LMI Conditions for the Stability of 2D State-Space Models

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Abstract

In this paper we consider the problem of stability of two-dimensional linear systems. New sufficient conditions for the asymptotic stability are derived in terms of linear matrix inequalities.

1. Introduction

Two dimensional (2D) systems have received in recent decades considerable research interest. The most popular two-dimensional systems were introduced in the discrete-time case by Attasi [3], Roesser [19], Fornasini-Marchesini [6], [7] and Kurek [16]. Such 2D systems are systems which are characterized by two independent variables propagating information in two independent directions. We also consider two-dimensional linear continuous-discrete systems which have been studied in [13] (i.e. systems where one independent variable is continuous and the second one is discrete) and finally a class of 2D continuous time systems. Two-dimensional systems have applications in many areas like iterative learning, control synthesis or repetitive processes, image processing, seismological and geographical data processing, power transmission lines, X-ray image enhancement, etc. Stability problems have been considered by several authors [7], [13], [8], [1], [15], [12], [22]. Some algebraic aspects have been introduced in [12], [20], [4], [11], [2], [18], [21] and [5]. The aim of this paper is to develop new sufficient conditions for asymptotic stability of two-dimensional state space systems. Based on [12], [9], [10], we derive a new sufficient LMI condition for asymptotic stability of the considered systems.

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2. Problem formulation and Preliminaries

We denote by $\mathbb{R}^{m\times n}$, ($\mathbb{C}^{m\times n}$), the set of real (complex) matrices with m rows and n columns and by $\mathbb{R}^m$, ($\mathbb{C}^m$), the set of real (complex) vectors. Also, $\mathbb{Z}_+$ denotes the positive integers and $\mathbb{R}_+$ the positive real line, and j is reserved for the square root of -1.

2.1. General 2D Discrete time System

We consider the 2D discrete system proposed in [14] as a generalisation of the 2D state-space system given in [16] and [13]

$$\begin{aligned} Ex(i+1,k+1) &= \\ A_1x(i+1,k)+A_2x(i,k+1)&+A_0x(i,k) \\ +B_0u(i,k)+B_1u(i+1,k)&+B_2u(i,k+1), \qquad (1) \\ y(i,k) &= Cx(i,k)+Du(i,k), \qquad (2) \end{aligned}$$

where $x(i,k) \in \mathbb{R}^n$ is the state vector of the system, $u(i,k) \in \mathbb{R}^m$ is the input vector, $y(i,k) \in \mathbb{R}^p$ the output vector of the system, $A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times m}, i = 0,1,2, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$. Boundary conditions of (1) are given by the known functions $x(0,k)$, $k \in \mathbb{Z}_+$ and $x(i,0)$, $i \in \mathbb{Z}_+$.

Remark 1 *For $B_1 = B_2 = 0$ (1) reduces to the first Fornasini-Marchesini model (FFMM) and for $A_0 = 0$ and $B_0 = 0$ to the second Fornasini-Marchesini model (SFMM). These models are somehow related and can also be recast to the Roesser model, as shown in [17].*

The characteristic polynomial of the system (1) is given by

$$B(z_1,z_2) = \det[z_1z_2E - z_1A_1 - z_2A_2 - A_0] \qquad (3)$$

and is obtained by applying a 2D z-transformation to the equations (1), (2). We then define asymptotic stability of the 2D system (1), (2) as in [13] and [7].

Definition 2 *The 2-D system (1) is asymptotically stable if the zero input response (i.e. $u(i,k) = 0$ for $i \geq 0$, $k \geq 0$) with any boundary conditions satisfying $sup_i \|x(i,0)\| < \infty$, $sup_k \|x(0,k)\| < \infty$ converges to zero, i.e. $lim_{i,k\to\infty} \|x(i,k)\| = 0$.*

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2.2. 2D Continuous-Discrete time System

Consider the 2D continuous-discrete time state-space system described by,

$$\begin{aligned} E\dot{x}(t,k+1) &= \\ A_1\dot{x}(t,k)+A_2x(t,k+1)+A_0x(t,k) & \\ +B_0u(t,k)+B_1\dot{u}(t,k)+B_2u(t,k+1) & \qquad (4) \\ y(t,k) = Cx(t,k)+Du(t,k) & \qquad (5) \end{aligned}$$

where, $\dot{x} = \frac{\partial x(t,k)}{\partial t}$, $x(t,k) \in \mathbb{R}^n$ is the state vector, $u(t,k) \in \mathbb{R}^m$ is the input vector and $y(t,k) \in \mathbb{R}^p$ is the output vector of the system, and where $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $i = 0, 1, 2$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$. Boundary conditions of (4) are given by the known functions $x(0,k), k \in \mathbb{Z}_+$ and $x(t,0), t \in \mathbb{R}_+$. The characteristic polynomial of the system (4) is defined as follows

$$B(s,z) = \det[szE - sA_1 - zA_2 - A_0] \qquad (6)$$

and is obtained by applying a 2D *sz*-transformation to the equation (4). We define asymptotic stability of 2D continuous-discrete systems (4), (5) as in [13] and [7].

