Model reduction of second order systems

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1 Introduction

The objective of this paper is to present a new method for model reduction of a second order linear time-invariant system of the type :

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t), \tag{1.1}$$

where the matrix $M \in \mathbb{R}^{N\times N}$ is assumed to be invertible. Models of mechanical systems are often of this type since (1.1) then represents the equation of motion of the system. For such a system $M = M^T$, $C = C^T$ and $K = K^T$ are respectively the mass, damping and stiffness matrices, $f(t) \in \mathbb{R}^{N\times 1}$ is the vector of external forces, and $x(t) \in \mathbb{R}^{N\times 1}$ is the vector of internal generalized coordinates (see [4] and [7] for more information on such models).

In civil engineering or aeronautics, the size N of the model (obtained using for instance finite elements techniques [4], [7]) is often so high that many analysis and design problems can not be solved anymore within a reasonable computing time. It is then advisable to construct a reduced order model [5] that nevertheless keeps the "mechanical" structure of the system. Since (1.1) is a particular case of a linear time-invariant system, one may consider its corresponding (linearized) state-space model (see section 2) and apply the techniques of model reduction known for state-space models. In doing so, the reduced-order system is generally not of the same type anymore and the symmetry of the data is lost. Since from a physical point of view it makes sense to impose the reduced-order system to be of the same type, we propose in this paper new methods of model reduction that preserve the second order form and (if needed) its symmetry.

When writing the motion equation in the Laplace domain, the *characteristic polynomial matrix* $P(s)$ appears :

$$P(s)X(s) = F(s), \quad P(s) \doteq Ms^2 + Cs + K. \tag{1.2}$$

The zeros of $\det(P(s))$ are also known as the *characteristic frequencies* of the system and play an important role in model reduction. Stability of the system e.g. implies that these zeros must lie in the open left half plane. In the next section, we present two possible state-space linearizations of a second order system. In section 3, we present new model reduction methods based on balanced truncation and modal approximation. We end with a few numerical examples comparing the different methods.

2 Modeling of a second-order system

We consider the following second order system of differential equations :

$$\left\{ \begin{array}{rcl} M\ddot{x} + C\dot{x} + Kx &=& Bu \\ y &=& Dx, \end{array} \right. \tag{2.3}$$

where $M \in \mathbb{R}^{N\times N}$ is assumed invertible. If this represents a mechanical system, we also assume M is symmetric and positive definite, C is symmetric and K symmetric and positive semi-definite. Because most of the model reduction techniques are developed for state-space models, we first consider the linearization of the above system (2.3) into state-space form. We consider two such linearizations, that will prove useful in later sections.

Using $\xi = \left[\begin{array}{cc} x^T & \dot{x}^T \end{array}\right]^T$, the system (2.3) can be written as

$$\left\{\begin{array}{rcl} \left[\begin{array}{cc} I & 0 \\ 0 & M \end{array}\right]\dot{\xi} &=& \left[\begin{array}{cc} 0 & I \\ -K & -C \end{array}\right]\xi + \left[\begin{array}{c} 0 \\ B \end{array}\right]u, \\ y &=& \left[\begin{array}{cc} D & 0 \end{array}\right]\xi \end{array}\right.$$

Since M is invertible, we can transform it to standard state-space form

$$\left\{\begin{array}{rcl} \dot{\xi} &=& \left[\begin{array}{cc} 0 & I \\ -M^{-1}K & -M^{-1}C \end{array}\right]\xi + \left[\begin{array}{c} 0 \\ M^{-1}B \end{array}\right]u, \\ y &=& \left[\begin{array}{cc} D & 0 \end{array}\right]\xi. \end{array}\right. \tag{2.4}$$

The controllability Gramian $\mathcal{P}$ and observability Gramian $\mathcal{Q}$ of the state-space model (2.4) are given respectively by:

$$\mathcal{P} \doteq \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\begin{array}{c} I \\ j\omega I \end{array}\right] P^{-1}(j\omega)BB^TP^{-T}(-j\omega) \left[\begin{array}{cc} I & -j\omega I \end{array}\right] d\omega, \tag{2.5}$$

$$\mathcal{Q} \doteq \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\begin{array}{c} -j\omega M^T + C^T \\ M^T \end{array}\right] P^{-T}(-j\omega)D^TDP^{-1}(j\omega) \left[\begin{array}{cc} j\omega M + C & M \end{array}\right] d\omega, \tag{2.6}$$

and can be computed via the solution of two Lyapunov equations. One also easily checks that the transfer function of the system is given by

$$H(s) = DP^{-1}(s)B. \tag{2.7}$$

If M, C and K are symmetric and $D = B^T$, then the transfer function is clearly symmetric. To keep symmetry in the state space model equations, we can use the following formulation :

$$\left\{\begin{array}{rcl} \underbrace{\left[\begin{array}{cc} C & M \\ M & 0 \end{array}\right]}_{\mathcal{E}}\dot{\xi} &=& \underbrace{\left[\begin{array}{cc} -K & 0 \\ 0 & M \end{array}\right]}_{\mathcal{A}}\xi + \underbrace{\left[\begin{array}{c} B \\ 0 \end{array}\right]}_{\mathcal{B}}u, \\ y &=& \underbrace{\left[\begin{array}{cc} B^T & 0 \end{array}\right]}_{\mathcal{C}}\xi \end{array}\right.$$

The transfer function is now $H(s) = B^TP^{-1}(s)B$ and the Gramians of the state-space model $\{\mathcal{E}^{-1}\mathcal{A}, \mathcal{E}^{-1}\mathcal{B}, \mathcal{C}\}$ are defined in terms of the solution $\mathcal{G}$ of a single generalized Lyapunov equation :

$$\mathcal{A}\mathcal{G}\mathcal{E} + \mathcal{E}\mathcal{G}\mathcal{A} + \mathcal{B}\mathcal{B}^T = 0, \quad \mathcal{P} = \mathcal{G}, \quad \mathcal{Q} = \mathcal{E}\mathcal{G}\mathcal{E}. \tag{2.8}$$

3 Model reduction via balanced truncation

Most model reduction methods use a projection to build the reduced-order model : given a generalized state-space model $\{\mathcal{E}, \mathcal{A}, \mathcal{B}, \mathcal{C}\}$, the reduced-order model is given by $\{W^T\mathcal{E}X, W^T\mathcal{A}X, W^T\mathcal{B}, \mathcal{C}X\}$ where W and Y are matrices of dimension $2N \times k$, with k the order of the reduced system. The widely used balanced truncation technique chooses X and $W \doteq \mathcal{E}^{-1}Y$ such that

$$\left\{ \begin{array}{rcl} \mathcal{P}\mathcal{Q}X & = & X\Lambda_+, \\ Y^T\mathcal{P}\mathcal{Q} & = & \Lambda_+ Y^T, \end{array} \right. \tag{3.9}$$

where Λ_+ is a $k \times k$ matrix containing the largest eigenvalues of $\mathcal{P}\mathcal{Q}$. This technique cannot be applied directly to a second order system since, in general, the resulting reduced order system is not a second order system anymore. The objective of this paper is to find a method that approximates the balanced truncation and preserves the second order structure of the system.

