{array}\right.. \tag{21}$$

where, moreover,

$$U_{j,j-1} \text{ has full row rank}, \quad i+1 \le j \le k. \tag{22}$$

We proceed by solving the equation

$$R_{i,i}^i = -B_i^{i-1}X. \tag{23}$$

Since B_i^{i-1} has full row rank, this equation has a minimum norm solution G_i given by

$$G_i = -\left(B_i^{i-1}\right)^+ R_{i,i}^i \tag{24}$$

where $(B_i^{i-1})^+$ denotes the Moore–Penrose inverse of B_i^{i-1}. Define the feedback matrix F_s^i by

$$F_s^i=[G_i,0,\ldots,0](U_s^i)'. \tag{25}$$

Note that F_s^i has dimensions $m\times\Sigma_{j=i}^k r_j$ $(=m\times(n-d_{i-1}))$. Then we have

$$(A_s^{i-1}+B_s^{i-1}F_s^i)U_s^i=\begin{array}{|cccc|l}
0 & R^i_{i,i+1} & \cdots & R^i_{i,k} & \}\,r_i\\
0 & R^i_{i+1,i+1} & & R^i_{i+1,k} & \}\,r_{i+1}\\
\vdots & & \ddots & \vdots & \\
0 & & & R^i_{k,k} & \}\,r_k\\
\multicolumn{1}{c}{\underbrace{\quad}_{r_i}} & \multicolumn{1}{c}{\underbrace{\quad}_{r_{i+1}}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\underbrace{\quad}_{r_k}} &
\end{array}. \tag{26}$$

We recall that the same transformations have to be applied to the matrices A and E and therefore to A_s^{i-1} and E_s^{i-1} (see (16)). Hence, matrix E_s^{i-1} is postmultiplied by U_s^i since A_s^{i-1} is in (26). Note that the resulting matrix $E_s^{i-1}U_s^i$ may no longer be upper (block) triangular. However, we can construct an orthogonal matrix Q_s^i such that $E_s^i:=Q_s^iE_s^{i-1}U_s^i$ is again upper (block) triangular, i.e.

$$E_s^i:=Q_s^iE_s^{i-1}U_s^i=\begin{array}{|ccc|l}
E^i_{i,i} & \cdots & E^i_{i,k} & \}\,r_i\\
 & \ddots & \vdots & \\
 & & E^i_{k,k} & \}\,r_k\\
\multicolumn{1}{c}{\underbrace{\quad}_{r_i}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\underbrace{\quad}_{r_k}} &
\end{array}. \tag{27}$$

Notice that the $E^i_{j,j}$ $(i\le j\le k)$ are invertible since Q_s^i and U_s^i are orthogonal and E_s^{i-1} is invertible. Moreover, the $E^i_{j,j}$ $(i\le j\le k)$ are upper triangular due to the special construction of Q_s^i. Since E_s^{i-1} and E_s^i are upper block triangular and $(U_s^i)'$ is upper block Hessenberg, it follows from $E_s^i(U_s^i)'=Q_s^iE_s^{i-1}$ that Q_s^i has the same structure as $(U_s^i)'$ (using the same reasoning as for $(U_s^i)'$) in (21)). Thus, we have that

$$Q_s^i=\begin{array}{|cccc|l}
Q_{i,i} & \cdots & Q_{i,k-1} & Q_{i,k} & \}\,r_i\\
Q_{i+1,i} & & & & \}\,r_{i+1}\\
 & \ddots & & \vdots & \\
 & & Q_{k,k-1} & Q_{k,k} & \}\,r_k\\
\multicolumn{1}{c}{\underbrace{\quad}_{r_i}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\underbrace{\quad}_{r_{k-1}}} & \multicolumn{1}{c}{\underbrace{\quad}_{r_k}} &
\end{array}. \tag{28}$$

By expressing the blocks $Q_{j,j-1}$ $(i+1\le j\le k)$ explicitly in terms of the blocks in E_s^{i-1}, E_s^i and $(U_s^i)'$ and using (21) and the fact that the diagonal blocks of E_s^{i-1} and E_s^i are invertible (see (19) and (27)), we find that

$$Q_{j,j-1}\text{ has full row rank,}\quad i+1\le j\le k. \tag{29}$$

We finish the transformations in step i by premultiplying the matrices B_s^{i-1} and $(A_s^{i-1}+B_s^{i-1}F_s^i)U_s^i$ by Q_s^i (see (19) and (26)). This gives

