Model Reduction of Second Order Systems

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1 Introduction

In this chapter, the problem of constructing a reduced order system while preserving the second order structure of the original system is discussed. After a brief introduction on second order systems and a review of first order model reduction techniques, two classes of second order structure preserving model reduction techniques – Krylov subspace-based and SVD-based – are presented. For the Krylov techniques, conditions on the projectors that guarantee the reduced second order system tangentially interpolates the original system at given frequencies are derived and an algorithm is described. For SVD-based techniques, a Second Order Balanced Truncation method is derived from *second order gramians*.

Second order systems arise naturally in many areas of engineering (see, for example, [13, 14, 18]) with the following form :

$$\begin{cases} M\ddot{q}(t) + D\dot{q}(t) + Sq(t) = F^{in}\, u(t), \\ \qquad\qquad\qquad y(t) = F^{out}\, q(t). \end{cases} \tag{1}$$

We assume that $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $q(t) \in \mathbb{R}^N$, $F^{in} \in \mathbb{R}^{N \times m}$, $F^{out} \in \mathbb{R}^{p \times N}$, and $M, D, S \in \mathbb{R}^{N \times N}$ with M invertible. For mechanical systems the matrices M, D and S represent, respectively, the *mass* (or *inertia*), *damping* and *stiffness* matrices, $u(t)$ corresponds to the vector of *external* forces, F^{in} is the input distribution matrix, $y(\cdot)$ is the output measurement vector, F^{out} is the output measurement matrix, and $q(t)$ to the vector of *internal generalized coordinates*.

The transfer function associated with the system (1) is

$$R(s) \doteq F^{out} P(s)^{-1} F^{in}, \tag{2}$$

where

$$P(s) \doteq Ms^2 + Ds + S \tag{3}$$

is the *characteristic polynomial matrix*. The zeros of $\det(P(s))$ are also known as the *characteristic frequencies* of the system and play an important role in model reduction, e.g., the system is stable if these zeros lie in the left half plane.

Often, the original system is too large to allow the efficient solution of various control or simulation tasks. In order to address this problem, techniques that produce a reduced system of size $n \ll N$ that possesses the essential properties of the full order model have been developed. Such a reduced model can then be used effectively, e.g., in real-time, for controlling or simulating the phenomena described by the original system. We therefore need to build a reduced model,

$$\begin{cases} \hat{M}\ddot{\hat{q}}(t) + \hat{D}\dot{\hat{q}}(t) + \hat{S}\hat{q}(t) = \hat{F}^{in}u(t) \\ \hat{y}(t) = \hat{F}^{out}\hat{q}(t) \end{cases} \tag{4}$$

where $\hat{q}(t) \in \mathbb{R}^n$, $\hat{M}$, $\hat{D}$, $\hat{S} \in \mathbb{R}^{n \times n}$, $\hat{F}^{in} \in \mathbb{R}^{n \times m}$, $\hat{F}^{out} \in \mathbb{R}^{p \times n}$, such that its transfer function is "close" to the original transfer function.

In contrast with second order systems, first order systems can be represented as follows :

$$\mathcal{S} \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{5}$$

where

$x(t) \in \mathbb{R}^N$ is the state vector, $y(t) \in \mathbb{R}^p$ is the output vector,
$u(t) \in \mathbb{R}^m$ is the input vector, N is the order of the system,
    m is the number of inputs, p is the number of outputs,
$C \in \mathbb{R}^{p \times N}$ is the output matrix, $A \in \mathbb{R}^{N \times N}$ is the state matrix,
$B \in \mathbb{R}^{N \times m}$ is the input matrix.

The transfer function associated with the system (5) is

$$R(s) \doteq C(sI_N - A)^{-1}B, \tag{6}$$

and links the inputs to the outputs in the Laplace domain.

Second order systems can be seen as a particular class of linear systems. Indeed, by rewriting the system (1) as follows

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & I_N \\ -S_M & -D_M \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ F_M^{in} \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} F_M^{out} & 0 \end{bmatrix} x(t) \end{cases} \tag{7}$$

where the state $x(t)$ is $\begin{bmatrix} q(t)^T & \dot{q}(t)^T \end{bmatrix}^T$, and where we have chosen a coordinate system in which the mass matrix M is the identity (for simplicity, the mass matrix M is assumed to be invertible, and one can write for example: $S_M = M^{-1}S$, $D_M = M^{-1}D$, $F_M^{in} = M^{-1}F^{in}$, $F_M^{out} = F^{out}$), one recovers the form (5). We can thus rewrite the transfer function defined in (2) as

$$R(s) = C(sI_{2N} - A)^{-1}B \qquad (8)$$

by defining

$$A \doteq \begin{bmatrix} 0 & I_N \\ -S_M & -D_M \end{bmatrix} \quad , \quad B \doteq \begin{bmatrix} 0 \\ F_M^{in} \end{bmatrix} \quad , \quad C \doteq \begin{bmatrix} F_M^{out} & 0 \end{bmatrix}. \qquad (9)$$

Note that if the dimension of the state $q(t)$ of the original second order system (1) is equal to N, the order of its corresponding linearized state space realization (9) (also called the Mc Millan degree of $R(s)$ if the (C, A, B) is minimal) is equal to $2N$.

A reduced model for the second order system (1) could be produced by applying standard linear model reduction techniques to (C, A, B) in (9) to yield a small linear system $(\hat{C}, \hat{A}, \hat{B})$. Unfortunately, there is no guarantee that the matrices defining the reduced system $(\hat{C}, \hat{A}, \hat{B})$ have the nonzero structure necessary to preserve the second order form of the original system. Such a guarantee requires the development of second order structure preserving model reduction techniques.

This chapter is organized as follows. In Section 2, general results concerning model reduction of first order systems are summarized. In Section 3, a simple sufficient condition for constructing reduced order systems that preserve the second order structure is developed. Generalizations of Balanced Truncation and Krylov subspace-based methods that enforce this sufficient condition for second order systems are presented in Sections 4 and and 5 respectively. After some numerical experiments in Section 6, concluding remarks are made in Section 7.

2 Model Reduction of Linear Systems

Most popular model reduction techniques for linear systems can be placed in one of two categories [1] : SVD-based and Krylov subspace-based techniques. Perhaps the most popular model reduction technique for linear systems is the Balanced Truncation method. This SVD-based technique has many advantages: the stability of the original system is preserved and there exists an a priori global bound on the error between the original and the reduced system. The main drawback is that the technique cannot be applied to large-scale systems of order N, i.e., those systems where $O(N^3)$ computations is an unacceptably large cost. On the other hand, Krylov subspace-based techniques that are based on imposing moment matching conditions between the original and the reduced transfer function, such as rational/tangential interpolation methods, can be applied to large-scale systems but do not provide global error bounds and depend significantly on the choice of certain parameters.

In this section, we present an overview of these examples of each category applied to a linear system described by (5). We assume that its transfer functions is *strictly proper*, i.e. $\lim_{s \to \infty} R(s) = 0$. Since M is invertible, the transfer function considered in (2) is also strictly proper.

2.1 Balanced Truncation

If A is stable, then the system $\mathcal{S}$ (5) is also a linear (convolution) operator mapping square integrable inputs $u(.) \in \mathcal{L}_2[-\infty, +\infty]$ to square integrable outputs $y(.) \in \mathcal{L}_2[-\infty, +\infty]$. Following the development in [4], we recall the concept of a dual operator to discuss the Balanced Truncation method.