Definition 3 *The 2D continuous-discrete time system (4) is asymptotically stable if the zero input response (i.e. $u(t,k) = 0$ for $t \geq 0$, $k \geq 0$) with any boundary conditions satisfying $sup_t \|x(t,0)\| < \infty$, $sup_k \|x(0,k)\| < \infty$ converges to zero, i.e. $lim_{t,k\to\infty} \|x(t,k)\| = 0$.*

2.3. General 2D Continuous time System

We finally look at the 2D continuous-time system considered in [13]:

$$\begin{aligned} E\frac{\partial^2 x(t_1,t_2)}{\partial t_1 \partial t_2} &= \\ A_1\frac{\partial x(t_1,t_2)}{\partial t_1} + A_2\frac{\partial x(t_1,t_2)}{\partial t_2} + A_0x(t_1,t_2) & \\ +B_0u(t_1,t_2) + B_1\frac{\partial u(t_1,t_2)}{\partial t_1} + B_2\frac{\partial u(t_1,t_2)}{\partial t_2} & \qquad (7) \\ y(t_1,t_2) = Cx(t_1,t_2)+Du(t_1,t_2) & \qquad (8) \end{aligned}$$

where $x(t_1,t_2) \in \mathbb{R}^n$ is the state vector, $u(t_1,t_2) \in \mathbb{R}^m$ is the input vector, $y(t_1,t_2) \in \mathbb{R}^p$ the output vector, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $i = 0, 1, 2$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$. Boundary conditions of (7) are given by the known functions $x(0,t_2), t_2 \in \mathbb{R}_+$ and $x(t_1,0), t_1 \in \mathbb{R}_+$. The characteristic polynomial of the 2D continuous system is defined as follows

$$B(s_1,s_2) = \det[s_1s_2E - s_1A_1 - s_2A_2 - A_0] \qquad (9)$$

and is obtained by applying a 2D *s*-transformation to the system. In this case asymptotic stability is defined as in [13].

Definition 4 *The 2D continuous-time model is asymptotically stable if the zero input response (i.e. $u(t_1,t_2) = 0$ for $t_1 \geq 0$, $t_2 \geq 0$) with any boundary conditions satisfying $sup_{t_1} \|x(t_1,0)\| < \infty$, $sup_{t_2} \|x(0,t_2)\| < \infty$ converges to zero, i.e. $lim_{t_1,t_2\to\infty} \|x(t_1,t_2)\| = 0$.*

3. Stability test

The following necessary and sufficient conditions for stability of such systems were derived in [20], [12] in terms of the characteristic polynomial.

Theorem 5 *The 2-D discrete time system (1) is asymptotically stable if and only if $B(z_1,z_2) \neq 0$ for every pair (z_1,z_2) such that $|z_1| \leq 1$ and $|z_2| \leq 1$.*

Theorem 6 *The 2-D continuous-discrete time system (4) is asymptotically stable if and only if $B(s,z) \neq 0$ for every pair (s,z) such that $\Re s \geq 0$ and $|z| \leq 1$.*

Theorem 7 *The 2-D continuous time system (7) is asymptotically stable if and only if $B(s_1,s_2) \neq 0$ for every pair (s_1,s_2) such that $\Re s_1 \geq 0$ and $\Re s_2 \geq 0$.*

Each of these necessary conditions imply checking the non singularity of a matrix of two variables in a connected 2D region. A clever argument presented first in [11], [12], and proved rigorously later on in [4] and [18], shows that this can be reduced to checking two simpler conditions. For the above three cases they are given below as theorems.

Theorem 8 *The 2-D discrete system (1) is asymptotically stable if and only if*

$$B(z_1,0) \neq 0 \; for \; |z_1| \leq 1, \qquad (10)$$

$$B(z_1,z_2) \neq 0 \; for \; |z_1| = 1 \; and \; |z_2| \leq 1. \qquad (11)$$

Theorem 9 *The 2-D continuous-discrete system (4) is asymptotically stable if and only if*

$$B(s,0) \neq 0 \quad \text{for} \quad \Re s \geq 0, \qquad (12)$$

$$B(j\omega,z) \neq 0 \quad \text{for} \quad \omega \in \mathbb{R} \, \text{and} \, |z| \leq 1. \qquad (13)$$

Theorem 10 *The 2-D continuous system (4) is asymptotically stable if and only if*

$$B(s_1,1) \neq 0 \quad \text{for} \quad \Re s_1 \geq 0, \qquad (14)$$

$$B(j\omega,s_2) \neq 0 \quad \text{for} \quad \omega \in \mathbb{R} \, \text{and} \, \Re s_2 \geq 0. \qquad (15)$$

In the next section we give new sufficient LMI conditions to verify these simpler conditions in polynomial time.

4. LMI conditions for Stability test

In order to further reduce this to an LMI formulation we will need the following theorems proved in [9], [10] to characterize positive polynomial matrices that depend on a real parameter ω and on the unit circle.

