GENERALIZATIONS OF THE SINGULAR VALUE AND QR DECOMPOSITION *

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We dedicate this paper to Gene Golub,
a true source of inspiration for our work,
but also a genuine friend,
at the occasion of his sixtieth birthday.

Abstract. In this paper, we discuss multi-matrix generalizations of two well-known orthogonal rank factorizations of a matrix: the generalized singular value decomposition and the generalized QR-(or URV-) decomposition. These generalizations can be obtained for any number of matrices of compatible dimensions. We discuss in detail the structure of these generalizations and their mutual relations and give a constructive proof for the generalized QR-decompositions.

Key words. singular value decomposition, QR-factorization, URV-decomposition, complete orthogonal decomposition

1. Introduction. In this paper, we present *multi-matrix generalizations* of some well-known orthogonal rank factorizations. We will show how the idea of a QR-decomposition (**QRD**), a URV-decomposition (**URVD**) and a singular value decomposition (**SVD**) for one matrix can be generalized to any number of matrices. While generalizations of the **SVD** for any number of matrices have been derived in [9], one of the main contributions of this paper is the constructive derivation of a generalization for the **QRD** (or **URD**) for any number of matrices of compatible dimensions. The idea is to reduce the set of matrices $A_1, A_2, \ldots, A_k$ to a simpler form using unitary transformations only. Hereby, we avoid explicit products and inverses of the matrices that are involved. We will show that these *generalized QR-decompositions* (**GQRD**) can be considered as a preliminary reduction for any *generalized singular value decomposition* (**GSVD**). The reason is that there is a certain one-to-one relation between the structure of a **GQRD** and the "corresponding" **GSVD**, which will be explained in detail.

This paper is organised as follows. In section 2, we provide a summary of *orthogonal rank factorizations* for one matrix. We briefly review the **SVD**, the **QRD** and the **URVD** as special cases. In section 3, we give a survey of existing generalizations of the **SVD** and **QRD** for two or three matrices. In section 4, we summarize the results on **GSVD**s for any number of matrices of compatible dimensions. Section 5, which contains the main new contribution of this paper, describes a generalization of the **QRD** and the **URVD** for any number of matrices. We derive a constructive, inductive proof which shows that a **GQRD** can be used as a preliminary reduction for a corresponding **GSVD**. In section 6, we analyse in detail the structure of the **GQRD**s and **GSVD**s and show that there is a one-to-one relation between the two generalizations. This relation is elaborated in more detail in section 7, where we illustrate how a **GQRD** can be used as a preliminary step in the derivation of a corresponding **GSVD**.

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While all results in this paper are stated for complex matrices, they can be specialized to the real case without much difficulty. This can be done in much the same was as with the **SVD** for complex and real matrices. In particular, it suffices to restate most results using the word *real orthonormal* instead of *unitary* and to replace a superscript "$*$" (which denotes the complex conjugate transpose of a matrix) by a superscript "t" (which is the transpose of a matrix).

2. Orthogonal rank factorizations . Any matrix $A \in C^{m\times n}$ can be factorized as

$$A = Q \begin{pmatrix} R \\ 0 \end{pmatrix} \Pi \tag{1}$$

where $R \in C^{n\times n}$ is upper trapezoidal and Π is a real $n \times n$ permutation matrix that permutes the columns of A so that the first $r_a = rank(A)$ columns are linearly independent. The matrix $Q \in C^{m\times m}$ is unitary and can be partitioned as

$$Q = \begin{array}{c} \begin{array}{cc} r_a & m - r_a \end{array} \\ \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \end{array}$$

If we partition R accordingly as $R = \begin{pmatrix} R_{11} & R_{12} \end{pmatrix}$ where $R_{11} \in C^{r_a\times r_a}$ is upper triangular and non-singular, we obtain

$$A = Q_{11}\begin{pmatrix} R_{11} & R_{12} \end{pmatrix}\Pi$$

which is sometimes called *the QR-factorization* of A.
If we rewrite (1) as

$$Q^* A = \begin{pmatrix} R \\ 0 \end{pmatrix} \Pi$$

we see that Q is an orthogonal transformation that compresses the rows of A. Therefore, it is called a *row compression.* A similar construction exists of course for a *column compression.*
An *complete orthogonal factorization* of an $m \times n$ matrix A is any factorization of the form

$$A = U \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} V^* \tag{2}$$

where T is $r_a \times r_a$ square nonsingular and $r_a = rank(A)$. One particular case is the *singular value decomposition* (**SVD**), which has become an important tool in the analysis and numerical solution of numerous problems, especially since the development of numerically robust algorithms by Gene Golub and his coworkers [15] [16] [17]. The **SVD** is a complete orthogonal factorization where the matrix T is diagonal with positive diagonal elements:

$$A = U\Sigma V^*$$

Here $U \in C^{m \times m}$ and $V \in C^{n \times n}$ are unitary and $\Sigma \in \Re^{m \times n}$ is of the form [1]

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & \dots & 0 & 0 \\ 0 & \sigma_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma_{r_a} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

The positive numbers $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{r_a} > 0$ are called the *singular values* of A while the columns of U and V are the *left* and *right singular vectors.*
In applications where $m \gg n$, it is often a good idea to use the **QRD** of the matrix as a *preliminary step* in the computation of its **SVD**. The **SVD** of A is obtained via the **SVD** of its triangular factor as:

$$A = QR = Q(U_r \Sigma_r V_r^*) = (QU_r) \Sigma_r V_r^*$$

This idea of combining the **QRD** and the **SVD** of the triangular matrix, in order to compute the **SVD** of the full matrix, is mentioned in [22, p.119] and was more fully analyzed in [3]. In [18] the method is referred to as R-bidiagonalization. Its flop count is (mn^2+n^3) as compared to $(2mn^2-2/3n^3)$ for a bidiagonalization of the full matrix. Hence, whenever $m \geq 5/3n$, it is more advantageous to use the R-bidiagonalization algorithm.
There exist still other complete orthogonal factorizations of the form (2) where only T is required to be triangular (upper or lower) (see e.g. [18]). Such a factorization has been called an *URV-decomposition* in [27]. Here

$$A = U \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} V^*$$

with $U \in C^{m \times m}$, $V \in C^{n \times n}$ are unitary matrices and $R \in C^{r_a \times r_a}$ is square nonsingular upper triangular.

