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On the stability of 2D state-space models

Djillali Bouagada [1] and Paul Van Dooren [2,*,†]

[1] *University Abdelhamid Ibn Badis of Mostaganem, Laboratory of Pure and Applied Mathematics, Department of Mathematics, Faculty SESN BP 227-Mostaganem 27000, Algeria*
[2] *CESAME, Université catholique de Louvain, Av Lemaitre 4, B-1348 Louvain-la-Neuve, Belgium*

SUMMARY

In this paper, we consider the problem of stability of two-dimensional linear systems. New sufficient conditions for the asymptotic stability are derived in terms of linear matrix inequalities. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Two-dimensional (2D) systems have received in recent decades considerable research interest. The most popular 2D models were introduced in the discrete-time case by Attasi [1], Roesser [2], Fornasini–Marchesini [3, 4], and Kurek [5]. Such 2D systems are systems that are characterized by two independent variables propagating information in two independent directions. We also consider 2D linear continuous-discrete model which have been studied in [6] (i.e., models where one independent variable is continuous and the second one is discrete) and finally a class of 2D continuous-time models. Two-dimensional models have applications in many areas such as iterative learning, control synthesis or repetitive processes, image processing, seismological and geographical data processing, power transmission lines, X-ray image enhancement, and so on. Stability problems have been considered by several authors [4, 6, 9–12, 22]. Several methods do exist to determine whether a 2D system is stable or not, and many approaches can be found in the literature [12, 16–19, 23–25]. In [16], Huang gave a stability test for 2D digital recursive filters, using a simplified version of a stability theorem of Shanks and proved that it is equivalent to stability results of Ansell [19]. In [18], Davis pointed out a small problem in Huang's proof and corrected it. Other stability criteria were also introduced by Jury [12] and Siljak [17] for 2D discrete-time and continuous-time systems. These approaches were all based on function theoretic criteria. Instead of that, several authors have attempted to use matrix algebraic techniques, such as Lyapunov matrix functions or linear matrix inequalities (LMIs) for testing stability of 2D systems, but only sufficient conditions were found so far. In [10], Anderson *et al.* constructed a 2D Lyapunov matrix equation that is sufficient for stability but not necessary. Sufficient conditions have also been derived in terms of LMIs by Galkowski *et al.* [22] who consider the problem of positive real control. In [20], Zou *et al.* gave sufficient LMI conditions for the internal stability of 2D singular systems, including acceptability and jump modes freeness. The aim of this paper is to develop new sufficient algebraic conditions for asymptotic stability of 2D state space models. On the basis of [12–14], we derive LMIs for guaranteed asymptotic stability of the considered models.

*Correspondence to: Paul Van Dooren, CESAME, Université catholique de Louvain, Av Lemaitre 4, B-1348 Louvain-la-Neuve, Belgium.
†E-mail: paul.vandooren@uclouvain.be

2. STABILITY OF TWO-DIMENSIONAL DISCRETE-TIME SYSTEMS

We denote by $\mathbb{R}^{m\times n}$, $(\mathbb{C}^{m\times n})$, the set of real (complex) matrices with m rows and n columns and by $\mathbb{R}^m$, $(\mathbb{C}^m)$, the set of real (complex) vectors. Also, $\mathbb{Z}_+$ denotes the non-negative integers and $\mathbb{R}_+$ the non-negative real line, and j is used to denote the square root of -1.

We consider the general 2D discrete model proposed in [7] as a generalization of the 2D state-space model given in [5],

$$z_1z_2Ex = z_1A_1x + z_2A_2x + A_0x + B_0u + z_1B_1u + z_2B_2u, \tag{1}$$

$$y = Cx + Du, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector of the model, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^p$ is the output vector of the model $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $i = 0, 1, 2$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, and $z_1x = x(k+1,l), z_2x = x(k,l+1)$. Boundary conditions of (1) are given by the known functions $x(0,j),\ j \in \mathbb{Z}_+$ and $x(i,0),\ i \in \mathbb{Z}_+$.

