A new approach for MOR of second order Dynamical Systems*

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Abstract

We consider a new idea for model reduction of second order dynamical systems. It is based on a new theorem which shows under which conditions one can recover the second order form of a dynamical system. This theorem adds some constraints on the projection matrices that will be used to construct the reduced model.

1 Introduction

We consider second order dynamical systems that govern the motion of large-scale structures

$$M\frac{d^2}{dt^2}q(t) + D\frac{d}{dt}q(t) + Kq(t) = Fu(t), \quad y(t) = Gq(t), \tag{1}$$

with input $u(t) \in \mathbb{R}^m$, state $q(t) \in \mathbb{R}^N$ and output $y(t) \in \mathbb{R}^p$, and $m, p \ll N$. We assume that the matrices M, D, K, F, and G are of appropriate dimensions. The number N is called the state-space dimension of (1), m is the number of inputs, and p is the number of outputs.

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Systems of the form (1) arise in important applications, for example in weather simulation, molecular dynamic simulations (e.g., modeling of bio-molecules and their identification), structural dynamics (e.g., flex models of the international space station, and structural response of high-rise building to wind and earthquakes), electronic circuit simulation (e.g., VLSI circuit simulation), and simulation and control of micro-electro-mechanical (MEMS) devices [1].
The matrix M can be singular such as for example in VLSI circuit simulation, and thus the equations (1) are indeed integro-differential algebraic equations. The only assumption we make here is that the quadratic matrix polynomial

$$Q(s) := s^2M + sD + K$$

is regular.
For systems of very large state-space dimension N, it is inefficient or even prohibitive to solve the original system (1). Furthermore, in applications such as VLSI circuit simulation, equations of the form (1) describe only linear subsystems of a much more complex, in general nonlinear, physical system. In these cases, the linear subsystems are coupled to the equations describing the remainder of the system via their input and output vectors $u(t)$ and $y(t)$. As a result, for the simulation of the complete system, it is usually sufficient to replace each system (1) by an approximate version of much smaller state-space dimension $n \ll N$, provided that the input-output behavior $u(t) \mapsto y(t)$ is well enough approximated. Such approximate versions are called reduced-order models [1, 2].
More precisely, a reduced-order model of (1) of state-space dimension n is a system of the form

$$\hat{M}\frac{d^2}{dt^2}\hat{q}(t) + \hat{D}\frac{d}{dt}\hat{q}(t) + \hat{K}\hat{q}(t) = \hat{F}u(t), \quad \hat{y}(t) = \hat{G}\hat{q}(t), \tag{2}$$

where $\hat{M}$, $\hat{D}$, $\hat{K}$ are $n \times n$ matrices, $\hat{F}$ is an $n \times m$ matrix, and $\hat{G}$ is a $p \times n$ matrix. The challenge of reduced-order modeling is to construct matrices $\hat{M}$, $\hat{D}$, $\hat{K}$, $\hat{F}$, $\hat{G}$ such that the input-output behavior $u(t) \mapsto \hat{y}(t)$ of the reduced-order model (2) is a good approximation of the input-output behavior $u(t) \mapsto y(t)$ of the original system (1).
The standard way of dealing with model order reduction (MOR) of a second order system is to reformulate it as a first order system and then apply model reduction techniques for first order systems [1]. With $x(t) = \begin{bmatrix} q(t) \\ \frac{d}{dt}q(t) \end{bmatrix}$, the first order system is

$$\begin{cases} E\frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{cases} \tag{3}$$

where

$$E = \begin{bmatrix} I_N & 0 \\ 0 & M \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I_N \\ -K & -D \end{bmatrix}, \quad C = \begin{bmatrix} G & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ F \end{bmatrix}. \tag{4}$$

Here $E, A \in \mathbb{R}^{2N\times 2N}$, $B \in \mathbb{R}^{2N\times m}$, and $C \in \mathbb{R}^{p\times 2N}$. A drawback of this approach is that the reduced system is of first order rather than second order.
Second order Balanced Truncation and Krylov subspaces methods that perform model reduction directly for the second order system have been derived. This happens to have many advantages as compared with the transformation to the first order system [2, 3, 5, 6, 9]. Here we consider another approach based on a new theorem which shows under which conditions one can recover a reduced second order form of a dynamical system from a first order reduced form. This theorem adds some constraints on the projection matrices used to construct the reduced model.

2 New Reduction method of second order systems

Most model reduction methods destroy the second order structure of the underlying equations, but recently some adaptations of these methods were proposed, like second order balanced truncation [3, 5] and Krylov subspaces structure preserving model reduction [7, 8, 9].
Basically, the new idea is to impose constraints during the construction of the reduced model in order that the second order structure is preserved. it was first introduced in [8]. It consists of three steps: conversion of the second order model into first order representation, reduction by any method preserving the second order character inside, then back conversion into a second order representation. This conversion is based on the following theorem.
Theorem
The generalized state space system (of state dimension $2n$)

$$\left(\begin{array}{c|c} \lambda\hat{E}-\hat{A} & \hat{B} \\ \hline \hat{C} & 0 \end{array}\right) \tag{5}$$

is system equivalent to the so-called second order form

$$\left(\begin{array}{cc|c} \lambda I_n & -I_n & 0 \\ \hat{K} & \lambda\hat{M}+\hat{D} & \hat{F} \\ \hline \hat{G} & 0 & 0 \end{array}\right) \tag{6}$$

if and only if there exists an $n\times 2n$ matrix R such that

$$\operatorname{rank}\begin{bmatrix} R\hat{E} \\ R\hat{A} \end{bmatrix} = 2n, \quad R\hat{B}=0, \quad \operatorname{rank}\begin{bmatrix} R\hat{E} \\ \hat{C} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} R\hat{E} \end{bmatrix}. \tag{7}$$

