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## Second-order balanced truncation

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### Abstract

We consider second-order linear time-invariant systems. The objective of this paper is to present a new method for constructing a reduced system by preserving the second-order structure of the original system. This new model reduction method uses a variant of the well-known balanced truncation technique applied to *second-order* gramians. We also compare it with another existing technique.

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### 1. Introduction

The objective of this paper is to present a new method for model reduction of second-order linear time-invariant systems of the type:

$$\begin{cases} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Bu(t), \\ y(t) = Cq(t), \end{cases} \quad (1)$$

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where  $u(t) \in \mathbb{R}^m$ ,  $y(t) \in \mathbb{R}^p$ ,  $q(t) \in \mathbb{R}^n$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $M, D, K \in \mathbb{R}^{n \times n}$  with  $M$  assumed to be invertible. Models of this form often come from mechanical systems, where the matrices  $M$ ,  $D$  and  $K$  are respectively called the mass, the damping and the stiffness matrices (in these applications, the matrices are also typically symmetric). The transfer function associated with the system (1) in the Laplace domain is given by

$$T(s) \doteq C \left( Ms^2 + Ds + K \right)^{-1} B, \quad (2)$$

where  $P(s) \doteq Ms^2 + Ds + K$  is the *characteristic polynomial matrix*.

In civil engineering or aeronautics, the size  $n$  of the model (obtained using for instance finite elements techniques [1,2]) is often so high that many analysis and design problems can not be solved anymore within a reasonable computing time. It is then advisable to construct a reduced model of size  $k \ll n$  that nevertheless keeps the “second-order structure” of the system. We thus need to build a reduced model,

$$\begin{cases} \hat{M}\ddot{\hat{q}}(t) + \hat{D}\dot{\hat{q}}(t) + \hat{K}\hat{q}(t) = \hat{B}u(t), \\ \hat{y}(t) = \hat{C}\hat{q}(t), \end{cases} \quad (3)$$

where  $\hat{q}(t) \in \mathbb{R}^k$ ,  $\hat{M}, \hat{D}, \hat{K} \in \mathbb{R}^{k \times k}$ ,  $\hat{B} \in \mathbb{R}^{k \times m}$ ,  $\hat{C} \in \mathbb{R}^{p \times k}$ , such that its transfer function is “close” to the original transfer function.

Since (1) is a particular case of a linear time-invariant system, one may consider its corresponding state-space model and apply the techniques of model reduction known for state-space models. In doing so, the reduced system is generally not of the same type anymore. Since from a physical point of view it makes sense to impose the reduced system to be of the same type, we propose in this paper a new method of model reduction that preserves the second-order form.

The outline of this paper is as follows. In Section 2, the well-known balanced truncation technique for model reduction of linear systems is presented. In Section 3, our model reduction technique called the *second-order balanced truncation* is presented after deriving two new pairs of second-order gramians. In Section 4, our model reduction technique is compared to a previous second-order structure preserving model reduction technique developed in [3]. Finally, concluding remarks are given in Section 6. A preliminary version of this work is available in [4].

## 2. Model reduction by balanced truncation

We consider here linear time-invariant systems modelled by the following system of equations

$$\mathcal{L} \begin{cases} \dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \\ y(t) = \mathcal{C}x(t), \end{cases} \quad u(t) \in \mathbb{R}^m, x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^p \quad (4)$$

which therefore have a transfer function  $T(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B}$  that links the inputs to the outputs in the Laplace domain. Such transfer functions are *strictly proper*,

i.e.  $\lim_{s \rightarrow \infty} T(s) = 0$ , which happens to be the case also for the transfer function considered in (2) since  $M$  was assumed to be invertible.

If  $A$  is Hurwitz, then the system  $\mathcal{S}$  is also a linear (convolution) operator mapping square integrable inputs  $u(\cdot) \in \mathcal{L}_2[-\infty, +\infty]$  to square integrable outputs  $y(\cdot) \in \mathcal{L}_2[-\infty, +\infty]$ . We will need the concept of dual operator which we briefly recall here.

**Definition 1.** Let  $L$  be a linear operator acting from a Hilbert space  $U$  to a Hilbert space  $Y$  equipped respectively with the inner products  $\langle \cdot, \cdot \rangle_U$  and  $\langle \cdot, \cdot \rangle_Y$ . The dual of  $L$ , denoted by  $L^*$ , is defined as the linear operator acting from  $Y$  to  $U$  such that  $\langle Lu, y \rangle_Y = \langle u, L^*y \rangle_U$  for all  $y \in Y$  and all  $u \in U$ .