Remark 1
For $B_1 = B_2 = 0$, (1) reduces to the first Fornasini–Marchesini model and for $A_0 = 0$ and $B_0 = 0$ to the second Fornasini–Marchesini model. These models are somehow related and can also be recast in the Roesser model, as shown in [8, 15].

We first introduce the notion of asymptotic stability of 2D discrete-time systems.

Definition 2
The 2D general model (1) is asymptotically stable if the zero input response (i.e., $u(i,j) = 0$ for $i \geqslant 0,\ j \geqslant 0$) with any boundary conditions satisfying $sup_i\,\|x(i,0)\| < \infty$, $sup_j\,\|x(0,j)\| < \infty$ converges to zero, that is, $lim_{i,j\to\infty}\,\|x(i,j)\| = 0$.

The characteristic polynomial of the model (1) is given by

$$B(z_1,z_2) = \det\left[z_1z_2E - z_1A_1 - z_2A_2 - A_0\right] \tag{3}$$

and is obtained by applying a 2D z-transformation to the Equations (1) and (2).

The following necessary and sufficient conditions for stability of such systems were derived in [12, 17, 18] in terms of the characteristic polynomial. We repeat here the basic ideas because they will be useful in the sequel.

Theorem 3
The general 2D discrete system (1) is asymptotically stable if and only if $B(z_1,z_2) \neq 0$ for every pair (z_1,z_2) such that $|z_1| \leqslant 1$ and $|z_2| \leqslant 1$.

Proof
Let

$$H(z_1,z_2) = \frac{A(z_1,z_2)}{B(z_1,z_2)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{mn}z_1^m z_2^n, \tag{4}$$

where the coefficients h_{mn} represent the impulse response of the filter. A rational function $H(z_1,z_2)$ is stable if and only if the impulse response converges to zero for all pairs (z_1,z_2) in the polydisc $|z_1| \leqslant 1$ and $|z_2| \leqslant 1$. Using properties of convergent series (such as the Weierstrass M-test), one then shows that it is equivalent to $H(z_1,z_2)$ being analytic for every pair (z_1,z_2) such that $|z_1| \leqslant 1$ and $|z_2| \leqslant 1$. Finally, for rational functions, this is shown to be equivalent to the absence of roots of $B(z_1,z_2)$ inside the unit polydisc. □

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Remark 4
Some authors represent the delay operator of discrete-time systems by z^{-1} rather than by z, which explains why different conditions are encountered in the literature. These results can easily be adapted when changing from one convention to another.

The condition of Theorem 3 implies checking the nonsingularity of a matrix of two variables in a connected 2D region. A clever argument presented first in [12, 16], and proved rigorously later on in [18] shows that this can be reduced to checking two simpler conditions. For the above case, this is given below as theorem.

Theorem 5
The general 2D discrete system (1) is asymptotically stable if and only if

$$B(z_1, 0) \neq 0 \text{ for } |z_1| \leqslant 1, \tag{5}$$

$$B(z_1, z_2) \neq 0 \text{ for } |z_1| = 1 \text{ and } |z_2| \leqslant 1. \tag{6}$$

Proof
This theorem was proved (incorrectly) in [16] and later in [12, 17]. Modified (and corrected) proofs appeared later in [18, 23]. All proofs are based on the fact that the functions implicitly relating z_1 and z_2 via $B(z_1, z_2) = 0$ are algebraic functions and that some form of the maximum modulus theorem then applies. □

Remark 6
Notice that the role of z_1 and z_2 can easily be interchanged and that this yields equivalent sufficient conditions.

In the next section, we convert these simpler conditions to a set of equivalent LMI conditions, which can be checked in polynomial time.

3. LINEAR MATRIX INEQUALITY CONDITIONS FOR STABILITY TEST

3.1. *Stability of two-dimensional discrete models*

In order to further reduce this to an LMI formulation, we will need the following theorems proved in [13] and [14] to characterize positive polynomial matrices that depend on a real parameter ω and on the unit circle.