Theorem 11 *A hermitian polynomial matrix $P(\omega) = \sum_{i=0}^{2} P_i\omega^i$ with $P_i = P_i^*$, is positive definite on $\omega \in \mathbb{R}$ if and only if there exists a hermitian matrix X such that*

$$\begin{bmatrix} P_0 & (P_1 + jX)/2 \\ (P_1 - jX)/2 & P_2 \end{bmatrix} \succ 0, \quad X = X^*. \quad (16)$$

Theorem 12 *A hermitian polynomial matrix $P(z) = \sum_{i=0}^{2} P_i z^i$ with $P_{-i} = P_i^*$, is positive definite on the unit circle if and only if there exists a hermitian matrix X such that*

$$\begin{bmatrix} P_0 + X & P_1 \\ P_1^* & -X \end{bmatrix} \succ 0, \quad X = X^*. \quad (17)$$

Based on the above definitions and theorems we now propose new sufficient LMI conditions for the asymptotic stability of 2D systems described by (1), (4) and (7) respectively.

Theorem 13 *The 2D discrete time system (1) is asymptotically stable if there exists hermitian matrices X_0, X_1, X_2 such that the following LMIs are feasible*

$$X_1 \succ 0, X_2 \succ 0, \quad (18)$$

$$A_1^T X_1 A_1 - A_0^T X_1 A_0 \succ 0, \quad (19)$$

$$\begin{bmatrix} A_2^T \\ -E^T \end{bmatrix} X_2 \begin{bmatrix} A_2 & -E \end{bmatrix} - \begin{bmatrix} A_0^T \\ A_1^T \end{bmatrix} X_2 \begin{bmatrix} A_0 & A_1 \end{bmatrix} - \begin{bmatrix} -X_0 & 0 \\ 0 & X_0 \end{bmatrix} \succ 0. \quad (20)$$

Theorem 14 *The 2D continuous-discrete time system (4) is asymptotically stable if there exists hermitian matrices X_0, X_1, X_2 such that the following LMIs are feasible*

$$X_1 \succ 0, X_2 \succ 0, \quad (21)$$

$$A_1^T X_1 A_0 + A_0^T X_1 A_1 \succ 0, \quad (22)$$

$$\begin{bmatrix} A_2^T \\ E^T \end{bmatrix} X_2 \begin{bmatrix} A_2 & E \end{bmatrix} - \begin{bmatrix} A_0^T \\ -A_1^T \end{bmatrix} X_2 \begin{bmatrix} A_0 & -A_1 \end{bmatrix} - \begin{bmatrix} 0 & X_0 \\ X_0 & 0 \end{bmatrix} \succ 0. \quad (23)$$

Theorem 15 *The 2D continuous time system (7) is asymptotically stable if there exists hermitian matrices X_0, X_1, X_2 such that the following LMIs are feasible*

$$X_1 \succ 0, X_2 \succ 0, \quad (24)$$

$$(A_1 - E)^T X_1 (A_2 + A_0) + (A_2 + A_0)^T X_1 (A_1 - E) \succ 0, \quad (25)$$

$$\begin{bmatrix} A_2^T X_2 A_0 + A_0^T X_2 A_2 & -A_1^T X_2 A_2 - A_0^T X_2 E \\ -A_2^T X_2 A_1 - E^T X_2 A_0 & E^T X_2 A_1 + A_1^T X_2 E \end{bmatrix} + \begin{bmatrix} 0 & X_0 \\ X_0 & 0 \end{bmatrix} \succ 0. \quad (26)$$

The proof of Theorems 13, 14 and 15 follows from applying Theorem 11 and 12 to the conditions given in Theorems 8, 9 and 10. The fact that we choose the matrices X_i independent of the variables s and z make these conditions conservative, which is why we only obtain sufficient conditions. Detailed proofs will be given in a full version of this paper.

Remark 16 *It is interesting to note that in the theorems 13, 14 and 15 all known matrices are real. Therefore the unknown matrices $X_i, i = 0, 1, 2$ can be chosen to be real as well. This follows from the following observation. Let a matrix $M \in \mathbb{C}^{n\times n}$ be hermitian and let $M = M_r + jM_i$ where $M_r, M_i \in \mathbb{R}^{n\times n}$ are its real and imaginary parts. Then $M_r = M_r^T$ and $M_i = -M_i^T$. The compound matrix*

$$\hat{M} := \begin{bmatrix} M_r & M_i \\ -M_i & M_r \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$$

is symmetric and its eigenvalues are the same as those of M (each eigenvalue appears twice). Therefore, $\hat{M}$ is positive definite iff M is positive definite. Since M_r is a submatrix of $\hat{M}$, it will also be positive definite if M is positive definite. If an LMI is used as feasibility condition, then the existence of a complex hermitian solution guarantees that there also exists a real symmetric solution. Conversely, the existence of a real symmetric solution implies that that there also exists a solution in the larger class of complex hermitian solutions.

Remark 17 *The same technique can also be applied to derive LMI conditions for the stability of 2D systems of Roesser type or of delay-differential equations. In the latter case, though, there is a connection between the two variables s and z since $z = e^{-s\delta}$ is the Laplace transform of the delay operator. In that special case the above conditions are still sufficient.*

5. Concluding remarks

In this paper we derived new sufficient conditions for 2D systems to be asymptotically stable. The conditions that we developed here yield a new test described by simple LMIs.

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