It is not difficult to verify [5] that the transfer function associated to the dual operator of (4) is  $\mathcal{B}^T(sI - \mathcal{A}^T)^{-1}\mathcal{C}^T$ .

Two matrices are associated with the linear system (4). These are the “controllability gramian”  $\mathcal{P}$  and the “observability gramian”  $\mathcal{Q}$ . If  $\mathcal{A}$  is Hurwitz, they are the unique solutions of the following Lyapunov equations:

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^T + \mathcal{B}\mathcal{B}^T = 0, \quad \mathcal{A}^T\mathcal{Q} + \mathcal{Q}\mathcal{A} + \mathcal{C}^T\mathcal{C} = 0. \tag{5}$$

If we apply an input  $u(\cdot) \in \mathcal{L}_2[-\infty, 0]$  to the system (4) for  $t < 0$ , the position of the state at time  $t = 0$  (by assuming the zero initial condition  $x(-\infty) = 0$ ) is equal to

$$x(0) = \int_{-\infty}^0 e^{-\mathcal{A}t} \mathcal{B}u(t) dt \doteq \mathcal{C}_o u(t).$$

By assuming that a zero input is applied to the system for  $t > 0$ , then for all  $t \geq 0$ , the output  $y(\cdot) \in \mathcal{L}_2[0, +\infty]$  of the system (4) is equal to

$$y(t) = \mathcal{C}e^{\mathcal{A}t}x(0) \doteq \mathcal{O}_b x(0).$$

The so-called controllability operator  $\mathcal{C}_o : \mathcal{L}_2[-\infty, 0] \mapsto \mathbb{R}^n$  (mapping past inputs  $u(\cdot)$  to the present state) and observability operator  $\mathcal{O}_b : \mathbb{R}^n \mapsto \mathcal{L}_2[0, +\infty]$  (mapping the present state to future outputs  $y(\cdot)$ ) also have dual operators, respectively  $\mathcal{C}_o^*$  and  $\mathcal{O}_b^*$ . It is easy to show that the controllability and observability gramians are related to those via the identities  $\mathcal{P} = \mathcal{C}_o^* \mathcal{C}_o$  and  $\mathcal{Q} = \mathcal{O}_b \mathcal{O}_b^*$  [5].

Another physical interpretation of the gramians is the following. The controllability matrix arises from the following optimization problem. Let

$$J(v(t), a, b) \doteq \int_a^b v(t)^T v(t) dt$$

be the energy of the vector function  $v(t)$  in the interval  $[a, b]$ . Then (see [6])

$$\min_{\mathcal{C}_o u(t)=x_0} J(u(t), -\infty, 0) = x_0^T \mathcal{P}^{-1} x_0, \tag{6}$$

and symmetrically, we have the dual property

$$\min_{\mathcal{C}_b^* y(t)=x_0} J(y(t), -\infty, 0) = x_0^T \mathcal{Q}^{-1} x_0. \tag{7}$$

Two essential algebraic properties of gramians  $\mathcal{P}$  and  $\mathcal{Q}$  are as follows. First, under a coordinate transformation  $x(t) = S\bar{x}(t)$ , the new gramians  $\bar{\mathcal{P}}$  and  $\bar{\mathcal{Q}}$  corresponding to the state-space realization  $(\bar{\mathcal{C}}, \bar{\mathcal{A}}, \bar{\mathcal{B}}) = (\mathcal{C}S, S^{-1}\mathcal{A}S, S^{-1}\mathcal{B})$  undergo the following (so-called *contragradient*) transformation:

$$\bar{\mathcal{P}} = S^{-1}\mathcal{P}S^{-T}, \quad \bar{\mathcal{Q}} = S^T\mathcal{Q}S. \tag{8}$$

This implies that the eigenvalues of the product  $\bar{\mathcal{P}}\bar{\mathcal{Q}} = S^{-1}\mathcal{P}\mathcal{Q}S$  depends only on the transfer function  $T(s)$  and not on a particular choice of state-space realization. This implies also that there exists a state-space realization  $(C_{\text{bal}}, A_{\text{bal}}, B_{\text{bal}})$  of  $T(s)$  such that the corresponding gramians are equal and diagonal  $\bar{\mathcal{P}} = \bar{\mathcal{Q}} = \Sigma$  [5]. Secondly, because these gramians appear in the solutions of the optimization problems (6) and (7), they tell something about the energy that goes through the system, and more specifically, about the distribution of this energy among the state variables.

