The H_∞-norm calculation for large sparse systems[‡]

Y. Chahlaoui[*]   K. A. Gallivan[*]   P. Van Dooren[†]

Abstract

In this paper, we describe an algorithm for estimating the H_∞-norm of a large linear time invariant dynamical system described by a discrete time state-space model. The algorithm is designed to be efficient for state-space models defined by $\{A, B, C, D\}$ where A is a large sparse matrix of order n which is assumed very large relative to the input and output dimensions of the system.

Keywords: H_∞-norm calculation, Chandrasekhar equations.

1 Introduction

In this paper, we consider the computation of the H_∞-norm of a $p \times m$ real rational transfer function

$$G(z) := C(zI_n - A)^{-1}B + D \tag{1.1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times p}$, with $n \gg m, p$. The algorithm is restricted to discrete-time systems and makes use of the associated Chandrasekhar equations. It can be adapted for use with continuous-time systems. Its main advantage over other methods for approximating the H_∞-norm is that it is designed to be efficient for systems where A is a large sparse matrix.

The H_∞-norm of a rational transfer function $G(z)$, $\gamma^* := \|G(z)\|_\infty$ is bounded if and only if it is stable [9]. We therefore assume that the given quadruple $\{A, B, C, D\}$ is a real and minimal realization of a stable transfer function $G(z)$. The stability of $G(z)$ implies that all of the eigenvalues of A are strictly inside the unit circle, and hence that $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of A. An important result upon which we rely is the *bounded real lemma* that states $\gamma > \|G(z)\|_\infty$ if and only if there exists a solution $P \succ 0$ to the linear matrix inequality (LMI) [3] :

$$H(P) := \begin{bmatrix} P - A^T P A - C^T C & -A^T P B - C^T D \\ -B^T P A - D^T C & \gamma^2 I_m - B^T P B - D^T D \end{bmatrix} \succ 0. \tag{1.2}$$

[*]School of Computational Science and Information Technology, Florida State University, Tallahassee FL 32306-4120, USA.

[†]CESAME, Université catholique de Louvain, Louvain-la-Neuve, Belgium.

This suggests that γ^* could be calculated by starting with an initial large value $\gamma > \gamma^*$ for which the LMI has a solution and iterate by decreasing γ until the LMI does not have a solution. Unfortunately, even if A is sparse, P is dense and this approach has a complexity that is more than cubic in n and therefore unacceptable for large sparse systems.

A second approach is to use the link with the zeros of the para-hermitian transfer function

$$\Phi_\gamma(z) := \gamma^2 I_m - G_*(z)\, G(z), \quad \text{where} \ \ G_*(z) := D^T + z B^T (I - z A^T)^{-1} C^T. \tag{1.3}$$

It is well-known that

$$\gamma > \gamma^* \iff \Phi_\gamma(z) \succ 0 \quad \forall |z| = 1, \tag{1.4}$$

which implies that $\Phi_\gamma(z)$ has no zeros on the unit circle [9]. These zeros can also be shown to be the generalized eigenvalues of the (symplectic) pencil:

$$\begin{bmatrix} 0 & A - zI_n \\ zA^T - I_n & -C^T C \end{bmatrix} - \begin{bmatrix} B \\ -C^T D \end{bmatrix} (\gamma^2 I_m - D^T D)^{-1} \begin{bmatrix} zB^T & -D^T C \end{bmatrix} \tag{1.5}$$

and which must appear as pairs that are mirror images with respect to the unit circle, i.e., $(z_i, 1/\bar{z}_i)$.

Several methods with linear [5], quadratic [2] and even quartic convergence [8] have been based on the calculation of these zeros. The latter are the so-called level set methods that are probably the method of choice for dense problems of small to moderate order. The order is restricted by that fact that these methods all rely on the accurate calculation of the generalized eigenvalues of (1.5) on the unit circle. Since these eigenvalues are not necessarily the largest or smallest eigenvalues, this typically requires transformation-based techniques and hence a complexity of $O(n^3)$ which is unacceptable for large systems.

In this paper, we therefore follow a third approach that involves the solution of a discrete-time algebraic Riccati equation (DARE) :

$$P = A^T P A + C^T C - (A^T P B + C^T D)(B^T P B + D^T D - \gamma^2 I_m)^{-1}(B^T P A + D^T C). \tag{1.6}$$

Since we are interested in solutions where γ is large, the matrix $R := B^T P B + D^T D - \gamma^2 I_m$ will be negative definite, which is a non-standard discrete-time Riccati equation. This equation is linked to the LMI (1.2). Note that as $H_{22}(P) = -R$ is positive definite and the Schur complement of $H(P)$ with respect to $H_{22}(P)$ must be non-negative definite. It is easily verified that this amounts to the so-called Riccati matrix inequality introduced in [15] :

$$\gamma^2 I_m - B^T P B - D^T D \succ 0, \tag{1.7}$$

$$P - A^T P A - C^T C + (A^T P B + C^T D)(B^T P B + D^T D - \gamma^2 I_m)^{-1}(B^T P A + D^T C) \succ 0. \tag{1.8}$$

If rank $H_{22}(P) = $ rank $H(P)$ for an appropriate choice of P, then it follows that the Schur complement of $H_{22}(P)$ in $H(P)$ must be zero. This is precisely the left hand side of (1.8), which then becomes an equality known as the Discrete Time Riccati Equation (DARE). Its solution P can be obtained from the calculation of an appropriate eigenspace of the symplectic pencil (1.5). However, in order to exploit the sparsity of the matrix A to yield an efficient algorithm for large systems, we solve (1.8) using an iterative scheme known as the Chandrasekhar equations.