It is well known that the QR-factorization of a singular matrix A and of its transpose A^* can be used for finding the image and kernel of A (URV-decompositions actually give both at once). In this paper, we try to extend these ideas to several matrices. Suppose we have a sequence of matrices $A_i, i = 1, \ldots, k$ and we want to know the kernels (or null spaces) of each partial product $A_1 \cdot A_2 \ldots A_j$. Then one could form these products and compute QR-decompositions of each of them. That can in fact be avoided as shown in the sequel. Let us take the "special" example $A_i = A,\ i = 1, 2, 3$ with :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is well known that the null spaces of A^i in fact give the Jordan structure of A and this structure is already obvious from the form of A. But let us reconstruct it from a

[1] In this paper, we use the convention that zero blocks may be "empty" matrices, i.e. certain block dimensions may be 0.

sequence of QR-decompositions (in fact we need here RQ-decompositions of A). The first one is of course a column compression of A_1, for which we use the permutation of columns 2 and 4 (denoted by the matrix P_{24}) :

$$A_1 P_{24} = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The separation line here indicates that the first 2 columns of P_{24} (i.e. e_1 and e_4) span the kernel of $A = A_1$. For the kernel of $A^2 = A_1 A_2$ we do not form this product but apply the inverse of the orthogonal transform P_{24} (which is again P_{24}) to the rows of $A_2 = A$:

$$P_{24} A_2 = \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since $A_1 A_2 = (A_1 P_{24})(P_{24} A_2)$ it is clear that the kernel of $A_1 A_2$ will also be the kernel of the bottom part of $P_{24} A_2$. The following column compression of $P_{24} A_2$ actually yields both the kernel of A_2 and of the product $A_1 A_2$. Perform indeed the orthogonal transformation $P_{24} P_{35}$:

$$P_{24} A_2 P_{24} P_{35} = \left[\begin{array}{cc|cc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

One sees that the kernel of A_2 are the first 2 columns of $P_{24} P_{35}$ (i.e. e_1 and e_4 as before) and the kernel of $A_1 A_2$ are the first 4 columns of $P_{24} P_{35}$, i.e. e_1, e_2, e_4 and e_5. An additional step of this procedure will finally show that the kernel of the product $A_1 A_2 A_3 = (A_1 P_{24})(P_{24} A_2 P_{24} P_{35})(P_{35} P_{24} A_3)$ is that of the bottom part of the matrix :

$$P_{35} P_{24} A_3 = \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and that is a zero matrix. Hence the kernel of A^3 is the whole space as expected. The interesting part of this simple example is the fact that we have not formed the intermediate products to get their corresponding kernels. The case treated here of equal matrices A_i is a simple one (and could be solved using e.g. the results of [19]) but in the next few sections we show how this can also be done for arbitrary sequences of matrices. The key idea is that at each step we do a number of QR-factorizations on the blocks of a partitioned matrix (column blocks in our case). This then induces a new partitioning on the rows of this matrix and on the columns of the next matrix and so on.

3. Generalizations for two or three matrices . In the last decade or so, several generalizations for the **SVD** have been derived. The motivation is basically the necessity to avoid the explicit formation of products and matrix quotients in the computation of the **SVD** of products and quotients of matrices. Let A and B be nonsingular square matrices and assume that we need the **SVD** of [2] $AB^{-*} = USV^*$. It is well known that the explicit calculation of B^{-1} followed by the computation of the product, may result in loss of numerical precision (digit cancellation), even before any factorization is attempted! The reason is the finite machine precision of any calculator. Therefore, it seems more appropriate to come up with an implicit combined factorization of A and B separately, such as

$$\begin{aligned} A &= UD_1X^{-1} \\ B &= X^{-*}D_2V^* \end{aligned} \tag{3}$$

where U and V are unitary and X nonsingular. The matrices D_1 and D_2 are real but "sparse" (*quasi-diagonal* as we will call them), and the product $D_1D_2^{-t}$ is diagonal with positive diagonal elements. Then we find

$$AB^{-*} = UD_1X^{-1}XD_2^{-t}V^* = U(D_1D_2^{-t})V^*$$

A factorization as in (3) is always possible for two square non-singular matrices. As a matter of fact, it is always possible for two matrices $A \in C^{m\times n}$ and $B \in C^{n\times p}$ (as long as the number of columns of A is the same as the number of rows of B, which we will refer to as a *compatibility condition*). In general, the matrices A and B may even be *rank deficient.* The combined factorization (3) is called the *quotient singular value decomposition* (**QSVD**) and was first suggested in [32] and refined in [23] (originally it was called the generalized SVD but we have suggested a standardized nomenclature in [6]).

A similar idea might be exploited for the **SVD** of the product of two matrices $AB = USV^*$, via the so-called *product singular value decomposition* (**PSVD**):

$$\begin{aligned} A &= UD_1X^{-1} \\ B &= XD_2V^* \end{aligned} \tag{4}$$

so that

$$AB = U(D_1D_2)V^*$$

which is an **SVD** of AB. The combined factorization (4) was proposed in [13] as a formalization of ideas in [21]. In the general case, for two compatible matrices A and B (that may be rank deficient), the **PSVD** as in (4) always exists and provides the **SVD** of AB without the explicit construction of the product. Similarly, if A and B are compatible, the **QSVD** always exists. However, it does not always deliver the **SVD** of $AB^\dagger$ when B is rank deficient ($B^\dagger$ is the pseudo-inverse of B).