$$[Q_s^iB_s^{i-1}\,|\,Q_s^i(A_s^{i-1}+B_s^{i-1}F_s^i)U_s^i]=\begin{array}{|c|cccc|l}
B_i^i & 0 & A^i_{i,i+1} & \cdots & A^i_{i,k} & \}\,r_i\\
B^i_{i+1} & 0 & A^i_{i+1,i+1} & & A^i_{i+1,k} & \}\,r_{i+1}\\
0 & 0 & A^i_{i+2,i+1} & & \vdots & \\
 & & & \ddots & \vdots & \\
0 & 0 & & A^i_{k,k-1} & A^i_{k,k} & \}\,r_k\\
\multicolumn{1}{c}{\underbrace{\quad}_{m}} & \multicolumn{1}{c}{\underbrace{\quad}_{r_i}} & \multicolumn{1}{c}{\underbrace{\quad}_{r_{i+1}}} & \multicolumn{1}{c}{\underbrace{\quad}_{r_{k-1}}} & \multicolumn{1}{c}{\underbrace{\quad}_{r_k}} &
\end{array}. \tag{30}$$

It can readily be verified from (18), (26) and (28) that

$$B^i_{i+1} = Q_{i+1,i} B^{i-1}_i \quad \text{and} \quad A^i_{j,j-1} = Q_{j,j-1} R^i_{j-1,j-1}, \quad i+2 \le j \le k. \tag{31}$$

Since B^{i-1}_i and $Q_{j,j-1}$ $(i+1 \le j \le k)$ have full row rank (see (18) and (29)) and the $R^i_{j,j}$ $(i+1 \le j \le k)$ are invertible (see (20)), it follows from (31) that

$$B^i_{i+1} \text{ and } A^i_{j,j-1} \ (i+2 \le j \le k) \text{ have full row rank.} \tag{32}$$

Inserting the results (27), (30) and (32) in the formulas (16)–(19) (with i replaced by $i-1$) we see that at the end of step i the forms (16)–(19) are retrieved. It can easily be checked that the $n \times n$ transformation matrices U_i and Q_i defined in (16) are given by

$$U_i := U_{i-1} \hat{U}_i, \quad i \ge 1, \qquad U_0 := I_n, \qquad Q_i := \hat{Q}_i Q_{i-1}, \quad i \ge 1, \qquad Q_0 := I_n,$$

where

$$\hat{U}_i := \left|\begin{array}{c|c} I_{d_{i-1}} & \\ \hline & U^i_s \end{array}\right|, \quad \hat{Q}_i := \left|\begin{array}{c|c} I_{d_{i-1}} & \\ \hline & Q^i_s \end{array}\right|, \qquad i \ge 1. \tag{33}$$

Let us now consider the update of the feedback matrices F_i. To this end, we introduce the $m \times n$ matrices $\hat{F}_i$ $(1 \le i \le k)$ by

$$\hat{F}_i := [\underbrace{\ 0\ }_{d_{i-1}} \mid F^i_s]. \tag{34}$$

Using (25) and (33) we find

$$\hat{F}_i \hat{U}_i = [\underbrace{\ 0\ }_{d_{i-1}} \mid F^i_s U^i_s] = [0 \mid G_i, 0, \dots, 0], \quad 1 \le i \le k. \tag{35}$$

At the end of step i $(i \ge 1)$ the feedback matrix F_i is then given by

$$F_i U_i = (F_{i-1} U_{i-1} + \hat{F}_i) \hat{U}_i \quad \text{with} \quad F_0 := 0. \tag{36}$$

Notice that the last $n - d_{i-1}$ columns of $F_{i-1} U_{i-1}$ are zero. Hence, in (36) we have $F_{i-1} U_{i-1} \hat{U}_i = F_{i-1} U_{i-1}$ and from this it follows that

$$F_i U_i = F_{i-1} U_{i-1} + \hat{F}_i \hat{U}_i = F_{i-1} U_{i-1} + [0, \dots, 0, G_i, 0, \dots, 0] \tag{37}$$

$$= [G_1, \dots, G_{i-1}, 0, \dots, 0] + [0, \dots, 0, G_i, 0, \dots, 0] = [G_1, \dots, G_{i-1}, G_i, 0, \dots, 0]. \tag{38}$$

Here the description of step i ends.

For completeness, we still have to define $\tilde{Q} := Q_k$, $\tilde{U} := U_k$ and $F := F_k$ where k denotes the total number of steps of the algorithm.

Remark 1. By carefully comparing the above results and those in [12] we see that the matrices U_i and F_i defined above are the same as the corresponding ones in [12]. Therefore, we also find here the minimum Frobenius norm solution F.

We end this section by summarizing the procedure presented above.