The more  $x_0^T \mathcal{P}^{-1} x_0$  is small, the more “controllable” the state  $x_0$  is, because it can be reached with a input of small energy. By duality, the more  $x_0^T \mathcal{Q}^{-1} x_0$  is small, the more “observable” the state  $x_0$  is. Thus when both gramians are equal and diagonal, the order of magnitude of a diagonal value of the product  $\mathcal{P}\mathcal{Q}$  is a good measure for the influence of the corresponding state variable in the mapping  $y(\cdot) = \mathcal{O}_b \mathcal{C}_o u(\cdot)$  which maps past inputs  $u(t) \in \mathcal{L}_2[-\infty, 0]$  to future outputs  $y(t) \in \mathcal{L}_2[0, +\infty]$  passing via that particular state at time  $t = 0$ .

Given a transfer function  $T(s)$ , the popular balanced truncation model reduction method consists in finding a state-space realization  $(\mathcal{C}_{\text{bal}}, \mathcal{A}_{\text{bal}}, \mathcal{B}_{\text{bal}})$  of  $T(s)$  such that the gramians are equal and diagonal (this is the so-called *balanced realization*) and then the reduced model is constructed by keeping the states corresponding to the largest eigenvalues of the product  $\mathcal{P}\mathcal{Q}$  in it. In other words, the widely used balanced truncation technique chooses  $Z$  and  $V$  such that  $Z^T V = I$ , and

$$\begin{cases} \mathcal{P}\mathcal{Q}V = V\Lambda_+, \\ \mathcal{Q}\mathcal{P}Z = Z\Lambda_+, \end{cases} \tag{9}$$

where  $\Lambda_+$  is a square diagonal matrix containing the largest eigenvalues of  $\mathcal{P}\mathcal{Q}$ . Then a state-space realization of the reduced transfer function is given by  $(\mathcal{C}V, Z^T\mathcal{A}V, Z^T\mathcal{B})$ . The idea of the balanced truncation technique thus consists in keeping those states that are most controllable and observable according to the gramians defined in (6) and (7).

**Remark 2.** The Hankel operator that maps the past input to the future output is defined as follows:  $\mathcal{H} \doteq \mathcal{O}_b \mathcal{C}_o$ . Since  $\mathcal{P}\mathcal{Q} = \mathcal{C}_o \mathcal{C}_o^* \mathcal{O}_b^* \mathcal{O}_b$  and  $\mathcal{Q}\mathcal{P} = \mathcal{O}_b^* \mathcal{O}_b \mathcal{C}_o \mathcal{C}_o^*$ , the dominant eigenspaces  $\mathcal{V}$  of  $\mathcal{P}\mathcal{Q}$  and  $\mathcal{Z}$  of  $\mathcal{Q}\mathcal{P}$  are linked with the dominant eigenspaces  $\mathcal{X}$  of  $\mathcal{H}\mathcal{H}^*$  and  $\mathcal{Y}$  of  $\mathcal{H}^*\mathcal{H}$ : indeed it holds  $\mathcal{X} = \mathcal{O}_b \mathcal{V}$  and  $\mathcal{Y} = \mathcal{C}_o^* \mathcal{Z}$ . Therefore projecting on the spaces  $\mathcal{V}$  and  $\mathcal{Z}$  also approximates the Hankel map  $\mathcal{H}$  well. We refer to [5,6] for a deeper study of the balanced truncation technique.

Second-order systems can be seen as a particular class of linear systems. Indeed, by rewriting the system (1) as follows

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & I \\ -K_M & -D_M \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ B_M \end{bmatrix} u(t), \\ y(t) = [C_M \quad 0] x(t), \end{cases} \tag{10}$$

where the state  $x(t)$  is  $[q(t)^T \quad \dot{q}(t)^T]^T$ , and where we have chosen a coordinate system in which the mass matrix  $M$  is the identity, one recovers the form (4). We can thus rewrite the transfer function defined in (2) as

$$H(s) = \mathcal{C}(sI - \mathcal{A})^{-1} \mathcal{B} \tag{11}$$

by defining

$$\mathcal{A} \doteq \begin{bmatrix} 0 & I \\ -K_M & -D_M \end{bmatrix}, \quad \mathcal{B} \doteq \begin{bmatrix} 0 \\ B_M \end{bmatrix}, \quad \mathcal{C} \doteq [C_M \quad 0]. \tag{12}$$

Unfortunately the classic balanced truncation technique cannot be applied directly to the state-space realization  $(\mathcal{C}, \mathcal{A}, \mathcal{B})$  (10) of the second-order system since, in general, the resulting reduced system is not a second-order system anymore. The objective of this paper is to find a method that performs balanced truncation in some sense *and* preserves the second-order structure of the system.