Theorem 7
A hermitian polynomial matrix $P(\omega) = \sum_{i=0}^{2} P_i \omega^i$ with $P_i = P_i^*$ is positive definite on $\omega \in \mathbb{R}$ if and only if there exists a hermitian matrix X such that

$$\begin{bmatrix} P_0 & (P_1 + jX)/2 \\ (P_1 - jX)/2 & P_2 \end{bmatrix} \succ 0, \; X = X^*. \tag{7}$$

Theorem 8
A hermitian polynomial matrix $P(z) = \sum_{i=-1}^{1} P_i z^i$ with $P_{-i} = P_i^*$ is positive definite on the unit circle if and only if there exists a hermitian matrix X such that

$$\begin{bmatrix} P_0 + X & P_1 \\ P_1^* & -X \end{bmatrix} \succ 0, \; X = X^*. \tag{8}$$

On the basis of the above definitions and theorems, we now propose sufficient LMI conditions for the asymptotic stability of 2D models described in (1).

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Theorem 9
The model (1) is asymptotically stable if there exists hermitian matrices X_0, X_1, and X_2 such that the following LMIs are feasible:

$$X_1 \succ 0,\ X_2 \succ 0, \tag{9}$$

$$A_0^T X_1 A_0 - A_1^T X_1 A_1 \succ 0, \tag{10}$$

$$\begin{bmatrix} A_0^T X_2 A_0 - A_2^T X_2 A_2 - X_0 & A_2^T X_2 E + A_0^T X_2 A_1 \\ E^T X_2 A_2 + A_1^T X_2 A_0 & A_1^T X_2 A_1 - E^T X_2 E + X_0 \end{bmatrix} \succ 0. \tag{11}$$

Proof
Condition (5) on the characteristic polynomial reduces to

$$B(z_1, 0) = \det[-z_1 A_1 - A_0] \neq 0 \text{ for } |z_1| \leqslant 1 \tag{12}$$

that is satisfied if and only if the following LMI is feasible:

$$A_0^T X_1 A_0 - A_1^T X_1 A_1 \succ 0\,,\ X_1 \succ 0\,,\ X_1 = X_1^*. \tag{13}$$

Condition (6) expresses that for all $\omega \in \mathbb{R}$ and $|z_2| \leqslant 1$, we have

$$B(e^{j\omega}, z_2) = \det[e^{j\omega} z_2 E - e^{j\omega} A_1 - z_2 A_2 - A_0] = \det[z_2(e^{j\omega} E - A_2) - (e^{j\omega} A_1 + A_0)] \neq 0. \tag{14}$$

This is equivalent to $\det(z_2 M - N) \neq 0$, where $M = e^{j\omega} E - A_2$ and $N = e^{j\omega} A_1 + A_0$, which holds if and only if the following LMI is feasible:

$$N^* X_2 N - M^* X_2 M \succ 0\,,\ X_2 \succ 0\,,\ X_2 = X_2^*, \tag{15}$$

where X_2 will, in general, also depend on ω. If we impose X_2 to be constant, then relation (15) is equivalent to

$$e^{j\omega} P_1 + e^{-j\omega} P_1^* + P_0 \succ 0, \tag{16}$$

where

$$P_1 := A_2^T X_2 E + A_0^T X_2 A_1, \tag{17}$$
$$P_0 := A_0^T X_2 A_0 + A_1^T X_2 A_1 - E^T X_2 E - A_2^T X_2 A_2, \tag{18}$$

with $P_j^* = P_{-j}$. Applying Theorem 8 then yields the condition

$$\begin{bmatrix} P_0 + X & P_1 \\ P_1^* & -X \end{bmatrix} \succ 0, \tag{19}$$

for some hermitian matrix X. Let us now define a new hermitian matrix X_0 via the identity

$$X = X_0 - E^T X_2 E + A_1^T X_2 A_1,$$

then we obtain the equivalent condition

$$\begin{bmatrix} A_0^T X_2 A_0 - A_2^T X_2 A_2 - X_0 & A_2^T X_2 E + A_0^T X_2 A_1 \\ E^T X_2 A_2 + A_1^T X_2 A_0 & A_1^T X_2 A_1 - E^T X_2 E + X_0 \end{bmatrix} \succ 0. \tag{20}$$

