VERSITA

Fractional Calculus & Applied Analysis

An International Journal for Theory and Applications

VOLUME 15, NUMBER 3 (2012)

(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

STATE SPACE SOLUTION OF IMPLICIT FRACTIONAL CONTINUOUS TIME SYSTEMS

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Abstract

In this work we extend a result from the literature on fractional continuous-time linear systems to the case of implicit fractional continuous-time state space models, based on the Caputo fractional derivative. The solution of the problem is derived using the Laplace transform.

MSC 2010: 26A33, 45F15, 44A10, 42B10, 42A38

Key Words and Phrases: singular system, fractional system, Laplace transform

1. Introduction

Fractional systems appear in several applications: in mechanical modeling (veloelasticity), in chemistry of polymers, in electrochemistry (Ichise et al., 1971), electromagnetism and electrical machines (Lin et al., 2000), in thermal systems and heat conduction (Battaglia et al., 2001; Cois et al., 2002), in transmission and acoustics (Matignon, 1994; Matignon et al., 1994), in robotics (Valerio and Costa, 2004), and in systems and control theory.

The solution of fractional continuous-time systems is derived in Kaczorek [3], [4]; see also [5]. In this paper we use Laplace transforms to extend the results of [3] to the case of fractional implicit continuous-time systems. The authors already announced these results in the conference paper [1].

pp. 356–361 , DOI: 10.2478/s13540-012-0026-z

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2. Problem formulation and preliminaries

We denote by $\mathbb{R}^{m\times n}$, the set of real matrices with m rows and n columns and by $\mathbb{R}^m$, the set of real vectors. For properties of fractional systems, we refer the reader to [7], [8], [9], [2].

Definition **2.1**. The Caputo fractional derivative of real order $\alpha > 0$ of a continuous function given by $x : (0,\infty) \longrightarrow \mathbb{R}$ is defined as

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{2.1}$$

where $n-1 < \alpha \leq n \in \mathbb{N}^*$, $x^{(n)}(\tau) = d^n x(\tau)/d\tau^n$ and Γ is the gamma function.

Based on the definition of (2.1), the Laplace transform of the Caputo fractional derivative is [8]

$$\mathcal{L}\left(D^\alpha x(t)\right) = \lambda^\alpha X(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} x^{(k)}(0). \tag{2.2}$$

We will consider fractional implicit continuous-time systems of the form

$$D^\alpha E x(t) = Ax(t) + Bu(t), \tag{2.3}$$

$$y(t) = Cx(t) + Du(t), \tag{2.4}$$

where D^α is the Caputo derivative, $x(t) \in \mathbb{R}^n$ is the state vector of the model, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$ the output vector of the model, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. The boundary conditions of (2.3) are given by $x(0) = x_0$, with the following compatibility conditions: $Ex(0)$ and $\lambda^{\alpha-k-1} Ex^{(k)}(0)$ for all k such that $0 \leq k \leq n-1$ are known and $u(t)$ specified for $t \geq 0$, which implies that the solution is impulse free. In this formulation, the order α does not appear in the derivatives of $x(t)$, but rather in the preceding multiplier $\lambda^{\alpha-k-1}$, so quite conveniently, integer order derivatives of $x(0)$ (i.e, $x(0)$, $x^{(1)}(0)$, $x^{(2)}(0)$ and so on) are used as the initial conditions, for which a certain physical interpretation exists. Using (2.1), we may write equation (2.3) in the form

$$ED^\alpha x(t) = Ax(t) + Bu(t), \tag{2.5}$$

$$y(t) = Cx(t) + Du(t). \tag{2.6}$$

If we put

$$h_0(\lambda) := \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} x^{(k)}(0), \tag{2.7}$$

and apply the Laplace transform to system (2.5), we obtain

$$\mathcal{L}\left(ED^\alpha x(t)\right) = \mathcal{L}\left(Ax(t)\right) + \mathcal{L}\left(Bu(t)\right)$$

and, using $X(\lambda)$ and $U(\lambda)$ as the Laplace transforms of $x(t)$ and $u(t)$, respectively :

$$(E\lambda^\alpha - A)\, X(\lambda) = Eh_0(\lambda) + BU(\lambda). \tag{2.8}$$

If we also assume that $(E\lambda^\alpha - A)$ is regular, then (2.8) is equivalent to

$$X(\lambda) = (E\lambda^\alpha - A)^{-1}\left[Eh_0(\lambda) + BU(\lambda)\right]. \tag{2.9}$$

3. The regular implicit case

When E is invertible, we have

$$\begin{aligned}(E\lambda^\alpha - A)^{-1} &= \left[(E\lambda^\alpha)\left(I - (E\lambda^\alpha)^{-1}A\right)\right]^{-1} \\ &= \sum_{i=0}^{\infty}(E^{-1}A)^i E^{-1}\lambda^{-(i+1)\alpha}.\end{aligned} \tag{3.1}$$

Remark **3.1**. Notice that if we put $E = I$ and $\alpha = 1$, then we recover the standard continuous-time case.

If we now substitute (3.1) into (2.9), this yields

$$X(\lambda) = \sum_{i=0}^{\infty}(E^{-1}A)^i\lambda^{-(i+1)\alpha}h_0(\lambda) + \sum_{i=0}^{\infty}(E^{-1}A)^i E^{-1}\lambda^{-(i+1)\alpha}BU(\lambda). \tag{3.2}$$

Now, using inverse Laplace transforms and the convolution theorem we obtain the following theorem.