Since second order systems always have an even order, we choose $k = 2n$. We then partition the matrices $X, Y \in \mathbb{R}^{2N \times 2n}$ as follows, where each block is $N \times n$:

$$Y = \left[\begin{array}{cc} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{array} \right], \quad X = \left[\begin{array}{cc} X_{11} & X_{12} \\ X_{21} & X_{22} \end{array} \right]. \tag{3.10}$$

Starting from the general linearization of a second-order system (see section 2), one proves that one will obtain a reduced second order system via the truncation technique explained above by choosing the projecting matrices X and Y such that

$$Y^T \left[\begin{array}{cc} I & 0 \\ 0 & M \end{array} \right] X = \left[\begin{array}{cc} T_1 & 0 \\ 0 & \hat{M} \end{array} \right], \quad Y^T \left[\begin{array}{cc} 0 & I \\ -K & -C \end{array} \right] X = \left[\begin{array}{cc} 0 & T_2 \\ -\hat{K} & -\hat{C} \end{array} \right],$$

$$Y^T \left[\begin{array}{c} 0 \\ B \end{array} \right] = \left[\begin{array}{c} 0 \\ \hat{B} \end{array} \right], \quad \left[\begin{array}{cc} D & 0 \end{array} \right] X = \left[\begin{array}{cc} \hat{D} & 0 \end{array} \right], \tag{3.11}$$

where $T_i, i = 1, 2$ are invertible matrices. Sufficient conditions to obtain this for all M, K, C, B, D, are to choose X and Y block-diagonal, i.e. :

$$X_{12} = 0,\ X_{21} = 0,\ Y_{12} = 0,\ Y_{21} = 0, \tag{3.12}$$

provided $T_1 \doteq Y_{11}^T X_{11}$ and $T_2 \doteq Y_{11}^T X_{22}$ are invertible. In such a case,

$$\hat{M} = Y_{22}^T M X_{22},\ \hat{C} = Y_{22}^T C X_{22},\ \hat{K} = Y_{22}^T K X_{11},\ \hat{B} = Y_{22}^T B,\ \hat{D} = D X_{11}.$$

In order to obtain a reduced order model in standardized form, it suffices to choose $\tilde{X} \doteq X.T^{-1}$ where $T = \mathrm{diag}\{T_1, T_2\}$. The reduced order model equations (3.11) then have $T_1 = T_2 = I_n$ and

$$\hat{M} = Y_{22}^T M \tilde{X}_{22},\ \hat{C} = Y_{22}^T C \tilde{X}_{22},\ \hat{K} = Y_{22}^T K \tilde{X}_{11},\ \hat{B} = Y_{22}^T B,\ \hat{D} = D \tilde{X}_{11}.$$

For the symmetric case, the projecting equations with block diagonal X and W become

$$W^T \left[\begin{array}{cc} C & M \\ M & 0 \end{array} \right] X = \left[\begin{array}{cc} \hat{C} & \hat{M}_3 \\ \hat{M}_1 & 0 \end{array} \right],\ W^T \left[\begin{array}{cc} -K & 0 \\ 0 & M \end{array} \right] X = \left[\begin{array}{cc} -\hat{K} & 0 \\ 0 & \hat{M}_2 \end{array} \right],\ W^T \left[\begin{array}{c} B \\ 0 \end{array} \right] = \left[\begin{array}{c} \hat{B} \\ 0 \end{array} \right]$$

for which we use the simplifying assumption $W = X$. This automatically preserves symmetry since then

$$\hat{M}_1 = X_{22}^T M X_{11},\ \hat{M}_3 = X_{11}^T M X_{22} = \hat{M}_1^T,\ \hat{M}_2 = X_{22}^T M X_{22},$$
$$\hat{C} = X_{11}^T C X_{11},\ \hat{K} = X_{11}^T K X_{11},\ \hat{B} = X_{11}^T B,\ \hat{D} = D X_{11} = \hat{B}^T.$$

Again, in order to obtain a reduced order model in standardized form, it suffices to choose $\tilde{W} = \tilde{X} \doteq X.T^{-1}$ where $T = \mathrm{diag}\{T_1, T_2\}$ and T_1 and T_2 satisfy $\hat{M}_1 T_1^{-1} = \hat{M}_2 T_2^{-1}$. The reduced order model equations (3.11) then have $\hat{M}_1 = \hat{M}_2 = \hat{M}_3$ and

$$\hat{M} = \tilde{X}_{22}^T M \tilde{X}_{22},\ \hat{C} = \tilde{X}_{11}^T C \tilde{X}_{11},\ \hat{K} = \tilde{X}_{11}^T K \tilde{X}_{11},\ \hat{B} = \tilde{X}_{11}^T B,\ \hat{D} = D \tilde{X}_{11} = \hat{B}^T.$$

Clearly, the block diagonal form of X, Y and W is not *necessary*. Furthermore, they are not compatible with the condition that X and Y span invariant subspaces of $\mathcal{PQ}$. The latter condition has thus to be relaxed.

3.1 Block-diagonal approximation of a basis (method CS)

A simple modification is to compute the spaces X and Y via the usual approach, and then to approximate the computed basis of the invariant subspace by a block diagonal one. To obtain such a block diagonal structure in X, we can use the CS decomposition [2]. We assume that an orthogonal basis for X was calculated, and we want to find another orthogonal basis for the same subspace, i.e.

$$\hat{X} = XV, \quad V^T V = I_{2n}$$

and partition $\hat{X}$ into four $N \times n$ blocks as follows :

$$\hat{X} = \left[\begin{array}{c|c} \hat{X}_{11} & \hat{X}_{12} \\ \hline \hat{X}_{21} & \hat{X}_{22} \end{array}\right].$$

We are looking for a transformation V that minimizes the squared norms of the off-diagonal blocks :

$$\|\hat{X}_{12}\|_F^2 + \|\hat{X}_{21}\|_F^2$$

since these blocks will be discarded in the block diagonal approximation. The solution to this problem is given by the CS decomposition of an orthogonal submatrix. Assuming for simplicity $N \geq 2n$, we have:

$$X = \left[\begin{array}{c|c} U_1 & 0 \\ \hline 0 & U_2 \end{array}\right] \left[\begin{array}{c} C \\ 0 \\ \hline 0 \\ S \end{array}\right] V^T, \text{ with } C = diag(c_i = \cos\theta_i) \text{ and } S = diag(s_i = \sin\theta_i).$$

In fact, the singular values of $\left[\begin{array}{c|c} X_{11} & X_{12} \end{array}\right]$ are the $c_1, \ldots, c_{2n}$ in decreasing order and those of $\left[\begin{array}{c|c} X_{21} & X_{22} \end{array}\right]$ are the $s_1, \ldots, s_{2n}$, in increasing order. From this we obtain the following bounds :

$$\|\hat{X}_{12}\|_F^2 \geq \sum_{i=n+1}^{2n} c_i^2, \quad \|\hat{X}_{21}\|_F^2 \geq \sum_{i=1}^{n} s_i^2$$

and it follows from the CS decomposition that its transformation V achieves this lower bound. If $N < 2n$, the CS decomposition has to be slightly modified, but the results are essentially the same. Applying this also to Y yields the requested block diagonal projection matrices. Moreover, in the symmetric case the left projector $W = X$ and only one approximation has to be performed.