Algorithm for solving the generalized deadbeat problem.

1. Compute the staircase form of $(\lambda E - A,\ B)$ as indicated in (10)–(12).

2. Perform the following recursive procedure:

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comment initialization;
$E_s^0 := \tilde{E}$, $A_s^0 := \tilde{A}$, $B_s^0 := \tilde{B}$, $F_0 := 0$, $U_0 := I_n$, $Q_0 := I_n$, $i := 1$;
while $i \leq k$ do
    determine $U_s^i$ such that $A_s^{i-1} U_s^i = R^i$ (see (20), (21));
    determine $G_i$ in (24) by solving (23);
    compute $E_s^{i-1} U_s^i$;
    construct $Q_s^i$ such that $Q_s^i E_s^{i-1} U_s^i$ is upper triangular (see (27));
    construct $\hat{U}_i$, $\hat{Q}_i$ and determine $U_i$, $Q_i$ (see (33));
    compute $F_i U_i$ using (37);
    set $E_s^i$, $A_s^i$ and $B_s^i$ respectively equal to $Q_s^i E_s^{i-1} U_s^i$, $Q_s^i B_s^{i-1}$ and
    $Q_s^i (A_s^{i-1} + B_s^{i-1} F_s^i) U_s^i$, after leaving out the first $r_i$ rows and/or columns;
    $i := i + 1$
end while-clause;
```

Remark 2. An important feature of the above algorithm which was not discussed here is its complexity. Results from [2,3] show that the feedback matrix F can be computed in $O(n^3)$ 'flops' (floating point operations). This is one of the major advantages of this method over e.g. the similar method presented in [5].

4. Numerical example

In this last section we give a numerical example illustrating the method. The system $(\lambda E - A, B)$ was generated randomly and then put in its generalized staircase form (10), (11). The system has state dimension $n = 5$ and input dimension $m = 2$. The stairs have the 'generical' sizes $r_1 = r_2 = 2$, $r_3 = 1$ and the controllability indices are thus equal to $c_1 = 2$, $c_2 = 3$. The test was run in double precision on a VAX/VMS machine with relative precision $\varepsilon = 2^{-56} \approx 1.4\ \mathrm{D} - 17$.

The input data to the deadbeat procedure were:

$$\tilde{E} = \begin{bmatrix} 6.9440\,\mathrm{D}{-01} & 9.7292\,\mathrm{D}{-01} & 9.4832\,\mathrm{D}{-01} & 1.1460\,\mathrm{D}{+00} & 1.4519\,\mathrm{D}{+00} \\ 0 & 8.7778\,\mathrm{D}{-02} & 1.2337\,\mathrm{D}{-01} & 7.3615\,\mathrm{D}{-01} & 5.9330\,\mathrm{D}{-01} \\ 0 & 0 & 6.9191\,\mathrm{D}{-08} & 7.4280\,\mathrm{D}{-02} & 3.0143\,\mathrm{D}{-01} \\ 0 & 0 & 0 & 1.3039\,\mathrm{D}{-01} & 2.1961\,\mathrm{D}{-01} \\ 0 & 0 & 0 & 0 & 1.0209\,\mathrm{D}{-01} \end{bmatrix},$$

$$\tilde{A} = \begin{bmatrix} 6.7866\,\mathrm{D}{-01} & 5.6127\,\mathrm{D}{-01} & 6.7376\,\mathrm{D}{-02} & -6.2656\,\mathrm{D}{-01} & 3.8263\,\mathrm{D}{-01} \\ 1.0581\,\mathrm{D}{-01} & -5.4205\,\mathrm{D}{-02} & -1.0989\,\mathrm{D}{-01} & -3.5375\,\mathrm{D}{-01} & 5.6899\,\mathrm{D}{-01} \\ 2.4558\,\mathrm{D}{-07} & -4.3672\,\mathrm{D}{-02} & -1.2704\,\mathrm{D}{-02} & -4.4709\,\mathrm{D}{-01} & 2.3386\,\mathrm{D}{-01} \\ 0 & -7.6664\,\mathrm{D}{-02} & -2.2302\,\mathrm{D}{-02} & -2.5967\,\mathrm{D}{-01} & 1.9381\,\mathrm{D}{-01} \\ 0 & 0 & 0 & -1.7322\,\mathrm{D}{-01} & 7.1478\,\mathrm{D}{-02} \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} -3.4110\,\mathrm{D}{-01} & -9.2072\,\mathrm{D}{-02} \\ 0 & -1.4339\,\mathrm{D}{+00} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and the resulting matrices are:

