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Calculating the H_∞ norm of a fractional system given in state-space form

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ABSTRACT

We present an efficient algorithm to compute the H_∞ norm of a fractional system. The algorithm is based on the computation of level sets of the maximum singular value of the transfer function, as a function of frequency. Numerical examples are given to illustrate the new method.

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1. Introduction

In the last few years many efforts have been made to develop fractional systems in different fields of research. Fractional calculus is a generalization of ordinary derivation and integration to non integer orders. Many phenomena can be modeled by fractional systems, such as thermal diffusion [1] and electrochemical diffusion of charges in acid batteries [2], which was shown to have a tight relation with derivatives of order 0.5 [3]. Another application is car suspension design, which can be viewed as a robust controller synthesis problem [4]. Some other applications of fractional order systems can be found in [5] and [6,7]. Mathematical fundamentals of fractional calculus are given in the monographs [8–13,7]. Stability results of fractional order system are also investigated in [11,6] and [14].

The H_∞ norm of a stable transfer function arises often in control theory [15,16]. Recently a method for the computation of the L_2-gain and H_∞-norm for fractional systems has been proposed in [17–19,4]. In the present paper, we describe another method for computing the H_∞-norm, using the concepts of parahermitian transfer functions [20] and level sets [21]. This allows us to converge in a few steps to the frequency ω^α

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(for $0 < \alpha \leq 1$) where the maximum singular value of the transfer function equals the H_∞ norm of the fractional system.

2. Parahermitian matrix functions

The H_∞-norm of a rational transfer function matrix is a very natural norm for a linear time invariant dynamical system. It represents the induced 2-norm of the linear convolution which describes the input/output map of a dynamical system. The H_∞-norm of a stable rational continuous-time transfer function $G(s)$ is known to be equal to the maximum of the largest singular value of the transfer function $G(j\omega)$ evaluated on the $j\omega$ axis [15]:

$$\|G\|_{H_\infty} = \sup_{\omega\in\mathbb{R}} \sigma_{max} G(j\omega).$$

Clearly $G(s)$ must be proper for $\|G\|_{H_\infty}$ to be bounded since otherwise $G(j\omega)$ will be unbounded for $\omega \to \infty$. If we define the so-called *parahermitian* matrix:

$$\Phi_G(\gamma, s) := \gamma^2 I - G(s)[G^*(-s)] \tag{1}$$

where $\gamma \in \mathbb{R}^+$ and $G^*(-s) := [G(-\overline{s})]^*$, then

$$\Phi_G(\gamma, \omega) := \gamma^2 I - G(j\omega)[G^*(-j\omega)] \tag{2}$$

is hermitian for all $\omega \in \mathbb{R}$ since $G^*(-j\omega) := [G(j\omega)]^*$. and it is not difficult to see that the above definition of the H_∞-norm is equivalent to :

$$\|G\|_{H_\infty} = \gamma_{min} := \inf_{\gamma\in\mathbb{R}} \left\{ \Phi_G(\gamma, \omega) \succ 0,\ \forall\omega \in \mathbb{R} \right\}. \tag{3}$$

Indeed, $\gamma > \sigma_{max} G(j\omega)\ \forall\omega \in \mathbb{R}$ iff the parahermitian matrix $\Phi_G(\gamma, \omega) \succ 0,\ \forall\omega \in \mathbb{R}$. In [20] it is shown how to transform this problem for rational matrices $G(s)$ to the solution of a parahermitian generalized eigenvalue problem. Rational transfer matrices can always be represented as simple expressions involving first order polynomial matrix functions (i.e. pencils). As shown in [20], every rational transfer matrix of dimension $p \times m$ is known from realization theory to admit a state space model $\{A, B, C, D, E\}$ such that $G(s) = C(sE - A)^{-1}B + D$, which is also the Schur complement of the so-called system matrix

$$S_G(s) = \left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right].$$

Since $G(s)$ is proper, the pencil $A - sE$ can be at most index 1 if E is singular (see also example 5.2). The so-called paraconjugate transfer function $G^*(-s) = B^T(-sE^T - A^T)^{-1}C^T + D^T$ is also the Schur complement of the corresponding system matrix

$$S_G^*(-s) = \left[\begin{array}{c|c} A^T + sE^T & C^T \\ \hline B^T & D^T \end{array}\right].$$