Because we imposed X_2 to be constant, this is only a sufficient condition for 2D stability. □

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Remark 10
Notice that the conditions (46), (10), and (11) are only sufficient and that different conditions will be obtained when interchanging the role of z_1 and z_2 and hence of A_1 and A_2. This would yield then the following necessary conditions:

$$X_1 \succ 0,\ X_2 \succ 0, \tag{21}$$

$$A_0^T X_1 A_0 - A_2^T X_1 A_2 \succ 0, \tag{22}$$

$$\begin{bmatrix} A_0^T X_2 A_0 - A_1^T X_2 A_1 - X_0 & A_1^T X_2 E + A_0^T X_2 A_2 \\ E^T X_2 A_1 + A_2^T X_2 A_0 & A_2^T X_2 A_2 - E^T X_2 E + X_0 \end{bmatrix} \succ 0. \tag{23}$$

The conditions for stability of 2D continuous system and continuous-discrete systems can be readily derived in the same way as the stability conditions for discrete systems discussed earlier.

3.2. *Stability of two-dimensional continuous-discrete models*

We consider now a 2D continuous-discrete state space model described by the following equations:

$$szEx = sA_1x + zA_2x + A_0x + B_0u + sB_1u + zB_2u, \tag{24}$$
$$y = Cx + Du, \tag{25}$$

where $x \in \mathbb{R}^n$ is the state vector of the model, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^p$ is the output vector of the model $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $i = 0, 1, 2$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$ and $zx = x(t, k+1)$, $sx = \frac{\partial x(t,k)}{\partial t}$.

Boundary conditions of (24) are given by the known functions $x(0,k)$, $k \in \mathbb{Z}_+$, and $x(t,0)$, $t \in \mathbb{R}_+$.

The characteristic polynomial of the system (24) is defined as

$$B(s,z) = \det\left[szE - sA_1 - zA_2 - A_0\right] \tag{26}$$

and is obtained by applying a 2D sz-transformation to the equation (24). We define asymptotic stability of 2D continuous-discrete systems (24) and (54) as in [6].

Definition 11
The 2D continuous-discrete model (24) is asymptotically stable if the zero input response (i.e., $u(t,k) = 0$ for $t \geqslant 0$, $k \geqslant 0$) with any boundary conditions satisfying $sup_t \|x(t,0)\| < \infty$, $sup_k \|x(0,k)\| < \infty$ converges to zero, that is, $lim_{t,k\to\infty} \|x(t,k)\| = 0$.

We can test stability via the following theorem.

Theorem 12
The 2D continuous-discrete system (24) is asymptotically stable if and only if $B(s,z) \neq 0$ for every pair (s,z) such that $\Re s \geqslant 0$ and $|z| \leqslant 1$.

The desired above condition is then replaced by two conditions as in [12, 17].

Theorem 13
The 2D continuous-discrete system (24) is asymptotically stable if and only if

$$B(s,0) \neq 0 \quad \text{for} \quad \Re s \geqslant 0, \tag{27}$$
$$B(j\omega, z) \neq 0 \quad \text{for} \quad \omega \in \mathbb{R} \text{ and } |z| \leqslant 1. \tag{28}$$

We now derive an LMI condition for the model of the type (24).