3.2 Relaxing the Trace Max condition (method trace)

It is well-known (see [2]) that X will span the dominant invariant subspace of a *positive definite generalized eigenvalue problem* $(sB - A)$ (i.e. with B symmetric positive definite and A symmetric) if and only if it satisfies a "trace max" condition. In other words, one has that :

$$B^{-1}AX = X\Lambda_+, \tag{3.13}$$

if and only if

$$\max \ trace\, X^T AX = \Lambda_+, \quad s.t.\ X^T BX = I. \tag{3.14}$$

Problem (3.9) is thus equivalent to the following optimization problems :

$$\max \ trace\, X^T \mathcal{Q} X = \Lambda_+, \quad s.t.\ X^T \mathcal{P}^{-1} X = I_{2n}, \tag{3.15}$$

$$\max \ trace\, Y^T \mathcal{P} Y = \Lambda_+, \quad s.t.\ Y^T \mathcal{Q}^{-1} Y = I_{2n}. \tag{3.16}$$

If we want also to impose the conditions (3.12) we have to relax the equality conditions in (3.15) and (3.16). We propose to replace them by their diagonal blocks. Using conditions (3.12), then (3.15) and (3.16) simplify to

$$\max \ trace\, X_{11}^T [\mathcal{Q}]_{11} X_{11} + X_{22}^T [\mathcal{Q}]_{22} X_{22}; \quad s.t.\ X_{11}^T [\mathcal{P}^{-1}]_{11} X_{11} = I_n,\ X_{22}^T [\mathcal{P}^{-1}]_{22} X_{22} = I_n \tag{3.17}$$

$$\max \ trace\, Y_{11}^T [\mathcal{P}]_{11} Y_{11} + Y_{22}^T [\mathcal{P}]_{22} Y_{22}; \quad s.t.\ Y_{11}^T [\mathcal{Q}^{-1}]_{11} Y_{11} = I_n,\ Y_{22}^T [\mathcal{Q}^{-1}]_{22} Y_{22} = I_n. \tag{3.18}$$

These then decouple into four independent problems defining X_{11}, X_{22}, Y_{11} and Y_{22}, which can be solved separately as a positive definite generalized eigenvalue problem since they are each of the type (3.14). We can also solve them as a singular value problem as follows. Let

$$\mathcal{Q} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} L_{11}^T & L_{21}^T \\ 0 & L_{22}^T \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \begin{bmatrix} U_{11}^T & 0 \\ U_{12}^T & U_{22}^T \end{bmatrix},$$

(i.e. the Cholesky factorization of the Gramians) be given, then

$$[\mathcal{P}^{-1}]_{11} = [U_{11} U_{11}^T]^{-1}, \quad \mathcal{Q}_{11} = L_{11} L_{11}^T,$$

and the first subproblem in (3.17) is equivalent to finding the dominant eigenspace of $[U_{11}U_{11}^T.L_{11}L_{11}^T]$. This can be obtained from the dominant singular subspace of the matrix $A \doteq L_{11}^T.U_{11}$. Indeed, let U_A, V_A be a pair of dominant singular subspaces of A :

$$AV_A = U_A\Sigma_+, \quad A^T U_A = V_A\Sigma_+.$$

Then V_A is the dominant eigenspace of A^TA since $A^TAV_A = V_A\Lambda_+$ where $\Lambda_+ = \Sigma_+^2$. Applying this to $A \doteq L_{11}^T.U_{11}$ yields finally $X_{11} = U_{11}V_A$ as the dominant eigenspace of $U_{11}A^TAU_{11}^{-1} = U_{11}U_{11}^T.L_{11}L_{11}^T$. Similar results hold for the other subproblems. The advantage of this SVD approach

is twofold : (1) the best numerical methods to compute Gramians in fact compute directly the above Cholesky factors rather than the Gramians themselves [3] and (2) the SVD is a more reliable tool than the generalized eigenvalue approach since in the latter case the positivity (or even the realness) of the eigenvalues can get lost due to numerical round-off.

In the symmetric case we apply the projection theorem to the standard state space system $\{\mathcal{E}^{-1}\mathcal{A}, \mathcal{E}^{-1}\mathcal{B}, \mathcal{C}\}$ (with $\mathcal{B}^T = \mathcal{C}$) described in section 2.2. This gives the reduced order model $\{\hat{\mathcal{E}}, \hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}\}$ via the formulas

$$\hat{\mathcal{E}} \doteq Y^T X,\ \hat{\mathcal{A}} \doteq Y^T \mathcal{E}^{-1}\mathcal{A} X,\ \hat{\mathcal{B}} \doteq Y^T \mathcal{E}^{-1}\mathcal{B},\ \hat{\mathcal{C}} \doteq \mathcal{C} X, \tag{3.19}$$

where X and Y are supposed to be the solutions of (3.9) :

$$\left\{ \begin{array}{rcl} \mathcal{G}\mathcal{E}\mathcal{G}\mathcal{E}X & = & X\Lambda_+, \\ \mathcal{E}\mathcal{G}\mathcal{E}\mathcal{G}Y & = & Y\Lambda_+ \end{array} \right.$$

This last system of equations indicates that we have to choose $Y = \mathcal{E}X$. This simplifies (3.19) to

$$\hat{\mathcal{E}} = X^T \mathcal{E} X,\ \hat{\mathcal{A}} = X^T \mathcal{A} X,\ \hat{\mathcal{B}} = X^T \mathcal{B},\ \hat{\mathcal{C}} = \mathcal{C}X = \hat{\mathcal{B}}^T, \tag{3.20}$$

which clearly preserves symmetry. If we now add condition (3.12) that X should have blocks $X_{12} = X_{21} = 0$ then X_{11} and X_{22} can be computed from

$$\begin{array}{c} \max trace X_{11}^T [\mathcal{E}\mathcal{G}\mathcal{E}]_{11} X_{11} + X_{22}^T [\mathcal{E}\mathcal{G}\mathcal{E}]_{22} X_{22} \\ s.t.\ X_{11}^T [\mathcal{G}^{-1}]_{11} X_{11} = I_n,\ X_{22}^T [\mathcal{G}^{-1}]_{22} X_{22} = I_n, \end{array}$$

which again can be solved separately as a positive definite generalized eigenvalue problem. In order to compute this via a singular value decomposition, we again start from the factorized Gramian $\mathcal{G}$. If

$$\mathcal{G} = \left[\begin{array}{cc} L_{11} & 0 \\ L_{21} & L_{22} \end{array} \right] \left[\begin{array}{cc} L_{11}^T & L_{21}^T \\ 0 & L_{22}^T \end{array} \right] = \left[\begin{array}{cc} U_{11} & U_{12} \\ 0 & U_{22} \end{array} \right] \left[\begin{array}{cc} U_{11}^T & 0 \\ U_{12}^T & U_{22}^T \end{array} \right], \tag{3.21}$$

then

$$\left[\mathcal{G}^{-1}\right]_{11} = \left[U_{11} U_{11}^T\right]^{-1}, \quad \left[\mathcal{G}^{-1}\right]_{22} = \left[L_{22} L_{22}^T\right]^{-1}.$$

We also have the identities

$$[\mathcal{E}\mathcal{G}\mathcal{E}]_{11} = \left[\begin{array}{cc} CL_{11} + ML_{21} & ML_{22} \end{array} \right] \left[\begin{array}{c} L_{11}^T C + L_{21}^T M \\ L_{22}^T M \end{array} \right], \quad [\mathcal{E}\mathcal{G}\mathcal{E}]_{22} = ML_{11}L_{11}^T M. \tag{3.22}$$

If V_A denotes the right dominant singular subspace of

$$A \doteq \left[\begin{array}{c} L_{11}^T C + L_{21}^T M \\ L_{22}^T M \end{array} \right] U_{11},$$

then $X_{11} = U_{11} V_A$. Analogously, if V_A denotes the right dominant singular subspace of

$$A \doteq L_{11}^T M L_{22},$$

then $X_{22} = L_{22} V_A$.

