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Applied Mathematics Letters 19 (2006) 451–457

Applied Mathematics Letters

www.elsevier.com/locate/aml

Stability margins for generalized state space systems

Djilali Bouagada[a], Paul Van Dooren[b,*]

[a]Faculté des Sciences et Sciences de Ingénieur, Université de Mostaganem, 2700 Algeria
[b]CESAME, Université catholique de Louvain, Belgium, Av Lemaitre 4, B-1348 Louvain-la-Neuve, Belgium

Received 13 June 2005; accepted 20 June 2005

Abstract

In this work we extend results from the literature on H_∞ design with pole placement constraints to the case of generalized state space models, for both continuous-time and discrete-time systems. We also propose tests using linear matrix inequalities of reduced dimension.

Keywords: Stability; Generalized state space systems; Linear matrix inequalities

1. Introduction

In controller design, one is typically concerned with the robustness of certain properties of a nominal system, in the presence of uncertainties in the model parameters. Linear matrix inequalities are often used in this context because they yield sufficient conditions expressing that a certain class of perturbations does not stabilize a nominal system.

In this work we study results obtained previously by Chilali and Gahinet regarding H_∞ design with pole placement constraints in a certain region of the complex plane. These conditions describe a class of convex regions in which the poles are constrained to lie for the given perturbations. The results derived in that work are formulated in terms of a standard state space model. In the present work, we extend them to the case of generalized state space models, for both continuous-time and discrete-time systems. We also propose modified tests using linear matrix inequalities of reduced dimension. These extensions

* Corresponding author.
E-mail addresses: bouagada@univ-mosta.dz (D. Bouagada), vdooren@csam.ucl.ac.be (P. Van Dooren).

0893-9659/$ - see front matter
doi:10.1016/j.aml.2005.06.015

have the advantage of reducing the complexity of the approach and of yielding numerical tests that are more reliable since the reduction to a standard state space model is not required any longer.

2. Stability margins for continuous- and discrete-time systems

We will consider linear time-invariant dynamical systems of the form

$$\begin{aligned} \lambda E x(t) &= A x(t) + B u(t), \\ y(t) &= C x(t) + D u(t). \end{aligned} \tag{1}$$

Here $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$ are given matrices, $x(t)$ is the vector of n state variables, $u(t)$ the vector of m inputs and $y(t)$ the vector of p outputs. The operator λ stands for the differential operator s (in the Laplace domain) when (1) is a continuous-time system and for the delay operator z (in the transformed domain) when (1) is a discrete-time system. We will suppose that the above realization is minimal and that the open loop system (E, A) is strictly stable, meaning that the generalized eigenvalue problem $\lambda E - A$ has all its eigenvalues in a prescribed open set Γ of the complex plane, which is the open left half-plane for continuous-time systems, and the open unit disc for discrete-time systems. This implies that the matrix E is non-singular since otherwise the system would have a pole at $\lambda = \infty$ [4].

If we now close the loop with $u = \Delta y$, we obtain

$$\begin{aligned} \lambda E x(t) &= A x(t) + B\Delta y(t), \\ y(t) &= C x(t) + D\Delta y(t), \end{aligned} \tag{2}$$

or, after elimination of $y(t)$,

$$\lambda E x(t) = A(\Delta) x(t), \qquad A(\Delta) := [A + B(I_m - \Delta D)^{-1}\Delta C]. \tag{3}$$

This follows from $(I_m - \Delta D)^{-1}\Delta = \Delta(I_p - D\Delta)^{-1}$ which is easily verified by the relation $\Delta(I_p - D\Delta) = (I_m - \Delta D)\Delta$.

We then want to know conditions to guarantee that the closed loop system $(E, A(\Delta))$ is also strictly stable. We therefore define the corresponding *stability radius* of the perturbed system $(E, A(\Delta))$ as the smallest perturbation Δ destabilizing the system:

$$r_C(E, A, B, C, D) := \inf_{\Delta}\{\|\Delta\|_2 : (\lambda E - A(\Delta)) \text{ has unstable eigenvalues}\} \tag{4}$$

where we use the 2-norm $\|\Delta\|_2 = \sup_{x\neq 0} \frac{\|\Delta x\|_2}{\|x\|_2}$ for measuring the *complex* perturbation $\Delta \in \mathbb{C}^{m\times p}$. Since eigenvalues are continuous functions of the elements of a pencil $\lambda E - A(\Delta)$, stability will be lost only when one of the eigenvalues crosses the boundary $\partial\Gamma$ of the stability region Γ. An equivalent formulation of this stability radius is thus given by

$$r_C(E, A, B, C, D) := \inf_{\lambda\in\partial\Gamma}\left\{\inf_{\Delta}\{\|\Delta\|_2 : \det[\lambda E - A(\Delta)] = 0\}\right\}. \tag{5}$$