### 3. Balanced truncation for second-order systems

The idea of our balance and truncate technique for second-order systems (called SOBT for second-order balanced truncation) is the following. First, we need to define *two* pairs of  $n \times n$  gramians (“second-order gramians”) that satisfy the two features we mentioned in the previous sections, i.e. they have to change according to contragradient transformations, and they must have some energetic interpretation (only then a balance and truncate process makes sense). The first pair  $(\mathcal{P}_{\text{pos}}, \mathcal{Q}_{\text{pos}})$  will correspond to an energy optimization problem depending only on the *positions*  $q(t)$  and not on the *velocities*  $\dot{q}(t)$ . Reciprocally, the second pair  $(\mathcal{P}_{\text{vel}}, \mathcal{Q}_{\text{vel}})$  will be associated to an optimization problem depending only on the velocities  $\dot{q}(t)$  and not on the positions  $q(t)$ . By analogy to the first-order case, the gramians  $\mathcal{Q}_{\text{pos}}$  and  $\mathcal{Q}_{\text{vel}}$  will be defined from the dual systems. After these definitions we then come to the balancing part of the method. For this we transform to a balanced coordinate system in which the second-order gramians are equal and diagonal:  $\bar{\mathcal{P}}_{\text{pos}} = \bar{\mathcal{Q}}_{\text{pos}} = \Sigma_{\text{pos}}$ ,  $\bar{\mathcal{P}}_{\text{vel}} = \bar{\mathcal{Q}}_{\text{vel}} = \Sigma_{\text{vel}}$ . Their diagonal values will enable us to point out what the *important* positions and the *important* velocities are, i.e. those with (hopefully) large effect on the I/O map. Hence to get a reduced second-order model we keep only the part of the system that depends on these variables. This is the truncation part of the method.

Let us first define a pair of second-order gramians measuring the contribution of the position coordinates (independently of the velocities) with respect to the I/O map.

A natural optimization problem (see [3]) associated with the second-order form is the following

$$\min_{\dot{q}_0 \in \mathbb{R}^n} \min_{u(t)} J(u(t), -\infty, 0) \tag{13}$$

subject to

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Bu(t), \quad q(0) = q_0.$$

One easily sees that the optimum is  $q_0^T \mathcal{P}_{11}^{-1} q_0$ , where  $\mathcal{P}_{11}$  is the  $n \times n$  left upper block of  $\mathcal{P}$  (see Eq. (5)). Indeed starting with 6 we have to solve

$$\min_{\dot{q}_0 \in \mathbb{R}^n} J_{q_0}(\dot{q}_0) = [q_0^T \quad \dot{q}_0^T] \mathcal{P}^{-1} \begin{bmatrix} q_0 \\ \dot{q}_0 \end{bmatrix}.$$

Partitioning  $\mathcal{P}^{-1}$  as follows

$$\mathcal{P}^{-1} = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}$$

and annihilating the gradient of  $J_{q_0}(\dot{q}_0)$  gives the following relation  $\dot{q}_0 = -R_3^{-1} R_2^T q_0$ . The value of  $J_{q_0}$  at this point is then  $q_0^T (R_1 - R_2 R_3^{-1} R_2^T) q_0$ . This is nothing but the Schur complement of  $R_3$  which is  $\mathcal{P}_{11}^{-1}$ .

The solution of the dual problem will correspond to  $q_0^T \mathcal{Q}_{11}^{-1} q_0$ , where  $\mathcal{Q}_{11}$  is the  $n \times n$  left upper block of  $\mathcal{Q}$  (5). It should be pointed that the transfer function is seen as a linear operator acting between two Hilbert spaces. The dual of such an operator is defined in Definition 1. It follows that the dual of a second-order transfer function might not be a second-order transfer function. This has no consequences because what only matters is the energy transfer interpretation between the inputs, the outputs, the initial positions and velocities. Under the change of coordinates  $q(t) = \Phi \bar{q}(t)$ , it is immediate to verify that this pair of gramians undergo a contragradient transformation:

$$(\bar{\mathcal{P}}_{11}, \bar{\mathcal{Q}}_{11}) = (\Phi^{-1} \mathcal{P}_{11} \Phi^{-T}, \Phi^T \mathcal{Q}_{11} \Phi).$$

This implies that there exists a new coordinate system such that both  $\mathcal{P}_{11}$  and  $\mathcal{Q}_{11}$  are equal end diagonal. Their energetic interpretation is given by looking at the underlying optimization problem. In (13), one minimizes the necessary energy to reach the given position  $q_0$  over all past inputs and initial velocities. Hence these gramians really describe how the I/O energy is distributed among the positions.