3.3 Block diagonal Gramians (method diag G)

It is clear that if the Gramians $\mathcal{P}$ and $\mathcal{Q}$ are block diagonal to start with, then automatically one obtains the optimal block diagonal projection matrices from the equations

$$\left\{ \begin{array}{rcl} \mathcal{P}_{11}\mathcal{Q}_{11}X_{11} & = & X_{11}\Lambda_{+1}, \\ \mathcal{P}_{22}\mathcal{Q}_{22}X_{22} & = & X_{22}\Lambda_{+2}, \\ \mathcal{Q}_{11}\mathcal{P}_{11}Y_{11} & = & Y_{11}\Lambda_{+1}, \\ \mathcal{Q}_{22}\mathcal{P}_{22}Y_{22} & = & Y_{22}\Lambda_{+2}. \end{array} \right. \tag{3.23}$$

where Λ_{+1} and Λ_{+2} contain the largest eigenvalues fo the respective matrices $\mathcal{P}_{11}\mathcal{Q}_{11}$ and $\mathcal{P}_{22}\mathcal{Q}_{22}$. A simple relaxation is therefore to neglect the off diagonal blocks of the Gramians $\mathcal{P}$ and $\mathcal{Q}$ and to solve for X_{11}, X_{22}, Y_{11} and Y_{22} via the above equations.

The computation of the dominant eigenspaces can again be obtained via a singular value decomposition rather than an eigendecomposition. As before, let

$$\mathcal{Q} = \left[\begin{array}{cc} L_{11} & 0 \\ L_{21} & L_{22} \end{array} \right] \left[\begin{array}{cc} L_{11}^T & L_{21}^T \\ 0 & L_{22}^T \end{array} \right], \quad \mathcal{P} = \left[\begin{array}{cc} R_{11} & 0 \\ R_{21} & R_{22} \end{array} \right] \left[\begin{array}{cc} R_{11}^T & R_{21}^T \\ 0 & R_{22}^T \end{array} \right],$$

then X_{11} is the dominant eigenspace of the matrix $[L_{11}L_{11}^T.R_{11}R_{11}^T]$. This can be obtained from the dominant right singular subspace V_A of $A \doteq R_{11}^T.L_{11}$ via $X_{11} = L_{11}V_A$. Similar results hold for the other blocks.

In the symmetric case we again consider the decompositions (3.21) which now yield

$$[\mathcal{G}]_{11} = \left[L_{11}L_{11}^T\right], \quad [\mathcal{G}]_{22} = \left[U_{22}U_{22}^T\right].$$

Using (3.22) we then obtain $X_{11} = L_{11}V_A$ from the dominant right singular subspace V_A of

$$A \doteq \left[\begin{array}{c} L_{11}^T C + L_{21}^T M \\ L_{22}^T M \end{array} \right] L_{11},$$

and $X_{22} = U_{22}V_A$ from the dominant right singular subspace V_A of

$$A \doteq L_{11}^T M U_{22}.$$

3.4 Modal approximation (method amod)

An apparently different approach is to perform modal approximation of the polynomial matrix $P(s)$. One then computes the eigenvectors x_i and y_i from the generalized eigenvalue problems :

$$\left[\begin{array}{cc} -\lambda_i K & K \\ K & \lambda_i M + C \end{array} \right] \left[\begin{array}{c} x_i \\ \lambda_i x_i \end{array} \right] = 0, \quad \left[\begin{array}{cc} y_i^T & \lambda_i y_i^T \end{array} \right] \left[\begin{array}{cc} -\lambda_i K & K \\ K & \lambda_i M + C \end{array} \right] = 0$$

which are equivalent to $P(\lambda_i)x_i = 0$ and $y_i^T P(\lambda_i) = 0$. A selection of n of these left and right eigenvectors are then put in the $N \times n$ matrices X_{11} and Y_{11}. The selection of characteristic frequencies λ_i can be based on several criteria : those lying in a particular frequency range or the rightmost ones. One should point out that real bases X_{11} and Y_{11} can be obtained when simultaneously selecting complex conjugate characteristic frequencies of $P(s)$. Moreover, if $P(s)$

is symmetric, then $Y_{11} = X_{11}$. One easily checks that constructing a reduced order model via $\hat{P}(s) \doteq Y_{11}^T P(s) X_{11}$ is precisely a projection method of the type described in the previous sections, except for the choice of bases X_{11} and Y_{11}. It follows also that

$$P(s)X_{11} = X_{11}\hat{P}(s), \quad Y_{11}^T P(s) = \hat{P}(s)Y_{11}^T,$$

provided we normalized the bases using $Y_{11}^T X_{11} = I_n$. These equations also imply that the reduced order model has the selected eigenvalues λ_i as eigenfrequencies since then $\det \hat{P}(\lambda_i) = 0$.

4 Numerical tests

Matlab codes implementing the model reduction methods discussed above have been developed and tested. We now describe numerical experiments performed on four models: the CD player, the International space station, a Building model and a Clamped beam model. A detailed description of these models can be found in [1]. We distinguish between the non-symmetric and the symmetric cases.

In the tables given below $\|\mathcal{H}\|_2$ denotes the Hankel norm of the original model and $\|\mathcal{H}-\hat{\mathcal{H}}_{method}\|_2$ denotes the Hankel norm of the difference between the original model and the reduced one, for each *method* : balanced truncation (bt), block-diagonal approximation of the projective matrices (CS), trace method (trace), block-diagonal approximation of Gramians (diagG) and modal approximation (amod). The ratio $\sigma_{2n+1}/\|\mathcal{H}\|_2$ (where σ_{2n+1} is the $(2n+1)$-st Hankel singular value of the original model) is the best relative error one can obtain for any reduced order model (see [8]). It is listed here as an indication of the efficiency of the other methods.

We also point out that none of the four approximate methods guarantee the stability of the reduced order model. Since for unstable systems the Hankel norm does not bound the system response of the error, we marked the corresponding entries in the above table as "UNST". The table shows also that for all above experiments, method diagG produced stable reduced order models.

Among the second order approximation schemes (CS, trace, diagG and amod) it was typically diagG which performed best and its results are very similar to those given by balanced truncation. We therefore compare only these two methods in the pictures given below : the poles and the frequency response of the original model and those taken by the reduced models (bt and diagG).