Testing whether or not

$$\det[\lambda E - A(\Delta)] = 0$$

is equivalent to testing

$$\det \begin{bmatrix} \lambda E - A & B \\ \Delta C & I_m - \Delta D \end{bmatrix} = 0 \tag{6}$$

since $\lambda E - A(\Delta)$ is the Schur complement of (6) with respect to $\lambda E - A$, which is assumed non-singular in the considered region Γ and its boundary $\partial\Gamma$. Notice that condition (6) can also be written as

$$\det \left(\begin{bmatrix} \lambda E - A & B \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} \Delta \begin{bmatrix} C & -D \end{bmatrix} \right) = 0 \tag{7}$$

and since $\lambda E - A$ is invertible this is equivalent to testing

$$\det \left(I_{n+m} + \begin{bmatrix} -(\lambda E - A)^{-1} B \\ I_m \end{bmatrix} \Delta \begin{bmatrix} C & -D \end{bmatrix} \right) = 0. \tag{8}$$

Since $\det[I + ST] = 0$ implies $\det[I + TS] = 0$ for any two conformable matrices S and T, this finally yields

$$\det[I_m - \Delta G(\lambda)] = 0, \qquad G(\lambda) := C(\lambda E - A)^{-1} B + D. \tag{9}$$

We can thus rephrase the stability radius as follows:

$$r_C(E, A, B, C, D) = \inf_{\lambda \in \partial\Gamma} \left\{ \inf_{\Delta} \{ \|\Delta\|_2 : \det[I_m - \Delta G(\lambda)] = 0 \} \right\}$$

and this is known to be equal to the so-called H_∞ norm of the system $G(\cdot)$:

$$r_C(E, A, B, C, D) = \left[\sup_{\lambda \in \partial\Gamma} \|G(\lambda)\|_2 \right]^{-1} = [\|G(\cdot)\|_\infty]^{-1}. \tag{10}$$

For continuous-time systems $\partial\Gamma = j\omega, \omega \in \mathbb{R}$, and (10) further simplifies to

$$r_C(E, A, B, C, D) := \left[\sup_{\omega \in \mathbb{R}} \|G(j\omega)\|_2 \right]^{-1}$$

and for discrete-time systems $\partial\Gamma = \mathrm{e}^{j\omega}, \omega \in \mathbb{R}$, which simplifies to

$$r_C(E, A, B, C, D) := \left[\sup_{\omega \in \mathbb{R}} \|G(\mathrm{e}^{j\omega})\|_2 \right]^{-1}.$$

For the case $E = I_n$ these connections are rather standard and we recall them in the following theorem given for arbitrary E.

Theorem 1. *Let (E, A) be a strictly stable open loop system; then the closed loop system $(E, A(\Delta))$ is strictly stable if and only if $\Delta \in \mathbb{C}^{m \times p}$ satisfies*

$$\|\Delta\|_2 < \gamma_\star^{-1}, \qquad \gamma_\star := \|G(\cdot)\|_\infty := \sup_{\lambda \in \partial\Gamma} \|G(\lambda)\|_2 \tag{11}$$

where $\partial\Gamma = j\mathbb{R}$ in the continuous-time case and $\partial\Gamma = \mathrm{e}^{j\mathbb{R}}$ in the discrete-time case.

We point out here that when imposing the condition that Δ is *real*, (11) becomes only a sufficient condition for stability. But the theorem implies that stability is guaranteed for all Δ (real or complex) satisfying (11). The key issue for the computation of $\gamma_\star$ is constructing computable conditions for an upper bound γ of $\gamma_\star$. Such a $\gamma > \gamma_\star$ must satisfy

$$G_\star(\lambda)G(\lambda) + \gamma^2 I_m \succ 0, \qquad \forall \lambda \in \partial\Gamma$$

where $G_\star(\lambda), \lambda \in \partial\Gamma$ means $G_\star(j\omega) := [G(-j\omega)]^{\mathrm{T}}$ in the continuous-time case and $G_\star(\mathrm{e}^{j\omega}) := [G(\mathrm{e}^{-j\omega})]^{\mathrm{T}}$ in the discrete-time case.