Definition 1. *Let L be a linear operator acting from a Hilbert space U to a Hilbert space Y equipped respectively with the inner products $< \ , \ >_U$ and $< \ , \ >_Y$. The dual of L, denoted by L^*, is defined as the linear operator acting from Y to U such that $< Lu, y >_Y \ = \ < u, L^*y >_U$ for all $y \in Y$ and all $u \in U$.* $\square$

It is easily verified that the transfer function associated with the dual operator of (6) is $B^T(sI_N - A^T)^{-1}C^T$, (see [19]).

Now consider the input/output behavior of the system (5). If we apply an input $u(.) \in \mathcal{L}_2[-\infty, 0]$ to the system for $t < 0$, the position of the state at time $t = 0$, assuming the zero initial condition $x(-\infty) = 0$, is equal to

$$x(0) = \int_{-\infty}^{0} e^{-At} Bu(t)dt \doteq \mathcal{C}_o u(t).$$

If a zero input is applied to the system for $t > 0$, then for all $t \geq 0$, the output $y(.) \in \mathcal{L}_2[0, +\infty]$ of the system (5) is equal to

$$y(t) = Ce^{At}x(0) \doteq \mathcal{O}_b x(0).$$

So the mapping of past inputs to future outputs is characterized by two operators – the so-called controllability operator $\mathcal{C}_o : \mathcal{L}_2[-\infty, 0] \mapsto \mathbb{R}^n$ (mapping past inputs $u(.)$ to the present state) and observability operator $\mathcal{O}_b : \mathbb{R}^n \mapsto \mathcal{L}_2[0, +\infty]$ (mapping the present state to future outputs $y(.)$).

Both $\mathcal{C}_o$ and $\mathcal{O}_b$ have dual operators, $\mathcal{C}_o^*$ and $\mathcal{O}_b^*$, respectively. The operators and their duals are related by two fundamental matrices associated with the linear system (5). These are the "controllability gramian" $\mathcal{P}$ and the "observability gramian" $\mathcal{Q}$. If A is stable, they are the unique solutions of the Lyapunov equations :

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0 \quad , \quad A^T\mathcal{Q} + \mathcal{Q}A + C^TC = 0. \tag{10}$$

It follows that $\mathcal{C}_o$ and $\mathcal{O}_b$ are related to their dual operators by the identities $\mathcal{P} = \mathcal{C}_o^*\mathcal{C}_o$ and $\mathcal{Q} = \mathcal{O}_b\mathcal{O}_b^*$ [19].

Another physical interpretation of the gramians arises from two optimization problems. Let

$$J(v(t), a, b) \doteq \int_a^b v(t)^T v(t)dt$$

be the *energy* of the vector function $v(t)$ in the interval $[a, b]$. It can be shown that (see [10])

$$\min_{\mathcal{C}_o u(t) = x_0} J(u(t), -\infty, 0) = x_0^T \mathcal{P}^{-1} x_0, \tag{11}$$

and, symmetrically, we have the dual property

$$\min_{\mathcal{O}_b^* y(t) = x_0} J(y(t), -\infty, 0) = x_0^T \mathcal{Q}^{-1} x_0. \tag{12}$$

Two algebraic properties of gramians $\mathcal{P}$ and $\mathcal{Q}$ are essential to the development of Balanced Truncation. First, under a coordinate transformation $x(t) = T\bar{x}(t)$, the new gramians $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$ corresponding to the state-space realization $(\bar{C}, \bar{A}, \bar{B}) = (CT, T^{-1}AT, T^{-1}B)$ undergo the following (so-called *contragradient*) transformation :

$$\bar{\mathcal{P}} = T^{-1} \mathcal{P} T^{-T}, \quad \bar{\mathcal{Q}} = T^T \mathcal{Q} T. \tag{13}$$

This implies that the eigenvalues of the product $\bar{\mathcal{P}}\bar{\mathcal{Q}} = T^{-1}\mathcal{P}\mathcal{Q}T$ depend only on the transfer function $R(s)$ and not on a particular choice of state-space realization. It implies also that there exists a state-space realization $(C_{bal}, A_{bal}, B_{bal})$ of $R(s)$ such that the corresponding gramians are equal and diagonal $\mathcal{P} = \mathcal{Q} = \Sigma$ [19].

Second, because the gramians appear in the solutions of the optimization problems (11) and (12), they give information about the energy that goes through the system, more specifically, about the distribution of this energy among the state variables. The smaller $x_0^T \mathcal{P}^{-1} x_0$ is, the more "controllable" the state x_0 is, since it can be reached with a input of small energy. By duality, the smaller $x_0^T \mathcal{Q}^{-1} x_0$ is, the more "observable" the state x_0 is. Thus when both gramians are equal and diagonal, the order of magnitude of a diagonal value of the product $\mathcal{P}\mathcal{Q}$ is a good measure for the influence of the corresponding state variable on the mapping $y(.) = \mathcal{O}_b \mathcal{C}_o u(.)$ that maps past inputs $u(t) \in \mathcal{L}_2[-\infty, 0]$ to future outputs $y(t) \in \mathcal{L}_2[0, +\infty]$ passing via that particular state at time $t = 0$.

Given a transfer function $R(s)$, the Balanced Truncation model reduction method consists of finding a state-space realization $(C_{bal}, A_{bal}, B_{bal})$ of $R(s)$ such that the gramians are equal and diagonal (this is the *balanced realization* of the system) and then constructing the reduced model by keeping the states corresponding to the largest eigenvalues of the product $\mathcal{P}\mathcal{Q}$ and discarding the others. In other words, the balanced truncation technique chooses Z and V such that $Z^T V = I$, and

$$\begin{cases} \mathcal{P}\mathcal{Q}V = V\Lambda_+ \\ \mathcal{Q}\mathcal{P}Z = Z\Lambda_+ \end{cases} \tag{14}$$

where Λ_+ is a square diagonal matrix containing the largest eigenvalues of $\mathcal{P}\mathcal{Q}$. A state-space realization of the reduced transfer function is given by $(CV, Z^T AV, Z^T B)$. The idea of the balanced truncation technique thus consists in keeping those states that are most controllable and observable according to the gramians defined in (11) and (12).

Finally, we note that Balanced Truncation can be related to the Hankel operator that maps the past inputs to the future outputs and is defined : $\mathcal{H} \doteq \mathcal{O}_b \mathcal{C}_o$. Since $\mathcal{P}\mathcal{Q} = \mathcal{C}_o \mathcal{C}_o^* \mathcal{O}_b^* \mathcal{O}_b$ and $\mathcal{Q}\mathcal{P} = \mathcal{O}_b^* \mathcal{O}_b \mathcal{C}_o \mathcal{C}_o^*$, the dominant eigenspaces $\mathcal{V}$ of $\mathcal{P}\mathcal{Q}$ and $\mathcal{Z}$ of $\mathcal{Q}\mathcal{P}$ are linked with the dominant eigenspaces $\mathcal{X}$ of $\mathcal{H}\mathcal{H}^*$ and $\mathcal{Y}$ of $\mathcal{H}^*\mathcal{H}$ via the equalities $\mathcal{X} = \mathcal{O}_b\mathcal{V}$ and $\mathcal{Y} = \mathcal{C}_o^*\mathcal{Z}$. Therefore, projecting onto the spaces $\mathcal{V}$ and $\mathcal{Z}$ also approximates the Hankel map $\mathcal{H}$ well. We refer the reader to [10] and [19], for a more detailed study and discussion of the Balanced Truncation method.