4.1 Non-symmetric case

	N	n	$\frac{\sigma_{2n+1}}{\|\mathcal{H}\|_2}$	$\frac{\|\mathcal{H}-\hat{\mathcal{H}}_{bt}\|_2}{\|\mathcal{H}\|_2}$	$\frac{\|\mathcal{H}-\hat{\mathcal{H}}_{CS}\|_2}{\|\mathcal{H}\|_2}$	$\frac{\|\mathcal{H}-\hat{\mathcal{H}}_{trace}\|_2}{\|\mathcal{H}\|_2}$	$\frac{\|\mathcal{H}-\hat{\mathcal{H}}_{diagG}\|_2}{\|\mathcal{H}\|_2}$	$\frac{\|\mathcal{H}-\hat{\mathcal{H}}_{amod}\|_2}{\|\mathcal{H}\|_2}$
CD	60	6	3.13e-06	3.16e-06	7.52e-06	8.41e-06	3.57e-06	1.48e-03
ISS	135	13	5.588e-03	5.594e-03	5.594e-03	5.594e-03	5.594e-03	1e-00
Build	24	4	6.1e-02	9.4e-02	2.0e-01	2.2e-01	7.9e-02	3.1e-01
Beam	174	17	1.35e-05	2.88e-05	*UNST*	*UNST*	1.83e-04	*UNST*

4.1.1 CD-player model

N	m	p	n	$\sigma_1 = \|\mathcal{H}\|_2$	σ_{2n+1}
60	2	2	6	1.17e+06	3.67

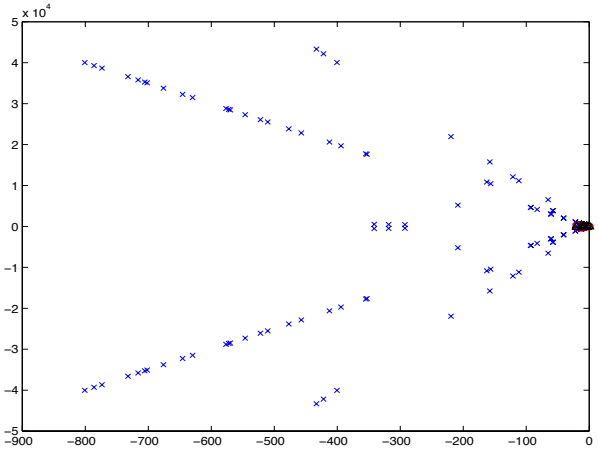

Figure 1: Poles of original model ($\star$),
bt reduced model ($\triangle$)
and diagG reduced model ($\circ$)

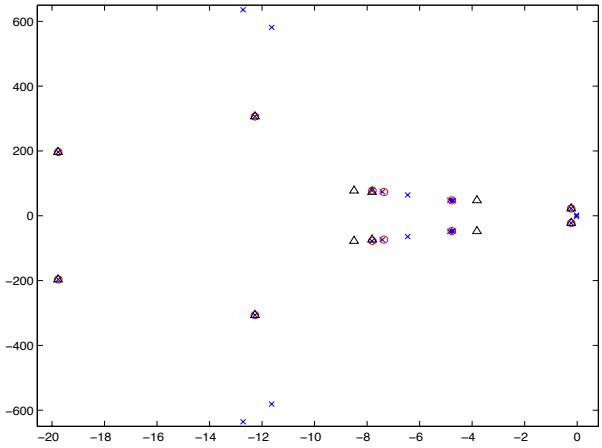

Figure 2: Zoom Figure 1

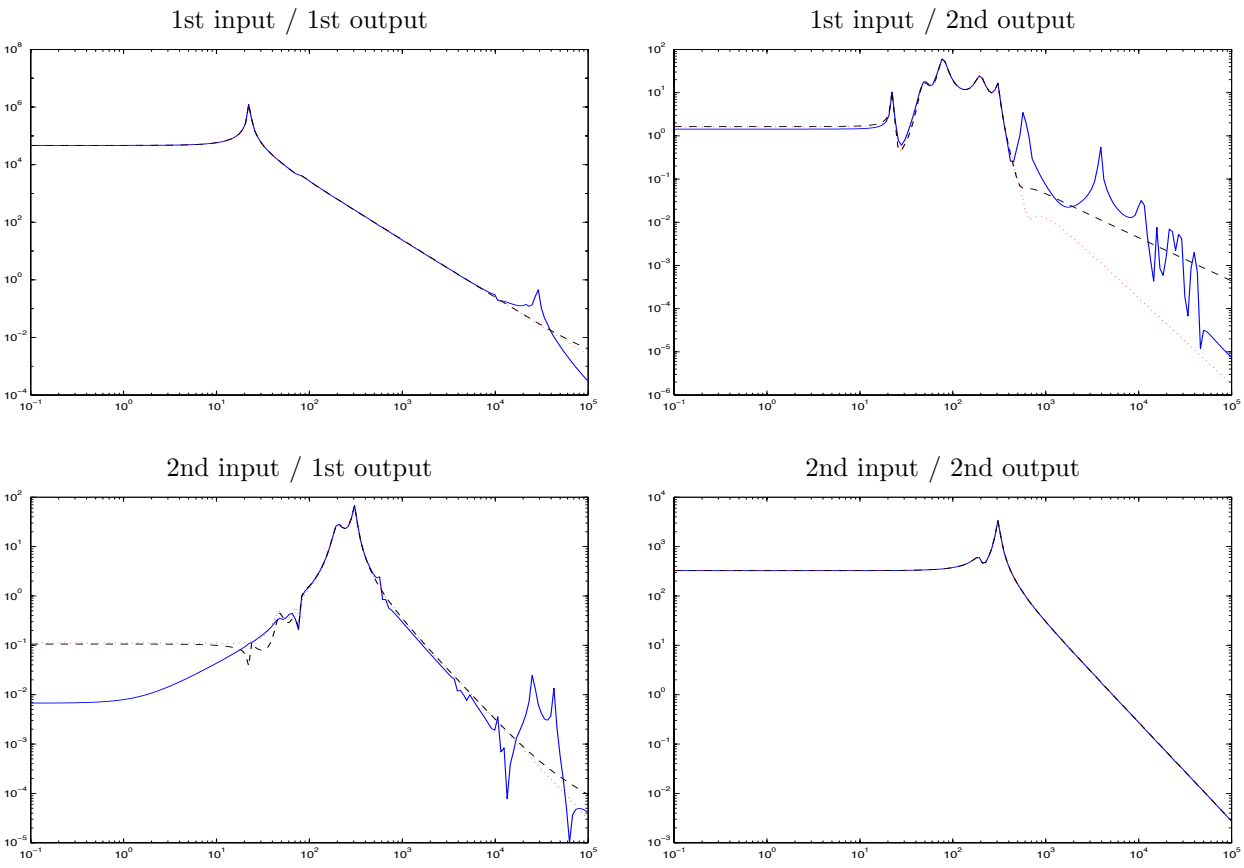

Figure 3: Frequency response
—— original model, - - bt reduced model and $\cdots$ diagG reduced model