Analogously, let us define a pair of second-order gramians that would give the contribution of the velocities with respect to the I/O map. The optimization problem associated is the following

$$\min_{q_0 \in \mathbb{R}^n} \min_{u(t)} J(u(t), -\infty, 0) \tag{14}$$

subject to

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Bu(t), \quad \dot{q}(0) = \dot{q}_0.$$

By exactly following the same reasoning as in [3] for the optimization problem (13), one can show that the solution of (14) is  $\dot{q}_0^T \mathcal{P}_{22}^{-1} \dot{q}_0$ , where  $\mathcal{P}_{22}$  is the  $n \times n$  right lower block of  $\mathcal{P}$ . The solution of the dual problem will correspond to  $\dot{q}_0^T \mathcal{Q}_{22}^{-1} \dot{q}_0$ , where  $\mathcal{Q}_{22}$  is the  $n \times n$  right lower block of  $\mathcal{Q}$ . Again under the change of coordinates  $q(t) = \Phi \bar{q}(t)$  one can check that this pair of gramians performs a contragradient transformation. Here too the energetic interpretation is given by looking at the underlying optimization problem. In (14), one minimizes the necessary energy to reach the given velocity  $\dot{q}_0$  over all past inputs and initial positions. Hence these gramians really describe how the I/O energy is distributed among the velocities.

The conclusion is that these second-order gramians are good candidates for our problem. We make thus the choice:

$$(\mathcal{P}_{\text{pos}}, \mathcal{Q}_{\text{pos}}) = (\mathcal{P}_{11}, \mathcal{Q}_{11}) \quad \text{and} \quad (\mathcal{P}_{\text{vel}}, \mathcal{Q}_{\text{vel}}) = (\mathcal{P}_{22}, \mathcal{Q}_{22}). \tag{15}$$

In the new model reduction technique that we propose here, we want to be able to balance both pairs of second-order gramians at the same time, and this is not possible with a change of coordinates of the type  $q(t) = \Phi \bar{q}(t)$ . For these reasons we work in a state-space context, starting with the system (10). The method SOBT proceeds then as follows:

- (1) Gramians computation: Compute both pairs of second-order gramians  $(\mathcal{P}_{\text{pos}}, \mathcal{Q}_{\text{pos}})$  and  $(\mathcal{P}_{\text{vel}}, \mathcal{Q}_{\text{vel}})$  and put them into block diagonal matrices:

$$\mathcal{X} = \begin{bmatrix} \mathcal{P}_{\text{pos}} & 0 \\ 0 & \mathcal{P}_{\text{vel}} \end{bmatrix}, \quad \mathcal{Y} = \begin{bmatrix} \mathcal{Q}_{\text{pos}} & 0 \\ 0 & \mathcal{Q}_{\text{vel}} \end{bmatrix}.$$

(Notice that these are the block diagonal parts of  $\mathcal{P}$  and  $\mathcal{Q}$ , respectively.)

- (2) Balancing: Compute the contragradient transformation

$$S = \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix}$$

making  $\mathcal{X}$  and  $\mathcal{Y}$  equal and diagonal. The transformed system is then

$$\begin{cases} \dot{\bar{x}}(t) = \begin{bmatrix} \Phi_1^{-1} & 0 \\ 0 & \Phi_2^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ -K_M & -D_M \end{bmatrix} \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 \\ \Phi_2^{-1} B_M \end{bmatrix} u(t), \\ y(t) = [C_M \Phi_1 \quad 0] \bar{x}(t). \end{cases} \tag{16}$$

- (3) Truncation: Partition  $\bar{x}$  as  $\begin{bmatrix} \bar{q}_+^T & \bar{q}_-^T & \dot{\bar{q}}_+^T & \dot{\bar{q}}_-^T \end{bmatrix}^T$  where  $\bar{q}_+$  (resp.  $\dot{\bar{q}}_+$ ) of dimension  $k$  corresponds to the  $k$  largest eigenvalues  $\mathcal{P}_{\text{pos}} \mathcal{Q}_{\text{pos}}$  (resp.  $\mathcal{P}_{\text{vel}} \mathcal{Q}_{\text{vel}}$ ), i.e. to the most controllable *and* observable positions (resp. velocities) with respect to (15), and keep the part of the system (16) that only depends on

the variables  $\bar{q}_+(t)$ ,  $\dot{\bar{q}}_+(t)$ . This yields the following  $k$ -dimensional reduced second-order system<sup>3</sup>  $\{\hat{M}, \hat{D}, \hat{K}, \hat{B}, \hat{C}\}$ :

$$\left\{ I_k, W^{-1} \left[ \Phi_2^{-1} D_M \Phi_2 \right]_{11}, W, W^{-1} \left[ \Phi_2^{-1} K_M \Phi_1 \right]_{11}, W^{-1} \left[ \Phi_2^{-1} B_M \right]_{1\cdot}, [C_M \Phi_1]_{1\cdot} \right\},$$

where  $W^{-1} = \left[ \Phi_1^{-1} \Phi_2 \right]_{11}$  is introduced in order to bring the reduced state matrix  $\hat{A}$  back to a companion form.