It then follows from simple algebraic manipulations that $\Phi_G(\gamma, \omega)$ is the Schur complement of

$$S_\Phi(\omega) := \left[\begin{array}{cc|c} 0 & A^T + \omega j E^T & C^T \\ A - \omega j E & -BB^T & -BD^T \\ \hline C & -DB^T & \gamma^2 I - DD^T \end{array}\right]. \tag{4}$$

For every fixed value of γ, this is a Hermitian pencil in the variable ω. As a consequence, its generalized eigenvalues $\lambda_i(\omega)$ are real analytical functions of the real variable ω [22]. The zeros ω_j of the largest eigenvalue

$\lambda_{max}(\omega)$ are therefore points at which $\sigma_{max}G(j\omega_j) = \gamma$. It is shown in [23,16,20] that computing $\|G\|_{H_\infty}$ therefore amounts to finding the smallest value of γ such that $\lambda_{max}(\omega)$ has no real zeros anymore. This can then be used in several different ways. The first method proposed in [23] was to use bisection on the calculation of γ_{min}, by verifying the existence of real roots ω to the generalized eigenvalue problem

$$\det S_\Phi(\omega) = 0.$$

Indeed, if $\gamma > \gamma_{min}$, then the pencil $S_\Phi(\omega)$ has no real eigenvalues ω. Later methods improved a lot on the convergence of this method by using the smoothness properties of the eigenvalues $\lambda_i(\omega)$ of the hermitian matrix $S_\Phi(\omega)$ [16,20,21].

3. Fractional systems

In this paper, we consider fractional continuous-time systems of the form

$$D^\alpha Ex(t) = Ax(t) + Bu(t) \tag{5}$$

$$y(t) = Cx(t) + Du(t) \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$ the output vector of the system, and $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$; and $\alpha \in \mathbb{R}$ denotes the fractional order of the system. For our purpose we use the Caputo's fractional differentiation which is defined by

$$D^\alpha x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau$$

where $m - 1 < \alpha < m$, $m \in \mathbb{N}$. Note that this differentiation is in the same form as for the integer order differential systems. We consider here a commensurate order case.

Using the same techniques as for rational matrices, we write the transfer function of such a fractional system as

$$G(s^\alpha) = C(s^\alpha E - A)^{-1}B + D \tag{7}$$

which can be described by a generalized state space realization consisting of the parameters $\{A, B, C, D, E, \alpha\}$, and provided $0 < \alpha \leq 1$, its H_∞-norm is then obtained from the following definition [17] :

$$\|G\|_{H_\infty} = \sup_{\omega\in\mathbb{R}} \sigma_{max} G((j\omega)^\alpha)$$

with $s = j\omega$. Since we denote $\tilde{\omega} = \omega^\alpha$ and $\beta = j^{\alpha-1}$, the corresponding transfer function is then written as

$$G((j\omega)^\alpha) = C((j\omega)^\alpha E - A)^{-1}B + D = C(j\tilde{\omega}\beta E - A)^{-1}B + D = \tilde{G}(j\tilde{\omega})$$

with generalized state space realization $\{A, B, C, D, \beta E\}$, and the corresponding H_∞-norm is then given as

$$\|\tilde{G}\|_{H_\infty} = \sup_{\omega\in\mathbb{R}} \sigma_{max} \tilde{G}((j\tilde{\omega})).$$

The following theorem then easily follows from this definition.

Theorem 1. *Assume $0 < \alpha \leq 1$ and define $\beta := j^{(\alpha-1)}$ and $\tilde{\omega} := \omega^\alpha$, then the H_∞-norm of the fractional transfer function $G(s^\alpha)$, with generalized state space realization $\{A, B, C, D, E, \alpha\}$ is equal to the H_∞-norm of the rational transfer function $\tilde{G}(\tilde{s})$, with $\tilde{s} := j\tilde{\omega}$ and with generalized state space realization $\{A, B, C, D, \beta E\}$.*

Proof. Inserting the change of parameters $\beta := j^{(\alpha-1)}$, $\tilde{\omega} := \omega^\alpha$ in Eq. (7), yields

$$G((j\omega)^\alpha) = C((j\omega)^\alpha E - A)^{-1}B + D = C(j\tilde{\omega}\beta E - A)^{-1}B + D = \tilde{G}(j\tilde{\omega}).$$