4.1.2 ISS model

N	m	p	n	$\sigma_1 = \|\mathcal{H}\|_2$	σ_{2n+1}
135	3	3	13	5.79e-02	3.24e-04

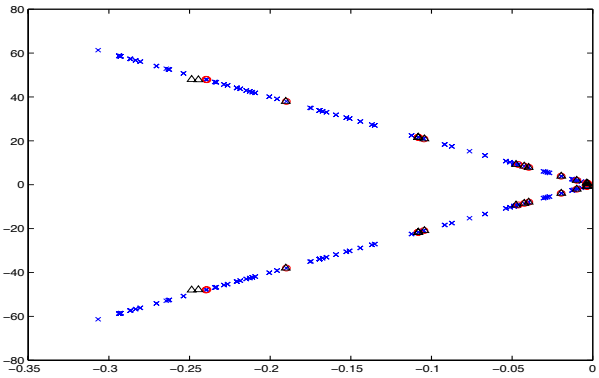

Figure 4: Poles of original model ($\star$),
bt reduced model ($\triangle$)
and diagG reduced model ($\circ$)

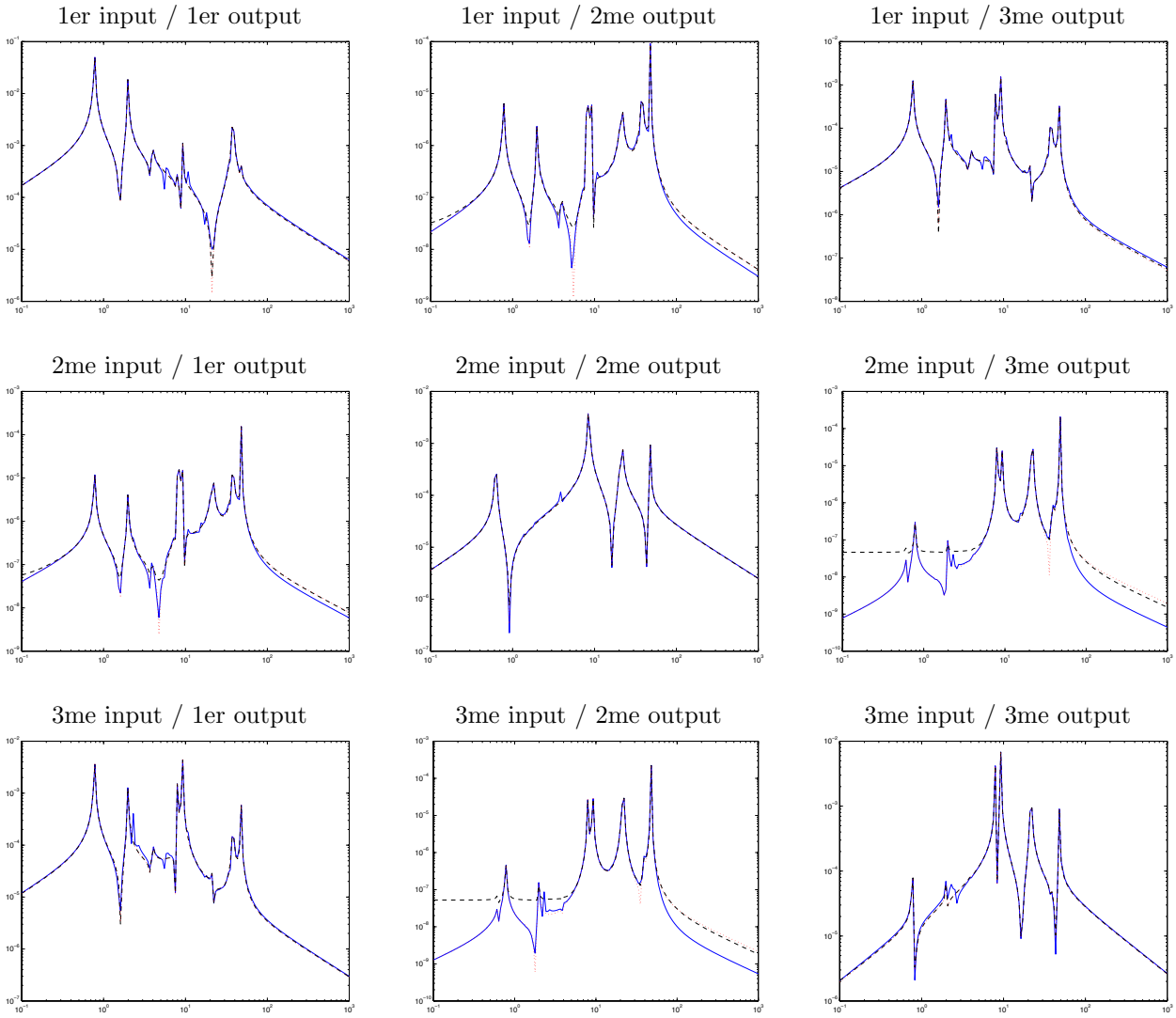

Figure 5: Frequency response
—— original model, - - bt reduced model and ⋯ diagG reduced model

4.1.3 Building model

N	m	p	n	$\sigma_1 = \|\mathcal{H}\|_2$	σ_{2n+1}
24	1	1	4	5.04e-04	3.06e-05

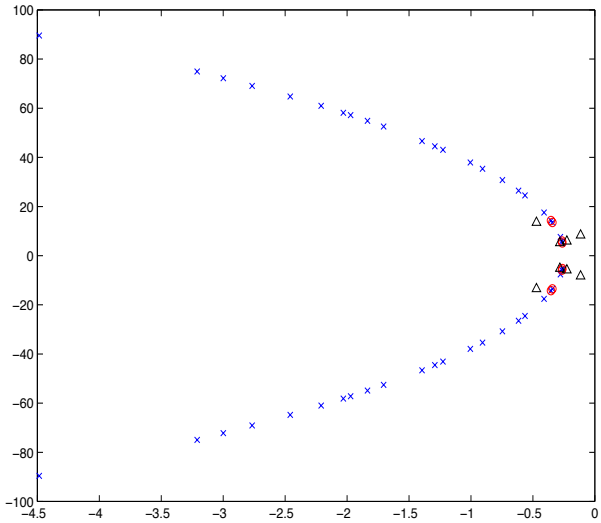

Figure 6: Poles of original model ($\star$),
bt reduced model ($\triangle$)
and diagG reduced model ($\circ$)

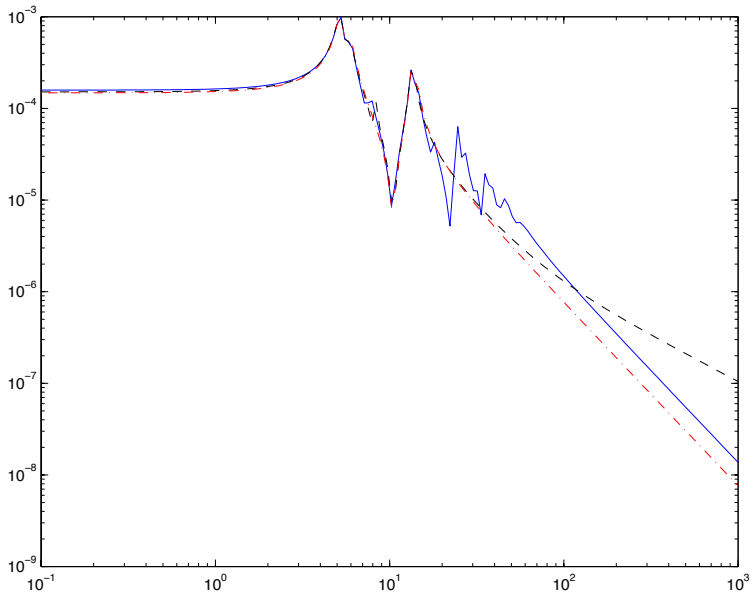

Figure 7: Frequency response
—— original model, - - bt reduced model and $\cdots$ diagG reduced model