The remainder of this paper is organized as follows. The relationship between the system zeros and the symplectic eigenvalues is presented in Section 2 and used to the describe the basic idea of level set methods in Section 3. The use of the Chandrasekhar equations to solve the DARE parameterized by γ is then discussed in Section 4 including relevant complexity results. In Section 5, details of crucial convergence issues are given. Section 6 contains a brief description of techniques to adapt γ in the search for γ^* and the behavior of two of these approaches is illustrated on an example in Section 7 and compared to a level set method. Conclusions and future work are discussed in Section 8.

2 Realizations and system zeros

Every rational transfer matrix $G(z)$ of dimension $p \times m$ is known from realization theory to admit a generalized state-space model [12] of the form

$$G(z) := (C - zF)(A - zE)^{-1}B + D, \tag{2.1}$$

which is the Schur complement of the so-called system matrix $S(z)$ of dimension $(n+p) \times (n+m)$

$$S(z) = \begin{bmatrix} A - zE & B \\ C - zF & D \end{bmatrix} \tag{2.2}$$

with respect to its top left block entry. This special case of the state-space models were e.g. studied in [14]. When they have minimal dimension, they possess the nice property that all structural properties of the transfer function can be recovered from the solution of a generalized eigenstructure problem, for which there are numerically reliable algorithms available [12] : the minimum dimension n of the invertible pencil $(A - zE)$ is the McMillan degree of $G(z)$ [12], and the generalized eigenvalues of $A - zE$ are then the poles of $G(z)$ [12]. A test for the minimality of the realization $S(z)$ is the following set of conditions :

$$
\begin{aligned}
(i) \quad & \mathrm{rank} \begin{bmatrix} A - zE & B \end{bmatrix} = \mathrm{rank} \begin{bmatrix} A - zE \\ C - zF \end{bmatrix} = n, \quad \forall\, |z| < \infty \\
(ii) \quad & \mathrm{rank} \begin{bmatrix} E & B \end{bmatrix} = \mathrm{rank} \begin{bmatrix} E \\ F \end{bmatrix} = n;
\end{aligned}
\tag{2.3}
$$

If these conditions are not all satisfied, then the system matrix (2.2) is not minimal and the state space dimension can always be reduced so as to achieve minimality [12].

When D is invertible, the zeros of $G(z)$ can also be computed as generalized eigenvalues of a smaller pencil, derived from $S(z)$. Clearly the Schur complement

$$(A - zE) - BD^{-1}(C - zF) \tag{2.4}$$

has the same generalized eigenvalues as $S(z)$, except for infinity (see [14] for a more elaborate discussion on this).

One easily verifies that the system matrix $S(z)$ of the transfer function $\Phi_\gamma(z)$ (1.3) is given by :

$$S(z) = \begin{bmatrix} 0 & A - zI_n & B \\ zA^T - I_n & -C^TC & -C^TD \\ zB^T & -D^TC & \gamma^2 I_m - D^TD \end{bmatrix}. \tag{2.5}$$

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The zeros of the transfer function are the eigenvalues of (1.5) which are also the finite eigenvalues of (2.5). The eigenvalues are mirror images with respect to the unit circle (i.e. they come in pairs $z_i, 1/\bar{z}_i$). This eigenvalue characterization leads to a straightforward but powerful method for the computation of the $\mathcal{H}_\infty$ norm discussed in Section 3.

3 Level set methods

The basic idea of level set methods is related to the unit circle zeros of $\Phi_\gamma(z)$. If we define $\Phi(z) := G_*(z)G(z)$, then both matrix functions $\Phi_\gamma(z)$ and $\Phi(z)$ are para-hermitian which implies they are hermitian for every point $z = e^{j\omega}$, i.e.,

$$[\Phi(e^{-j\omega})]^T = \Phi(e^{j\omega}), \quad \omega \in [-\pi, +\pi]. \tag{3.1}$$

If we define the extremal eigenvalues of $\Phi(e^{j\omega})$ on $\omega \in [-\pi, +\pi]$ as

$$\lambda_* := \min_{\omega \in [-\pi, +\pi]} \lambda_{min} \Phi(e^{j\omega}), \quad \lambda^* := \max_{\omega \in [-\pi, +\pi]} \lambda_{max} \Phi(e^{j\omega}) \tag{3.2}$$

and let $\gamma_* = \sqrt{\lambda_*}$ and $\gamma^* = \sqrt{\lambda^*}$ then we have that $\Phi(e^{j\omega})$ is non-negative on $\omega \in [-\pi, +\pi]$ if and only if $\gamma^2 \geq \lambda^*$ and thus the H_∞ norm of $G(z)$ equals γ^*.

It is shown in [10] that a hermitian matrix $\Phi(e^{j\omega})$ of a real variable ω has real analytic eigenvalues as a function of ω, if $\Phi(e^{j\omega})$ is itself analytic in ω. Since we consider here rational functions of $e^{j\omega}$ – where the "frequency" ω is real – this is certainly the case. In Figure 1, we show these functions $z_i(\omega)$ for a 2×2 matrix $\Phi(e^{j\omega})$. We also indicate a level λ_0 for which we want to check if there is any eigenvalue $z_i(\omega) = \lambda_0$. Clearly these z_i are the intersections of the level λ_0 with the eigenvalues of $\Phi(e^{j\omega})$. Assume that these intersections occur at frequencies ω_i. Since

$$\det(\lambda_0 I_m - \Phi(e^{j\omega_i})) = 0$$

each frequency ω_i is an imaginary axis zero of the shifted transfer function $\lambda_0 I_m - \Phi(z)$. These can be computed as the eigenvalues of the corresponding zero pencil (2.5) or the corresponding symplectic pencil (1.5), that are located on the $e^{j\omega}$ axis. Note that if there are no imaginary axis eigenvalues, then the level λ_0 does not intersect the eigenvalue plots and hence

$$\lambda_0 < \lambda_* \quad \text{or} \quad \lambda_0 > \lambda^*.$$

In order to find a value λ_0 for which there are eigenvalue crossings one can, e.g., choose $\lambda_0 = \sigma_{max} \Phi(e^{j\omega_0})$ for an arbitrary value ω_0.

