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An affine eigenvalue problem on the nonnegative orthant

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Abstract

In this paper, we consider the conditional affine eigenvalue problem

$$\lambda x = Ax + b, \quad \lambda \in \mathbb{R}, \ x \geqslant 0, \ \|x\| = 1,$$

where A is an $n \times n$ nonnegative matrix, b a nonnegative vector, and $\|\cdot\|$ a monotone vector norm. Under suitable hypotheses, we prove the existence and uniqueness of the solution (λ_*, x_*) and give its expression as the Perron root and vector of a matrix $A + bc_*^{\mathrm{T}}$, where c_* has a maximizing property depending on the considered norm. The equation $x = (Ax + b)/\|Ax + b\|$ has then a unique nonnegative solution, given by the unique Perron vector of $A + bc_*^{\mathrm{T}}$.

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1. Introduction

Nonnegative matrices have applications in many areas [1], including economics, statistics and network theory. For a nonnegative matrix A, at least one of the eigenvalues of maximal magnitude is nonnegative and hence equal to the spectral radius ρ of the matrix A. The corresponding eigenvectors x satisfy $Ax = \rho x$ and are called Perron vectors of A if they are nonnegative. There always exists at least one Perron vector, and in most applications the Perron vectors play an important role (they describe e.g. an equilibrium, a probability distribution or an optimal network property [1]). One is then often interested in verifying uniqueness and strict positivity of the Perron vector [1].

The motivation of the problem analyzed in this paper comes from graph theory. One can define a measure of similarity between nodes of two graphs via the calculation of a particular extremal nonnegative vector of the so-called product graph [2,8]. Such a vector can be defined as the limit of the iterates

$$x_{t+1} = \frac{Ax_t}{\|Ax_t\|},$$

where A is a nonnegative matrix derived from the adjacency matrix of the product graph [2]. The matrix A may have several eigenvectors associated to eigenvalues of maximal magnitude and so these iterates may fail to converge. To cope with this lack of convergence, Blondel et al. [2] suggest to look at the limit of a particular convergent subsequence of the iterates. On the other hand, Melnik et al. [8], propose to change the iteration formula for

$$x_{t+1} = \frac{Ax_t + b}{\|Ax_t + b\|}, \tag{1}$$

where b is the vector of ones and $\|\cdot\|$ is the ℓ_∞ norm. They observe experimentally the convergence of their algorithm.

The convergence and the fixed point of such a normalized affine iteration in the nonnegative orthant can be analyzed theoretically.

In the following, we write $x \geqslant y$ or $x - y \in \mathbb{R}^n_{\geqslant 0}$ if the vector $x - y$ is nonnegative (all its entries are nonnegative); $x \gneqq y$ if $x - y$ is nonnegative and nonzero, and $x > y$ or $x - y \in \mathbb{R}^n_{>0}$ if $x - y$ is positive (all its entries are positive). The same notations apply to matrices. A vector norm is said to be monotone if for all $x, y \in \mathbb{R}^n$, $|x| \geqslant |y|$ implies $\|x\| \geqslant \|y\|$.

Under the hypotheses that A and b are a nonnegative matrix and vector such that $Ax + b > 0$ for any $x \gneqq 0$, and that the norm is monotone, the *existence* and the *uniqueness* of the fixed point of the normalized affine iteration (1) can be easily proved (see Appendix A, p. 83). It is not so easy to prove its *global convergence*. This convergence, as well as the existence and uniqueness of the fixed point, can be deduced from the work of Krause [6,7] on nonlinear mappings on cones.

Krause's Theorem. *Let $\|\cdot\|$ be a monotone norm on $\mathbb{R}^n$. For a concave mapping $f : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{\geqslant 0}$ with $f(x) > 0$ for $x \gneqq 0$, the following statements hold.*

The conditional eigenvalue problem $f(x) = \lambda x$, $\lambda \in \mathbb{R}$, $x \geqslant 0$, $\|x\| = 1$, has a unique solution (λ_*, x_*), and $\lambda_* > 0$, $x_* > 0$. Furthermore, $\lim_{k\to\infty} \tilde{f}^k(x) = x_*$ for all $x \gneqq 0$, where $\tilde{f}$ is the normalized mapping $\tilde{f}(x) = \frac{f(x)}{\|f(x)\|}$.

The idea of Krause is to prove that the metric space $X = \{x \in \mathbb{R}^n_{>0} : \|x\| = 1\}$ for the Hilbert's projective metric is complete, and that the mapping $\tilde{f} : X \to X$ is a contraction. He then applies Banach's fixed point theorem to $\tilde{f}$.

In this paper, we do not deal with convergence questions but we provide an alternative proof of the existence and the uniqueness of the fixed point x_* of the normalized affine iteration (1), with more general assumptions: for our proof, the hypothesis $Ax + b > 0$ for any $x \gneqq 0$ will be relaxed. Moreover, we will show that this *fixed point* can be *characterized* as the Perron vector of a matrix $A + bc_*^{\mathrm{T}}$, where c_* maximizes the spectral radius of a particular set of matrices. Our main result can be stated as follows.

Theorem. *Let A be a nonnegative matrix and b a nonnegative vector. Let $\|\cdot\|^{\mathrm{D}}$ be the dual norm of a monotone norm $\|\cdot\|$. If $\max_{\|c\|^{\mathrm{D}}=1} \rho(A + bc^{\mathrm{T}}) > \rho(A)$, then the problem*

$$\lambda x = Ax + b, \quad \lambda \in \mathbb{R}, \quad x \geqslant 0, \quad \|x\| = 1,$$

has a unique solution (λ_*, x_*). Moreover, this solution is given by the spectral radius λ_* and the unique normalized Perron vector x_* of $A + bc_*^{\mathrm{T}}$, where $c_* \geqslant 0$ is a maximizer of $\rho(A + bc^{\mathrm{T}})$, $\|c\|^{\mathrm{D}} = 1$.

As a consequence, the equation

$$x = \frac{Ax + b}{\|Ax + b\|}$$

has a unique nonnegative solution, which is x_*.

Let us point out a problem whose formulation seems similar but which is actually very different. Let A a nonnegative matrix, b a nonnegative vector and λ a positive scalar be given. The solvability of

$$\lambda x = Ax + b, \quad x \geqslant 0, \tag{2}$$

has been studied for a long time. In 1963, Carlson [3] gave equivalent conditions of solvability of this equation for a given $\lambda \geqslant \rho(A)$. He showed that the existence of a nonnegative solution x of $\lambda x = Ax + b$ is solely determined by the location of the zero and nonzero entries in the matrix $\lambda I - A$ and the vector b, and by the set of indices of singular irreducible submatrices on the diagonal in a standard form of $\lambda I - A$. Several authors have then found other equivalent conditions of solvability of (2), with extensions to the case where $0 < \lambda < \rho(A)$ and to particular classes of operators on Banach spaces. For more recent results on this subject, see Tam and Schneider

[9] and the references therein. Let us illustrate that the equation $\lambda x = Ax + b$ can have a nonnegative solution for $0 < \lambda \leqslant \rho(A)$. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If $\lambda = 4 = \rho(A)$, then $\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ is a nonnegative solution of (2), and if $\lambda = 3 < \rho(A)$, then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a nonnegative solution.