4.1.4 Beam model

N	m	p	n	$\sigma_1 = \|\mathcal{H}\|_2$	σ_{2n+1}
174	1	1	17	2.39e+03	3.23e-02

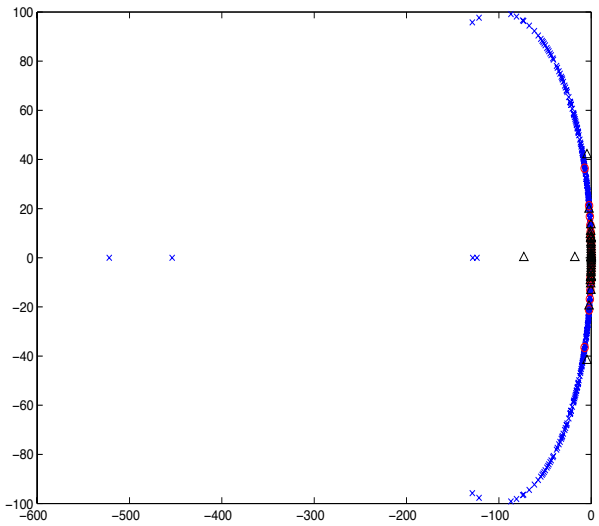

Figure 8: Poles of original model ($\star$),
bt reduced model ($\triangle$)
and diagG reduced model ($\circ$)

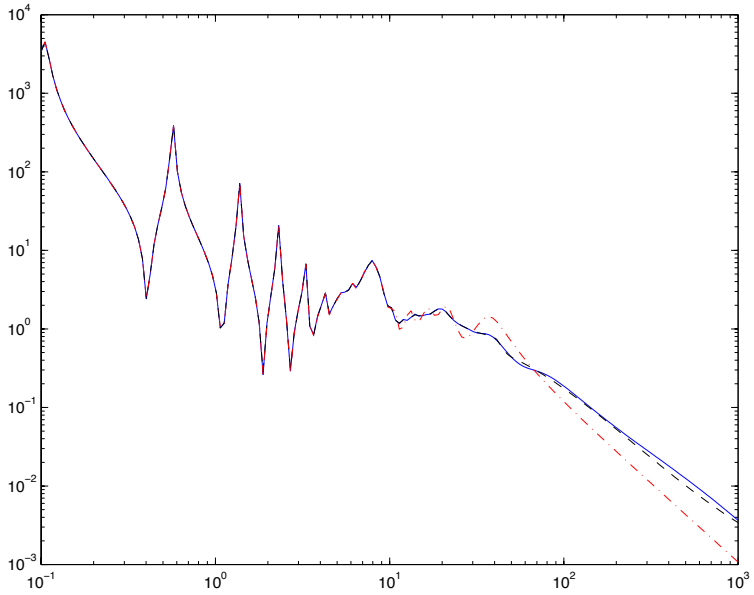

Figure 9: Frequency response
—— original model, - - bt reduced model and $\cdots$ diagG reduced model

4.2 Symmetric case

Sym	N	n	$\frac{\sigma_{2n+1}}{\|\mathcal{H}\|_2}$	$\frac{\|\mathcal{H}-\hat{\mathcal{H}}_{bt}\|_2}{\|\mathcal{H}\|_2}$	$\frac{\|\mathcal{H}-\hat{\mathcal{H}}_{CS}\|_2}{\|\mathcal{H}\|_2}$	$\frac{\|\mathcal{H}-\hat{\mathcal{H}}_{trace}\|_2}{\|\mathcal{H}\|_2}$	$\frac{\|\mathcal{H}-\hat{\mathcal{H}}_{diagG}\|_2}{\|\mathcal{H}\|_2}$	$\frac{\|\mathcal{H}-\hat{\mathcal{H}}_{amod}\|_2}{\|\mathcal{H}\|_2}$
CD	60	6	5.61e-05	8.83e-05	1.82e-03	1.56e-04	1.06e-04	2.02e-02
ISS	135	13	7.749e-04	7.757e-04	7.758e-04	7.758e-04	7.758e-04	2.42e-02

4.2.1 CD player model

N	$m=p$	n	$\sigma_1 = \|\mathcal{H}\|_2$	σ_{2n+1}
60	2	6	2.64e+07	1.48e+03

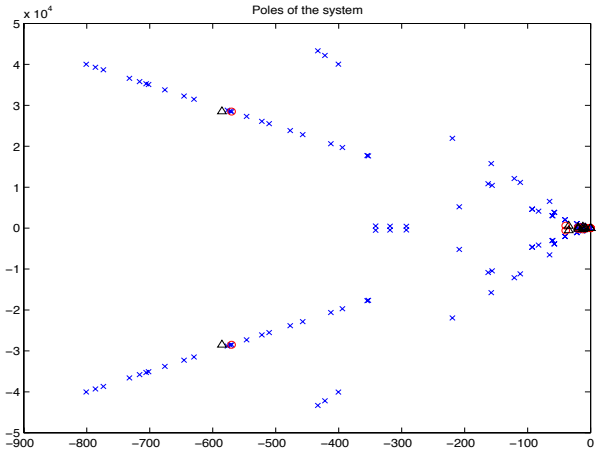

Figure 10: Poles of original model ($\star$), bt reduced model ($\triangle$) and diagG reduced model ($\circ$)