Using these ingredients, a bisection-based algorithm to find λ_{max} is easily derived: each interval must contain an upper bound λ_{up} and a lower bound λ_{lo} for λ_{max} and the bisection method checks whether there are eigenvalues on the $e^{j\omega}$ axis equal to the new level $(\lambda_{lo} + \lambda_{up})/2$ [5]. This algorithm has linear convergence.

A method with more rapid convergence can be obtained by using information on the eigenvalue functions (see [4, 8]). Start from a point λ_{old} which intersects the eigenvalues of $\Phi(e^{j\omega})$ as in Figure 1, and obtain from this the intervals for which $z_{max}(\omega) > \lambda_{old}$ (these are called the level sets for
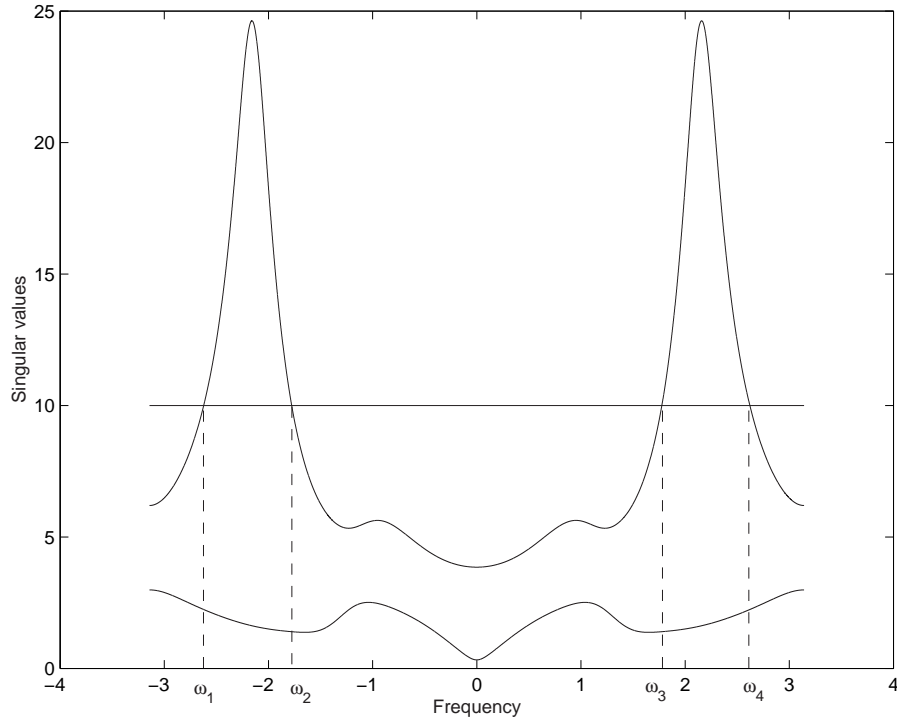
Figure 1: Level set iterations

λ_{old}). In Figure 1 these are the intervals $[\omega_1, \omega_2]$ and $[\omega_3, \omega_4]$ (in this context we need to define $z_{max}(\omega)$ as the piecewise analytic function that is maximal at each frequency ω). In [8] it is shown how to use the information of the derivative of $z_{max}(\omega)$ at each point in order to determine the relevant "level sets". It is also shown how to obtain these derivatives at little extra cost from the eigenvalue problem of the underlying zero pencil. Using these level sets and the derivative of $z_{max}(\omega)$ at their endpoints, one then constructs a new frequency ω_{new} that is a good estimate of an extremal frequency ω_{max} :

$$\lambda_{max} = z_{max}[\Phi(e^{j\omega_{max}})] = \max_{\omega} z_{max}[\Phi(e^{j\omega})].$$

It is shown in [8] that such a scheme has global linear convergence and at least cubic asymptotic convergence. Each step requires the calculation of the largest eigenvalue λ_{new} of $\Phi(e^{j\omega_{new}})$ and the eigenvalues and eigenvectors of the zero pencil defining the zeros of $\lambda_{new} I_m - \Phi(z)$. The complexity of each iteration is thus cubic in the dimensions of the system matrix of $\Phi(z)$.

4 Chandrasekhar equations

Efficient algorithms to solve the DARE have been proposed in the literature [9]. The so-called Chandrasekhar equations amount to calculating the solution of the discrete-time Riccati difference equation

$$P_{i+1} = A^T P_i A + C^T C - (A^T P_i B + C^T D)(B^T P_i B + D^T D - \gamma^2 I_m)^{-1}(B^T P_i A + D^T C) \qquad (4.1)$$

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in an efficient manner. Defining the matrices

$$K_i := B^T P_i A + D^T C, \quad R_i := B^T P_i B + D^T D - \gamma^2 I_m, \tag{4.2}$$

this becomes

$$P_{i+1} = A^T P_i A + C^T C - K_i^T R_i^{-1} K_i. \tag{4.3}$$

Clearly the difference matrices

$$\delta P_i := P_{i+1} - P_i \tag{4.4}$$

satisfy

$$\delta P_{i+1} = A^T \delta P_i A - K_{i+1}^T R_{i+1}^{-1} K_{i+1} + K_i^T R_i^{-1} K_i. \tag{4.5}$$

Using (4.2-4.5), one obtains the following identity :

$$\begin{bmatrix} R_{i+1} & K_{i+1} \\ K_{i+1}^T & \delta P_{i+1} + K_{i+1}^T R_{i+1}^{-1} K_{i+1} \end{bmatrix} = \begin{bmatrix} B^T \delta P_i B + R_i & B^T \delta P_i A + K_i \\ A^T \delta P_i B + K_i^T & A^T \delta P_i A + K_i^T R_i^{-1} K_i \end{bmatrix}, \tag{4.6}$$

and the Schur complement with respect to the (2,2) block is equal to δP_{i+1}.