Unfortunately, the Balanced Truncation method cannot be applied directly to the state-space realization (C, A, B) (7) of the second order system without destroying its second order structure in the reduced realization. An approach that solves this problem is discussed in Section 4. Also note that, due to its dependence on transformations with $O(N^3)$ complexity, the Balanced Truncation method cannot be applied, as described, to large-scale systems. Recent work by Antoulas and Sorensen considers this problem and describes an Approximate Balanced Truncation approach for large-scale linear systems [15].

2.2 Krylov subspace-based model reduction

The Krylov subspace-based model reduction methods have been developed in order to produce reduced order models of large-scale linear systems efficiently and stably via projection onto subspaces that satisfy specific conditions. These conditions are based on requiring the reduced order transfer function to match selected moments of the transfer function $R(s)$ of the original system.

Rational Interpolation

Krylov subspaces play an important role in the development of these methods and are defined as follows:

Definition 2. *Let $M \in \mathbb{C}^{n \times n}$ and $X \in \mathbb{C}^{n \times m}$. A Krylov subspace of order k of the pair (M, X), denoted $\mathcal{K}_k(M, X)$, is the image of the matrix $\begin{bmatrix} M & MX & \dots & M^{k-1}X \end{bmatrix}$.*

If A is stable, $R(s)$ expanded around infinity gives

$$R(s) = C(sI_N - A)^{-1}B = \sum_{i=0}^{\infty} CA^i B s^{-i-1} \doteq \sum_{i=0}^{\infty} R_i^{\infty} s^{-i-1},$$

where the coefficients R_i^{∞} are called the Markov parameters of the system. One intuitive way to approximate $R(s)$ is to construct a transfer function $\hat{R}(s)$ of Mc Millan degree $n \ll N$,

$$\hat{R}(s) \doteq \hat{C}(sI_n - \hat{A})^{-1}\hat{B} \doteq \sum_{i=1}^{\infty} \hat{R}_i^{\infty} s^{-i} \tag{15}$$

such that $\hat{R}_i^\infty = R_i^\infty$ for $1 \leq i \leq r$, where r is as large as possible and is generically equal to $2n$. The resulting reduced transfer function $\hat{R}(s)$ generally approximates quite well the original transfer function for large values of s.

If a good approximation for low frequencies is desired, one can construct a transfer function

$$\hat{R}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B} = \sum_{k=0}^{\infty} \hat{R}_k^{(\lambda)}(\lambda - s)^k,$$

such that

$$\hat{R}_k^{(\lambda)} = R_k^{(\lambda)} \qquad for \ 1 \leq k \leq K, \tag{16}$$

with

$$R_k^{(\lambda)} \doteq C(\lambda I_N - A)^{-k}B, \quad \hat{R}_k^{(\lambda)} \doteq \hat{C}(\lambda I_n - \hat{A})^{-k}\hat{B}.$$

More generally, one can choose a transfer function $\hat{R}(s)$ that interpolates $R(s)$ at several points in the complex plane, up to several orders. The main results concerning this problem for MIMO standard state space systems are summarized in the Theorem 1.

Theorem 1. *Let the original system be*

$$R(s) \doteq C(sI_N - A)^{-1}B, \tag{17}$$

and the reduced system be

$$\hat{R}(s) \doteq CV\left(Z^T(sI_N - A)V\right)^{-1}Z^TB, \tag{18}$$

with $Z^TV = I_n$. If

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{b_k}}((\lambda_k I - A)^{-1}, (\lambda_k I - A)^{-1}B) \subseteq Im(V) \tag{19}$$

and

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{c_k}}((\lambda_k I - A)^{-T}, (\lambda_k I - A)^{-T}C^T) \subseteq Im(Z) \tag{20}$$

where the interpolation points λ_k are chosen such that the matrices $\lambda_k I_N - A$ are invertible $\forall k \in \{1, \ldots, K\}$ then the moments of the systems (17) and (18) at the points λ_k satisfy

$$R_{j_k}^{(\lambda_k)} = \hat{R}_{j_k}^{(\lambda_k)} \tag{21}$$

for $j_k = 1, 2, \ldots, J_{b_k} + J_{c_k}$ and $k = 1, 2, \ldots, K$, provided these moments exist, i.e. provided the matrices $\lambda_k I_n - \hat{A}$ are invertible.

For a proof, see [5] and [11]. A proof for MIMO generalized state space systems is given in [9].   □

Matching Markov parameters, i.e., $\lambda = \infty$, is known as *partial realization*. When $\lambda = 0$, the corresponding problem is known as *Padé approximation*. If λ takes a finite number of points λ_i, it is called a *multi-point Padé approximation*. In the general case, the problem is known as *rational interpolation*. Rational interpolation generally results in a good approximation of the original transfer function in a region near the expansion points (and increasing the order at a point tends to expand the region), but may not be accurate at other frequencies (see for instance [1]).

The advantage of these moment matching methods is that they can be implemented in a numerically stable and efficient way for large-scale systems with sparse coefficient matrices (see for example [9] and [11]). Also, the local approximation property means that good approximations can be achieved in specific regions over a wide dynamic range typically at the cost of larger global error. This requires however, that the interpolation points and their corresponding order of matching must be specified. For some applications, the user may have such information but for blackbox library software a heuristic automatic selection strategy is needed (see [11]) and the design of such a strategy is still an open question. The other main drawback is the lack of an error bound on the global quality of the approximation, e.g., the H_∞-norm of the difference between original and reduced transfer functions. Recent research has begun to address the evaluation of the H_∞-norm given a reduced order model that may help in selecting points [3].

One could apply these methods to the state space realization (9) of a second order transfer function. Unfortunately, if the methods are used in the forms described, the resulting reduced order transfer function will generically not be in second order form. An approach to maintain second order form is discussed in Section 5.

Tangential Interpolation

The Krylov subspace-based methods that produce reduced order models based on rational interpolation can be applied to MIMO systems efficiently as long as the number of inputs and outputs, m and p, stay suitably moderate in size. For MIMO systems where m and p are too large, a more general *tangential interpolation* problem has recently been considered (see [8]). Instead of imposing interpolation condition of the form $R(\lambda_i) = \hat{R}(\lambda_i)$, one could be interested, for example, in only imposing interpolation conditions of the following form :

$$\hat{R}(\lambda_i)x_i = R(\lambda_i)x_i \quad , \quad y_i\hat{R}(\lambda_{i+n}) = y_iR(\lambda_{i+n}), \quad 1 \le i \le n, \quad (22)$$

where the n column vectors x_i are called the right interpolation directions and the n row vectors y_i are called the left interpolation directions. As with

rational interpolation, higher order tangential interpolation conditions can be imposed at each point to improve the approximation.

Stable and efficient methods for tangential interpolation of MIMO systems can be developed using theorems and techniques similar to those used for Krylov subspace-based rational interpolation. However, the problem of constructing a reduced transfer function that satisfies a set of tangential interpolation conditions and that preserves the second order structure of the original transfer function requires additional consideration as discussed in Section 5.

3 Second order structure preserving model reduction

In this section, a simple sufficient condition for obtaining a second order reduced system from a second order system is presented. The following result can be found in a slightly different form in [4].