It was shown in [3] that for the continuous-time case $\gamma > \gamma_\star \geq 0$ if and only if

$$\begin{bmatrix} -E^{\mathrm{T}}YA - A^{\mathrm{T}}YE & -E^{\mathrm{T}}YB \\ -B^{\mathrm{T}}YE & \gamma^2 I_m \end{bmatrix} - \begin{bmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \succ 0, \qquad Y = Y^{\mathrm{T}} \tag{12}$$

and for the discrete-time case $\gamma > \gamma_\star \geq 0$ if and only if

$$\begin{bmatrix} E^{\mathrm{T}}YE - A^{\mathrm{T}}YA & -A^{\mathrm{T}}YB \\ -B^{\mathrm{T}}YA & \gamma^2 I_m - B^{\mathrm{T}}YB \end{bmatrix} - \begin{bmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \succ 0, \qquad Y = Y^{\mathrm{T}}. \tag{13}$$

For standard state-space systems (i.e. for $E = I_n$), other linear matrix inequalities were derived in [2] for the continuous-time case:

$$\begin{bmatrix} -XA - A^{\mathrm{T}}X & -XB & C^{\mathrm{T}} \\ -B^{\mathrm{T}}X & \gamma I_m & D^{\mathrm{T}} \\ C & D & \gamma I_p \end{bmatrix} \succ 0, \qquad X = X^{\mathrm{T}} \tag{14}$$

and for the discrete-time case:

$$\begin{bmatrix} X - A^{\mathrm{T}}XA & -A^{\mathrm{T}}XB & C^{\mathrm{T}} \\ -B^{\mathrm{T}}XA & \gamma I_m - B^{\mathrm{T}}XB & D^{\mathrm{T}} \\ C & D & \gamma I_p \end{bmatrix} \succ 0, \qquad X = X^{\mathrm{T}}. \tag{15}$$

But for invertible E, one can write the above conditions for the standard state space model $\{E^{-1}A, E^{-1}B, C, D\}$. Replacing then X by $E^{\mathrm{T}}YE$, one finally obtains the following equivalent conditions for the continuous-time case:

$$\begin{bmatrix} -E^{\mathrm{T}}YA - A^{\mathrm{T}}YE & -E^{\mathrm{T}}YB & C^{\mathrm{T}} \\ -B^{\mathrm{T}}YE & \gamma I_m & D^{\mathrm{T}} \\ C & D & \gamma I_p \end{bmatrix} \succ 0, \qquad Y = Y^{\mathrm{T}}, \tag{16}$$

and for the discrete-time case:

$$\begin{bmatrix} E^{\mathrm{T}}YE - A^{\mathrm{T}}YA & -A^{\mathrm{T}}YB & C^{\mathrm{T}} \\ -B^{\mathrm{T}}YA & \gamma I_m - B^{\mathrm{T}}YB & D^{\mathrm{T}} \\ C & D & \gamma I_p \end{bmatrix} \succ 0, \qquad Y = Y^{\mathrm{T}}. \tag{17}$$

We point out that the pair of conditions (12), (13) and (16), (17) essentially are equivalent to the bounded real lemma and that they can also be derived from each other via the use of Schur complements and appropriate scalings.

3. LMI regions and $\mathcal{D}$-stability

We first recall here definitions taken from [1], which we will need later on.

Definition 2. An LMI region is any subset $\mathcal{D}$ of the complex plane that can be defined as

$$\mathcal{D} = \{z \in \mathbb{C} \mid D_0 + zD_1 + \bar{z}D_1^{\mathrm{T}} \prec 0\} \tag{18}$$

where D_0 and D_1 are real matrices and $D_0^{\mathrm{T}} = D_0$.