Theorem **3.1**. *The solution of the implicit fractional dynamical system (2.3) with E invertible is given by*

$$\begin{aligned}x(t) = &\sum_{i=0}^{\infty}\sum_{k=0}^{n-1}\frac{t^{i\alpha+k}}{\Gamma(i\alpha+k+1)}(E^{-1}A)^i x_0^{(k)} \\ &+ \int_0^t \sum_{i=0}^{\infty}(E^{-1}A)^i\frac{(t-\tau)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}E^{-1}Bu(\tau)d\tau.\end{aligned} \tag{3.3}$$

Remark **3.2**. If $E = I$, it reduces to the standard state space result given in [3].

4. The singular implicit case

For the implicit fractional state space system with E not invertible, there exists a Laurent series expansion about zero, which is given by

$$(E\lambda^\alpha - A)^{-1} = \sum_{i=-\mu}^{\infty} \phi_i \lambda^{-(i+1)\alpha}, \tag{4.1}$$

where the ϕ_i satisfy (δ_{ij} is the Kronecker delta) :

$$\phi_i E - \phi_{i-1} A = \delta_{i0} = E\phi_i - A\phi_{i-1}. \tag{4.2}$$

This follows easily from [6] with $\lambda^\alpha = s \in \mathbb{C}$. Assuming that the fractional system is regular and impulse free, then the substitution of (4.1) into (2.9) yields

$$X(\lambda) = \sum_{i=-\mu}^{\infty} \phi_i \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} x^{(k)}(0) + \sum_{i=-\mu}^{\infty} \phi_i \lambda^{-(i+1)\alpha} BU(\lambda). \tag{4.3}$$

Applying now the inverse Laplace transform and the convolution theorem to positive and negative sums separately :

$$\begin{aligned} X(\lambda) &= \sum_{i=0}^{\infty} \phi_i \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} x^{(k)}(0) + \sum_{i=0}^{\infty} \phi_i \lambda^{-(i+1)\alpha} BU(\lambda) \\ &+ \sum_{i=1}^{\mu} \phi_{-i} \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} x^{(k)}(0) + \sum_{i=1}^{\mu} \phi_{-i} \lambda^{(i-1)\alpha} BU(\lambda), \end{aligned} \tag{4.4}$$

we then obtain the following theorem.

Theorem **4.1**. *The solution of the regular and impulse free implicit fractional dynamical system of equation (2.3) is given by*

$$\begin{aligned} x(t) &= \sum_{i=0}^{\infty} \sum_{k=0}^{n-1} \frac{t^{i\alpha+k}}{\Gamma(i\alpha+k+1)} \phi_i E x_0^{(k)} \phi_i \\ &+ \sum_{i=0}^{\infty} \int_0^t \phi_i \frac{(t-\tau)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)} Bu(\tau) d\tau \\ &+ \sum_{i=1}^{\mu} \phi_{-i} \left(Bu(t)^{(i-1)\alpha} + \sum_{k=0}^{n-1} \delta^{(i\alpha-1)} E x_0^{(k)} \right). \end{aligned} \tag{4.5}$$

Note that it follows from [6] that

$$\phi_i = (\phi_0 A)^i \phi_0, \quad \text{for } i \geq 0,$$

so that a substitution in (4.5) yields

$$\begin{aligned} x(t) &= \sum_{i=0}^{\infty}\sum_{k=0}^{n-1}(\phi_0 A)^i\phi_0\frac{t^{i\alpha+k}}{\Gamma(i\alpha+k+1)}Ex_0^{(k)} \\ &+ \sum_{i=0}^{\infty}\int_0^t(\phi_0 A)^i\phi_0\frac{(t-\tau)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}Bu(\tau)d\tau \\ &+ \sum_{i=1}^{\mu}\phi_{-i}\left(Bu(t)^{(i-1)\alpha}+\sum_{k=0}^{n-1}\delta^{(i\alpha-1)}Ex_0^{(k)}\right). \end{aligned} \tag{4.6}$$

Let us now consider the case when $n = 1$, which corresponds to $0 < \alpha \leq 1$, then the solution becomes

$$\begin{aligned} x(t) &= \sum_{i=0}^{\infty}(\phi_0 A)^i\phi_0\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}Ex_0 \\ &+ \sum_{i=0}^{\infty}\int_0^t(\phi_0 A)^i\phi_0\frac{(t-\tau)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}Bu(\tau)d\tau \\ &+ \sum_{i=1}^{\mu}\phi_{-i}\left(Bu(t)^{(i-1)\alpha}+\delta^{(i\alpha-1)}Ex_0\right). \end{aligned} \tag{4.7}$$

Remark **4.1**. Note that in the equation (4.7), one defines

$$\sum_{i=0}^{\infty}\frac{(\phi_0 At^{\alpha})^i}{\Gamma((i\alpha+1))} \tag{4.8}$$

as the extended Mittag-Leffler matrix function. This follows from the following observation. If we choose $E = I$, then $\phi_0 = I$ and we obtain the known Mittag-Leffler matrix function.

Remark **4.2**. If $\alpha = 1$, then

$$\sum_{i=0}^{\infty}\frac{(\phi_0 At)^i}{\Gamma(i+1)} = e^{\phi_0 At} \tag{4.9}$$

which reduces (4.7) to the solution of the implicit continuous-time system.

5. Concluding remarks

In this paper we give explicit formulas for the solution of implicit fractional continuous time state space models, based on the Caputo definition of fractional derivatives, this definition involves only a finite number of initial conditions and compatibility requirements. It is also useful for several practical applications.

Acknowledgment: This paper presents research results of the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office and by the PNR code 50/03 ENSET Oran. The scientific responsibility rests with its authors.

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Please cite to this paper as published in:
Fract. Calc. Appl. Anal., Vol. **15**, No 3 (2012), pp. 356–361;
DOI: 10.2478/s13540-012-0026-z