**Remark 3.** In practice we do not compute explicitly the state-space realization (16). Instead we compute the following dominant left and right invariant subspaces  $X_1, X_2, Y_1, Y_2 \in \mathbb{R}^{n \times k}$  such that  $Y_1^T X_1 = I_k, Y_2^T X_2 = I_k$  and

$$\mathcal{Q}_{\text{pos}} \mathcal{P}_{\text{pos}} Y_1 = Y_1 A_{\text{pos}}^+, \quad \mathcal{P}_{\text{pos}} \mathcal{Q}_{\text{pos}} X_1 = X_1 A_{\text{pos}}^+, \tag{17}$$

$$\mathcal{Q}_{\text{vel}} \mathcal{P}_{\text{vel}} Y_2 = Y_2 A_{\text{vel}}^+, \quad \mathcal{P}_{\text{vel}} \mathcal{Q}_{\text{vel}} X_2 = X_2 A_{\text{vel}}^+, \tag{18}$$

where  $A_{\text{pos}}^+$  is a  $k \times k$  matrix containing the largest eigenvalues of  $\mathcal{P}_{\text{pos}} \mathcal{Q}_{\text{pos}}$  and  $A_{\text{vel}}^+$  is a  $k \times k$  matrix containing the largest eigenvalues of  $\mathcal{P}_{\text{vel}} \mathcal{Q}_{\text{vel}}$ . Defining  $\tilde{X}_2 \doteq X_2 W$  and  $\tilde{Y}_2^T \doteq W^{-1} Y_2^T$  where  $W^{-1} = (Y_1^T X_2)$ , the reduced second-order model  $\{\hat{M}, \hat{D}, \hat{K}, \hat{B}, \hat{C}\}$  of dimension  $k \ll n$  is then given by

$$\hat{M} = I_k, \quad \hat{D} = \tilde{Y}_2^T D_M \tilde{X}_2, \quad \hat{K} = \tilde{Y}_2^T K_M X_1, \quad \hat{B} = \tilde{Y}_2^T B_M, \quad \hat{C} = C_M X_1.$$

**Remark 4.** This method can easily be extended to  $\ell$ th order linear time-invariant systems. Indeed one can define  $\ell$  pairs of  $n \times n$  gramians exactly in the same way we did for second-order systems, i.e. from optimization problems. The  $i$ th pair contains information about the distribution of the I/O energy among the  $i$ th derivative variables  $q^{(i)}(t)$ . One sees easily that this pair is given by  $(\mathcal{P}_{ii}, \mathcal{Q}_{ii})$ , where  $\mathcal{P}_{ii}$  and  $\mathcal{Q}_{ii}$  are the  $i$ th  $n \times n$  diagonal block of the  $\ell n \times \ell n$  gramians  $\mathcal{P}$  and  $\mathcal{Q}$  (5). Then considering a state-space realization of the system, one balances these  $\ell$  pairs simultaneously using a  $\ell$ -blocks diagonal transformation, in order to be able to determine the important part of each “component”  $q^{(i-1)}(t)$  of the state  $x(t)$ . One obtains then a reduced model by keeping the subsystem that only depends on these variables.

#### 4. Comparison with an existing technique

In this section, we compare our method (SOBT) with the method presented in [3]. This method produces a second-order system and is also inspired from a balanced truncation technique. The main ideas of [3] are the following. First one has to define second-order gramians. To do so the following optimization problems analogous to (6) are proposed. The first problem is

<sup>3</sup> For a square matrix  $A$ ,  $A_{11}$  denotes its left upper  $k \times k$  block, for a rectangular matrix  $B$ ,  $B_{1\cdot}$  (resp.  $B_{\cdot 1}$ ) denotes its  $k$  first rows (resp. columns).