Assume now that for each step i we define the matrices L_i, S_i and G_i according to $\delta P_i = L_i^T \Sigma_2 L_i$, $R_i = S_i^T \Sigma_1 S_i$ and $K_i = S_i^T \Sigma_1 G_i$. This, of course, implies that the signature of the matrices R_i and δP_i remains constant for all i. It is shown in [9] that this condition is in fact necessary and sufficient for the Riccati difference equation (4.1) to converge. An obvious choice is to take $P_0 = 0$, which yields

$$\delta P_0 = P_1 = C^T C - C^T D (D^T D - \gamma^2 I_m)^{-1} D^T C.$$

It also follows from the LMI (1.2) that we must take $\gamma^2 I_m - D^T D \succ 0$, which implies $\delta P_0 = C^T [I + D(\gamma^2 I_m - D^T D)^{-1} D^T] C \succ 0$. We thus have that $\Sigma_1 = -I_m$ and $\Sigma_2 = I_\alpha$ with $\alpha \le p$.

Under these conditions the above matrix may be factored as

$$\begin{bmatrix} R_{i+1} & K_{i+1} \\ K_{i+1}^T & \delta P_{i+1} + K_{i+1}^T R_{i+1}^{-1} K_{i+1} \end{bmatrix} = \begin{bmatrix} S_{i+1}^T & 0 \\ G_{i+1}^T & L_{i+1}^T \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} S_{i+1} & G_{i+1} \\ 0 & L_{i+1} \end{bmatrix}. \tag{4.7}$$

One also easily checks the identity

$$\begin{bmatrix} B^T \delta P_i B + R_i & B^T \delta P_i A + K_i \\ A^T \delta P_i B + K_i^T & A^T \delta P_i A + K_i^T R_i^{-1} K_i \end{bmatrix} = \begin{bmatrix} S_i^T & B^T L_i^T \\ G_i^T & A^T L_i^T \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} S_i & G_i \\ L_i B & L_i A \end{bmatrix}. \tag{4.8}$$

It follows from the comparison of (4.7) and (4.8) that there exists a pseudo-orthogonal transformation Q satisfying

$$\Sigma := \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad Q^T \Sigma Q = \Sigma, \quad Q \begin{bmatrix} S_i & G_i \\ L_i B & L_i A \end{bmatrix} = \begin{bmatrix} S_{i+1} & G_{i+1} \\ 0 & L_{i+1} \end{bmatrix}.$$

Notice that as the matrix R_i is nonsingular and so is S_i, we have a simple expression for the feedback matrix $F_i := S_i^{-1} G_i$ which yields the closed loop matrix $A - BF_i = A - BR_i^{-1} K_i$ whose spectral radius $\rho_i := \rho(A - BF_i)$ determines essentially the convergence of the Riccati difference equation (4.1). So if γ is overestimated then ρ_i will be smaller than 1, while if γ is underestimated, and especially $\gamma_* \le \gamma \le \gamma^*$, ρ_i will become larger or equal to 1, i.e., $\rho_i \ge 1$, and eventually the

signature Σ will not be constant, since the DARE does not have a symmetric steady state solution. Since $A - BF_i$ is stable and converges to $A - BF$, one can track the spectral radius ρ, e.g., by the power method applied to $A - BF_i$ or by monitoring $\|\delta P_i\|_2 = \|L_i\|_2$ since $\Sigma_2 = I_m$. We discuss in more details the convergence of the Chandrasekhar equations in Section 5. Note finally that the complexity is acceptable for large sparse systems since the transformation $Q \in \mathbb{R}^{m+\alpha \times m+\alpha}$ and any use of A or A^T involves a matrix times a number of columns, k, where $n \gg k$.

5   Convergence of Chandrasekhar equations

The convergence of the Riccati difference equation (4.1) depends on whether or not the signature of Σ is constant. Indeed, for $\gamma < \sqrt{\lambda_*}$ or $\gamma > \sqrt{\lambda^*}$ this is the case and the Riccati difference equation will converge. In the level set plot, these values correspond to the levels where there are no imaginary axis eigenvalues. Notice also that the resulting feedback for the case $\gamma < \sqrt{\lambda_*}$ does not stabilize the system.

The Riccati difference equation (4.1) has many formulations. One useful formulation results from the two-point boundary value problem (see [6] and the references in). For our case, we have the following result.

Lemma 5.1. *The Riccati difference equation* (4.1) *can be rewritten as follows, under condition that* $\bar{R}_\gamma := D^T D - \gamma^2 I_m$ *is nonsingular :*

$$\underbrace{\begin{bmatrix} A - B\bar{R}_\gamma^{-1}D^T C & 0 \\ -C^T C + C^T D \bar{R}_\gamma^{-1} D^T C & I \end{bmatrix}}_{M_1} \begin{bmatrix} I \\ P_{i+1} \end{bmatrix} = \underbrace{\begin{bmatrix} I & B\bar{R}_\gamma^{-1}B^T \\ 0 & A^T - C^T D \bar{R}_\gamma^{-1} B^T \end{bmatrix}}_{M_2} \begin{bmatrix} I \\ P_i \end{bmatrix}(A - BF_i). \quad (5.1)$$

Proof. The result is very similar to Lemma 1 in [6]. We have to show the two identities

$$A - B\bar{R}_\gamma^{-1}D^T C = A - BF_i + B\bar{R}_\gamma^{-1}B^T P_i(A - BF_i),$$

$$P_{i+1} - C^T C + C^T D \bar{R}_\gamma^{-1} D^T C = (A^T - C^T D \bar{R}_\gamma^{-1} B^T) P_i(A - BF_i),$$

which follow from the definitions of K_i, $\bar{R}_\gamma$ and of the Riccati difference equation (4.1).   □

Note that the condition of non-singularity of $\bar{R}_\gamma$ is generically true (i.e., it is singular on a set of measure 0). For $\gamma \geq \gamma^*$ the matrix $\bar{R}_\gamma$ is nonsingular since it must satisfy the first condition of the Riccati matrix inequality (1.7) and hence $\bar{R}_\gamma - B^T P_i B \succ 0$.