Proof. The only if part is trivial since if both generalized state space systems are equivalent, then there exist invertible matrices S and T such that

$$\left[\begin{array}{c|c} S(\lambda\hat{E}-\hat{A})T & S\hat{B} \\ \hline \hat{C}T & 0 \end{array}\right] = \left[\begin{array}{cc|c} \lambda I_n & -I_n & 0 \\ \hat{K} & \lambda\hat{M}+\hat{D} & \hat{F} \\ \hline \hat{G} & 0 & 0 \end{array}\right].$$

But then clearly the matrix R made from the first n rows of S satisfies

$$\begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} R\hat{E} \\ R\hat{A} \end{bmatrix} T = I_{2n}, \quad R\hat{B} = 0, \quad \begin{bmatrix} R\hat{E} \\ \hat{C} \end{bmatrix} T = \begin{bmatrix} I_n & 0 \\ \hat{G} & 0 \end{bmatrix} \tag{8}$$

which clearly satisfies (7). The if part follows the converse reasoning. Construct the inverse of the matrix T from the given matrix R, and partition it as follows

$$T^{-1} := \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} R\hat{E} \\ R\hat{A} \end{bmatrix}, \quad \left[\, T_1 \mid T_2 \,\right] := T.$$

Now choose $S_1 = R$ and S_2 such that $S_2\hat{E}T_1 = 0$ where

$$S := \begin{bmatrix} S_1 \\ S_2 \end{bmatrix},$$

is of full rank. This can always be done since $S_1\hat{E}T_1 = I_n$ implies that none of the rows of S_1 are orthogonal to $\hat{E}T_1$ while $S_2\hat{E}T_1 = 0$ implies that all the rows of S_2 are orthogonal to $\hat{E}T_1$. We now obtain with this construction the required equivalence (7) by putting

$$\hat{G} := \hat{C}T_1, \quad \hat{F} := S_2\hat{B}, \quad \hat{K} := S_2\hat{A}T_1, \quad \hat{M} := S_2\hat{E}T_2, \quad \hat{D} := S_2\hat{A}T_2.$$

$\square$

Notice that we have not assumed any special properties of the systems (5) and (6). It easily follows that

1. The system (6) is regular if and only if the system (5) is regular
2. The system (6) is minimal if and only if the system (5) is minimal
3. The matrix M in (6) is nonsingular if and only if the matrix $\hat{E}$ in (5) is nonsingular

since equivalence transformations do not change these properties. Moreover, if the system is regular, then the transfer function is also given by

$$G(\lambda) = \hat{G}(\lambda^2\hat{M} + \lambda\hat{D} + \hat{K})^{-1}\hat{F}.$$

In this talk we will present this method and show how to construct the matrix R in the theorem. We will discuss the limitations, the advantages and drawbacks of a such method.

References

[1] Antoulas A.C. *Approximation of Large-Scale Dynamical Systems.* SIAM, Philadelphia, (2005).

[2] Chahlaoui Y. *Low-rank approximation and model reduction. PhD thesis number 14/2003*, Louvain-la-Neuve: UCL, (2003).

[3] Chahlaoui Y. and Gallivan K. and Vandendorpe A. and Van Dooren P. *Model reduction of second order systems.* Model Reduction of Dynamical Systems, Eds. P. Benner et al., LNCSE Vol. **45**, Springer Verlag, (2005), 149 - 172.

[4] Chahlaoui Y. and Lemonnier D. and Meerbergen K and Vandendorpe A and Van Dooren P.*Model reduction of second order systems,* International Symposium Math. Th. Netw. Syst., Paper 26984, Univ. Notre Dame, USA, (2002).

[5] Chahlaoui Y. and Lemonnier D. and Vandendorpe A. and Van Dooren P. *Second Order Balanced Truncation.* Lin. Alg. Appl., **415**, (2006), 373 - 384.

[6] Chahlaoui Y. and Lemonnier D. and Vandendorpe A. and Van Dooren P. *Second Order Structure Preserving Balanced Truncation.* International Symposium Math. Th. Netw. Syst., Leuven, (2004).

[7] Meyer D. and Srinivasan S. *Balancing and model reduction for second-order form linar systems.* IEEE Trans. Automat. Control, **41** (11), (1996) 1632 – 1644.

[8] Salimbahrami B. and Eid R. and Lohmann B. and Rudnyi E.B. and Korvink J.G. *Order Reduction of Second Order Systems with Proportional Damping.* ALA 2006, Joint GAMM-SIAM Conference on Applied Linear Algebra, University of Düsseldorf, Germany.

[9] Chahlaoui Y. *Model reduction of linear systems, an interpolation point of view. PhD thesis number 51/2004*, Louvain-la-Neuve: UCL, (2004).