$$\min_{q_0 \in \mathbb{R}^n} \min_{u(t)} J(u(t), -\infty, 0)$$

subject to

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Bu(t), \quad q(0) = q_0$$

for which the optimum is proved to be  $q_0^T \mathcal{P}_{11}^{-1} q_0$ , where  $\mathcal{P}_{11}$  is the  $n \times n$  left upper block of  $\mathcal{P}$ . The second problem is

$$\min_{u(t)} J(u(t), -\infty, 0)$$

subject to

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Bu(t), \quad q(0) = q_0, \quad \dot{q}(0) = 0$$

for which the optimum is proved to be  $q_0^T (S_c(\mathcal{P}_{22}))^{-1} q_0$  where  $S_c(\mathcal{P}_{22})$  is the Schur complement of the  $n \times n$  right bottom block of  $\mathcal{P}$ . Then based on this, two pairs of second-order gramians are defined. The second-order “free velocity” gramians are  $\mathcal{P}_{FV} \doteq \mathcal{P}_{11}$ ;  $\mathcal{Q}_{FV} \doteq \mathcal{Q}_{11}$ , and the second-order “zero velocity” gramians are  $\mathcal{P}_{ZV} \doteq S_c(\mathcal{P}_{22})$ ;  $\mathcal{Q}_{ZV} \doteq S_c(\mathcal{Q}_{22})$ . The reduction process is then given below. We give here only the free velocity reduction method since the zero velocity version follows by analogy.

- (1) By a change of coordinates (preserving the symmetry of the data if any), put the matrix  $M$  equal to the identity. The second-order model is then  $\{I, D_M, K_M, B_M, C_M\}$ .
- (2) Compute the contragradient transformation  $q(t) = \Phi \bar{q}(t)$  such that

$$\Phi^{-1} \mathcal{P}_{FV} \Phi^{-T} = \Sigma_{FV} = \Phi^T \mathcal{Q}_{FV} \Phi,$$

where  $\Sigma_{FV}$  is a positive diagonal matrix with diagonal values sorted in decreasing order. Define  $V \in \mathbb{R}^{n \times k}$  to be the first  $k$  columns of  $\Phi$ .

- (3) The reduced system is then given by

$$\hat{M} = V^T V, \quad \hat{D} = V^T D_M V, \quad \hat{K} = V^T K_M V, \quad \hat{B} = V^T B_M, \quad \hat{C} = C_M V.$$

We point out some drawbacks of this method. From  $\{I, D_M, K_M, B_M, C_M\}$  a balanced realization

$$\{\Phi, D_M \Phi, K_M \Phi, B_M, C_M \Phi\}$$

is first computed with respect to free velocity gramians. Truncation (i.e. selection of  $k \ll n$  coordinates) is performed on the system matrices after multiplication by  $\Phi^T$ ! So the reduced model is given by

$$\left\{ [\Phi^T \Phi]_{11}, [\Phi^T D_M \Phi]_{11}, [\Phi^T K_M \Phi]_{11}, [\Phi^T B_M]_{:,1}, [C_M \Phi]_{:1} \right\}. \tag{19}$$

It is not clear that the above truncation indeed selects the dominant state vectors transferring input energy to output energy. One would rather have expected a left

multiplication by  $\Phi^{-1}$  to normalize the mass matrix to the identity, followed by a truncation:

$$\{I, [\Phi^{-1}D_M\Phi]_{11}, [\Phi^{-1}K_M\Phi]_{11}, [\Phi^{-1}B_M]_{1:}, [C_M\Phi]_{:1}\}. \tag{20}$$

This modified method would be equivalent to our method when imposing  $\Phi_2 = \Phi_1$  (and hence “freezes” one degree of freedom). Obviously the motivation for multiplying the equations by  $\Phi^T$  before truncating is to obtain a reduced model that preserves the symmetry properties of the original system. But a clear motivation from the point of view of the projection error is lacking. A second drawback is that, with the method proposed in [3], one implicitly assumes that influential velocities are also influential positions, and again there is no reason why this should give good results since one does not use all degrees of freedom at hand. The last drawback lies in the definition of the “zero velocity” gramians  $\mathcal{P}_{ZV}$ ,  $\mathcal{Q}_{ZV}$ . Contrarily to the “free velocity” gramians, the definition of  $\mathcal{P}_{ZV}$ ,  $\mathcal{Q}_{ZV}$  has not been well justified from an energetic point of view. As illustrated in the test examples of the next section, we believe that these drawbacks have a negative effect on the approximation error of that approach.

### 5. Numerical examples

In Table 1, we compare four model reduction methods: the classic balanced truncation (‘BT’), the method of [3] (free velocity: ‘FV’, zero velocity: ‘ZV’), and our method: the second-order balanced truncation (‘SOBT’). The comparison is made on the basis of the relative reduction error measured according to the Hankel norm, i.e. the two norm of the “error” between the Hankel maps of the true ( $\mathcal{H}$ ) and approximated system ( $\hat{\mathcal{H}}$ ):

$$\|\mathcal{E}\|_2 / \|\mathcal{H}\|_2 \doteq \|\mathcal{H} - \hat{\mathcal{H}}\|_2 / \|\mathcal{H}\|_2.$$