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Theorem 14

The model (24) is asymptotically stable if there exists hermitian matrices X_0, X_1, and X_2 such that the following LMIs are feasible:

$$X_1 \succ 0,\ X_2 \succ 0, \tag{29}$$

$$A_1^T X_1 A_0 + A_0^T X_1 A_1 \succ 0, \tag{30}$$

$$\begin{bmatrix} A_0^T X_2 A_0 - A_2^T X_2 A_2 & A_2^T X_2 E + A_0^T X_2 A_1 - X_0 \\ E^T X_2 A_2 + A_1^T X_2 A_0 - X_0 & A_1^T X_2 A_1 - E^T X_2 E \end{bmatrix} \succ 0. \tag{31}$$

Proof

Condition (27) on the characteristic polynomial reduces to

$$B(s,0) = \det[-sA_1 - A_0] \neq 0 \text{ for } \Re s \geqslant 0 \tag{32}$$

that is satisfied if and only if the following LMI is feasible:

$$A_1^T X_1 A_0 + A_0^T X_1 A_1 \succ 0\,,\ X_1 \succ 0\,,\ X_1 = X_1^*. \tag{33}$$

Condition (28) expresses that for all $\omega \in \mathbb{R}$ and $|z| \geqslant 1$, we have

$$B(j\omega, z) = \det[j\omega z E - j\omega A_1 - zA_2 - A_0] = \det[z(j\omega E - A_2) - (j\omega A_1 + A_0)] \neq 0. \tag{34}$$

This is equivalent to $\det(zM - N) \neq 0$, where $M = j\omega E - A_2$; $N = j\omega A_1 + A_0$, which holds if and only if the following LMI is feasible:

$$N^* X_2 N - M^* X_2 M \succ 0\,,\ X_2 \succ 0\,,\ X_2 = X_2^*, \tag{35}$$

where X_2 will in general depend on ω. If we choose X_2 to be constant, then relation (35) is equivalent to

$$\omega^2 P_2 + \omega P_1 + P_0 = \sum_{i=0}^{2} P_i \omega^i \succ 0, \tag{36}$$

where

$$P_0 := A_0^T X_2 A_0 - A_2^T X_2 A_2, \tag{37}$$

$$P_1 := j\left[\left(A_2^T X_2 E - E^T X_2 A_2\right) + \left(A_0^T X_2 A_1 - A_1^T X_2 A_0\right)\right], \tag{38}$$

$$P_2 := A_1^T X_2 A_1 - E^T X_2 E, \tag{39}$$

with $P_i^* = P_i,\ i = 0,\ 1,\ 2$. Applying Theorem 7 then yields the condition

$$\begin{bmatrix} P_0 & (P_1 + jX)/2 \\ (P_1 - jX)/2 & P_2 \end{bmatrix} \succ 0, \tag{40}$$

for some hermitian matrix X. Let us now define a new hermitian matrix X_0 via the identity $X = 2X_0 - \left(E^T X_2 A_2 + A_2^T X_2 E\right) - \left(A_1^T X_2 A_0 + A_0^T X_2 A_1\right)$, then we obtain the equivalent condition

$$\begin{bmatrix} A_0^T X_2 A_0 - A_2^T X_2 A_2 & j\left(A_2^T X_2 E + A_0^T X_2 A_1 - X_0\right) \\ -j\left(E^T X_2 A_2 + A_1^T X_2 A_0 - X_0\right) & A_1^T X_2 A_1 - E^T X_2 E \end{bmatrix} \succ 0. \tag{41}$$

Applying a block diagonal congruence transformation diag $\{I_n, jI_n\}$ finally yields the desired result. Because we imposed X_2 to be constant, this is only a sufficient condition for stability. □

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Remark 15
Notice that when interchanging the role of z and s, one uses the conditions

$$B(1,z) \neq 0 \quad \text{for} \quad |z| \leqslant 1, \tag{42}$$
$$B(s,e^{j\omega}) \neq 0 \quad \text{for} \quad \omega \in \mathbb{R} \text{ and } \Re s \geqslant 0 \tag{43}$$

as necessary and sufficient conditions for stability. This finally yields the sufficient conditions

$$X_1 \succ 0,\ X_2 \succ 0, \tag{44}$$

$$(A_1 + A_0)^T X_1 (A_1 + A_0) - (E - A_2)^T X_1 (E - A_2) \succ 0, \tag{45}$$

$$\begin{bmatrix} -A_2^T X_2 E + A_0^T X_2 A_1 + X_0 & A_1^T X_2 A_2 - A_0^T X_2 E \\ A_2^T X_2 A_1 - E^T X_2 A_0 & -E^T X_2 A_2 + A_1^T X_2 A_0 - X_0 \end{bmatrix} \succ 0. \tag{46}$$