The rest of this paper is organized as follows. First, some preliminaries are introduced in Section 2. Then, in Section 3, we prove the main result: the existence, uniqueness and expression of the solution of the conditional eigenvalue problem. In Section 4, we derive a graph-theoretic condition which implies the hypotheses of our theorem, and we show that Krause's assumption $Ax + b > 0$ for $x \gneqq 0$ is a particular case of this condition. Finally, Section 5 particularizes the result for the ℓ_1, ℓ_∞ and ℓ_2 norms.

2. Preliminaries

In this section, we present some preliminaries that will be useful in the sequel.

Let $\mathcal{I}$ be a subset of $\{1, \ldots, n\}$. We denote by $x_{\mathcal{I}}$ the corresponding subvector of a vector $x \in \mathbb{R}^n$ and by $M_{\mathcal{I}}$ the corresponding principal submatrix of a matrix $M \in \mathbb{R}^{n\times n}$. By e_i, we denote the ith column of the $n \times n$ identity matrix I, and $e \in \mathbb{R}^n$ is the vector of all ones. In particular, $e_{\mathcal{I}}$ is the vector of all ones of length $|\mathcal{I}|$.

By *Perron vector* of a nonnegative matrix $M \in \mathbb{R}^{n\times n}_{\geqslant 0}$, we mean a nonnegative vector $x \gneqq 0$ such that $Mx = \rho(M)x$. The Perron–Frobenius theory ensures that every nonnegative and nonzero matrix always has a Perron vector, but this is not necessarily unique.

We will need the following well known results on nonnegative matrices (see for example Chapter 8 of [4] and Chapter 2 of [1]).

Proposition 1. *If M is a nonnegative matrix and if $M_{\mathcal{I}}$ is any principal submatrix of M, then $\rho(M_{\mathcal{I}}) \leqslant \rho(M)$.*

Proposition 2. *Let M be a nonnegative matrix, $x \gneqq 0$ a nonnegative vector and α, β two nonnegative scalars. If $\alpha x \leqslant Mx$ then $\alpha \leqslant \rho(M)$, and if $\alpha x < Mx$ then $\alpha < \rho(M)$. Moreover, if x is positive, then $Mx \leqslant \beta x$ implies $\rho(M) \leqslant \beta$, and $Mx < \beta x$ implies $\rho(M) < \beta$.*

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Its *dual norm* $\|\cdot\|^{\mathrm{D}}$ is defined by

$$\|y\|^{\mathrm{D}} = \max_{\|x\|=1} |y^{\mathrm{T}}x|.$$

For a fixed $x \in \mathbb{R}^n$, the nonempty set

$$\{y \in \mathbb{R}^n : \|y\|^{\mathrm{D}}\|x\| = y^{\mathrm{T}}x = 1\}$$

is the *dual of x with respect to* $\|\cdot\|$. A pair (x, y) of vectors of $\mathbb{R}^n$ is said to be a *dual pair with respect to* $\|\cdot\|$ if $\|y\|^{\mathrm{D}}\|x\| = y^{\mathrm{T}}x = 1$. It can be shown that if $\|\cdot\|^{\mathrm{D}}$ is the dual norm of $\|\cdot\|$, then $\|\cdot\|$ is the dual norm of $\|\cdot\|^{\mathrm{D}}$ (see Sections 5.4 and 5.5 in [4]).

3. Solution of the conditional affine eigenvalue problem

In this section, we give the expression for the fixed point of the normalized affine iteration (1), and a proof of the existence and uniqueness of this solution.

Let A be a nonnegative matrix and b a nonnegative vector. The first stage is to prove the uniqueness of the Perron vector corresponding to a spectral radius $\rho(A + bc^{\mathrm{T}}) > \rho(A)$, for a nonnegative vector c.

Lemma 3. *Let A be a nonnegative matrix and b, c two nonnegative vectors. If $\rho(A + bc^{\mathrm{T}}) > \rho(A)$, then the matrix $A + bc^{\mathrm{T}}$ has only one Perron vector. Moreover, for any nonnegative vector d, if the matrices $A + bc^{\mathrm{T}}$ and $A + bd^{\mathrm{T}}$ have the same spectral radius $\rho(A + bd^{\mathrm{T}}) = \rho(A + bc^{\mathrm{T}}) > \rho(A)$, then their normalized Perron vector are equal.*

Proof. Let $u \gneqq 0$ such that $\rho(A + bc^{\mathrm{T}})u = (A + bc^{\mathrm{T}})u$. We must have $c^{\mathrm{T}}u > 0$, since otherwise $\rho(A + bc^{\mathrm{T}})u = Au$ with $\rho(A + bc^{\mathrm{T}}) > \rho(A)$. So, from $\rho(A + bc^{\mathrm{T}})u = Au + (c^{\mathrm{T}}u)b$, it follows

$$\frac{u}{c^{\mathrm{T}}u} = (\rho(A + bc^{\mathrm{T}})I - A)^{-1}b,$$

which shows that the Perron vector of $A + bc^{\mathrm{T}}$ is unique.

Similarly, if $\rho(A + bd^{\mathrm{T}}) = \rho(A + bc^{\mathrm{T}})$, then, for any Perron vector v of $A + bd^{\mathrm{T}}$,

$$\frac{u}{c^{\mathrm{T}}u} = (\rho(A + bc^{\mathrm{T}})I - A)^{-1}b = (\rho(A + bd^{\mathrm{T}})I - A)^{-1}b = \frac{v}{d^{\mathrm{T}}v},$$

and hence u and v are equal, up to a scalar factor. $\square$

Example 1. Let us illustrate that two matrices $A + bc^{\mathrm{T}}$ and $A + bd^{\mathrm{T}}$ which have the same spectral radius larger than $\rho(A)$, have also the same Perron vector. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Then $\rho(A) = 1$ and, for instance, $\rho(A + be_1^{\mathrm{T}}) = \rho(A + be_2^{\mathrm{T}}) = 3$. Therefore, by Lemma 3, the corresponding normalized Perron vectors of $A + be_1^{\mathrm{T}}$ and $A + be_2^{\mathrm{T}}$ are equal:

$$u_1 = u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Furthermore, it is easily proved that if $\rho(A + bc^{\mathrm{T}}) = \rho(A + bd^{\mathrm{T}}) > \rho(A)$, then the Perron vector of these matrices is also the Perron vector of any matrix $A + b(\alpha c^{\mathrm{T}} + (1 - \alpha)d^{\mathrm{T}})$, with $0 \leqslant \alpha \leqslant 1$, which has moreover the same spectral radius.