This is done for six different benchmark models obtained from [7]: the *building* model (‘B’), the *cd-player* model and its symmetric version (‘CD’), the *international space station* model and its symmetric version (‘ISS’), and the *clamped beam* model (‘CB’). Each method corresponds to one column in the table. When a method gives a unstable reduced model, we write ‘UNST’. In the symmetric case, the method [3] seems to

Table 1  
Comparison of the projection error

Mod	$n$	$k$	$m$	$p$	$\ \mathcal{H}\ _2$	$\frac{\ \mathcal{E}_{BT}\ _2}{\ \mathcal{H}\ _2}$	$\frac{\ \mathcal{E}_{FV}\ _2}{\ \mathcal{H}\ _2}$	$\frac{\ \mathcal{E}_{ZV}\ _2}{\ \mathcal{H}\ _2}$	$\frac{\ \mathcal{E}_{SOBT}\ _2}{\ \mathcal{H}\ _2}$
B	24	4	1	1	5.1e-04	9.4e-02	9.9e-01	9.9e-01	7.9e-02
CD	60	6	2	2	1.2e+06	3.2e-06	4.1e-06	8.4e-05	3.6e-06
ISS	135	13	3	3	7.5e-02	3.4e-04	3.4e-04	3.4e-04	3.4e-04
CB	174	17	1	1	2.4e+03	2.9e-05	6.6e-01	UNST	1.8e-04
CD <sub>sym</sub>	60	6	2	2	2.6e+07	8.8e-05	1.1e-04	5.5e-04	1.1e-04
ISS <sub>sym</sub>	135	13	3	3	2.2e+01	7.8e-04	7.8e-04	7.8e-04	7.8e-04

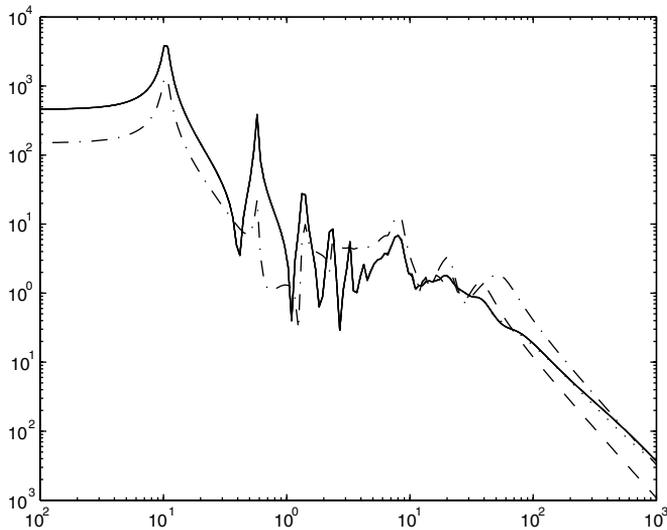


Fig. 1. Amplitude of the frequency response (---: Original model, ···: BT, -·-·: SOBT, - - - -: FV).

work quite well, but in general our new projection method behaves better. Further experiments can also be found in [4] where this method is also compared with three other less efficient techniques, including “modal approximation”.

For the *clamped beam* model, we give Fig. 1 that shows the amplitude of the frequency response of both the original and the reduced systems.

## 6. Concluding remarks

In this paper we have introduced a structure preserving model reduction method for second-order systems. It is a balance and truncate method that is defined based on two pairs of  $n \times n$  gramians ( $\mathcal{P}_{\text{pos}}, \mathcal{Q}_{\text{pos}}$ ), and ( $\mathcal{P}_{\text{vel}}, \mathcal{Q}_{\text{vel}}$ ). Following the idea of [3], these gramians are derived from optimization problems, and are shown to contain information about the capacity of positions  $q_0$  and velocities  $\dot{q}_0$  to transfer energy between inputs and outputs. Working in a state-space model enables us to balance both pairs of gramians simultaneously, which allows to determine the  $k$  most controllable and observable position components and the  $k$  most controllable and observable velocity components, where  $k \ll n$ . The reduced model is then obtained by keeping that part of the balanced system that only depends on these variables, which then automatically gives a reduced model of second-order type.

We have shown that the method in [3] is not a real balance and truncate method, and that it has a few drawbacks. Numerically speaking our method performs better than the method of [3], but for symmetric models, both methods work well.

In a theoretical point of view, many questions remain open. For instance, does the SOBT technique preserve stability? If this turns out to be true, then does there exist a global error bound between the original and the reduced order model depending on the eigenvalues of the product of the gramians that we neglect, as in the case of the standard balanced truncation? If this turns out to be false, does there exist other pairs of gramians that provide a global error bound?

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