It then follows that for $0 < \alpha \leq 1$

$$\sup_{\omega\in\mathbb{R}} \sigma_{max} G((j\omega)^\alpha) = \sup_{\tilde{\omega}\in\mathbb{R}} \sigma_{max} \tilde{G}((j\tilde{\omega}))$$

since both ω and $\tilde{\omega}$ vary over the whole real line. Therefore, the transfer functions $G(s^\alpha)$ and $\tilde{G}(\tilde{s})$ have the same infinity norm. □

We can now apply again the same techniques to show that $G((j\omega)^\alpha)$ and $[G((j\omega)^\alpha)]^*$ are the Schur complements of

$$\left[\begin{array}{c|c} A - \omega^\alpha j\beta E & B \\ \hline C & D \end{array}\right], \quad \text{and} \quad \left[\begin{array}{c|c} A^T + \omega^\alpha j\bar{\beta} E^T & C^T \\ \hline B^T & D^T \end{array}\right] \tag{8}$$

respectively, and that

$$\Phi(\gamma,\omega) := \gamma^2 I - G((j\omega)^\alpha)[G((j\omega)^\alpha)]^* \tag{9}$$

is the Schur complement of

$$S_\Phi(\omega) := \left[\begin{array}{cc|c} 0 & A^T + \omega^\alpha j\bar{\beta}E^T & C^T \\ A - \omega^\alpha j\beta E & -BB^T & -BD^T \\ \hline C & -DB^T & \gamma^2 I - DD^T \end{array}\right] \tag{10}$$

which represents a complex pencil of the form

$$\mathcal{E}_\gamma - \tilde{\omega}\mathcal{F} = \begin{bmatrix} 0 & A^T & C^T \\ A & -BB^T & -BD^T \\ C & -DB^T & \gamma^2 I - DD^T \end{bmatrix} - j\tilde{\omega} \begin{bmatrix} 0 & -\bar{\beta}E^* & 0 \\ \beta E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

where $\mathcal{E}_\gamma$ and $\mathcal{F}$ are both Hermitian. With the substitution $\tilde{\omega} := \omega^\alpha$ and $\tilde{E} = \beta E$ this can also be written as the pencil

$$\mathcal{E}_\gamma - \tilde{\omega}\mathcal{F} = \begin{bmatrix} 0 & A^T & C^T \\ A & -BB^T & -BD^T \\ C & -DB^T & \gamma^2 I - DD^T \end{bmatrix} - \tilde{\omega} \begin{bmatrix} 0 & -j\tilde{E}^* & 0 \\ j\tilde{E} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

which is Hermitian for every fixed value of the *real* variable γ. It then follows again that the eigenvalues $\lambda_i(\tilde{\omega})$ of this pencil, are real analytical functions of the real variable $\tilde{\omega}$ [22] and that techniques borrowed from H_∞-norm calculations can be applied here as well. This fact was observed by the presentation in FCPNLO [24] and by Liang et al. in [25].

4. Level set methods

Note

$$\gamma_* = \sup_{\tilde{\omega}\in\mathbb{R}} \sigma_{max} \tilde{G}(j\tilde{\omega}). \tag{13}$$

The computation of (13) can be performed iteratively using a test for the existence of real zeros $\tilde{\omega}$ of the matrix function

$$\gamma_0^2 I - G(j\tilde{\omega})[G(j\tilde{\omega})]^*. \tag{14}$$

It turns out that $\tilde{\omega}_i$ is a real zero of Eq. (14) if and only if σ_0 is a singular value of $G(j\tilde{\omega}_i)$, which then leads to a test for a bisection algorithm to find the maximum of the scalar function $\sigma(\tilde{\omega})=\sigma_{max}G(j\tilde{\omega})$. As in [21] and [23], one introduced a more efficient algorithm by taking benefit of the fact that the above eigenvalue problems not only give the intersection points of the function $\sigma(\tilde{\omega})$ with a particular level σ_0 but also the derivative of the function in these points, which is obtained at a little extra cost from the generalized eigenvectors of the pencil. In the first phase, the set of subintervals of the real axis among which the optimum $\tilde{\omega}_*$ necessarily belongs, is computed by the algorithm for a given σ_0. These subintervals are determined by computing the real zeros of (14) corresponding to this value of σ_0. In the second phase, σ_0 is increased to the largest value obtained from considering the successive midpoints of the above subintervals. This two phase process can then be iterated up to convergence and therefore delivers the supremum σ_* in a finite number of steps within any required degree of accuracy.