The next stage is to show that, for a nonnegative vector c such that $\rho(A + bc^{\mathrm{T}}) > \rho(A)$, we can compare $\rho(A + bc^{\mathrm{T}})$ with $\rho(A + bd^{\mathrm{T}})$ for any $d \geqslant 0$ by comparing the scalar product of the Perron vector of $A + bc^{\mathrm{T}}$ with c or d, and reciprocally. In the following lemma, the sign of a scalar $\alpha \in \mathbb{R}$ is denoted by $\operatorname{sign}(\alpha)$.

Lemma 4. *Let A be a nonnegative matrix and b, c nonnegative vectors. If $\rho(A + bc^{\mathrm{T}}) > \rho(A)$ then, for any nonnegative vector d,*

$$\operatorname{sign}(\rho(A + bc^{\mathrm{T}}) - \rho(A + bd^{\mathrm{T}})) = \operatorname{sign}(c^{\mathrm{T}}u - d^{\mathrm{T}}u),$$

where u is the Perron vector of $A + bc^{\mathrm{T}}$.

Proof. Let $\mathcal{J} = \{j : e_j^{\mathrm{T}}u > 0\}$ be the set of indices for which the jth entry of u is positive, and let $\bar{\mathcal{J}} = \{j : e_j^{\mathrm{T}}u = 0\}$ be its complementary subset. From $(A + bc^{\mathrm{T}})u = \rho(A + bc^{\mathrm{T}})u$ with $u_{\mathcal{J}} > 0$ and from $u_{\bar{\mathcal{J}}} = 0$, it follows that, up to a permutation, A is block upper triangular with diagonal blocks $A_{\mathcal{J}}$ and $A_{\bar{\mathcal{J}}}$. Moreover, since $\rho((A + bc^{\mathrm{T}})_{\mathcal{J}}) = \rho(A + bc^{\mathrm{T}}) > \rho(A) \geqslant \rho(A_{\mathcal{J}})$, it follows that $c_{\mathcal{J}} \neq 0$ and hence $b_{\bar{\mathcal{J}}} = 0$.

Suppose first that $\rho(A + bc^{\mathrm{T}}) > \rho(A + bd^{\mathrm{T}})$. If we had $\rho(A + bc^{\mathrm{T}})u \leqslant (A + bd^{\mathrm{T}})u$, we would have $\rho(A + bc^{\mathrm{T}}) \leqslant \rho(A + bd^{\mathrm{T}})$ by Proposition 2. Therefore there must exist an index i such that

$$e_i^{\mathrm{T}}(A + bc^{\mathrm{T}})u = e_i^{\mathrm{T}}\rho(A + bc^{\mathrm{T}})u > e_i^{\mathrm{T}}(A + bd^{\mathrm{T}})u,$$

and hence $c^{\mathrm{T}}u > d^{\mathrm{T}}u$.

Suppose now that $\rho(A + bc^{\mathrm{T}}) < \rho(A + bd^{\mathrm{T}})$. Then $\rho(A + bd^{\mathrm{T}}) > \rho(A) \geqslant \rho(A_{\bar{\mathcal{J}}})$ and hence $\rho(A + bd^{\mathrm{T}}) = \rho((A + bd^{\mathrm{T}})_{\mathcal{J}})$, since, up to a permutation, $A + bd^{\mathrm{T}}$ is block upper triangular with diagonal blocks $(A + bd^{\mathrm{T}})_{\mathcal{J}}$ and $A_{\bar{\mathcal{J}}}$. If we had $\rho(A + bc^{\mathrm{T}})u \geqslant (A + bd^{\mathrm{T}})u$, then we would have $\rho(A + bc^{\mathrm{T}})u_{\mathcal{J}} \geqslant (A + bd^{\mathrm{T}})_{\mathcal{J}}u_{\mathcal{J}}$ with $u_{\mathcal{J}} > 0$ and $\rho(A + bc^{\mathrm{T}}) \geqslant \rho(A + bd^{\mathrm{T}})$ by Proposition 2. Therefore, there must exist an index i such that

$$e_i^{\mathrm{T}}(A + bc^{\mathrm{T}})u = e_i^{\mathrm{T}}\rho(A + bc^{\mathrm{T}})u < e_i^{\mathrm{T}}(A + bd^{\mathrm{T}})u,$$

and hence $c^{\mathrm{T}}u < d^{\mathrm{T}}u$.

Finally, if $\rho(A + bc^{\mathrm{T}}) = \rho(A + bd^{\mathrm{T}})$, then u is also a Perron vector of $A + bd^{\mathrm{T}}$ by Lemma 3. Therefore

$$(A + bc^{\mathrm{T}})u = \rho(A + bc^{\mathrm{T}})u = \rho(A + bd^{\mathrm{T}})u = (A + bd^{\mathrm{T}})u,$$

and $c^{\mathrm{T}}u = d^{\mathrm{T}}u$. $\square$

Example 2. Let us illustrate that two spectral radii $\rho(A + bd^{\mathrm{T}})$ and $\rho(A + bc^{\mathrm{T}}) > \rho(A)$ can be compared by comparing the scalar products $d^{\mathrm{T}}u$ and $c^{\mathrm{T}}u$, where u is the Perron vector of $A + bc^{\mathrm{T}}$. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $c = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Then $\rho(A + bc^{\mathrm{T}}) = 5 > \rho(A)$ and $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is the Perron vector of $A + bc^{\mathrm{T}}$. If $d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, since $d^{\mathrm{T}}u = 3 < c^{\mathrm{T}}u = 4$, we know by Lemma 4 that $\rho(A + bd^{\mathrm{T}}) < \rho(A + bc^{\mathrm{T}})$. Indeed, $\rho(A + bd^{\mathrm{T}}) = 4$.

Noticing that, for a nonnegative matrix A, nonnegative vectors b, u, and a monotone norm $\|\cdot\|$,

$$\max_{\|c\|^{\mathrm{D}}=1} \rho(A + bc^{\mathrm{T}}) = \max_{\|c\|^{\mathrm{D}}=1, c \geqslant 0} \rho(A + bc^{\mathrm{T}}),$$
$$\max_{\|c\|^{\mathrm{D}}=1} c^{\mathrm{T}}u = \max_{\|c\|^{\mathrm{D}}=1, c \geqslant 0} c^{\mathrm{T}}u,$$

the following result is a direct consequence of Lemma 4.