Another generalization, this time for three matrices, is the *restricted singular value decomposition* (**RSVD**). It was proposed in [35] while numerous applications are reviewed in [7]. Soon after this it was found that all of these generalized **SVD**s for two or three matrices are special cases of a general theorem, presented in [9]. The main

[2] The notation B^{-*} refers to the complex conjugate transpose of the inverse of the matrix B.

result is that there exist *generalized singular value decompositions* (**GSVD**) for any number of matrices $A_1, A_2, \ldots, A_k$ of compatible dimensions. The general structure of these **GSVD**s was further analysed in [10]. The dimensions of the blocks that occur in any **GSVD** can be expressed as ranks of the matrices involved and certain products and concatenations of these. We will present a summary of the results below.

As for generalizations of the **QRD**, it is mainly Paige in [25] who pointed out the importance of generalized **QRD**s for two matrices as a basic conceptual and mathematical tool. The motivation is that in some applications, one needs the **QRD** of a product of two matrices AB where $A \in \Re^{m\times n}$ and $B \in \Re^{n\times p}$. For general matrices A and B such a computation avoids forming the product explicitly and transforms A and B separately to obtain the desired results. Paige [25] refers to such a factorization as a *product QR factorization*. Similarly, in some applications one needs the QR-factorization of AB^{-1} where B is square and nonsingular. A general numerically robust algorithm would not compute the inverse of B nor the product explicitly, but would transform A and B separately. Paige in [25] proposed to call such a combined decomposition of two matrices a *generalized QR factorization*, following [20]. We propose here to reserve the name *generalized QRD* for the *complete* set of generalizations of the QR-decompositions, which will be developed in this paper. We will also propose a novel nomenclature in a similar way as we have done for the generalizations of the **SVD** in [6].
Apparently, Stoer [28] appears to be the first to have given a reliable computation of this type of generalized QR-factorization for two matrices (see [14]). Computational methods for producing the two types of generalized QR factorizations for two matrices as described above, have appeared regularly in the literature as (intermediate) steps in the solution of some problems. In this paper, we will derive a constructive proof of generalizations of the **QRD** for any number of matrices. As we will see below, our generalized **QRD**s can also be considered as the appropriate generalization of the URV-decomposition of a matrix.

4. Generalized singular value decompositions. In this section, we present a general theorem which can be considered as the appropriate generalization for any number of matrices of the **SVD** of one matrix. It contains the existing generalizations of the **SVD** for two (i.e. the **PSVD** and the **QSVD**) and three matrices (i.e. the **RSVD**) as special cases. A constructive proof can be found in [9].

THEOREM 4.1. ***Generalized Singular Value Decompositions for k matrices.*** *Consider a set of k matrices with compatible dimensions: A_1 $(n_0 \times n_1)$, A_2 $(n_1 \times n_2)$, ..., A_{k-1} $(n_{k-2} \times n_{k-1})$, A_k $(n_{k-1} \times n_k)$. Then there exist*

- *Unitary matrices U_1 $(n_0 \times n_0)$ and V_k $(n_k \times n_k)$*

- *Matrices $D_j, j = 1, 2, \ldots, (k-1)$ of the form*

$$
(5) \qquad \underset{n_{j-1}\times n_j}{D_j} = \begin{array}{c|c}
 & \begin{array}{ccccccc} r_j^1 & r_j^2 & r_j^3 & \ldots & \ldots & r_j^j & n_j - r_j \end{array} \\
\begin{array}{l} r_j^1 \\ r_{j-1}^1 - r_j^1 \\ r_j^2 \\ r_{j-1}^2 - r_j^2 \\ r_j^3 \\ \ldots \\ \ldots \\ r_j^j \\ n_{j-1} - r_{j-1} - r_j^j \end{array} &
\begin{pmatrix}
I & 0 & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\
0 & I & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & I & \ldots & \ldots & 0 & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & I & 0 \\
0 & 0 & \ldots & \ldots & \ldots & 0 & 0
\end{pmatrix}
\end{array}
$$

where

$$
(6) \qquad r_0 = 0, \qquad r_j = \sum_{i=1}^{j} r_j^i = rank(A_j)
$$

- A matrix S_k of the form

$$
(7) \qquad \underset{n_{k-1}\times n_k}{S_k} = \begin{array}{c|c}
 & \begin{array}{ccccccc} r_k^1 & r_k^2 & r_k^3 & \ldots & \ldots & r_k^k & n_k - r_k \end{array} \\
\begin{array}{l} r_k^1 \\ r_{k-1}^1 - r_k^1 \\ r_k^2 \\ r_{k-1}^2 - r_k^2 \\ r_k^3 \\ \ldots \\ \ldots \\ r_k^k \\ n_{k-1} - r_{k-1} - r_k^k \end{array} &
\begin{pmatrix}
S_k^1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\
0 & S_k^2 & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & S_k^3 & \ldots & \ldots & 0 & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & S_k^k & 0 \\
0 & 0 & \ldots & \ldots & \ldots & 0 & 0
\end{pmatrix}
\end{array}
$$

where

$$
(8) \qquad r_k = \sum_{i=1}^{k} r_k^i = rank(A_k)
$$

and the $r_k^i \times r_k^i$ matrices S_k^i are diagonal with positive diagonal elements. Expressions for the integers r_j^i are given in section 6 [3].