The lemma above plays an important role in analyzing the convergence of (4.1). It is clear that the convergence to a (unique) stabilizing solution requires the pair (A, B) to be stabilizable, i.e., there exists a feedback F such that $A - BF$ has eigenvalues in the open unit disc. It then follows that the pencil $\lambda M_1 - M_2$ (or the matrix $M_1^{-1} M_2$) has no eigenvalues on the unit circle. This is used in [6] to prove convergence of (4.1) when (A, B) is stabilizable.

Now let us assume $\gamma > \gamma^*$ and that P is the steady state solution of (4.1) for that γ. Then

$$M_1 \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} = M_2 \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} A_F & X \\ 0 & A_F^{-T} \end{bmatrix}$$

where $X := P^{-1}((A - B\bar{R}_\gamma^{-1}D^TC)^{-T} - A_F^{-T})$ and $A_F := A - BF$. Equivalently we have

$$\begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} M_2^{-1} M_1 \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} = \begin{bmatrix} A_F & X \\ 0 & A_F^{-T} \end{bmatrix}. \tag{5.2}$$

Also, as P_i converges to P we can suppose that $\Delta_i := P - P_i$ is small. Using

$$\begin{bmatrix} I & 0 \\ P & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ P_i & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \Delta_i & I \end{bmatrix}, \tag{5.3}$$

and the block Schur form (5.2), we have the following result.

Lemma 5.2. *When the Riccati difference equation (4.1) converges, each iteration corresponds to an approximate Schur decomposition*

$$\begin{bmatrix} I & 0 \\ -P_i & I \end{bmatrix} M_2^{-1} M_1 \begin{bmatrix} I & 0 \\ P_i & I \end{bmatrix} = \begin{bmatrix} A_{F_i} & X \\ E_{21} & A_{F_i}^{-T} \end{bmatrix}$$

where

$$E_{21} := \Delta_i A_F - A_F^{-T}\Delta_i - \Delta_i X \Delta_i, \quad A_{F_i} := A - BF_i.$$

Proof. If one combines (5.2) and (5.3) we obtain

$$\begin{aligned}
\begin{bmatrix} I & 0 \\ -P_i & I \end{bmatrix} M_2^{-1} M_1 \begin{bmatrix} I & 0 \\ P_i & I \end{bmatrix} &= \begin{bmatrix} I & 0 \\ \Delta_i & I \end{bmatrix} \begin{bmatrix} A_F & X \\ 0 & A_F^{-T} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\Delta_i & I \end{bmatrix} \\
&= \begin{bmatrix} A_F - X\Delta_i & X \\ \Delta_i A_F - A_F^{-T}\Delta_i - \Delta_i X \Delta_i & A_F^{-T} + \Delta_i X \end{bmatrix}.
\end{aligned} \tag{5.4}$$

If Δ_i is small, the above form is indeed an approximate block Schur decomposition. It follows from (5.1) that $A_{F_i} = A_F - X\Delta_i$ and it also follows from the symplectic structure of (5.4) that $A_{F_i}^{-T} = A_F^{-T} + \Delta_i X$. The off-diagonal block then equals

$$E_{21} = \Delta_i A_F - A_F^{-T}\Delta_i - \Delta_i X \Delta_i = \Delta_i A_{F_i} - A_{F_i}^{-T}\Delta_i + \Delta_i X \Delta_i. \tag{5.5}$$

$\square$

It was shown in [13] that when there exists a positive definite solution P to the DARE (4.1) then upon convergence we have

$$\delta P_{i+1} \approx A_{F_i}^T \delta P_i A_{F_i},$$

and hence

$$\|L_{i+1}\|_2 \approx \|L_i A_{F_i}\|_2. \tag{5.6}$$

This implies that Δ_i can be approximated by the correction δP_i. Therefore, using the previous lemma, the matrix E_{21} can be estimated using the computed quantities δP_i and A_{F_i}.

It is also important to note that if $\gamma > \gamma^*$ then A_F is stable and since $A - BF_i \approx A_F$ we can estimate $\rho(A_F)$ using subspace iteration on $A - BF_i$, i.e.,

$$Q_i R_i = (A - BF_i)Q_{i-1}, \quad Q_i^T Q_i = I_k,$$

started with an arbitrary k-dimensional orthogonal basis Q_0. This can be performed at low cost since A is sparse and BF_i is relatively low rank. Moreover, even if $\gamma \leq \gamma^*$ and A_F is unstable, the eigenvalues of $Q_{i-1}^T(A - BF_i)Q_{i-1}$ will be close to the dominant spectrum of $A - BF_i$ and according to Lemma 5.2 this will be close to k eigenvalues of the symplectic pencil as long as Δ_i is small.