Inspired by [2], we describe here a few examples:

- Half plane $\Re(z) < \alpha$: $D_0 = -2\alpha$, $D_1 = 1$,
- Ellipse with main axes $1/(\alpha \pm \beta)$: $D_0 = -I_2$, $D_1 = \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}$,
- Parabola $-\Re(z) > (\alpha \Im(z))^2$: $D_0 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, $D_1 = \begin{bmatrix} 0 & \alpha \\ -\alpha & 2 \end{bmatrix}$.

Definition 3. A system is $\mathcal{D}$-stable if and only if all its poles are in the LMI region $\mathcal{D}$.

A first extension of the results of [2] is now given below.

Theorem 4. *The eigenvalue problem $\lambda E - A$, with E non-singular, is $\mathcal{D}$-stable if there exists a symmetric matrix Y such that*

$$M_{\mathcal{D}}(Y) := D_0 \otimes (E^{\mathrm{T}} Y E) + D_1 \otimes (E^{\mathrm{T}} Y A) + D_1^{\mathrm{T}} \otimes (A^{\mathrm{T}} Y E) \prec 0, \qquad Y \succ 0. \tag{19}$$

Proof. This follows easily from applying the result of [1] to the standard eigenvalue problem $E^{-1}A$ and then substituting in $X = E^{\mathrm{T}} Y E$. □

Let us now suppose that $\lambda E - A$ is $\mathcal{D}$-stable. We are looking for a sufficient condition to guarantee that $\lambda E - A(\Delta)$ is also $\mathcal{D}$-stable. As shown above, this pencil describes the poles of the closed loop matrix (2). We therefore need to check that $\det[I - \Delta G(\lambda)] = 0$ for $\lambda \in \mathcal{D}$, as given in the following theorem [1].

Theorem 5. *The eigenvalue problem $\lambda E - A(\Delta)$ is strictly $\mathcal{D}$-stable if and only if $\|\Delta\|_2 < \gamma_{\mathcal{D}}^{-1}$ where $\gamma_{\mathcal{D}} := \|G(\cdot)\|_\infty^{\mathcal{D}} := \sup_{\lambda \in \partial\mathcal{D}} \|G(\lambda)\|_2$.*

Sufficient conditions can be derived from a similar result reported in [1].

Theorem 6. *The pencil $\lambda E - A(\Delta)$ is $\mathcal{D}$-stable for all $\|\Delta\|_2 < \gamma^{-1}$ if there exist matrices $Y \in \mathbb{R}^{n \times n}$ and $P \in \mathbb{R}^{k \times k}$ such that*

$$\begin{bmatrix} M_{\mathcal{D}}(Y) & M_1^{\mathrm{T}} \otimes E^{\mathrm{T}} Y B & M_2^{\mathrm{T}} P \otimes C^{\mathrm{T}} \\ M_1 \otimes B^{\mathrm{T}} Y E & -\gamma P \otimes I & P \otimes D^{\mathrm{T}} \\ P M_2 \otimes C & P \otimes D & -\gamma P \otimes I \end{bmatrix} \prec 0, \qquad P \succ 0, Y \succ 0 \tag{20}$$

where $D_1 = M_1^{\mathrm{T}} M_2$ is a factorization with M_1 and M_2 of full row rank k.

Proof. This follows easily from applying the result of [2] to the standard state space realization $\{E^{-1}A, E^{-1}B, C, D\}$ of the system (1) and then substituting in $X = E^{\mathrm{T}} Y E$. □

We now derive a new equivalent linear matrix inequality constraint of smaller size.

Theorem 7. *The pencil $\lambda E - A(\Delta)$ is $\mathcal{D}$-stable for all $\|\Delta\|_2 < \gamma^{-1}$ if there exist matrices $Y \in \mathbb{R}^{n \times n}$ and $P \in \mathbb{R}^{k \times k}$ such that $P \succ 0$, $Y \succ 0$ and*

$$\begin{bmatrix} M_{\mathcal{D}}(Y) & M_1^{\mathrm{T}} \otimes E^{\mathrm{T}} Y B \\ M_1 \otimes B^{\mathrm{T}} Y E & -\gamma^2 P \otimes I \end{bmatrix} + \begin{bmatrix} M_2^{\mathrm{T}} \otimes C^{\mathrm{T}} \\ I \otimes D^{\mathrm{T}} \end{bmatrix} P \otimes I \begin{bmatrix} M_2 \otimes C & I \otimes D \end{bmatrix} \prec 0. \tag{21}$$