- Nonsingular matrices X_j $(n_j \times n_j)$ and Z_j, $j = 1, 2, \ldots, (k-1)$ where Z_j is either $Z_j = X_j^{-*}$ or either $Z_j = X_j$ (i.e. both choices are always possible).

such that the given matrices can be factorized as

$$
\begin{array}{rcl}
A_1 & = & U_1 D_1 X_1^{-1} \\
A_2 & = & Z_1 D_2 X_2^{-1} \\
A_3 & = & Z_2 D_3 X_3^{-1} \\
\ldots & = & \ldots \\
A_i & = & Z_{i-1} D_i X_i^{-1} \\
\ldots & = & \ldots \\
A_k & = & Z_{k-1} S_k V_k^*
\end{array}
$$

Observe that the matrices D_j in (5) and S_k in (7) are in general not diagonal. Their

[3] In [9], these block dimensions follow from the constructive proof

only non-zero blocks however are diagonal block matrices. We propose to label them as *quasi-diagonal* matrices. The matrices $D_j, j = 1, \ldots, k-1$ are quasi-diagonal, their only nonzero blocks being identity matrices. The matrix S_k is quasi-diagonal and its nonzero blocks are diagonal matrices with positive diagonal elements. Observe that we always take the last factor in every factorization as the inverse of a nonsingular matrix, which is only a matter of convention (Another convention would result in a modified definition of the matrices Z_i). As to the name of a certain **GSVD**, we propose to adopt the following convention (see also [9]):

Definition 4.2. The nomenclature for GSVDs. If $k = 1$ in Theorem 1, then the corresponding factorization of the matrix A_1 will be called the (ordinary) singular value decomposition.
If for a matrix pair $A_i, A_{i+1}, 1 \leq i \leq k-1$ in Theorem 1, we have

$$Z_i = X_i$$

then, the factorization of the pair will be said to be of P-type.
If on the other hand, for a matrix pair $A_i, A_{i+1}, 1 \leq i \leq k-1$ in Theorem 1, we have

$$Z_i = X_i^{-*}$$

the factorization of the pair will be said to be of Q-type.
The name of a **GSVD** of the matrices A_i, $i = 1, 2, \ldots k > 1$ as in Theorem 1, is then obtained by simply enumerating the different factorization types. Let us give some examples.
Example:
Consider two matrices A_1 $(n_0 \times n_1)$ and A_2 $(n_1 \times n_2)$. Then, we have two possible **GSVD**s:

	P-type	Q-type
A_1	$U_1 D_1 X_1^{-1}$	$U_1 D_1 X_1^{-1}$
A_2	$X_1 S_2 V_2^*$	$X_1^{-*} S_2 V_2^*$

The P-type factorization is called the *Product Singular Value Decomposition* (**PSVD**) (see [8] and references therein), while the Q-type factorization is called the **QSVD**.
Example:
Let us write down a **PQQP-SVD** for 5 matrices:

$$\begin{aligned} A_1 &= U_1 D_1 X_1^{-1} \\ A_2 &= X_1 D_2 X_2^{-1} \\ A_3 &= X_2^{-*} D_3 X_3^{-1} \\ A_4 &= X_3^{-*} D_4 X_4^{-1} \\ A_5 &= X_4 S_5 V_5^* \end{aligned}$$

We also introduce a notation using powers which symbolize a certain repetition of a letter or of a sequence of letters:

- $P^3 Q^2$**-SVD = PPPQQ-SVD**
- $(PQ)^2 Q^3 (PPQ)^2$**-SVD = PQPQQQQPPQPPQ-SVD**

Despite the fact that there are 2^{k-1} different sequences of letters **P** and **Q** at level $k > 1$, not all of these sequences correspond to different **GSVD**s. The reason for this

is that for instance the **QP-SVD** of (A^1, A^2, A^3) can be obtained from the **PQ-SVD** of $((A^3)^*, (A^2)^*, (A^1)^*)$. Similarly, the $P^2(QP)^3$**-SVD** of $(A^1, \ldots, A^9)$ is essentially the same as the $(PQ)^3P^2$**-SVD** of $((A^9)^*, \ldots, (A^1)^*)$. The number of *different* factorizations for k matrices is in fact $\frac{1}{2}(2^{k-1} + 2^{k/2})$ for k even and $\frac{1}{2}(2^{k-1} + 2^{(k-1)/2})$ for k odd.
A possible way to visualize Theorem 1 is to build a tree with all *different* factorizations for 1, 2, 3, etc ... matrices as follows:

$$\begin{array}{ccccccccccccc}
1 & & & & & & O & & & & & & \\
2 & & & & & P & & Q & & & & & \\
3 & & & & P^2 & & PQ & & Q^2 & & & & \\
4 & & P^3 & & P^2Q & PQP & & PQ^2 & & QPQ & & Q^3 & \\
\ldots & \ldots & & \ldots & \ldots & & \ldots & & \ldots & & \ldots & & \ldots
\end{array}$$

5. Generalized *URV*-decompositions . In this section, we derive a generalization for several matrices, of the *URV*-decomposition of one matrix. We will proceed in several stages. First, we show how k matrices can be reduced to block triangular matrices using unitary transformations only. Next it is shown how the block triangular factors can be triangularized further to triangular factors.