Figure 2 describes the convergence properties of the Riccati difference equation and the H_∞ approximation algorithm in terms of the spectral radius $\rho(A_F)$ as a function of γ. One can define a region of acceptance for the approximation of γ^* and the width of this region will depend mainly of a tolerance value associated with the convergence/divergence decision. As long as the δP_i remain reasonably small, so will E_{21} and the spectrum of A_{F_i} then gives a reasonable approximation of half of the spectrum of the symplectic pencil. This can be used to detect the symplectic eigenvalues closest to or on the unit circle.
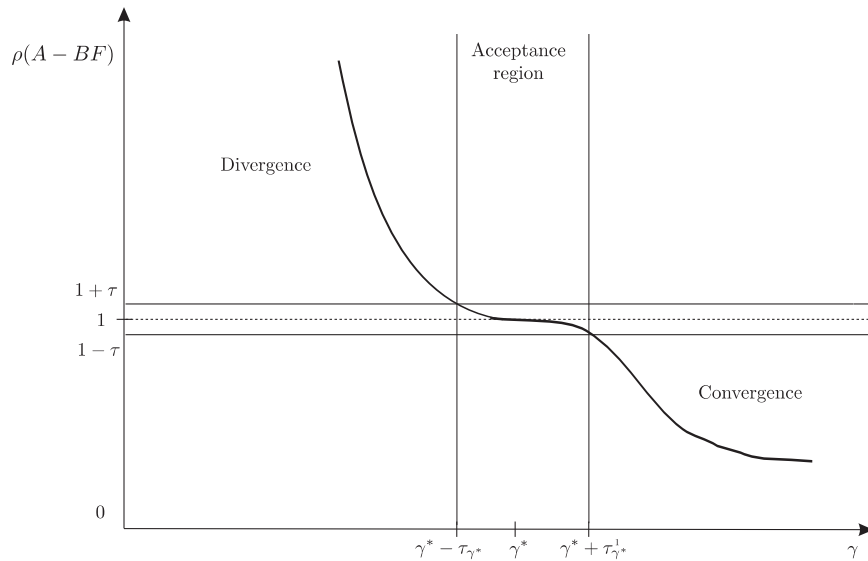

Figure 2: Evolution of $\phi(\gamma) := \rho(A - BF)$ as a function of γ

Clearly, the convergence/divergence decision plays a crucial role in the choice of the direction of the adaptation of γ. Recall that for a given initial condition P_0, the solution of the discrete-time Riccati equation is given at each instant i by

$$P_i = P_0 + \sum_{k=1}^{i-1} \delta P_k.$$

For a given tolerance τ, we will say that the discrete-time Riccati equation diverges, if one of the following is true:

- the spectral radius $\rho(A - BF_i)$ (estimated by subspace iteration) is larger than $1 + \tau$

- the ratio $\|\delta P_{i+1}\|_2/\|\delta P_i\|_2 = \|L_{i+1}\|_2^2/\|L_i\|_2^2$ is larger than $(1+\tau)^2$. Notice that this is similar to the previous as one has the relation $\|L_{i+1}\|_2/\|L_i\|_2 \approx \rho(A - BF_i)$,

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- the inequality (1.7) does not hold, i.e., $\gamma^2 I_m - B^T P B - D^T D \preceq 0$.

By monitoring the convergence using one of these criteria, one can decide if at the steady state we have relative convergence or not and adapt γ in the appropriate direction using one of the approaches discussed in Section 6.

6   Adapting γ

In order for the algorithm to approximate γ^* to work efficiently there must be an effective method to adapt the value of γ given the observed behavior of the Riccati difference equation. The simplest method is the combination of the Chandrasekhar equations with a bisection method to estimate γ^*. A lower bound for $\gamma^* = \|G(\ .\ )\|_\infty$ is easily obtained from $\gamma_{lo} := G(e^{j\omega_{lo}})$ for any frequency ω_{lo}, as pointed out in Section 3. The idea for estimating γ^* is to run the Chandrasekhar equations for a given $\gamma > \gamma_{lo}$ and check whether or not it converges. If it converges, then $\gamma_{up} := \gamma$ is an overestimate for γ^* and we repeat the process for a new value of γ (say $(\gamma_{lo} + \gamma_{up})/2$) in the interval $[\gamma_{lo}, \gamma_{up}]$ which is known to contain γ^*. If divergence is observed then $\gamma_{lo} := \gamma$ before choosing the next value of γ.

Another general strategy, and one that is preferred in practice, is to start from an overestimate γ_{up} and let it decrease until convergence of the Chandrasekhar equation fails. There are several possible avenues to consider to get an effective strategy to update γ to approach γ^* from above. All share the need to model, based on data observed while executing the algorithm, how $\phi(\gamma) := \rho(A - BF_\gamma)$ evolves with γ where F_γ is the steady state feedback matrix obtained from the DARE. Essentially, this is an attempt to determine the function in Figure 2 for $\gamma > \gamma^*$. We have found that for each value of γ, $\rho(A - BF_\gamma)$ can be estimated reasonably well with a few subspace iteration steps. We can therefore estimate the value of the function $\phi(\gamma)$ for several values of γ which can be used to produce a $\gamma > \gamma^*$ closer to γ^* via inverse interpolation.

We are also investigating an approach that uses the estimate of $\phi(\gamma)$ and estimates of left and right eigenvectors associated with the dominant eigenspaces of $A - BF_\gamma$ to estimate the derivative of $\phi(\gamma)$ to produce a new value of γ [7]. Finally, the relationship given by Lemma 5.1 between the eigenvalues of $A - BF_i$ and eigenvalues on the unit circle of the associated symplectic pencil can be used to estimate a subset of the unit circle eigenvalues when $\gamma < \gamma^*$. These estimates could be used to develop an update strategy similar to that used in the level set method [8].

For any of these strategies, once $\gamma_{old} \to \gamma_{new}$ we must make a choice as to how to proceed with the Chandrasekhar iteration. The simplest approach is to simply restart the iteration with $P_0 = 0$ and monitor convergence/divergence in preparation for the next update of γ. A more efficient way is to consider the time varying coefficient equation

$$P_{i+1} = A^T P_i A + C^T C - (A^T P_i B + C^T D)(B^T P_i B + D^T D - \gamma_i^2 I_m)^{-1}(B^T P_i A + D^T C), \quad (6.1)$$

where $\gamma_i \neq \gamma_{i+1}$ is modified only when it is obvious from the evolution of the Chandrasekhar recurrence that it converges.