Proposition 5. *Let A be a nonnegative matrix and b a nonnegative vector. Let $\|\cdot\|$ be a monotone vector norm and let $\|\cdot\|^{\mathrm{D}}$ be its dual norm. If there exists a nonnegative vector d, with $\|d\|^{\mathrm{D}} = 1$ such that $\rho(A + bd^{\mathrm{T}}) > \rho(A)$, then*

$$\rho(A + bc_*^{\mathrm{T}}) = \max_{\|c\|^{\mathrm{D}}=1} \rho(A + bc^{\mathrm{T}}),$$

with $c_* \in \mathbb{R}^n_{\geqslant 0}$, $\|c_*\|^{\mathrm{D}} = 1$, if and only if

$$c_*^{\mathrm{T}}u_* = \max_{\|c\|^{\mathrm{D}}=1} c^{\mathrm{T}}u_*,$$

with $c_* \in \mathbb{R}^n_{\geqslant 0}$, $\|c_*\|^{\mathrm{D}} = 1$ and where u_* is the Perron vector of $A + bc_*^{\mathrm{T}}$.

In other words, Proposition 5 says that c_* is a maximizer of the spectral radius $\rho(A + bc^{\mathrm{T}})$ among all c of dual norm $\|c\|^{\mathrm{D}} = 1$ if and only if (u_*, c_*) is a dual pair with respect to $\|\cdot\|$, where u_* is the normalized Perron vector of $A + bc_*^{\mathrm{T}}$.

Now we are ready to prove the result announced in the introduction: the existence, the uniqueness and the expression of the solution of a conditional eigenvalue problem or, equivalently, a normalized affine iteration.

Theorem 6. *Let A be a nonnegative matrix and b a nonnegative vector. Let $\|\cdot\|$ be a monotone vector norm and let $\|\cdot\|^{\mathrm{D}}$ be its dual norm. Let c_* be a nonnegative vector, with $\|c_*\|^{\mathrm{D}} = 1$, such that*

$$\rho(A+bc_*^{\mathrm{T}}) = \max_{\|c\|^{\mathrm{D}}=1} \rho(A+bc^{\mathrm{T}}).$$

If $\rho(A+bc_*^{\mathrm{T}}) > \rho(A)$, *then the conditional eigenvalue problem*

$$\lambda x = Ax + b, \quad \lambda \in \mathbb{R}, \quad x \geqslant 0, \quad \|x\| = 1, \tag{3}$$

has a unique solution (λ_*, x_*), *where* $\lambda_* = \rho(A+bc_*^{\mathrm{T}})$ *and* x_* *is the unique normalized Perron vector of* $A+bc_*^{\mathrm{T}}$. *Moreover* (x_*, c_*) *is a dual pair with respect to* $\|\cdot\|$.

As a consequence, the equation

$$x = \frac{Ax+b}{\|Ax+b\|}$$

has a unique nonnegative solution, which is x_*.

Proof. Let c_* be a nonnegative vector, with $\|c_*\|^{\mathrm{D}} = 1$ such that

$$\rho(A+bc_*^{\mathrm{T}}) = \max_{\|c\|^{\mathrm{D}}=1} \rho(A+bc^{\mathrm{T}}) > \rho(A),$$

and let $x_* \in \mathbb{R}^n_{\geqslant 0}$, $\|x_*\| = 1$ be the unique normalized Perron vector of $A+bc_*^{\mathrm{T}}$, by Lemma 3. Then, by Proposition 5 and by the property of dual norms,

$$c_*^{\mathrm{T}} x_* = \max_{\|c\|^{\mathrm{D}}=1} c^{\mathrm{T}} x_* = \|x_*\| = 1.$$

From $\rho(A+bc_*^{\mathrm{T}})x_* = (A+bc_*^{\mathrm{T}})x_*$, we have $\rho(A+bc_*^{\mathrm{T}})x_* = Ax_* + b$ with $x_* \geqslant 0$, $\|x_*\| = 1$ and therefore (λ_*, x_*), with $\lambda_* = \rho(A+bc_*^{\mathrm{T}})$, is a solution of the conditional eigenvalue problem (3). Moreover, (x_*, c_*) is a dual pair with respect to $\|\cdot\|$.

Now, let us prove that (λ_*, x_*) is the only solution to problem (3). Suppose there is another solution $(\tilde{\lambda}, \tilde{x})$ to this conditional eigenvalue problem. Let $\tilde{c} \in \mathbb{R}^n_{\geqslant 0}$, $\|\tilde{c}\|^{\mathrm{D}} = 1$ belong to the dual of $\tilde{x}$ with respect to $\|\cdot\|$. Then $(A+b\tilde{c}^{\mathrm{T}})\tilde{x} = A\tilde{x} + b = \tilde{\lambda}\tilde{x}$. Let $\mathcal{J} = \{j : e_j^{\mathrm{T}}\tilde{x} > 0\}$. As in the proof of Lemma 4, we deduce that, up to a permutation, $A+b\tilde{c}^{\mathrm{T}}$ is block upper triangular with $(A+b\tilde{c}^{\mathrm{T}})_{\mathcal{J}}$ and $A_{\bar{\mathcal{J}}}$ on its diagonal, and that $\tilde{\lambda}\tilde{x}_{\mathcal{J}} = (A+b\tilde{c}^{\mathrm{T}})_{\mathcal{J}}\tilde{x}_{\mathcal{J}}$, which leads to $\tilde{\lambda} = \rho((A+b\tilde{c}^{\mathrm{T}})_{\mathcal{J}})$.

Suppose first that $\rho(A+b\tilde{c}^{\mathrm{T}}) = \rho(A)$. If we had $(A+bc_*^{\mathrm{T}})\tilde{x} \leqslant \tilde{\lambda}\tilde{x}$, then we would have $(A+bc_*^{\mathrm{T}})_{\mathcal{J}}\tilde{x}_{\mathcal{J}} \leqslant \tilde{\lambda}\tilde{x}_{\mathcal{J}}$, which by Propositions 1 and 2 is inconsistent with $\rho((A+bc_*^{\mathrm{T}})_{\mathcal{J}}) = \rho(A+bc_*^{\mathrm{T}}) > \rho(A) = \rho(A+b\tilde{c}^{\mathrm{T}}) \geqslant \tilde{\lambda}$. Therefore, there must exist an index i such that

$$e_i^{\mathrm{T}}(A+bc_*^{\mathrm{T}})\tilde{x} > e_i^{\mathrm{T}}\tilde{\lambda}\tilde{x} = e_i^{\mathrm{T}}(A+b\tilde{c}^{\mathrm{T}})\tilde{x},$$

and hence $c_*^{\mathrm{T}}\tilde{x} > \tilde{c}^{\mathrm{T}}\tilde{x}$, which is impossible since $\tilde{c}$ belongs to the dual of $\tilde{x}$.