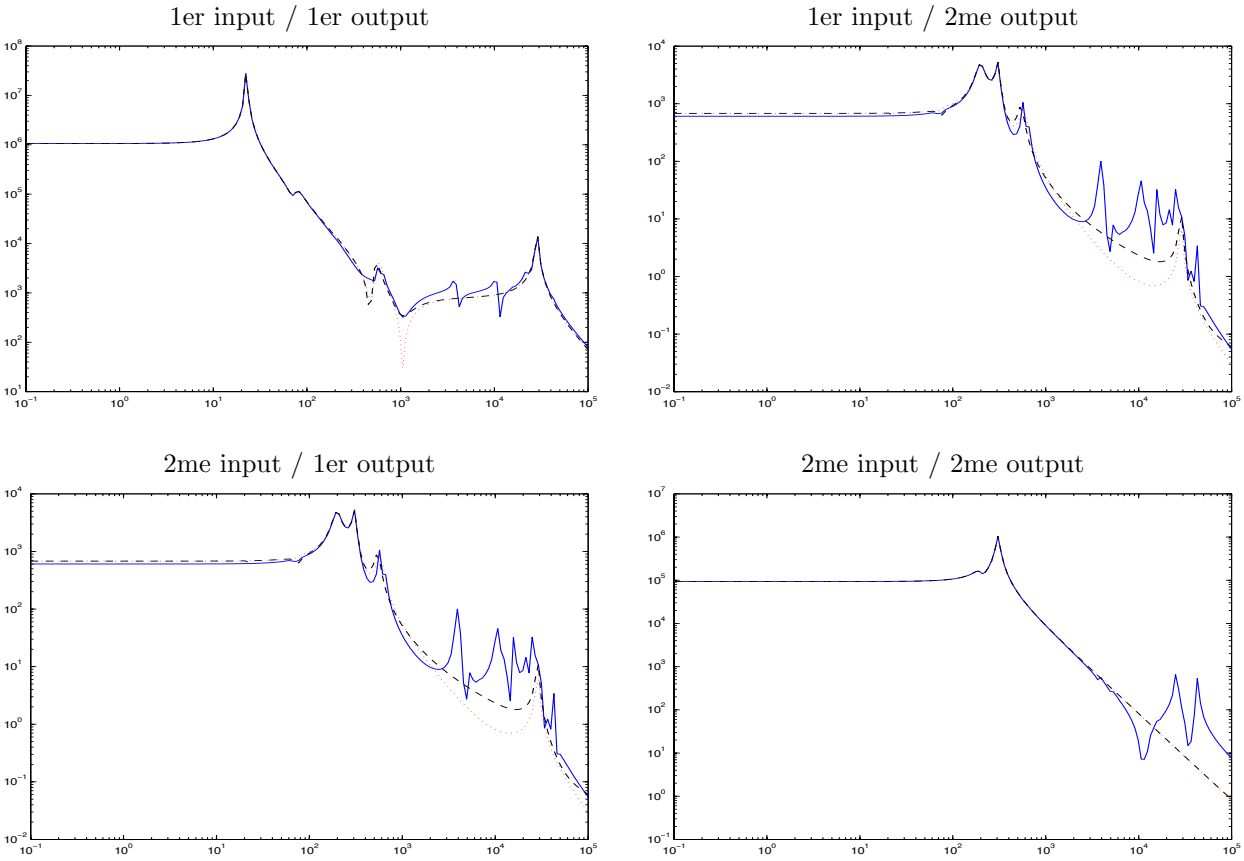

Figure 11: Frequency response
—— original model, - - bt reduced model and $\cdots$ diagG reduced model

4.2.2 ISS model

N	$m = p$	n	$\sigma_1 = \|\mathcal{H}\|_2$	σ_{2n+1}
135	3	13	21.58	1.67e-02

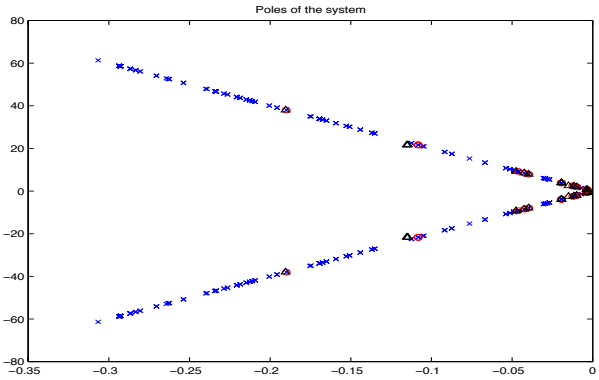

Figure 12: Poles of original model ($\star$),
bt reduced model ($\triangle$)
and diagG reduced model ($\circ$)

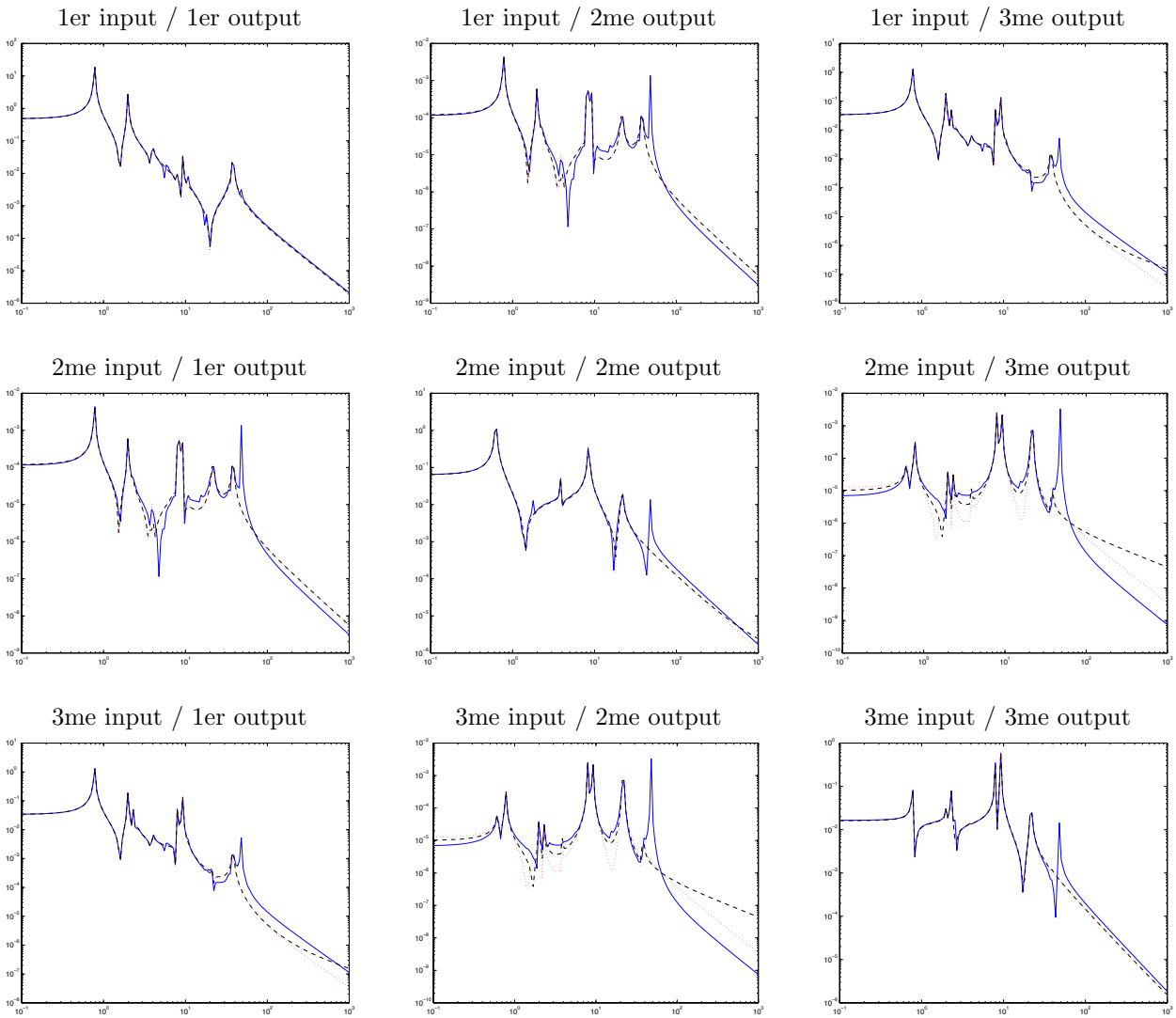

Figure 13: Frequency response
—— original model, - - bt reduced model and $\cdots$ diagG reduced model

4.3 Conclusion

In this paper we propose new model reduction methods which preserve the polynomial form of a given second order system. We also give numerical results to illustrate that even when imposing such restrictions, one still obtains approximation errors which are comparable to those obtained via balanced truncation. The advantage of our approach is that preserving the system structure may better reflect the physical properties of the system we want to approximate.

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