Suppose we have modified γ_i such that $\eta^2 := \gamma_i^2 - \gamma_{i+1}^2$. We can derive a modified update step of the Chandrasekhar equations that corresponds to solve the time varying difference equation. The

equation (4.6) becomes

$$
\left[\begin{array}{cc} R_{i+1} & K_{i+1} \\ K_{i+1}^T & \delta P_{i+1} + K_{i+1}^T R_{i+1}^{-1} K_{i+1} \end{array} \right] = \left[\begin{array}{cc} B^T \delta P_i B + R_i + \eta^2 I_m & B^T \delta P_i A + K_i \\ A^T \delta P_i B + K_i^T & A^T \delta P_i A + K_i^T R_i^{-1} K_i \end{array} \right],
$$

and the Schur complement with respect to the (2,2) block is still equal to δP_{i+1}. Reasoning in much the same way as before we find that the square root updating for those steps where $\eta \neq 0$ requires an additional correction

$$
Q^T \left[\begin{array}{cc} \Sigma_1 & 0 \\ 0 & I_m \end{array} \right] Q = \left[\begin{array}{cc} \Sigma_1 & 0 \\ 0 & I_m \end{array} \right], \quad Q \left[\begin{array}{c} \hat{S}_i \\ \eta I_m \end{array} \right] = \left[\begin{array}{c} S_i \\ 0 \end{array} \right].
$$

7   Numerical examples

In this section, we present numerical results for two sets of numerical experiments based on stable minimal discrete time systems with randomly generated coefficient matrices. In the first set of examples we empirically assess the convergence behavior of the Chandrasekhar iteration as a function of γ and its relationship to γ^*. The second set compares the bisection-based level set algorithm estimate of $\|G(z)\|_\infty$ to that produced by the bisection-based adaptation of γ and the Chandrasekhar iteration.

The data for the first set of experiments are shown in Figures 3 and 4. For a given stable discrete-time system, we computed its H_∞-norm γ^* using a set level method of Section 3 and tracked $\|\delta P_i\|_2 = \|L_i\|_2$ and $\|L_{i+1}\|_2/\|L_i\|_2 \approx \rho_i$ in order to decide when convergence occurs (see also [11] for a discussion on convergence of Riccati difference equations and stopping criteria). The values of $\|\delta P_i\|_2 = \|L_i\|_2$, and $\|L_{i+1}\|_2/\|L_i\|_2 \approx \rho_i$ for different values of γ are displayed in Figures 3 and 4.

When $\gamma > \gamma^*$ there exists a symmetric solution P and $\|L_{i+1}\|_2/\|L_i\|_2 \approx \rho_i < 1$ according to (5.6). The norm $\|\delta P_i\|_2$ decreases faster as the ratio γ/γ^* increases. As we decrease γ we obtain a ratio $\|L_{i+1}\|_2/\|L_i\|_2 \approx \rho_i$ that approaches 1. For $\gamma_* < \gamma < \gamma^*$ there is no symmetric solution P_∞ and the Chandrasekhar iteration does not converge and we observe the expected behavior that the smaller γ, the faster the divergence. Even in the case of divergence the dominant eigenspace of $A - BF_i$ will yield an eigenspace of the symplectic pencil for its unit circle eigenvalues (or those of interest outside the circle) that could be used to estimate the intersection of singular value plots of $\Phi(e^{j\omega})$ with the γ level of Section 3 and thereby update γ. Finally, when $\gamma \leq \gamma_*$ we observe the convergence of the Chandrasekhar iteration.
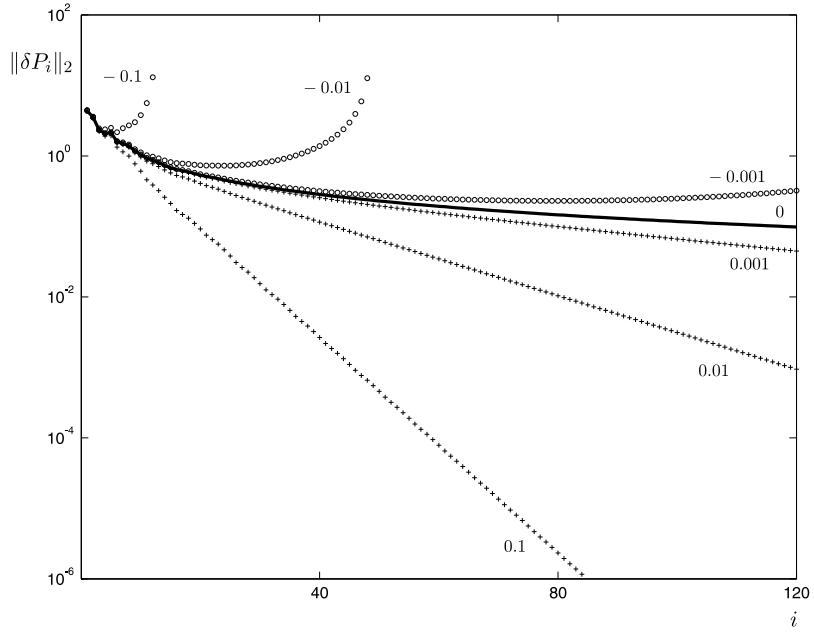
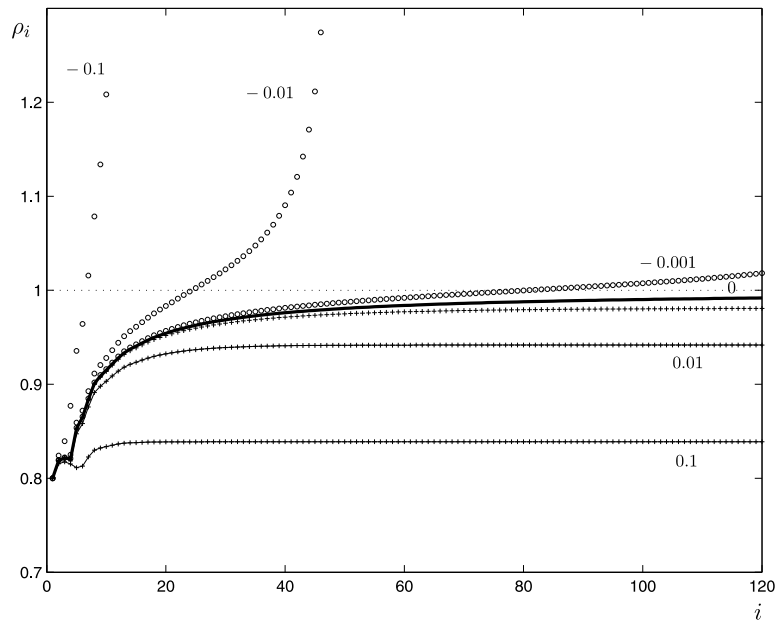Figure 3: Behavior of $\|\delta P_i\|_2 = \|L_i\|_2$ for different values of γ $(\gamma^* + \epsilon)$.