THEOREM 5.1. *Given k complex matrices A_1 $(n_0 \times n_1)$, A_2 $(n_1 \times n_2)$, ..., A_k $(n_{k-1} \times n_k)$. There always exist unitary matrices $Q_0, Q_1, \ldots, Q_k$ such that*

$$T_i = Q_{i-1}^* A_i Q_i$$

where T_i is a block lower triangular or block upper triangular matrix (both cases are always possible) with the following structure:

- *Lower block triangular (which will be denoted by a superscript l):*

$$(9) \qquad T_i^l = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{cccccc} r_i^1 & r_i^2 & \ldots & r_i^{i-1} & r_i^i & r_i^{i+1} \end{array} \\ \begin{pmatrix} T_{i,1} & 0 & \ldots & 0 & 0 & 0 \\ * & T_{i,2} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots & * & T_{i,i} & 0 \end{pmatrix} \end{array}$$

- *Upper block triangular (which will be denoted by a superscript u):*

$$(10) \qquad T_i^u = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{cccccc} r_i^1 & r_i^2 & \ldots & r_i^{i-1} & r_i^i & r_i^{i+1} \end{array} \\ \begin{pmatrix} T_{i,1} & * & \ldots & * & * & 0 \\ 0 & T_{i,2} & \ldots & * & * & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & 0 & T_{i,i} & 0 \end{pmatrix} \end{array}$$

where $T_{i,j}$, $j = 1, \ldots, i$ are full column rank matrices and each $*$ represents a nonzero block. The block dimensions coincide with those of Theorem 1. In particular,

$$\begin{aligned} r_0^1 &= n_0 \\ r_i^{i+1} &= nullity(A_i) = n_i - r_i \end{aligned}$$

and

$$\begin{aligned}\sum_{j=1}^{i} r_i^j &= rank(A_i) = r_i \\ \sum_{j=1}^{i} r_{i-1}^j &= n_{i-1}\end{aligned}$$

Our proof of Theorem 2 is inductive: We will obtain the required factorization of A_i from that of A_{i-1}.

Proof. The induction is initialized for $i = 1$ as follows. First take the case where T_1 is to be lower block triangular. Use a unitary column compression matrix Q_1 to reduce the matrix A_1 to

$$T_1^l = A_1 Q_1 = \begin{array}{cc} & \begin{array}{cc} r_1^1 & r_1^2 \end{array} \\ r_0^1 & \begin{pmatrix} T_{1,1} & 0 \end{pmatrix} \end{array}$$

where

$$\begin{aligned} r_1^1 &= rank(T_{1,1}) = rank(A_1) \\ r_1^2 &= nullity(A_1) = n_1 - r_1 \end{aligned}$$

and

$$r_0^1 = n_0$$

The case where T_1 is required to be upper block triangular is similar:

$$T_1^u = A_1 Q_1 = \begin{array}{cc} & \begin{array}{cc} r_1^2 & r_1^1 \end{array} \\ r_0^1 & \begin{pmatrix} 0 & T_{1,1} \end{pmatrix} \end{array}$$

Observe that we have taken $Q_0 = I_{n_0}$.
Now, we can start our induction. Assume that we have the required factorization for the first $i - 1$ matrices:

$$\begin{aligned} T_1 &= Q_0^* A_1 Q_1 \\ \vdots &= \vdots \\ T_{i-1} &= Q_{i-2}^* A_{i-1} Q_{i-1} \end{aligned}$$

where the matrices $T_j, j = 1, \ldots, i-1$ have the block structure as in Theorem 2. We now want to find a unitary matrix Q_i such that $T_i = Q_{i-1}^* A_i Q_i$ is either lower or upper block triangular. First consider the case where T_i is to be lower block triangular. The matrix $Q_{i-1}^* A_i$ can be partitioned according to the dimensions of the block columns of T_{i-1} as

$$(11) \qquad Q_{i-1}^* A_i = \begin{array}{cc} & n_i \\ \begin{array}{c} r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} & \begin{pmatrix} * \\ * \\ \vdots \\ * \end{pmatrix} \end{array}$$

It is always possible to construct a unitary matrix Q_i to compress the columns of each of the block rows to the left as

$$
(12) \qquad T_i^l = Q_{i-1}^* A_i Q_i = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{ccccc} r_i^1 & r_i^2 & \dots & r_i^i & r_i^{i+1} \end{array} \\ \left(\begin{array}{ccccc} T_{i,1} & 0 & \dots & 0 & 0 \\ * & T_{i,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & T_{i,i} & 0 \end{array} \right) \end{array}
$$

where the subblocks $T_{i,j}$ are of full column rank, denoted by r_i^l, $l = 1, \dots, i$ and $r_i^{i+1} = nullity(A_i)$. Hereto, we first compress the first block row of (11) to the left with unitary column transformations applied to the full matrix. Then we proceed with the second block row in the deflated matrix (i.e.without modifying the previous block column). By repeating this procedure i times, we find the required form (12). Obviously

$$
(13) \qquad r_i^l \leq r_{i-1}^l \quad l = 1, \dots, i
$$

The construction of T_i when it is required to be upper block triangular is similar. Construct a unitary matrix Q_i that compresses the columns of the block rows of $Q_{i-1}^* A_i$ to the right. The only difference is that we now start from the bottom to find that:

$$
(14) \qquad T_i^u = Q_{i-1}^* A_i Q_i = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{ccccc} r_i^{i+1} & r_i^1 & r_i^2 & \dots & r_i^i \end{array} \\ \left(\begin{array}{ccccc} 0 & T_{i,1} & * & \dots & * \\ 0 & 0 & T_{i,2} & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & T_{i,i} \end{array} \right) \end{array}.
$$

We can now apply an additional (block) column permutation to the right of the matrix T_i^u so as to find the matrix of (10). This completes the proof. □ ▯

We will now demonstrate that the matrices $T_{i,j}$ can always be further reduced to triangular form using unitary transformations into

$$
\left(\begin{array}{c} 0 \\ R_{i,j}^l \end{array} \right)
$$

in the case when T_i is lower block triangular. Here, $R_{i,j}^l$ is a lower triangular matrix. Similarly, we can always reduce $T_{i,j}$ to

$$
\left(\begin{array}{c} R_{i,j}^u \\ 0 \end{array} \right)
$$

in the case where T_i is upper block triangular. Here $R_{i,j}^u$ is an upper triangular matrix. In order to demonstrate this, we need the following result:

LEMMA 5.2.
Let $P_1, \dots, P_k$ be k given complex matrices where P_i has dimensions $p_{i-1} \times p_i$, $p_{i-1} \geq p_i$ and $rank(P_i) = p_i$. Then there always exist unitary matrices $Q_0, Q_1, \dots, Q_k$ such that

$$
R_i = Q_{i-1}^* P_i Q_i
$$

where R_i is either of the form

$$\text{(15)} \qquad R_i = \begin{array}{c} \\ p_{i-1}-p_i \\ p_i \end{array} \!\!\begin{array}{c} p_i \\ \left(\begin{array}{c} 0 \\ R_i^l \end{array}\right) \end{array}$$

with R_i^l a lower triangular matrix or

$$\text{(16)} \qquad R_i = \begin{array}{c} \\ p_i \\ p_{i-1}-p_i \end{array} \!\!\begin{array}{c} p_i \\ \left(\begin{array}{c} R_i^u \\ 0 \end{array}\right) \end{array}$$

with R_i^u upper triangular.
For every $i = 1, \ldots, k$, both choices, (15) and (16), are always possible. *Proof:* Again, the proof is by induction, but now for *decreasing* index i. For the initialization, start with $i = k$ and obtain a QR-decomposition of P_k with either an upper or a lower triangular factor as required. This defines the unitary matrix Q_{k-1}. We take $Q_k = I_{p_k}$. Hence, we find

- Lower triangular:

$$P_k = Q_{k-1}R_k = Q_{k-1}\left(\begin{array}{c} 0 \\ R_k^l \end{array}\right)$$

- Upper triangular:

$$P_k = Q_{k-1}R_k = Q_{k-1}\left(\begin{array}{c} R_k^u \\ 0 \end{array}\right)$$

We can now start the induction for $i = k-1, k-2, \ldots, 1$. Therefore, assume we have the required factorizations for the matrices $P_k, P_{k-1}, \ldots, P_{i+1}$:

$$\begin{array}{rcl} R_k & = & Q_{k-1}^* P_k Q_k \\ R_{k-1} & = & Q_{k-2}^* P_{k-1} Q_{k-1} \\ \vdots & = & \vdots \\ R_{i+1} & = & Q_i^* P_{i+1} Q_{i+1} \end{array}$$

Then, if R_i is to be lower triangular, obtain a QR-decomposition of the product P_iQ_i as:

$$P_iQ_i = Q_{i-1}R_i = Q_{i-1}\left(\begin{array}{c} 0 \\ R_i^l \end{array}\right)$$

so that

$$R_i = \left(\begin{array}{c} 0 \\ R_i^l \end{array}\right) = Q_{i-1}^* P_i Q_i$$

If R_i is required to be upper triangular, obtain a QR-decomposition as

$$P_iQ_i = Q_{i-1}R_i = Q_{i-1}\left(\begin{array}{c} R_i^u \\ 0 \end{array}\right)$$

so that

$$R_i = \begin{pmatrix} R_i^u \\ 0 \end{pmatrix} = Q_{i-1}^* P_i Q_i$$

This completes the construction. $\square$

We will now repeatedly apply Lemma 5.2 on the full rank blocks in the matrices T_i in (9) and (10). First, we will apply Lemma 5.2 to the sequence of k subblocks

$$T_{1,1} \;\; T_{2,1} \;\; \ldots \;\; T_{k,1}$$

Next, we will apply it to the sequence of the $k-1$ subblocks

$$T_{2,2} \;\; T_{3,2} \;\; \ldots \;\; T_{k,2}$$

In general, we will apply Lemma 5.2 k times to the k sequences of subblocks

$$T_{j,j}, T_{j+1,j}, \ldots, T_{k,j} \qquad for \;\; j = 1, \ldots, k$$

In applying Lemma 5.2 to the j-th of these sequences, we can find a sequence of unitary matrices $Q_0^{[j]}, Q_1^{[j]}, \ldots, Q_{k-j+1}^{[j]}$ and matrices $R_{i,j}$ such that:

$$T_{i,j} = Q_{i-j}^{[j]} R_{i,j} Q_{i-j+1}^{[j]*} \qquad\qquad i = j, \ldots, k$$

where

$$R_{i,j} = \begin{pmatrix} 0 \\ R_{i,j}^l \end{pmatrix}$$

or

$$R_{i,j} = \begin{pmatrix} R_{i,j}^u \\ 0 \end{pmatrix}$$

We now define the *unitary* matrices $\tilde{Q}_i$ for $i = 0, \ldots, k$, which are block-diagonal with blocks:

$$\tilde{Q}_i = diag(Q_i^{[1]}, Q_{i-1}^{[2]}, \ldots, Q_1^{[i]}, Q_0^{[i+1]}) \qquad\qquad i = 0, \ldots, k$$

with

$$Q_0^{[k+1]} = I$$

Next we define

$$\tilde{T}_i = \tilde{Q}_{i-1}^* T_i \tilde{Q}_i \qquad\qquad i = 0, \ldots, k$$

Then, it can be verified that for the lower triangular case we obtain

$$(17) \qquad\qquad \tilde{T}_i = \tilde{T}_i^l = \begin{array}{c c} & \begin{array}{ccccc} r_i^1 & r_i^2 & \ldots & r_i^i & r_i^{i+1} \end{array} \\ \begin{array}{c} r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} & \begin{pmatrix} R_{i,1} & 0 & \ldots & 0 & 0 \\ * & R_{i,2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots & R_{i,i} & 0 \end{pmatrix} \end{array}$$

and for the upper triangular case we find

$$
\tilde{T}_i = \tilde{T}_i^u = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array}
\begin{array}{c}
\begin{array}{cccccc} r_i^{i+1} & r_i^1 & \dots & r_i^{i-1} & r_i^i \end{array} \\
\begin{pmatrix} 0 & R_{i,1} & * & \dots & * \\ 0 & 0 & R_{i,2} & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & R_{i,i} \end{pmatrix}
\end{array} \tag{18}
$$