Therefore $\rho(A+b\tilde{c}^{\mathrm{T}}) > \rho(A)$. It follows that $\rho(A+b\tilde{c}^{\mathrm{T}}) = \rho((A+b\tilde{c}^{\mathrm{T}})_{\mathcal{J}}) = \tilde{\lambda}$, because of the block triangular structure of $A+b\tilde{c}^{\mathrm{T}}$, and therefore $\tilde{x}$ is a Perron vector of $A+b\tilde{c}^{\mathrm{T}}$. By Proposition 5, we then have that $\rho(A+b\tilde{c}^{\mathrm{T}}) = \max_{\|c\|^{\mathrm{D}}=1} \rho(A+bc^{\mathrm{T}})$ and hence $\rho(A+b\tilde{c}^{\mathrm{T}}) = \rho(A+bc_*^{\mathrm{T}})$. Lemma 3, therefore implies $\tilde{x} = x_*$, that is the problem (3) admits only one solution (λ_*, x_*). $\square$

Example 3. Let us illustrate that, if $\max_{\|c\|^{\mathrm{D}}=1} \rho(A + bc^{\mathrm{T}}) = \rho(A)$, then the conclusion of Theorem 6 does not hold in general. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and let $\|\cdot\|$ be the ℓ_1 norm, and $\|\cdot\|^{\mathrm{D}}$ the ℓ_∞ norm. Then, $\rho(A + bc^{\mathrm{T}}) = \rho(A)$ for any nonnegative vector c with $\|c\|^{\mathrm{D}} = 1$. The conditional eigenvalue problem (3) has two solutions:

$$\lambda_* = 4,\ x_* = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \quad \text{and} \quad \tilde{\lambda}_* = 3, \quad \tilde{x}_* = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and, as a consequence, the equation $x = (Ax + b)/\|Ax + b\|$ has also two nonnegative solutions.

Example 4. Let us also notice that our condition $\max_{\|c\|^{\mathrm{D}}=1} \rho(A + bc^{\mathrm{T}}) > \rho(A)$ is not necessary to obtain a unique solution to the problem (3). Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and let $\|\cdot\|$ be the ℓ_1 norm, and $\|\cdot\|^{\mathrm{D}}$ the ℓ_∞ norm. Then, $\rho(A + bc^{\mathrm{T}}) = \rho(A)$ for any nonnegative vector c with $\|c\|^{\mathrm{D}} = 1$, but the conditional eigenvalue problem has a unique solution $\lambda_* = 2$, $x_* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

4. Graph-theoretic condition

In this section, we derive a graph-theoretic condition that ensures the existence of a nonnegative vector c, satisfying $\|c\|^{\mathrm{D}} = 1$ and $\rho(A + bc^{\mathrm{T}}) > \rho(A)$. We will see that Krause's condition $Ax + b > 0$ for all $x \gneqq 0$ is a particular case of this graph-theoretic condition.

Let us remind that a nonzero matrix $M \in \mathbb{R}^{n\times n}$ is said to be *irreducible* if there exists *no* permutation matrix P such that $P^{\mathrm{T}}MP$ is block upper triangular with at least two diagonal blocks. If a nonnegative matrix M is irreducible, then the Perron–Frobenius theory ensures that M has a unique Perron vector, which is positive. For irreducible matrices, Proposition 2 can be improved as follows (see Chapter 2 of [1] and Chapter 8 of [4]).

Proposition 7. *Let M be a nonnegative irreducible matrix, $x \gneqq 0$ a nonnegative vector and α, β two nonnegative scalars. If $\alpha x \lneqq Mx \lneqq \beta x$ then $\alpha < \rho(M) < \beta$.*

The *directed graph* $\Gamma(M)$ of a nonnegative matrix M is a graph which has an edge from the node i to the node j whenever the (i, j) entry of M is positive. It is well known that M is irreducible if and only if $\Gamma(M)$ is *strongly connected*, that is, for every pair of nodes i, j of $\Gamma(M)$, there exists a directed path from i to j. By convention, we will also say that a graph of a single node is strictly connected, even if it has no edge.

In order to derive the graph-theoretic condition, it is useful to consider matrices written in a particular normal form. Up to a permutation, a nonnegative matrix M can always be written in a block upper triangular form, whose diagonal blocks $M_{\mathcal{I}_1}, \dots, M_{\mathcal{I}_p}$ are either irreducible or one-by-one zero matrices. Such a matrix will be said to be in a *block irreducible normal form*. The corresponding directed graph $\Gamma(M)$ of M can also be decomposed in p subgraphs $\Gamma(M_{\mathcal{I}_1}), \dots, \Gamma(M_{\mathcal{I}_p})$, each of them strongly connected, such that there does not exist a link from a node of the subgraph $\Gamma(M_{\mathcal{I}_k})$ to a node of $\Gamma(M_{\mathcal{I}_\ell})$ if $k > \ell$. Let us also point out that, since M is block upper triangular, there is at least an index k such as $\rho(M) = \rho(M_{\mathcal{I}_k})$.

We can now state the following graph-theoretic condition.

Path Condition. *The nonnegative matrix A and vector b are said to satisfy the Path Condition if there exist an index i with $e_i^{\mathrm{T}} b \neq 0$ and a principal submatrix $A_{\mathcal{J}}$ of A, corresponding to a strongly connected subgraph of $\Gamma(A)$ with $\rho(A_{\mathcal{J}}) = \rho(A)$, such that there is a directed path from any node of $\Gamma(A_{\mathcal{J}})$ to the node i.*

In order to show that the Path Condition ensures that the hypotheses of Theorem 6 are verified, we first need to prove the particular case when A is irreducible.