Figure 4: Behavior of $\|L_{i+1}\|_2 / \|L_i\|_2 \approx \rho_i$ for the values of γ $(\gamma^* + \epsilon)$used in Figure 3.

The second set of experimental results are also made using stable discrete time systems based on randomly generated coefficient matrices. These matrices are scaled in such a way that the $\mathcal{H}_\infty$ norm of each system is equal to 0.9, i.e., $\gamma^* = 0.9$. Each system is a SISO system of order

400. We approximate γ^* using a level set method (we denote this approximation by γ_{ls}), and a Chandrasekhar based method combined with a bisection method (we denote this approximation by γ_{Cb}). The convergence/divergence decision for the Chandrasekhar iteration is based an estimate spectral radius of $\phi(\gamma) = \rho(A - BF_\gamma)$ using a subspace iteration that is easily incorporated into the Chandrasekhar iteration.

Table 1 contains the estimates of $\|G(z)\|_\infty$ for each problem and method. The level set method is used here to indicate the best one could expect via an iterative approximation approach.

In order to interpret these results we also show in Figure 5 the behavior of the spectral radius of $A - BF$ as function of γ for each of the systems. Note that the acceptance region flattens significantly as the spectral radius $\rho(A)$ approaches 1. This indicates that for a fixed tolerance τ an increasingly wide interval on the γ axis will be considered an acceptable approximation to γ^*, i.e., the problem is becoming ill-conditioned. The numerical results in Table 1 are consistent with this observation. The quality of the approximation depends of the spectral radius of A and, as expected, the Chandrasekhar approach is more sensitive to this parameter than the level set method. This is due to the heavy dependence on the convergence/divergence decision in the Chandrasekhar approach and the fact that the convergence of the Chandrasekhar iteration is very slow. Fortunately, the quality of the approximation is very good for all but system 1 with the extreme value of $\rho(A) = 0.99$.

The level set method is, of course, a method based on dense matrix methods and is therefore not viable for large problems. The Chandrasekhar iteration with bisection exploits sparse matrix kernels and low order dense matrices to achieve efficiency for large problems. For the experiments presented here, MATLAB implementations of the Chandrasekhar iteration without substantial performance tuning was at least five time faster than the level set method.

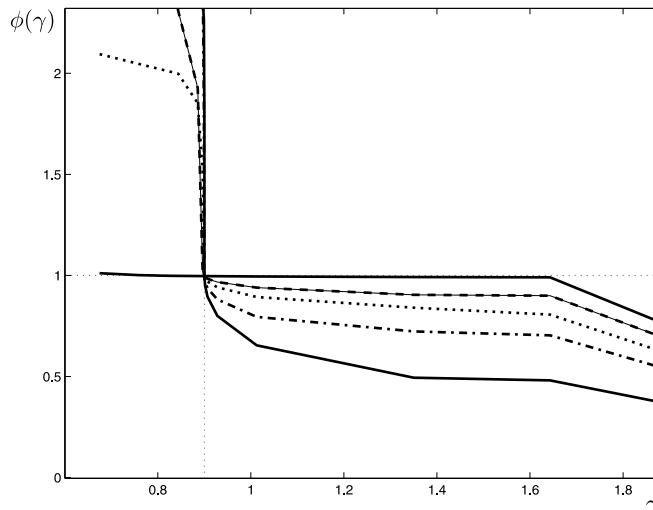|	system 1	system 2	system 3	system 4	system 5
$\rho(A)$	0.99	0.9	0.8	0.7	0.5
γ_{ls}	0.901185	0.900330	0.900205	0.900018	0.900337
γ_{Cb}	0.789265	0.898865	0.899783	0.899920	0.900303
$\phi(\gamma_{Cb})$	0.999999	1.000000	0.999999	1.000000	0.999999

Table 1: Approximations for $\gamma^* = 0.9$.

Figure 5: Behavior of $\phi(\gamma)$ in function of γ for different systems.
LEGEND: —— system 1, – – system 2, $\cdots$ system 3, – $\cdot$ – system 4, —— system 5.

8   Conclusion

We have presented an algorithm based on the Chandrasekhar iteration and initial empirical evidence that it can be used to estimate efficiently $\|G(z)\|_\infty$ for large discrete time linear time invariant dynamical systems. Of course, much remains to do in order to develop a reliable and efficient method. We are currently investigating the influence of the structure of the spectrum of A on the behavior of the algorithm particularly relative to the convergence/divergence decision. We are also investigating the design and behavior of the time-varying coefficient version of the Chandrasekhar iteration and the associated strategies for adapting γ. The algorithm is easily adapted to estimate the $\mathcal{H}_\infty$ norm of a continuous time system.

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