Lemma 1. *Let (C, A, B) be the state space realization defined in* (9). *If one projects such a state space realization with $2N \times 2n$ bloc diagonal matrices*

$$\mathcal{Z} \doteq \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}, \quad \mathcal{V} \doteq \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}, \quad \mathcal{Z}^T \mathcal{V} = I_{2n},$$

where $Z_1, V_1, Z_2, V_2 \in \mathbb{C}^{N \times n}$, then the reduced transfer function

$$\hat{R}(s) \doteq C\mathcal{V} \left(\mathcal{Z}^T (sI_{2N} - A)\mathcal{V} \right)^{-1} \mathcal{Z}^T B$$

is a second order transfer function, provided the matrix $Z_1^T V_2$ is invertible.

Proof. First, notice that the transfer function does not change under any similarity transformation of the system matrices. Let us consider the similarity transformation $M \in \mathbb{C}^{2n \times 2n}$ such that

$$M \doteq \begin{bmatrix} X & \\ & Y \end{bmatrix},$$

with $X, Y \in \mathbb{C}^{n \times n}$ verifying

$$X^{-1}(Z_1^T V_2)Y = I_n.$$

From the preceding results,

$$\hat{R}(s) \doteq C\mathcal{V}M \left(M^{-1}\mathcal{Z}^T (sI_{2N} - A)\mathcal{V}M \right)^{-1} M^{-1}\mathcal{Z}^T B$$
$$= F_M^{out} V_1 X \left(s^2 I_n + sY^{-1}Z_2^T D_M V_2 Y + Y^{-1}Z_2^T S_M V_1 X \right)^{-1} Y^{-1}Z_2^T F_M^{in}.$$

This is clearly a second order transfer function. $\square$

4 Second order Balanced Truncation

The earliest balanced truncation technique for second order systems known to the authors is described in [12]. Based on this work, an alternative technique was developed in [4]. In this section an overview of the latter method, called SOBT (Second Order Balanced Truncation), is given.

The first step in the development of SOBT, based on a balance and truncate process similar to that discussed in Section 2.1, involves the definition of *two* pairs of $N \times N$ gramians ("second order gramians") that change according to contragradient transformations, and that have some energetic interpretation. The first pair $(\mathcal{P}_{pos}, \mathcal{Q}_{pos})$ corresponds to an energy optimization problem depending only on the *positions* $q(t)$ and not on the *velocities* $\dot{q}(t)$. Reciprocally, the second pair $(\mathcal{P}_{vel}, \mathcal{Q}_{vel})$ correspond to an optimization problem depending only on the velocities $\dot{q}(t)$ and not the on the positions $q(t)$. By analogy to the first order case, the gramians $\mathcal{Q}_{pos}$ and $\mathcal{Q}_{vel}$ are defined from the dual systems. Given the gramians, a balancing step in the method is defined by transforming to a coordinate system in which the second order gramians are equal and diagonal : $\bar{\mathcal{P}}_{pos} = \bar{\mathcal{Q}}_{pos} = \Sigma_{pos}, \quad \bar{\mathcal{P}}_{vel} = \bar{\mathcal{Q}}_{vel} = \Sigma_{vel}$. Their diagonal values enable us to identify the *important* positions and the *important* velocities, i.e. those with (hopefully) large effect on the I/O map. Once identified the reduced second order model follows by truncation of all variables not identified as important.

In order to define a pair of second order gramians measuring the contribution of the position coordinates (independently of the velocities) with respect to the I/O map, consider an optimization problem naturally associated with the second order system (see [12]) of the form

$$\min_{\dot{q}_0 \in \mathbb{R}^n} \min_{u(t)} \ J(u(t), -\infty, 0), \tag{23}$$

subject to

$$\ddot{q}(t) + D_M \dot{q}(t) + S_M q(t) = F_M^{in} u(t), \ q(0) = q_0.$$

One easily sees that the optimum is $q_0^T \mathcal{P}_{11}{}^{-1} q_0$, where $\mathcal{P}_{11}$ is the $N \times N$ left upper block of $\mathcal{P}$, where $\mathcal{P}$ is the controllability Gramian satisfying equation (10) with (C, A, B) given in (9). Starting with (11) we must solve

$$\min_{\dot{q}_0 \in \mathbb{R}^n} \ J_{q_0}(\dot{q}_0) \ = \ \begin{bmatrix} q_0^T & \dot{q}_0^T \end{bmatrix} \mathcal{P}^{-1} \begin{bmatrix} q_0 \\ \dot{q}_0 \end{bmatrix}$$

Partitioning $\mathcal{P}^{-1}$ as follows

$$\mathcal{P}^{-1} \ = \ \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}$$

and annihilating the gradient of $J_{q_0}(\dot{q}_0)$ gives the relation $\dot{q}_0 = -R_3^{-1} R_2^T q_0$. The value of J_{q_0} at this point is $q_0^T (R_1 - R_2 R_3^{-1} R_2^T) q_0$. This is simply the

Schur complement of R_3 which is $\mathcal{P}_{11}{}^{-1}$. Similarly, the solution of the dual problem corresponds to $q_0^T \mathcal{Q}_{11}{}^{-1} q_0$, where $\mathcal{Q}_{11}$ is the $N \times N$ left upper block of $\mathcal{Q}$ (10).

Note that the transfer function is seen as a linear operator acting between two Hilbert spaces. The dual of such an operator is defined in Definition 1. It follows that the dual of a second order transfer function might not be a second order transfer function. This has no consequences here since only the energy transfer interpretation between the inputs, the outputs, the initial positions and velocities is important. Under the change of coordinates $q(t) = T\bar{q}(t)$, it is easy to verify that this pair of gramians undergo a contragradient transformation :

$$(\bar{\mathcal{P}}_{11}, \bar{\mathcal{Q}}_{11}) = (T^{-1}\mathcal{P}_{11}T^{-T}, T^T\mathcal{Q}_{11}T).$$

This implies that there exists a new coordinate system such that both $\mathcal{P}_{11}$ and $\mathcal{Q}_{11}$ are equal end diagonal. Their energetic interpretation is seen by considering the underlying optimization problem. In (23), the energy necessary to reach the given position q_0 over all past inputs and initial velocities is minimized. Hence, these gramians describe the distribution of the I/O energy among the positions.

A pair of second order gramians that give the contribution of the velocities with respect to the I/O map can be defined analogously. The associated optimization problem is

$$\min_{q_0 \in \mathbb{R}^n} \min_{u(t)} \; J(u(t), -\infty, 0) \tag{24}$$

subject to

$$\ddot{q}(t) + D_M \dot{q}(t) + S_M q(t) \;=\; F_M^{in} u(t), \quad \dot{q}(0) \;=\; \dot{q}_0.$$

Following the same reasoning as before for the optimization problem (23), one can show that the solution of (24) is $\dot{q}_0^T \mathcal{P}_{22}{}^{-1} \dot{q}_0$, where $\mathcal{P}_{22}$ is the $n \times n$ right lower block of $\mathcal{P}$. The solution of the dual problem is $\dot{q}_0^T \mathcal{Q}_{22}{}^{-1} \dot{q}_0$, where $\mathcal{Q}_{22}$ is the $n \times n$ right lower block of $\mathcal{Q}$. As before, under the change of coordinates $q(t) = T\bar{q}(t)$ one can check that this pair of gramians undergo a contragradient transformation and the energetic interpretation is given by considering the underlying optimization problem. In (24), the energy necessary to reach the given velocity $\dot{q}_0$ over all past inputs and initial positions is minimized. Hence, these gramians describe the distribution of the I/O energy among the velocities.