If we now combine (9)-(17) and (10)-(18), we obtain a combined factorization of the form:

$$
\tilde{T}_i = (Q_{i-1}\tilde{Q}_{i-1})^* A_i (Q_i \tilde{Q}_i)
$$

Hence, we have proved:

THEOREM 5.3. ***Generalized URV-decompositions.*** *Given k complex matrices A_1 $(n_0 \times n_1)$, A_2 $(n_1 \times n_2)$, $\dots$, A_k $(n_{k-1} \times n_k)$. There always exist unitary matrices $Q_0, Q_1, \dots, Q_k$ such that*

$$
\tilde{T}_i = Q_{i-1}^* A_i Q_i
$$

where $\tilde{T}_i$ is a lower triangular or upper triangular matrix (both cases are always possible) with the following structure:

- *Lower triangular (which will be denoted by a superscript l):*

$$
\tilde{T}_i^l = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array}
\begin{array}{c}
\begin{array}{ccccc} r_i^1 & r_i^2 & \dots & r_i^i & r_i^{i+1} \end{array} \\
\begin{pmatrix} R_{i,1} & 0 & \dots & 0 & 0 \\ * & R_{i,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & R_{i,i} & 0 \end{pmatrix}
\end{array}
$$

  where

$$
R_{i,j} = \begin{pmatrix} 0 \\ R_{i,j}^l \end{pmatrix}
$$

  and $R_{i,j}^l$ is a square nonsingular lower triangular matrix.

- *Upper triangular (which will be denoted by a superscript u):*

$$
\tilde{T}_i^u = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array}
\begin{array}{c}
\begin{array}{ccccc} r_i^{i+1} & r_i^1 & \dots & r_i^{i-1} & r_i^i \end{array} \\
\begin{pmatrix} 0 & R_{i,1} & * & \dots & * \\ 0 & 0 & R_{i,2} & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & R_{i,i} \end{pmatrix}
\end{array}
$$

  where

$$
R_{i,j} = \begin{pmatrix} 0 \\ R_{i,j}^u \end{pmatrix}
$$

  and $R_{i,j}^u$ is a square nonsingular upper triangular matrix.

The block dimensions coincide with those of Theorem 1.
As to the nomenclature of these *generalized URV-decompositions*, we propose the following:

Definition 5.4. Nomenclature for generalized *URV*
The name of a generalized *URV*-decomposition of k matrices of compatible dimensions is generated by enumerating the letters **L** (for lower) and **U** (for upper), according to the lower or upper triangularity of the matrices $T_i, i = 1, \ldots, k$ in the decomposition of Theorem 5.3. For k matrices, there are 2^k different sequences with two letters. For instance, for $k = 3$, there are 8 generalized *URV* decompositions (**LLL**, **LLU**, **LUL**, **LLU**, **ULL**, **ULU**, **UUL**, **UUU**).

Remarks
The decompositions in Theorems 2 and 3 both use column and row compressions of a matrix as a cornerstone for the rank determination of the individual blocks. As already pointed out in section 2, the rank determination can be done via an **OSVD** but a more economical method uses the **QRD** as initial step since typically, the matrices involved here have many more columns than rows or vice versa. A further alternative would be to replace the **OSVD** of the triangular matrix resulting from the initial **QRD** by a rank revealing **QRD**. Since the initial paper drawing attention to this [5], much progress has been made in this area and we only want to stress here that such alternatives can only benefit our decomposition.
The overall complexity of this **GQRD** is easily seen to be comparable to that of performing two **QRD**'s of each matrix A_i involved. For each A_i one indeed applies the left transformation Q_{i-1}^* derived from the previous matrix and then applies a "special" compression Q_i of the resulting matrix while respecting its block structure. Both steps have a complexity comparable to a **QRD** of a matrix of the same dimensions. For parallel machines one can check that the "block"-algorithms [18] for one sided orthogonal transformations as the **QRD** also can be applied to the present decomposition and that they will yield satisfactory speed-ups. The main reason for this, is that the two sided orthogonal transforms applied to each A_i are done separately and hence they can essentially be considered as one sided for parallellization purposes.

6. On the structure of the GSVD and the GQRD . In this section, we first point out how for each **GSVD** there are two generalized *URV*-decompositions and we clarify the correspondance between the two types of generalized decompositions. Next we give a summary of expressions for the block dimensions r_i^j in Theorem 1 and 2, in terms of the ranks of the matrices $A_1, \ldots, A_k$ and concatenations and products thereof. These expressions were derived in [10].

Recall the nomenclature for the generalized *URV*-decompositions (Definition 5) and the generalized singular value decompositions (Definition 4). The relation between these two definitions is the following. A pair of identical letters, i.e. **L-L** or **U-U** that occurs in the factorization of A_i, A_{i+1} corresponds to a *P-type* factorization of the pair. A pair of alternating letters, i.e. **L-U** or **U-L** that occurs in the factorization of A_i, A_{i+1} corresponds to a *Q-type* factorization of the pair. As an example, for a **PQP-SVD** of 4 matrices, there are two possible corresponding generalized *URV*-decompositions, namely an **LLUL**-decomposition and an **UULU**-decomposition. As with the **GSVD**, we can also introduce the convention to use powers of (a sequence of) letters. For instance, for a P^3Q^2**-SVD**, there are two **GURV**s, namely an L^4UL-

URV and an U^4LU-**URV**.