Lemma 8. *Let A be a nonnegative irreducible matrix, and $b, c \gneqq 0$ two nonnegative vectors. Then $\rho(A + bc^{\mathrm{T}}) > \rho(A)$.*

Proof. Since A is irreducible and $bc^{\mathrm{T}} \geqslant 0$, the matrix $A + bc^{\mathrm{T}}$ is also irreducible, and therefore it has a positive Perron vector u. It follows

$$\rho(A + bc^{\mathrm{T}})u = (A + bc^{\mathrm{T}})u = Au + (c^{\mathrm{T}}u)b \gneqq Au,$$

and then $\rho(A + bc^{\mathrm{T}}) > \rho(A)$. $\square$

Proposition 9. *Let A be a nonnegative matrix, b a nonnegative vector and $\|\cdot\|^{\mathrm{D}}$ any vector norm. If the Path Condition is satisfied, then for all $\gamma > 0$, there exists a nonnegative vector c, with $\|c\|^{\mathrm{D}} = \gamma$ such that $\rho(A + bc^{\mathrm{T}}) > \rho(A)$.*

Proof. For simplicity, we can assume that the matrix A is already written in block irreducible normal form. Suppose first that $i \in \Gamma(A_{\mathcal{J}})$, that is, $b_{\mathcal{J}} \gneqq 0$. Let c be a nonnegative vector, with $\|c\|^{\mathrm{D}} = \gamma$ and $c_{\mathcal{J}} \gneqq 0$. Then, by Proposition 1 and Lemma 8,

$$\rho(A + bc^{\mathrm{T}}) \geqslant \rho((A + bc^{\mathrm{T}})_{\mathcal{J}}) > \rho(A_{\mathcal{J}}) = \rho(A).$$

Now, suppose that $i \notin \Gamma(A_{\mathcal{J}})$. By hypothesis, there is a directed path from $\Gamma(A_{\mathcal{J}})$ to the node i, i.e. there exists a sequence $j_1, \dots, j_s$ such that the vector $a_{\mathcal{J}j_1} \neq 0$ and the scalars $a_{j_1 j_2}, \dots, a_{j_s i} \neq 0$, where we denote by $a_{\mathcal{J}j_1}$ the subvector corresponding to $\mathcal{J}$ of the j_1th column of A and by a_{rt} the (r, t) entry of A. Let $\mathcal{I}$ be the set $\mathcal{J} \cup \{j_1, \dots, j_s, i\}$. In order to use Proposition 7, we want to construct nonnegative vectors x and c such that $(A+bc^{\mathrm{T}})_{\mathcal{I}}$ is irreducible and $(A+bc^{\mathrm{T}})_{\mathcal{I}}x_{\mathcal{I}} \gneqq \rho(A)x_{\mathcal{I}}$. Since $A_{\mathcal{J}}$ is irreducible, it has a positive Perron vector $x_{\mathcal{J}}$ such that $\rho(A)x_{\mathcal{J}} = A_{\mathcal{J}}x_{\mathcal{J}}$, and we can assume that $e_{\mathcal{J}}^{\mathrm{T}}x_{\mathcal{J}} = 1$. We now let $c = \gamma e/\|e\|^{\mathrm{D}}$ and construct the positive vector $x_{\mathcal{I}}$, of length $|\mathcal{I}|$ by completing the vector $x_{\mathcal{J}}$ by scalars $x_{j_1}, \ldots, x_{j_s}, x_i > 0$ such that $x_i \leqslant (\rho(A))^{-1}\delta$ and

$$x_{j_k} \leqslant (\rho(A))^{k-s-2} a_{j_k j_{k+1}} \ldots a_{j_s i}\delta$$

for all $k = 1, \ldots, s$, and with $\delta = b_i\gamma/\|e\|^{\mathrm{D}}$. It is then easy to verify that $(A+bc^{\mathrm{T}})_{\mathcal{I}}x_{\mathcal{I}} \gneqq \rho(A)x_{\mathcal{I}}$, and since $\Gamma((A+bc^{\mathrm{T}})_{\mathcal{I}})$ is strongly connected, $(A+bc^{\mathrm{T}})_{\mathcal{I}}$ is irreducible and therefore $\rho((A+bc^{\mathrm{T}})_{\mathcal{I}}) > \rho(A)$. The result then follows by Proposition 1. □

The following corollary of Theorem 6 can now be derived.

Corollary 10. *Let A be a nonnegative matrix and b a nonnegative vector. Let $\|\cdot\|$ be a monotone vector norm and let $\|\cdot\|^{\mathrm{D}}$ be its dual norm. If the Path Condition is satisfied, then the conditional eigenvalue problem*

$$\lambda x = Ax + b, \quad \lambda \in \mathbb{R}, \quad x \geqslant 0, \quad \|x\| = 1,$$

has a unique solution (λ_*, x_*), where $\lambda_* = \rho(A + bc_*^{\mathrm{T}})$, with $c_* \geqslant 0$, $\|c_*\|^{\mathrm{D}} = 1$, such that

$$\rho(A + bc_*^{\mathrm{T}}) = \max_{\|c\|^{\mathrm{D}}=1} \rho(A + bc^{\mathrm{T}}),$$

and x_* is the unique normalized Perron vector of $A + bc_*^{\mathrm{T}}$. Moreover (x_*, c_*) is a dual pair with respect to $\|\cdot\|$. As a consequence, x_* is the unique nonnegative solution of the equation

$$x = \frac{Ax + b}{\|Ax + b\|}.$$

The following proposition shows that if the Path Condition is not verified, then the existence of a normalized nonnegative vector c such that $\rho(A + bc^{\mathrm{T}}) = \rho(A)$ depends on the norm of b and the spectral radii of the irreducible blocks of A.

Proposition 11. *Let A be a nonnegative matrix, b a nonnegative vector and $\|\cdot\|^{\mathrm{D}}$ a vector norm. If the Path Condition is not satisfied, then, there exists $\gamma_0 > 0$ such that $\rho(A + bc^{\mathrm{T}}) = \rho(A)$ for every nonnegative vector c with $\|c\|^{\mathrm{D}} \leqslant \gamma_0$.*

Proof. Let $\mathcal{I} = \{i : e_i^{\mathrm{T}}b > 0\}$, and let $\bar{\mathcal{I}} = \{i : e_i^{\mathrm{T}}b = 0\}$ be its complementary subset. Since there is no directed path from a strongly connected $\Gamma(A_{\mathcal{J}})$ with $\rho(A_{\mathcal{J}}) = \rho(A)$ to a node i with $e_i^{\mathrm{T}}b \neq 0$, the matrix A is block upper triangular, up to a permutation, with diagonal block $A_{\mathcal{I}}$ and $A_{\bar{\mathcal{I}}}$ such that $\rho(A_{\mathcal{I}}) < \rho(A_{\bar{\mathcal{I}}}) = \rho(A)$. Therefore, for any nonnegative vector c,

$$\rho(A + bc^{\mathrm{T}}) = \max\{\rho((A + bc^{\mathrm{T}})_{\mathcal{I}}), \rho(A_{\bar{\mathcal{I}}})\} = \max\{\rho(A_{\mathcal{I}} + (bc^{\mathrm{T}})_{\mathcal{I}}), \rho(A)\},$$

and since the spectral radius of a matrix is a continuous function of its entries, there exists $\gamma_0 > 0$ such that $\rho(A + bc^{\mathrm{T}}) = \rho(A)$ for every nonnegative vector c with $\|c\|^{\mathrm{D}} \leqslant \gamma_0$. $\square$

Krause's sufficient condition appears as a particular case of the Path Condition.