Given the interpretation above these second order gramians are good candidates for balancing and truncation. Therefore, we choose :

$$(\mathcal{P}_{pos}, \; \mathcal{Q}_{pos}) \;=\; (\mathcal{P}_{11}, \; \mathcal{Q}_{11}) \quad \text{and} \quad (\mathcal{P}_{vel}, \; \mathcal{Q}_{vel}) \;=\; (\mathcal{P}_{22}, \; \mathcal{Q}_{22}) \;. \tag{25}$$

It is not possible to balance both pairs of second order gramians at the same time with a single change of coordinates of the type $q(t) = T\bar{q}(t)$. A

change of coordinates is required for both positions and velocities (unlike the approach in [12]). Therefore, we work in a state-space context, starting with the system (9). The SOBT method, therefore, first computes both pairs of second order gramians, $(\mathcal{P}_{pos}, \mathcal{Q}_{pos})$ and $(\mathcal{P}_{vel}, \mathcal{Q}_{vel})$. Given the gramians, the contragradient transformations that make $\mathcal{P}_{pos} = \mathcal{Q}_{pos} = \Lambda_{pos}$ and $\mathcal{P}_{vel} = \mathcal{Q}_{vel} = \Lambda_{vel}$, where Λ_{pos} and Λ_{vel} are positive definite diagonal matrices, are computed. Finally, truncate the positions corresponding to the smallest eigenvalues of Λ_{pos} and the velocities corresponding to the smallest eigenvalues of Λ_{vel}.

At present, there exists no a priori global error bound for SOBT and the stability of the reduced system is not guaranteed. Nevertheless, SOBT yields good numerical results, providing reduced transfer functions with approximation error comparable with the traditional Balanced Truncation technique.

5 Second Order Structure Preserving Krylov Techniques

The Krylov subspace-based methods discussed in Section 2.2 do not preserve second order structure when applied to the linear system (9). It is possible to modify them to satisfy the constraint presented in Section 3 and thereby produce a second order reduced system. Section 5.1 summarizes the earliest Krylov subspace-based method for second order systems [16]. The simple technique constructs, via projection, a second order reduced transfer function that matches the Markov parameters ($\lambda = \infty$) of the original transfer function. The limitation of the technique when applied to a complex interpolation point is also discussed. Section 5.2, addresses this limitation using a generalization that allows multipoint rational interpolation. Finally, the problem of second order structure preserving tangential interpolation is solved in 5.3.

5.1 A particular case : Matching the Markov parameters

Su and Craig proposed a Krylov subspace-based projection method that preserves second order structure while matching the Markov parameters of the original transfer function [16]. The method is based on the observation the right Krylov subspaces corresponding to interpolation at $\lambda = \infty$ for the system (9) has the form

$$\begin{bmatrix} B \ AB \ A^2B \ \dots \end{bmatrix} = \begin{bmatrix} 0 & F_M^{in} & -D_M F_M^{in} & \dots \\ F_M^{in} & -D_M F_M^{in} & -S_M F_M^{in} + D_M^2 F_M^{in} & \dots \end{bmatrix} \qquad (26)$$

$$= \begin{bmatrix} 0 & Q_{v,0} & Q_{v,1} & \dots \\ Q_{v,0} & Q_{v,1} & Q_{v,2} & \dots \end{bmatrix}. \qquad (27)$$

and that if we write

$$\mathcal{K}_k(A, B) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix},$$

it follows that

$$Im(V_1) \subseteq Im(V_2).$$

So by projecting the state space realization (9) with

$$\mathcal{V} \doteq \begin{bmatrix} V_2 & 0 \\ 0 & V_2 \end{bmatrix}, \quad \mathcal{Z} \doteq \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}$$

such that $Z^T V_2 = I_n$, we obtain an interpolating second order transfer function of the form

$$\hat{R}(s) = F_M^{out} V_2 \left(Z^T (s^2 I_N + s D_M + S_M)^{-1} V_2 \right) Z^T F_M^{in}. \tag{28}$$

Hence, a second order system with the same n first Markov parameters as the original second order system can be constructed by projecting with $Z, V \in \mathbb{C}^{N \times n}$ such that $Z^T V = I_n$ and the image of V contains the image of $Q_{v,0}, \ldots, Q_{v,n-1}$. Since $\mathcal{K}_n(A, B) \subseteq \mathcal{V}$, it follows from Theorem 1 that the first n Markov parameters of $R(s)$ and $\hat{R}(s)$ are equal.

If we apply the construction for any interpolation points $\lambda \in \mathbb{C}$, the corresponding right Krylov space is such that

$$\mathcal{K}_k((\lambda I - A)^{-1}, (\lambda I - A)^{-1} B) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix},$$

with A and B defined in (9) and

$$Im(V_1) \subseteq Im(V_2).$$

Unfortunately, a similar statement can not be made for the *left* Krylov subspaces $\mathcal{K}_k((\lambda I - A)^{-T}, (\lambda I - A)^{-T} C^T)$. This implies that when the second order Krylov technique is extended to interpolation at arbitrary points in the complex plane by projecting as in (28), only n interpolation conditions can be imposed for a reduced second order system of Mc Millan degree $2n$.

5.2 Second Order Rational Interpolation

The projection technique of Su and Craig has been generalized independently in [6] and [17] to solve the rational interpolation problem that produces a *second order* transfer function of order n, i.e., of Mc Millan degree $2n$, $\hat{R}(s)$, that interpolates $R(s)$ at $2n$ points in the complex plane. (See also the chapters in this volume by Bai and Freund.) After some preliminary discussion of notation, the conditions that determine the projections are given in Theorem 2 and the associated algorithm is presented.

A rational matrix function $R(s)$ is said to be $O(\lambda - s)^k$ in s with $k \in \mathbb{Z}$ if its Taylor expansion about the point λ can be written as

$$R(s) = O(\lambda - s)^k \iff R(s) = \sum_{i=k}^{+\infty} R_i(\lambda - s)^i, \tag{29}$$

where the coefficients R_i are constant matrices. If $R_k \neq 0$, then we say that $R(s) = \Theta(\lambda - s)^k$. As a consequence, if $R(s) = \Theta(\lambda - s)^k$ and k is strictly negative, then λ is a pole of $R(s)$ and if k is strictly positive, then λ is a zero of $R(s)$. Analogously, we say that $R(s)$ is $O(s^{-1})^k$ if the following condition is satisfied :

$$R(s) = O(s^{-1})^k \iff R(s) = \sum_{i=k}^{+\infty} R_i s^{-i}, \tag{30}$$

where the coefficients R_i are constant matrices. It should be stressed that, in general, $R(s)$ being $O(s)^{-k}$ is not equivalent to $R(s)$ being $O(s^{-1})^k$.

By combining the results of Sections 2.2 and 3, the following Theorem can be proven.