Relation (45) is satisfied when condition (42) on the characteristic polynomial reduces to

$$B(1,z) = \det\left[z(E - A_2) - (A_1 + A_0)\right] \neq 0 \text{ for } |z| \leqslant 1. \tag{47}$$

Condition (43) expresses that for all $\omega \in \mathbb{R}$ and $|z| \geqslant 1$, we have

$$B(s,e^{j\omega}) = \det[-s(A_1 - e^{j\omega}E) - (A_2 e^{j\omega} + A_0)] \neq 0. \tag{48}$$

This is equivalent to $\det(-sM - N) \neq 0$, where $M = A_1 - e^{j\omega}E;\ N = A_2 e^{j\omega} + A_0$, which holds if and only if the following LMI is feasible:

$$M^* X_2 N + N^* X_2 M \succ 0,\ X_2 \succ 0,\ X_2 = X_2^*, \tag{49}$$

where X_2 will in general also depend on ω. If we impose X_2 to be constant, the relation (49) is equivalent to

$$e^{j\omega} P_1 + e^{-j\omega} P_1^* + P_0 \succ 0, \tag{50}$$

where

$$P_1 := A_1^T X_2 A_2 - A_0^T X_2 E, \tag{51}$$
$$P_0 := -E^T X_2 A_2 - A_2^T X_2 E + A_0^T X_2 A_1 + A_1^T X_2 A_0, \tag{52}$$

with $P_j^* = P_{-j}$. Applying Theorem 8 then yields the condition

$$\begin{bmatrix} P_0 + X & P_1 \\ P_1^* & -X \end{bmatrix} \succ 0, \tag{53}$$

for some hermitian matrix X. Let us now define a new hermitian matrix X_0 via the identity

$$X = X_0 + E^T X_2 A_2 - A_1^T X_2 A_0,$$

then we obtain the equivalent condition (46).

3.3. Stability of two-dimensional continuous models

We finally look at the general 2D continuous-time model considered in [6],

$$s_1 s_2 E x = s_1 A_1 x + s_2 A_2 x + A_0 x + B_0 u + s_1 B_1 u + s_2 B_2 u, \tag{54}$$
$$y = Cx + Du, \tag{55}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^p$ is the output vector, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $i = 0, 1, 2$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, and $s_1 s_2 x = \frac{\partial^2 x(t_1,t_2)}{\partial t_1 \partial t_2}$. Boundary conditions of (54) are given by the known functions $x(0,t_2)$, $t_2 \in \mathbb{R}_+$, and $x(t_1,0)$, $t_1 \in \mathbb{R}_+$.

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The characteristic polynomial of the 2D continuous system is defined as

$$B(s_1, s_2) = \det[s_1 s_2 E - s_1 A_1 - s_2 A_2 - A_0] \tag{56}$$

and is obtained by applying a 2D s-transformation to the system. In this case, asymptotic stability is defined as in [6].

Definition 16
The general 2D continuous-time model (54) is asymptotically stable if the zero input response (i.e., $u(t_1, t_2) = 0$ for $t_1 \geqslant 0$, $t_2 \geqslant 0$) with any boundary conditions satisfying $sup_{t_1} \|x(t_1, 0)\| < \infty$, $sup_{t_2} \|x(0, t_2)\| < \infty$ converges to zero, that is, $lim_{t_1,t_2 \to \infty} \|x(t_1, t_2)\| = 0$.

Theorem 17
The general 2D continuous system (54) is asymptotically stable if and only if $B(s_1, s_2) \neq 0$ for every pair (s_1, s_2) such that $\Re s_1 \geqslant 0$ and $\Re s_2 \geqslant 0$.