Proof. The inequality (21) is the Schur complement of

$$\begin{bmatrix} M_{\mathcal{D}}(Y) & M_1^{\mathrm{T}} \otimes E^{\mathrm{T}} Y B & M_2^{\mathrm{T}} P \otimes C^{\mathrm{T}} \\ M_1 \otimes B^{\mathrm{T}} Y E & -\gamma^2 P \otimes I & P \otimes D^{\mathrm{T}} \\ P M_2 \otimes C & P \otimes D & -P \otimes I \end{bmatrix} \prec 0, \tag{22}$$

with respect to the negative definite block $-P \otimes I$. And (22) is obtained from (20) by replacing P by γP and dividing the last row and column by γ. All conditions are thus equivalent. □

We end this work by giving a few examples for the above theorem.

Example 1. Guaranteed damping of a continuous-time system corresponds to the LMI region $D_0 + zD_1 + \bar{z}D_1^{\mathrm{T}} \prec 0$ with $D_0 = -2\alpha$, $D_1 = D_1^{\mathrm{T}} = 1$. We can choose $M_1 = M_2 = 1$ and P becomes a positive scalar p. The inequality (20) reduces to $Y = Y^{\mathrm{T}} \succ 0$, $p > 0$ and

$$\begin{bmatrix} -2\alpha E^{\mathrm{T}}YE + E^{\mathrm{T}}YA + A^{\mathrm{T}}YE & E^{\mathrm{T}}YB \\ B^{\mathrm{T}}YE & -\gamma^2 p \otimes I \end{bmatrix} + p \begin{bmatrix} C^{\mathrm{T}} \\ I \otimes D^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} C & I \otimes D \end{bmatrix} \prec 0. \tag{23}$$

Notice that if we replace Y by pY, choose $\alpha = 0$ and divide by p, then we recover from this the inequality (12) for continuous-time robust stability.

Example 2. Eigenvalues inside an ellipse with principal axes $1/(\alpha \pm \beta)$ correspond to the LMI region $D_0 + zD_1 + \bar{z}D_1^{\mathrm{T}} \prec 0$ with $D_0 = -I_2$, $D_1 = \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}$. If we choose $M_1^{\mathrm{T}} = D_1$ and $M_2 = I_2$, the inequality (20) then reduces to $Y = Y^{\mathrm{T}} \succ 0$, $P = P^{\mathrm{T}} \succ 0$ and

$$\begin{bmatrix} -E^{\mathrm{T}}YE & \alpha E^{\mathrm{T}}YA + \beta A^{\mathrm{T}}YE & 0 & \alpha E^{\mathrm{T}}YB \\ \beta E^{\mathrm{T}}YA + \alpha A^{\mathrm{T}}YE & -E^{\mathrm{T}}YE & \beta E^{\mathrm{T}}YB & 0 \\ 0 & \beta B^{\mathrm{T}}YE & -\gamma^2 P & 0 \\ \alpha B^{\mathrm{T}}YE & 0 & 0 & -\gamma^2 P \end{bmatrix}$$
$$+ \begin{bmatrix} I_2 \otimes C^{\mathrm{T}} \\ I_2 \otimes D^{\mathrm{T}} \end{bmatrix} (P \otimes I) \begin{bmatrix} I_2 \otimes C & I_2 \otimes D \end{bmatrix} \prec 0.$$

4. Concluding remarks

In this work we derived sufficient conditions for a perturbed pencil $\lambda E - A(\Delta)$ to have all its eigenvalues in a region $\mathcal{D}$ described by a simple LMI. The conditions are conservative for *real perturbations* Δ but strict for complex perturbations. It remains an open problem how to find necessary and sufficient conditions. The conditions we developed here are an extension of those of [1] to generalized state space models. Such models have often the advantage of being sparser that the equivalent standard state space models. Moreover, the LMI conditions developed in this work are of smaller dimension than those of [1] and should therefore lead to faster algorithms.

Acknowledgements

This research was supported by the Belgian Programme on Inter-university Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture and by a concerted research project on Algorithmic Challenges in Large Networks, funded by the Université catholique de Louvain. This work was also partially supported by the National Science Foundation under the Information Technology Research Grant ACI 0324944.

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