Proposition 12. *Let A be a nonnegative matrix, b a nonnegative vector and $\|\cdot\|^{\mathrm{D}}$ a vector norm. If $b \neq 0$ and if $Ax + b > 0$ for any $x \gneqq 0$ then the Path Condition is satisfied.*

Proof. First, let us note that $Ax + b > 0$ for any $x \gneqq 0$ if and only if, for every index i, either $e_i^{\mathrm{T}} A > 0$ or $e_i^{\mathrm{T}} b > 0$, or both $e_i^{\mathrm{T}} A$ and $e_i^{\mathrm{T}} b$ are positive. Let $A_{\mathcal{I}_1}, \ldots, A_{\mathcal{I}_p}$ be the diagonal blocks of the block irreducible normal form of A. If A does not have a positive row, then $b > 0$ and the Path Condition is satisfied.

We now assume that A has positive rows, that is the set $\{i : e_i^{\mathrm{T}} A > 0\}$ is not empty. Obviously, this set must be included in the first diagonal block, $\{i : e_i^{\mathrm{T}} A > 0\} \subset \mathcal{I}_1$. Two cases can occur. Suppose first that $\rho(A_{\mathcal{I}_1}) = \rho(A)$, then the Path Condition is satisfied, since there is a directed path from any node of $\Gamma(A_{\mathcal{I}_1})$ to every node of $\Gamma(A)$. If $\rho(A_{\mathcal{I}_1}) < \rho(A)$ then it must be another block $A_{\mathcal{I}_k}$ such that $\rho(A_{\mathcal{I}_k}) = \rho(A)$, and since $b_{\mathcal{I}_k} > 0$ for every $k \neq 1$, the Path Condition is also satisfied. $\square$

Example 5. Let us now illustrate by an example that Krause's condition is not necessary to ensure the existence of a nonnegative vector c, $\|c\|^{\mathrm{D}} = 1$, such that $\rho(A + bc^{\mathrm{T}}) > \rho(A)$. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The condition $Ax + b > 0$ for any $x \gneqq 0$ is not verified, but the Path Condition is satisfied, and therefore, if we choose the ℓ_1 norm for $\|\cdot\|$, and the ℓ_∞ norm for $\|\cdot\|^{\mathrm{D}}$, we have that $\rho(A + be^{\mathrm{T}}) = 3 > \rho(A) = 2$. Theorem 6 then implies that the equation $x = (Ax + b)/\|Ax + b\|$ has a unique nonnegative solution $x_* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

5. Particular norms

In this section, we will see how Theorem 6 can be particularized for ℓ_1, ℓ_∞ and ℓ_2 norms, denoted respectively by $\|\cdot\|_1$, $\|\cdot\|_\infty$ and $\|\cdot\|_2$.

Let us first consider the ℓ_1 norm. Let $u \gneqq 0$, $\|u\|_1 = 1$ be a nonnegative normalized vector. The dual of u with respect to $\|\cdot\|_1$ is given by

$$\{c \in \mathbb{R}^n : \|c\|_\infty = c^{\mathrm{T}} u = 1\},$$

since $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are dual to each other. Clearly, the vector $e \in \mathbb{R}^n$ of all ones belongs to the dual of u. Moreover $e \geqslant c$ for any vector $c \in \mathbb{R}^n$ such that $\|c\|_\infty = 1$, and hence, for a nonnegative matrix A and a nonnegative vector b,

$$\rho(A + be^{\mathrm{T}}) = \max_{\|c\|_\infty = 1} \rho(A + bc^{\mathrm{T}}).$$

Therefore, Theorem 6 can be particularized as follows.

Corollary 13. *Let A be a nonnegative matrix and b a nonnegative vector. If $\rho(A + be^{\mathrm{T}}) > \rho(A)$, then the conditional eigenvalue problem*

$$\lambda x = Ax + b, \quad \lambda \in \mathbb{R}, \quad x \geqslant 0, \quad \|x\|_1 = 1,$$

has a unique solution (λ_*, x_*), where $\lambda_* = \rho(A + be^{\mathrm{T}})$ and x_* is the unique normalized Perron vector of $A + be^{\mathrm{T}}$. Moreover, x_* is the unique nonnegative solution of the equation

$$x = \frac{Ax + b}{\|Ax + b\|_1}.$$

It means that, for the ℓ_1 norm, the solution of the conditional eigenvalue problem (3) is explicitly defined. Actually, under hypotheses of Theorem 6, the iteration

$$x_{t+1} = \frac{Ax_t + b}{\|Ax_t + b\|_1}$$

for $x_0 \geqslant 0$, is equivalent to the iteration

$$x_{t+1} = \frac{(A + be^{\mathrm{T}})x_t}{\|(A + be^{\mathrm{T}})x_t\|_1},$$

that is the power method applied to matrix $A + be^{\mathrm{T}}$.

Now, let us consider the ℓ_∞ norm. The dual of a vector $u \gneqq 0$, $\|u\|_\infty = 1$ with respect to $\|\cdot\|_\infty$ is

$$\{c \in \mathbb{R}^n : \|c\|_1 = c^{\mathrm{T}}u = 1\}.$$

Clearly, there exists at least a basis vector e_k in the dual of u, which satisfies $e_k^{\mathrm{T}}u = \max_i e_i^{\mathrm{T}}u = 1$. As it was noticed in Example 1, if $\rho(A + be_i^{\mathrm{T}}) = \rho(A + be_j^{\mathrm{T}}) > \rho(A)$, then the convex combination $A + b(\alpha e_i^{\mathrm{T}} + (1 - \alpha)e_j^{\mathrm{T}})$, $0 \leqslant \alpha \leqslant 1$, has also the same spectral radius and Perron vector. Therefore, Theorem 6 can be particularized in the following way.

Corollary 14. *Let A be a nonnegative matrix and b a nonnegative vector. If $\rho(A + be_\ell^{\mathrm{T}}) = \max_i \rho(A + be_i^{\mathrm{T}}) > \rho(A)$, then the conditional eigenvalue problem*

$$\lambda x = Ax + b, \quad \lambda \in \mathbb{R}, \quad x \geqslant 0, \quad \|x\|_\infty = 1,$$

has a unique solution (λ_*, x_*), where $\lambda_* = \rho(A + be_\ell^{\mathrm{T}})$ and x_* is the unique normalized Perron vector of $A + be_\ell^{\mathrm{T}}$. Moreover, x_* is the unique nonnegative solution of the equation

$$x = \frac{Ax + b}{\|Ax + b\|_{\infty}}.$$

Let us notice that in this case, contrary to the case of the ℓ_1 norm, it cannot be said a priori which matrix $A + be_i^{\mathrm{T}}$ will give the solution, but there are potentially n choices.