Theorem 2. *Let* $R(s) \doteq F_M^{out}(s^2 I_N + D_M s + S_M)^{-1} F_M^{in} = C(sI_{2N} - A)^{-1}B$, *with*

$$A \doteq \begin{bmatrix} 0 & I_N \\ -S_M & -D_M \end{bmatrix}, \quad B \doteq \begin{bmatrix} 0 \\ F_M^{in} \end{bmatrix}, \quad C \doteq \begin{bmatrix} F_M^{out} & 0 \end{bmatrix},$$

be a second order transfer function of Mc Millan degree $2N$, *i.e.* $S_M, D_M \in \mathbb{C}^{N \times N}$). *Let* $Z, V \in \mathbb{C}^{2N \times n}$ *be defined as*

$$V \doteq \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad , \quad Z \doteq \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

with V_1, V_2, Z_1 *and* $Z_2 \in \mathbb{C}^{N \times n}$ *such that*

$$Z_1^T V_1 = Z_2^T V_2 = I_n.$$

Let us define the $2N \times 2n$ *projecting matrices*

$$\mathcal{V} \doteq \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \quad , \quad \mathcal{Z} \doteq \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}.$$

Define the second order transfer function $\hat{R}(s)$ *of order* n *(and of Mc Millan degree* $2n$*) by*

$$\begin{aligned} \hat{R}(s) &\doteq C\mathcal{V} \left(\mathcal{Z}^T(sI_{2N} - A)\mathcal{V} \right)^{-1} \mathcal{Z}^T B \\ &\doteq \hat{C}(sI_{2n} - \hat{A})^{-1}\hat{B}. \end{aligned} \tag{31}$$

If

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{b_k}} ((\lambda_k I_{2N} - A)^{-1}, (\lambda_k I_{2N} - A)^{-1}B) \subseteq Im(V) \tag{32}$$

and

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{c_k}} ((\lambda_k I_{2N} - A)^{-T}, (\lambda_k I_{2N} - A)^{-T}C^T) \subseteq Im(Z) \tag{33}$$

where the interpolation points λ_k are chosen such that the matrices $\lambda_k I_{2N} - A$ are invertible $\forall k \in \{1, \ldots, K\}$ then, if the matrix $Z_1^T V_2$ is invertible,

$$R(s) - \hat{R}(s) = O(s - \lambda_k)^{J_{b_k} + J_{c_k}} \tag{34}$$

for the finite points λ_k, provided these moments exist, i.e. provided the matrices $\lambda_k I_{2n} - \hat{A}$ are invertible and

$$R(s) - \hat{R}(s) = O(s^{-1})^{J_{b_k} + J_{c_k}} \tag{35}$$

if $\lambda_k = \infty$.

Proof. Clearly, $\mathcal{Z}^T \mathcal{V} = I_{2n}$. The second order structure of $\hat{R}(s)$ follows from Lemma 1. It is clear that

$$Im(V) \subset Im(\mathcal{V}) \quad , \quad Im(Z) \subset Im(\mathcal{Z}).$$

The interpolation conditions are then satisfied because of Theorem 1.   □

The form of the projectors allows the development of an algorithm similar to the Rational Krylov family of algorithms for first order systems [11]. The algorithm, shown below, finds a *second order* transfer function of order n, i.e. of Mc Millan degree $2n$, $\hat{R}(s)$, that interpolates $R(s)$ at $2n$ interpolation points λ_1 up to λ_{2n}, i.e.,

$$R(s) - \hat{R}(s) = O(\lambda_i - s) \qquad for\ 1 \le i \le 2n, \tag{36}$$

We assume for simplicity that the interpolation points are finite, distinct and not poles of $R(s)$. The algorithm is easily modified to impose higher order conditions at the interpolation points.

Algorithm 1   *1. Construct Z and V such that*

$$V = \left[(\lambda_1 I_{2N} - A)^{-1} B \ldots (\lambda_n I_{2N} - A)^{-1} B \right] = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$Z^T = \begin{bmatrix} C(\lambda_{n+1} I_{2N} - A)^{-1} \\ \vdots \\ C(\lambda_{2n} I_{2N} - A)^{-1} \end{bmatrix} = \begin{bmatrix} Z_1^T & Z_2^T \end{bmatrix},$$

where $V_1, V_2 \in \mathbb{C}^{N \times n}$ are the first N rows and the last N rows of V respectively and $Z_1, Z_2 \in \mathbb{C}^{N \times n}$ are the first N rows and the last N rows of Z respectively. Choose the matrices $M_1, M_2, N_1, N_2 \in \mathbb{C}^{n \times n}$ such that $N_1^T Z_1^T V_1 M_1 = N_2^T Z_2^T V_2 M_2 = I_n$.
2. Construct

$$\mathcal{V} \doteq \begin{bmatrix} V_1 M_1 & \\ & V_2 M_2 \end{bmatrix} \quad , \quad \mathcal{Z} \doteq \begin{bmatrix} Z_1 N_1 & \\ & Z_2 N_2 \end{bmatrix}.$$

3. Construct the matrices

$$\hat{C} \doteq C\mathcal{V} \quad, \quad \hat{A} \doteq \mathcal{Z}^T A\mathcal{V} \quad, \quad \hat{B} \doteq \mathcal{Z}^T B.$$

and define the reduced transfer function

$$\hat{R}(s) \doteq \hat{C}(sI_{2n} - \hat{A})^{-1}\hat{B}.$$

From Theorem 2, $\hat{R}(s)$ is a second order transfer function of order n that satisfies the interpolation conditions (36). The algorithm above has all of the freedom in the method of forming the bases and selecting interpolation points and their associated orders found in the Rational Krylov family of algorithms [11]. As a result, the second order rational interpolation problem can be solved while exploiting the sparsity of the matrices and parallelism of the computing platform in a similar fashion.

5.3 Second order Structure Preserving Tangential Interpolation

It is possible to generalize the earlier results for MIMO systems to perform tangential interpolation and preserve second order structure. This is accomplished by replacing Krylov subspaces at each interpolation point, λ_i, with *generalized* Krylov subspaces as done in [8]. The spaces are defined as follows:

Definition 3. *Let $M \in \mathbb{C}^{n \times n}$, $X \in \mathbb{C}^{n \times m}$, $y^{[i]} \in \Re^m$, $i = 0, \ldots, k - 1$ and define $Y \in \Re^{km \times k}$ as*

$$Y = \begin{bmatrix} y^{[0]} & \cdots & y^{[k-1]} \\ & \ddots & \vdots \\ & & y^{[0]} \end{bmatrix}.$$

A generalized Krylov subspace of order k, denoted $\mathcal{K}_k(M, X, Y)$, is the image of the matrix $\begin{bmatrix} X & MX & \ldots & M^{k-1}X \end{bmatrix} Y$.

For example, by using Algorithm 2 below to compute bases for *generalized* Krylov subspaces and forming the appropriate projections, one can construct a second order transfer function $\hat{R}(s)$ of order n that satisfies the following interpolation conditions with respect to the second order transfer function $R(s)$ of order N:

$$x_i\left(R(s) - \hat{R}(s)\right) = O(\lambda_i - s) \quad, \quad \left(R(s) - \hat{R}(s)\right)x_{i+n} = O(\lambda_{i+n} - s), \quad (37)$$

where $x_1, \ldots, x_n \in \mathbb{C}^{1 \times p}$ and $x_{n+1}, \ldots, x_{2n} \in \mathbb{C}^{m \times 1}$.

Algorithm 2 *1. Construct Z and V such that*

$$V = \begin{bmatrix} (\lambda_{n+1}I_{2N} - A)^{-1}Bx_{n+1} & \ldots & (\lambda_{2n}I_{2n} - A)^{-1}Bx_{2n} \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$Z^T = \begin{bmatrix} x_1 C(\lambda_1 I_{2N} - A)^{-1} \\ \vdots \\ x_n C(\lambda_n I_{2N} - A)^{-1} \end{bmatrix} = \begin{bmatrix} Z_1^T & Z_2^T \end{bmatrix},$$

where $Z_1, Z_2, V_1, V_2 \in \mathbb{C}^{N \times n}$. Choose the matrices $M_1, M_2, N_1, N_2 \in \mathbb{C}^{n \times n}$ such that $N_1^T Z_1^T V_1 M_1 = N_2^T Z_2^T V_2 M_2 = I_n$.