5. Numerical examples

For numerical tests we consider a few examples where we can compute the H_∞ norm also analytically.

Example 1.

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D = 0, \; \alpha = 0.5.$$

The corresponding transfer function is

$$\tilde{G}_1(\tilde{\omega}) = \frac{e^{j\frac{\pi}{4}}\tilde{\omega} + 1}{(e^{j\frac{\pi}{4}}\tilde{\omega} + 1)^2 + 4}.$$

In this case the frequency at which $\left\|\tilde{G}_1(\tilde{\omega})\right\|_{H_\infty}$ is reached is $\tilde{\omega} = 1.4588$ and then the result of the H_∞ norm is $\gamma = 0.2774$.

Example 2. If we consider the system (6) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} -1 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \; D = 0, \; \alpha = 0.5.$$

The transfer function of the system is $\tilde{G}_2(\tilde{\omega}) = \tilde{G}_1(\tilde{\omega}) + \frac{1}{3}$. Here the maximizing frequency is $\tilde{\omega} = 1.4285$ and the result of the H_∞ norm is then $\gamma = 0.6105$.

Example 3. Consider the system (6) where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -0.2 & 0.4 \\ 0 & 0 & -0.4 & -0.2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \; D = 0, \; \alpha = 0.5.$$

For this example the maximizing frequency is $\tilde{\omega} = 0.2843$ and we obtain for the H_∞ norm the value $\gamma = 0.13823$.

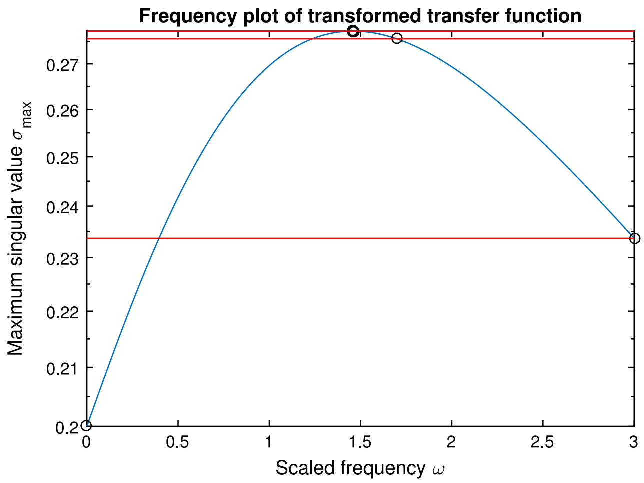

Fig. 1. Plot of $\sigma_{max}(\tilde{G}(j\tilde{\omega_i}))$ with points $\tilde{\omega_i}$ (marked by ∘) and levels $\gamma_i = \sigma_{max}(\tilde{G}(j\tilde{\omega_i}))$.

Notice that we use the procedure described in [16] and [21] where the function $\tilde{G}(j\tilde{\omega}_i)$ is evaluated only at a few points $\tilde{\omega}_i$. At each of these points we also calculate $\gamma(\tilde{\omega}_i) = \sigma_{max}(\tilde{G}(j\tilde{\omega}_i))$ and then find the next point $\tilde{\omega}_{i+1}$ by a quadratic [16] or quartic [21] fit (see Fig. 1). This turns out to be very economical. In our examples we used the algorithm described in [16] with tolerance 10^{-7} and initial interval $[0,3]$ for $\tilde{\omega}$. In all three examples the method converged in less than 5 steps. Fig. 1 shows the different γ levels and $\tilde{\omega}$ values for Example 5.1.

6. Conclusion

In this paper an efficient algorithm to compute the H_∞ norm of a fractional system is given. The algorithm is based on the computation of level sets of the maximum singular value of the transfer function. Illustrative examples are also given to illustrate the applicability of the proposed approach. This approach compares favorably with an LMI based method.

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