$$\tilde{Q}\tilde{E}\tilde{U} = \begin{bmatrix} 1.4032\,\mathrm{D}{+00} & 1.7430\,\mathrm{D}{+00} & 1.3579\,\mathrm{D}{-01} & -8.9658\,\mathrm{D}{-01} & 8.1582\,\mathrm{D}{-01} \\ 0 & 8.6965\,\mathrm{D}{-02} & -3.3344\,\mathrm{D}{-01} & 7.1104\,\mathrm{D}{-02} & 2.5033\,\mathrm{D}{-01} \\ 0 & 0 & -8.3049\,\mathrm{D}{-02} & 1.2083\,\mathrm{D}{-01} & 3.0554\,\mathrm{D}{-01} \\ 0 & 0 & 0 & -6.4753\,\mathrm{D}{-02} & -1.9468\,\mathrm{D}{-01} \\ 0 & 0 & 0 & 0 & 8.5555\,\mathrm{D}{-08} \end{bmatrix},$$

$$\tilde{Q}(\tilde{A}+\tilde{B}F)\tilde{U}=\begin{bmatrix} 0 & 0 & -1.3257\,\mathrm{D}+00 & -2.1160\,\mathrm{D}+00 & -7.8168\,\mathrm{D}-01 \\ 0 & 0 & -2.9842\,\mathrm{D}-01 & 8.3402\,\mathrm{D}-02 & -2.4620\,\mathrm{D}-01 \\ 0 & 0 & 0 & 0 & 2.9084\,\mathrm{D}-01 \\ 0 & 0 & 0 & 0 & -2.3972\,\mathrm{D}-01 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{Q}\tilde{B}=\begin{bmatrix} 3.2482\,\mathrm{D}-01 & 5.0086\,\mathrm{D}-01 \\ -7.6010\,\mathrm{D}-02 & 5.9753\,\mathrm{D}-01 \\ -6.5339\,\mathrm{D}-02 & 1.1085\,\mathrm{D}+00 \\ 2.8146\,\mathrm{D}-02 & -4.7751\,\mathrm{D}-01 \\ 8.2083\,\mathrm{D}-19 & -4.8860\,\mathrm{D}-07 \end{bmatrix},$$

$$F=\begin{bmatrix} -2.1388\,\mathrm{D}+00 & 4.2275\,\mathrm{D}+00 & -5.9674\,\mathrm{D}-01 & 1.5561\,\mathrm{D}+00 & -2.9458\,\mathrm{D}+00 \\ -1.6839\,\mathrm{D}-01 & 2.5017\,\mathrm{D}-01 & -8.4999\,\mathrm{D}-02 & -1.1897\,\mathrm{D}-01 & 3.7390\,\mathrm{D}-02 \end{bmatrix}.$$

Note that these four matrices are constructed *recursively* just as the transformation matrices $\tilde{Q}$ and $\tilde{U}$. When computing $\tilde{Q}\tilde{E}\tilde{U}$, $\tilde{Q}(\tilde{A}+\tilde{B}F)\tilde{U}$ and $\tilde{Q}\tilde{B}$ again from the matrices stored in computer, we retrieve the same results up to the relative precision of the machine. This shows the good numerical behaviour of the method (which is mainly due to the use of orthogonal transformations). The relative high condition number κ of E ($\kappa(E)=\|E\|_2\cdot\|E^{-1}\|_2\approx 1.4\cdot 10^8$) did *not* affect this property, because the inverse of E is never explicitly computed. In this example the good behaviour of the method is also due to the 'reasonable' norm of F, while in general this norm may be quite large. A method to ensure feedbacks that have lower norm than with deadbeat feedback is to relax the minimality condition on the Jordan chains while still insisting on the eigenvalues at $\lambda=0$ (see [4]). If the norm is significantly lower, a better numerical behaviour of the algorithm may then also be expected. Finally, when using an explicit method passing via $(E^{-1}A,\ E^{-1}B)$ (e.g. [12]) for the above example, rounding errors can be expected to be higher because of errors occurring while inverting E. When actually *computing* he eigenvalues of $\lambda\tilde{E}-(\tilde{A}+\tilde{B}F)$ one finds values that are rather far from $\lambda=0$ (we found $\approx 1.3698\ \mathrm{D}-06$ with the QZ-algorithm) but this is due to the occurrence of Jordan chains of length 2 and 3. The fact that our pencil $\lambda\tilde{E}-(\tilde{A}+\tilde{B}F)$ stored in computer is ε-close to one with *exact* eigenvalues at $\lambda=0$ (namely the recursively computed results above) is all we can guarantee. Similar results were also obtained for the method of [12].

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