It is known that the ℓ_2 norm is its own dual norm, and that the dual of a vector $u \geqslant 0$, $\|u\|_2$=1, with respect to $\|\cdot\|_2$ is the singleton $\{u\}$. Therefore, Proposition 5 and Theorem 6 can be particularized as follows.

Corollary 15. *Let A be a nonnegative matrix and b a nonnegative vector. If there exists a nonnegative vector d, with $\|d\|_2 = 1$, such that $\rho(A + bd^{\mathrm{T}}) > \rho(A)$, then*

$$\rho(A + bc_*^{\mathrm{T}}) = \max_{\|c\|_2=1} \rho(A + bc^{\mathrm{T}}),$$

with $c_* \in \mathbb{R}^n_{\geqslant 0}$, $\|c_*\|_2 = 1$, if and only if c_* is the Perron vector of $A + bc_*^{\mathrm{T}}$ and $c_* \in \mathbb{R}^n_{\geqslant 0}$, $\|c_*\|_2 = 1$.

Corollary 16. *Let A be a nonnegative matrix and b a nonnegative vector. Let c_* be a nonnegative vector, with $\|c_*\|_2 = 1$, such that*

$$\rho(A + bc_*^{\mathrm{T}}) = \max_{\|c\|_2=1} \rho(A + bc^{\mathrm{T}}).$$

If $\rho(A + bc_*^{\mathrm{T}}) > \rho(A)$, then the conditional eigenvalue problem

$$\lambda x = Ax + b, \quad \lambda \in \mathbb{R}, \quad x \geqslant 0, \quad \|x\|_2 = 1,$$

has a unique solution (λ_*, x_*), where $\lambda_* = \rho(A + bc_*^{\mathrm{T}})$ and $x_* = c_*$. Moreover, x_* is the unique nonnegative solution of the equation

$$x = \frac{Ax + b}{\|Ax + b\|_2}.$$

6. Conclusions

In this paper, we show that, for a nonnegative matrix A, a nonnegative vector b, and a monotone norm $\|\cdot\|$, the solution (λ_*, x_*) of the conditional eigenvalue problem

$$\lambda x = Ax + b, \quad \lambda \in \mathbb{R}, \quad x \geqslant 0, \quad \|x\| = 1,$$

can be expressed as the spectral radius and normalized Perron vector of a matrix $A + bc_*^{\mathrm{T}}$, where c_* is a maximizer of the spectral radius $\rho(A + bc^{\mathrm{T}})$ among all $c \geqslant 0$ such that $\|c\|^{\mathrm{D}} = 1$.

The assumption required is that $\rho(A + bc_*^{\mathrm{T}}) > \rho(A)$. In particular, if A and b are such that $Ax + b > 0$ for all $x \gneqq 0$ and if $b \neq 0$, this assumption is verified.

Within the context of the normalized affine iteration

$$x_{t+1} = \frac{Ax_t + b}{\|Ax_t + b\|}$$

for computing similarity scores between graphs, the limit point x_* is the normalized Perron vector of the matrix $A + bc_*^{\mathrm{T}}$, a rank one perturbation of the original iteration matrix A.

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Appendix A

Here we gave a simple proof of the existence and uniqueness of the fixed point of the normalized affine iteration (1) in case Krause's assumption is satisfied.

Proposition. *Let A be a nonnegative matrix and $b \gneqq 0$ a nonnegative vector such that $Ax + b > 0$ for all $x \gneqq 0$. Let $\|\cdot\|$ be a monotone vector norm. Then the equation*

$$x = \frac{Ax + b}{\|Ax + b\|}$$

has one and only one nonnegative solution.

Proof. For all $t > \rho(A)$, the matrix $tI - A$ is invertible, and $(tI - A)^{-1}$ is a nonnegative matrix (see Section 2.5 in [5]). Moreover, for $t > \rho(A)$, we have

$$(tI - A)^{-1}b = \frac{1}{t}\sum_{k\in\mathbb{N}} \left(\frac{A}{t}\right)^k b,$$

and since, by hypothesis, $A(b/t) + b > 0$, it follows that $(tI - A)^{-1}b > 0$. Let $r :]\rho(A), \infty[\to \mathbb{R}_{\geqslant 0} : t \mapsto \|(tI - A)^{-1}b\|$. This function is continuous. Moreover, since $\|\cdot\|$ is monotone, r is a nonincreasing function on its domain. Furthermore, we can compute that $\lim_{t\to\rho(A)} r(t) = \infty$ and $\lim_{t\to\infty} r(t) = 0$. Therefore, there exists a unique $t^* > \rho(A)$ such that $r(t^*) = 1$, and we can verify that $x^* = (t^*I - A)^{-1}b$ is a nonnegative solution of

$$x = \frac{Ax + b}{\|Ax + b\|}.$$

Let us now prove that this equation does not have another nonnegative solution. Suppose that $\tilde{x} \geqslant 0$ is a solution, that is $\tilde{t}\tilde{x} = A\tilde{x} + b$ with $\tilde{t} = \|A\tilde{x} + b\|$. Then $\tilde{x}$ must be positive, by hypothesis. Let $w \gneqq 0$ be a Perron vector of A^{T}, and let i be an index such that $e_i^{\mathrm{T}} w > 0$. Two cases can occur. If $e_i^{\mathrm{T}} A$ is not positive then $e_i^{\mathrm{T}} b > 0$, and hence $w^{\mathrm{T}} b > 0$. Else, if $e_i^{\mathrm{T}} A > 0$ then we have $\rho(A)w^{\mathrm{T}} = w^{\mathrm{T}} A \geqslant (w^{\mathrm{T}} e_i)(e_i^{\mathrm{T}} A) > 0$, and hence $w > 0$. Since $b \neq 0$, it follows that $w^{\mathrm{T}} b > 0$. Consequently,

$$\tilde{t} w^{\mathrm{T}} \tilde{x} = w^{\mathrm{T}} A\tilde{x} + w^{\mathrm{T}} b = \rho(A) w^{\mathrm{T}} \tilde{x} + w^{\mathrm{T}} b > \rho(A) w^{\mathrm{T}} \tilde{x}.$$

Therefore $\tilde{t} > \rho(A)$, with $\tilde{x} = (\tilde{t}I - A)^{-1}b$ and $\|(\tilde{t}I - A)^{-1}b\| = 1$. But we have seen above that t^* is the unique number which satisfies these conditions. It follows that $\tilde{t} = t^*$ and $\tilde{x} = x^*$, and therefore the solution is unique. $\square$

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