2. Construct

$$\mathcal{V} \doteq \begin{bmatrix} V_1 M_1 & \\ & V_2 M_2 \end{bmatrix} \quad , \quad \mathcal{Z} \doteq \begin{bmatrix} Z_1 N_1 & \\ & Z_2 N_2 \end{bmatrix}.$$

3. Construct the matrices

$$\hat{C} \doteq C\mathcal{V} \quad , \quad \hat{A} \doteq \mathcal{Z}^T A \mathcal{V} \quad , \quad \hat{B} \doteq \mathcal{Z}^T B.$$

and define the reduced transfer function

$$\hat{R}(s) \doteq \hat{C}(sI_{2n} - \hat{A})^{-1}\hat{B}.$$

It can be shown that $\hat{R}(s)$ is a second order transfer function of order n that satisfies the interpolation conditions (37) (see [8]).

It is also possible to impose higher order conditions while preserving the structure of the algorithm and the reduced order system. Consider, for instance, right tangential interpolation conditions of higher order (similar results hold for left tangential interpolation). Let the polynomial vector $x(s) \doteq \sum_{i=0}^{k-1} x^{[i]}(s - \lambda)^i$. To impose the tangential interpolation condition

$$\left(R(s) - \hat{R}(s) \right) x(s) = O(s - \lambda)^k,$$

we construct $\hat{R}(s)$ as in Algorithm 2 using the generalized Krylov subspace $\mathcal{K}((\lambda I - A)^{-1}, (\lambda I - A)^{-1}B, X)$ where X is formed from the $x^{[i]}$, $i = 0, \ldots, k-1$, i.e.,

$$Im\left\{ \begin{bmatrix} (\lambda I - A)^{-1}B \ldots (\lambda I - A)^{-k}B \end{bmatrix} \begin{bmatrix} x^{[0]} & \cdots & x^{[k-1]} \\ & \ddots & \vdots \\ & & x^{[0]} \end{bmatrix} \right\} \subseteq Im\left\{ \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \right\}.$$

We refer to [8] for more details on this topic.

6 Numerical Experiments

In this section, model reduction techniques are applied to a large scale second order system representing the vibrating structure of a building. The data are two sparse square matrices S and M of dimension $N = 26394$. The mass matrix M is diagonal and the stiffness matrix S is symmetric and sparse (S contains approximately 2×10^5 non zero elements). The input is equal to the output:

$$B = C^T = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

The damping matrix is *proportional*:

$$D \doteq \alpha M + \beta S.$$

The second order transfer function of Mc Millan degree $2N = 52788$ to be reduced is

$$R(s) \doteq B^T(s^2M + sD + S)^{-1}B = B^T(s^2M + s(\alpha M + \beta S) + S)^{-1}B.$$

Given the structure of M we normalize by transform the equation so that the mass matrix is the identity as follows :

$$R(s) = B^T M^{-0.5}\left(s^2I + s(\alpha I + \beta M^{-0.5}SM^{-0.5}) + M^{-0.5}SM^{-0.5}\right)^{-1}M^{-0.5}B.$$

So, we redefine $\bar{S} = M^{-0.5}SM^{-0.5}$ and $\bar{B} = M^{-0.5}B = \bar{C}^T$.

Reduced systems are directly computed from $R(s)$ using first order and second order Krylov techniques and the required interpolation points are chosen by the following procedure. First, an intermediate second order transfer function of order 200 (i.e. of Mc Millan degree 400) is constructed by projecting $\bar{S}$ onto its eigenspace corresponding to its 200 eigenvalues of smallest magnitude. This corresponds to keeping the 400 eigenvalues of $s^2I + s(\alpha I + \beta\bar{S}) + \bar{S}$ the closest to the imaginary axis. Let $V_{f200} \in \mathbb{R}^{26364 \times 200}$ be the projection matrix corresponding to the 200 eigenvalues of $\bar{S}$ the closest to the imaginary axis (with $V_p^T V_p = I_{200}$) (V_{f200} is computed with the Matlab function *eigs*). The intermediate transfer function is

$$R_{200}(s)$$
$$\doteq \tilde{B}^T\left(s^2I_{200} + s\tilde{D} + \tilde{S}\right)^{-1}\tilde{B} = B^T M^{-0.5}V_p$$
$$\left(s^2I + s(\alpha I + \beta V_p^T M^{-0.5}SM^{-0.5}V_p) + V_p^T M^{-0.5}SM^{-0.5}V_p\right)^{-1}V_p^T M^{-0.5}B.$$

By checking the difference between $R(s)$ and $R_{200}(s)$ at different points in the complex plane, it has been verified that both transfer functions are very close to each other. The Hankel singular values of $R_{200}(s)$ are shown in Figure 1.

From this intermediate transfer function, we compute the reduced transfer function of Mc Millan degree 20 obtained by using balanced truncation (with the *sysred* Matlab function of the Niconet library), called $R_{bt}(s)$. Note that $R_{bt}(s)$ is no longer in second order form. An interesting fact is that there are 42 interpolation points between $R_{200}(s)$ and $R_{bt}(s)$ that have a positive real part (among the 420 zeros of $R_{200}(s) - R_{bt}(s)$). In general, it has been observed that the number of interpolation points in the right-half plane is roughly equal to twice the Mc Millan degree of the reduced transfer function obtained by balanced truncation. The interpolation points in the right-half plane have the advantage that they are neither close to the poles of the system to be reduced nor to the poles of the Balanced Truncation reduced system because both transfer functions are stable. This implies that both transfer functions do not vary too much there and this is preferable in order to avoid numerical instability.
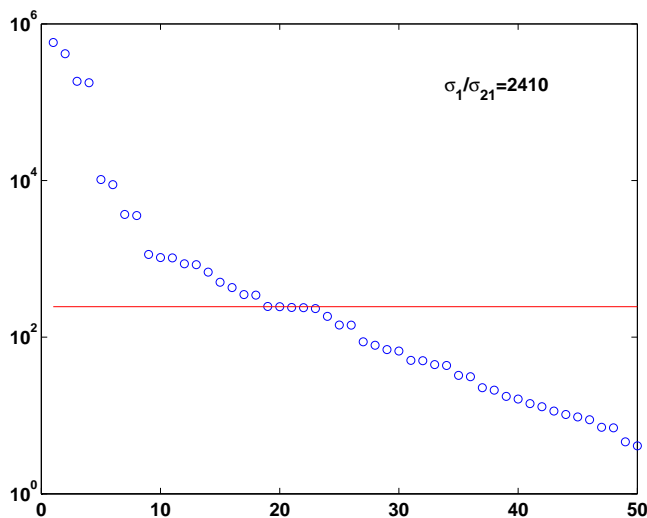
Fig. 1. Hankel singular values of $R_{200}(s)$

It is well known that 40 points are sufficient in order to describe $R_{bt}(s)$. In other words, the only transfer function of Mc Millan degree smaller than 20 that interpolates $R_{bt}(s)$ at 40 points in the complex plane is $R_{bt}(s)$ itself [7]. So, we take the 40 interpolation points between $R_{200}(s)$ and $R_{bt}(s)$ with largest real part as our choice for computing the transfer function of Mc Millan degree 20, denoted $R_{Krylov}(s)$, that interpolates the original transfer function $R(s)$ at these points. The poles and interpolation points are shown in Figure 2. Because $R_{200}(s)$ is very close to $R(s)$, $R_{Krylov}(s)$ should be close to $R_{bt}(s)$.