As mentioned in [12, 17], the equivalent conditions are as follows.

Theorem 18
The General 2D continuous system (54) is asymptotically stable if and only if

$$B(s_1, 1) \neq 0 \quad \text{for} \quad \Re s_1 \geqslant 0, \tag{57}$$

$$B(j\omega, s_2) \neq 0 \quad \text{for} \quad \omega \in \mathbb{R} \text{ and } \Re s_2 \geqslant 0. \tag{58}$$

Sufficient LMI conditions for the asymptotic stability of 2D continuous systems is derived as previously.

Theorem 19
The 2D continuous model (54) is asymptotically stable if there exists hermitian matrices X_0, X_1, and X_2 such that the following LMIs are feasible:

$$X_1 \succ 0,\ X_2 \succ 0, \tag{59}$$

$$(A_1 - E)^T X_1 (A_2 + A_0) + (A_2 + A_0)^T X_1 (A_1 - E) \succ 0, \tag{60}$$

$$\begin{bmatrix} A_2^T X_2 A_0 + A_0^T X_2 A_2 & -A_1^T X_2 A_2 - A_0^T X_2 E + X_0 \\ -A_2^T X_2 A_1 - E^T X_2 A_0 + X_0 & E^T X_2 A_1 + A_1^T X_2 E \end{bmatrix} \succ 0. \tag{61}$$

Proof
This follows from applying Theorem 7 and relations 57 and 58. Again, these are only sufficient conditions. □

Remark 20
Notice that when interchanging the role of s_1 and s_2, this would yield the following conditions.

$$X_1 \succ 0,\ X_2 \succ 0, \tag{62}$$

$$(A_2 - E)^T X_1 (A_1 + A_0) + (A_1 + A_0)^T X_1 (A_2 - E) \succ 0, \tag{63}$$

$$\begin{bmatrix} A_1^T X_2 A_0 + A_0^T X_2 A_1 & -A_2^T X_2 A_1 - A_0^T X_2 E + X_0 \\ -A_1^T X_2 A_2 - E^T X_2 A_0 + X_0 & E^T X_2 A_2 + A_2^T X_2 E \end{bmatrix} \succ 0. \tag{64}$$

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Remark 21
It is interesting to note that in the above theorem, all known matrices are real. Therefore, the unknown matrices $X_i, i = 0, 1, 2$ can be chosen to be real as well. This follows from the following observation. Let a matrix $M \in \mathbb{C}^{n\times n}$ be hermitian and let $M = M_r + jM_i$, where $M_r, M_i \in \mathbb{R}^{n\times n}$ are its real and imaginary parts. Then, $M_r = M_r^T$ and $M_i = -M_i^T$. The compound matrix

$$\hat{M} := \begin{bmatrix} M_r & M_i \\ -M_i & M_r \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$$

is symmetric, and its eigenvalues are the same as those of M (each eigenvalue appears twice). Therefore, $\hat{M}$ is positive definite if and only if M is positive definite. Because M_r is a submatrix of $\hat{M}$, it will also be positive definite if M is positive definite. If an LMI is used as feasibility condition, then the existence of a complex hermitian solution guarantees that there also exists a real symmetric solution. Conversely, the existence of a real symmetric solution implies that there also exists a solution in the larger class of complex hermitian solutions.

Remark 22
The same technique can also be applied to derive LMI conditions for the stability of 2D systems of Roesser type or of delay differential equations. In the latter case, though, there is a connection between the two variables s and z because $z = e^{-s\delta}$ is the Laplace transform of the delay operator. But in this special case, the above sufficient conditions still hold.

4. CONCLUDING REMARKS

In this paper, we derived sufficient conditions for 2D models to be asymptotically stable. The conditions we developed here are a new test described by a simple LMI. All of the obtained LMIs have dimension $2n \times 2n$ at most, where n is the state dimension of the 2D system.

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