Using the second order Krylov technique, a reduced second order transfer function R_{sokryl} of Mc Millan degree 28 is also constructed. It interpolates $R(s)$ at the 28 rightmost interpolation points between $R_{200}(s)$ and $R_{bt}(s)$. For comparison purposes a set of interpolation points randomly generated (with symmetry with respect to the real axis in order to obtain a real interpolating transfer function) in a rectangle delimited by the extreme zeros in the left half plane of $R_{200}(s) - R_{bt}(s)$ is also used in the second order Krylov method to generate $R_{randkryl}(s)$. These two sets of interpolation points are shown in Figure 3. Finally a second order transfer function of order 20 (and Mc Millan degree 40) is constructed from $R_{200}(s)$ using the SOBT algorithm [4].

The transfer functions $R_{200}(s)$, $R_{bt}(s)$, $R_{sobt}(s)$, $R_{randkryl}(s)$, $R_{kryl}(s)$ and $R_{sokryl}(s)$ are plotted in Figure 4. Recall, that $R_{200}(s)$ is used here as computationally tractable approximation of $R(s)$. As we can see, all the reduced transfer functions are very close to each other. More can be learned by considering the the H_{∞}-norm errors relative to $\|R_{200}(s)\|_{\infty}$ shown in Table 1.

By choosing random interpolation points, the error is 100 times larger than by taking the *balanced truncation* interpolation points. This indicates
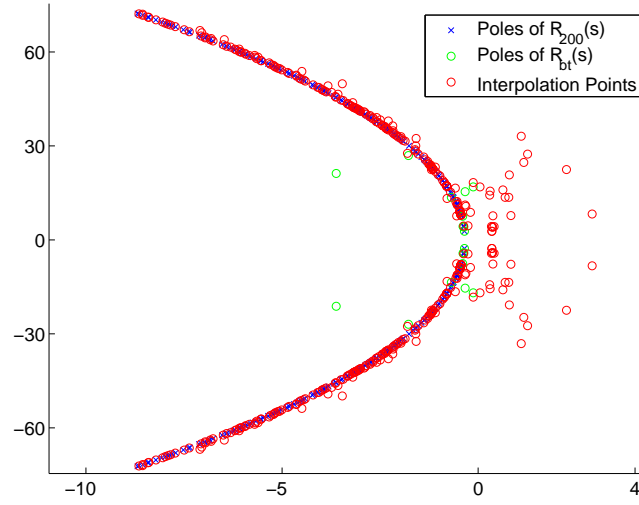
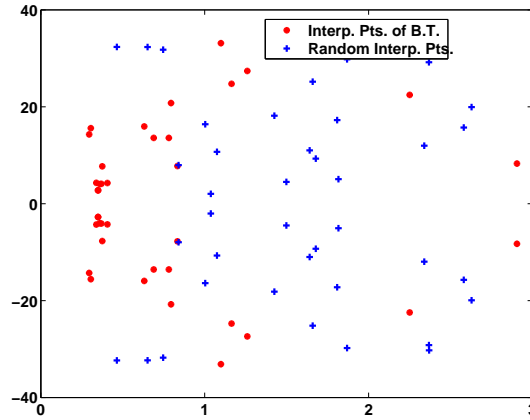**Fig. 2.** Poles and interpolation points for $R_{200}(s)$ and $R_{bt}(s)$

Fig. 3. Interpolation points for $R_{bt}(s)$, $R_{sokryl}(s)$ and $R_{randkryl}(s)$

clearly that the choice of interpolation points is very important in the model reduction procedure. The SOBT reduced transfer function gives the best error bound, but one should not forget that its Mc Millan degree is twice as large as the Mc Millan degree of $R_{bt}(s)$ and it is not applicable to large sparse systems such as $R(s)$ directly.

Finally, for this example, the second order Krylov approximating transfer function performs quite poorly compared to the first order techniques, perhaps indicating that a more sophisticated algorithm for choosing the interpolation
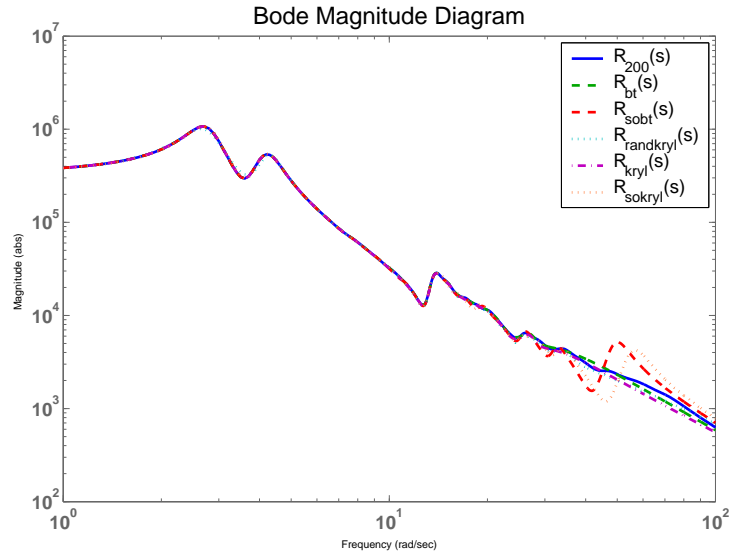
Fig. 4. The six transfer functions

Table 1. Relative errors for reduced order models

Model Reduction technique	Mc Millan degree	$\frac{\|R_{200}(s)-\hat{R}(s)\|_\infty}{\|R_{200}(s)\|_\infty}$
Balanced Truncation	20	$4.3\ 10^{-4}$
Second Order Balanced Truncation	40	$2.6\ 10^{-4}$
Krylov	20	$8.3\ 10^{-4}$
Second Order Krylov	28	$5.8\ 10^{-2}$
Random Krylov	20	$7\ 10^{-2}$

points for these methods is needed. However, one should not forgot that only the Krylov reduced transfer functions have been computed directly from the state space realization of the original transfer function $R(s)$, and not from the intermediate transfer function $R_{200}(s)$.

7 Concluding Remarks

Concerning the second order Krylov technique, the following observation is worth mentioning. For SISO systems, it has been shown in [2] and [12] that for every first order system (c, A, b) such that $cb = 0$, there exists a state space transformation that puts it into a second order form. In other words, every SISO system (with first Markov parameter equal to zero) can be rewritten as a second order system. This implies that in the SISO case, it is possible to

impose $4n - 1$ interpolation conditions for a reduced second order system of Mc Millan degree $2n$ by first using the classic Multipoint Padé technique of Theorem 1 and then reconstructing a second order form with an appropriate state space coordinate transformation. Currently, no proof is available for the MIMO case.

As for generalized state space realizations of first order systems, it is also possible to apply Krylov technique to second order systems without requiring the mass matrix M to be equal to the identity. Concerning the SOBT technique, special care must taken in deriving the second order Gramians.

For second order balanced truncation, numerical results are very encouraging, but many important questions remain open. For instance, does there exist an a priori global error bound with SOBT, as for Balanced Truncation? Even simpler, is stability of the reduced system always guaranteed? If the answer to the preceding questions is negative, does there exist a better choice of second order gramians? Also, the development of an approximate version applicable to large scale systems is needed.

Acknowledgement

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