We now derive expressions for the block dimensions r_i^j [4]. Let us first consider the case of a **GSVD** that consists only of $P-type$ factorizations. Denote the rank of the product of the matrices A_i,A_{i+1}, ..., A_j with $i \leq j$ by

$$r_{i(i+1)\ldots(j-1)j} = rank(A_i A_{i+1} \ldots A_{j-1} A_j)$$

THEOREM 6.1. ***On the structure of the P^{k-1}-SVD, the L^k-URV and the U^k-URV.*** *Consider any of the factorizations above for the matrices $A_1, A_2, \ldots, A_k$. Then, the block dimensions r_j^i that appear in Theorem 4.1 - 5.1 - 5.3 are given by:*

$$r_j^1 = r_{(1)(2)\ldots(j)} \tag{19}$$

$$r_j^i = r_{i(i+1)\ldots(j)} - r_{(i-1)(i)\ldots(j)} \tag{20}$$

with $r_j^i = r_i$ if $i = j$.
Next, consider the case of a **GSVD** that only consists of $Q-type$ factorizations. Denote the rank of the block bidiagonal matrix

$$\begin{pmatrix} A_i & 0 & 0 & \ldots & 0 & 0 & 0 \\ A_{i+1}^* & A_{i+2} & 0 & \ldots & 0 & 0 & 0 \\ 0 & A_{i+3}^* & A_{i+4} & \ldots & 0 & 0 & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & A_{j-3}^* & A_{j-2} & 0 \\ 0 & \ldots & \ldots & \ldots & 0 & A_{j-1}^* & A_j \end{pmatrix} \tag{21}$$

by $r_{i|i+1|\ldots|j-1|j}$.
THEOREM 6.2. ***On the structure of the Q^{k-1}-SVD, the $(LU)^{k/2}$-URV (k even), the $(UL)^{k/2}$-URV (k even), the $(UL)^{(k-1)/2}U$-URV (k odd) and the $(LU)^{(k-1)/2}L$-URV (k odd).*** *Consider any of the above factorizations for the matrices $A_1, A_2, \ldots, A_k$. Then*

- *If $j-i$ even*

$$r_{i|\ldots|j} = r_{i|\ldots|j-1} + (r_j^1 + r_j^2 + \ldots + r_j^i) + r_j^{i+2} + r_j^{i+4} + \ldots + r_j^{j-2} + r_j^j$$

- *If $j-i$ odd*

$$r_{i|\ldots|j} = r_{i|\ldots|j-1} + (r_j^{i+1} + r_j^{i+3} + \ldots + r_j^{j-2} + r_j^j)$$

For the general case, we shall need a mixture of the two preceding notations for block bidiagonal matrices, the blocks of which can be products of matrices, such as:

$$\begin{pmatrix} A_{i_0}A_{i_0+1}\ldots A_{i_1-1} & 0 & 0 & \ldots & 0 \\ (A_{i_1}\ldots A_{i_2-1})^* & A_{i_2}\ldots A_{i_3-1} & 0 & \ldots & 0 \\ 0 & (A_{i_3}\ldots A_{i_4-1})^* & A_{i_4}\ldots A_{i_5-1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & A_{i_l}\ldots A_j \end{pmatrix}$$

where $1 \leq i_0 < i_1 < i_2 < i_3 < \ldots < i_l \leq j \leq k$. Their rank will be denoted by

$$r_{(i_0)\ldots(i_1-1)|i_1\ldots(i_2-1)|\ldots|i_l\ldots(j)}$$

[4] Recall that the subscript i refers to the i-th matrix, while the superscript j refers to the j-th block in that matrix.

For instance, the rank of the matrix

$$\begin{pmatrix} A_2A_3 & 0 & 0 \\ A_4^* & A_5A_6A_7 & 0 \\ 0 & (A_8A_9)^* & A_{10} \end{pmatrix}$$

will be represented by

$$r_{(2)(3)|4|(5)(6)(7)|(8)(9)|(10)}$$

THEOREM 6.3. ***On the structure of a GSVD and a GURV.*** *The rank* $r_{(i_0)(i_0+1)\ldots(i_1-1)|i_1\ldots(i_2-1)|\ldots|i_l\ldots j}$ *can be derived as follows:*
1. *Calculate the following* $l+1$ *integers* s_j^i, $i = 1, 2, \ldots, l+1$*:*

$$\begin{aligned} s_j^1 &= r_j^1 + r_j^2 + \ldots + r_j^{i_0} \\ s_j^2 &= r_j^{i_0+1} + r_j^{i_0+2} + \ldots + r_j^{i_1} \\ \ldots &= \ldots \\ s_j^{l+1} &= r_j^{i_{l-1}+1} + r_j^{i_{l-1}+2} + \ldots + r_j^{i_l} \end{aligned}$$

2. *Depending on* l *even or odd there are two cases:*

l ***even:***

$$r_{i_0\ldots i_1-1|i_1\ldots i_2-1|\ldots|i_l\ldots j} = r_{i_0\ldots i_1-1|i_1\ldots i_2-1|\ldots|i_{l-1}\ldots i_l-1} + s_j^1 + s_j^3 + \ldots + s_j^{l+1}$$

l ***odd:***

$$r_{i_0\ldots i_1-1|i_1\ldots i_2-1|\ldots|i_l\ldots j} = r_{i_0\ldots i_1-1|i_1\ldots i_2-1|\ldots|i_{l-1}\ldots i_l-1} + s_j^2 + s_j^4 + \ldots + s_j^{l+1}$$

Observe that Theorems 6.1 and 6.2 are special cases of Theorem 6.3.

While Theorem 6.1 provides a direct expression of the dimensions r_j^i in terms of differences of ranks of products, Theorem 6.2 and 6.3 do so only implicitly. Let us illustrate this with a couple of examples.
Example:
Let us determine the block dimensions of the quasi-diagonal matrix S_4 in a **QPP-SVD** of the matrices A_1, A_2, A_3, A_4 (which will also be the block dimensions of a **LUUU** or a **ULLL**-decomposition). From Theorem 6.3 we find:

$$\begin{aligned} r_4^4 &= r_4 - r_{34} \\ r_3^4 &= r_{34} - r_{234} \end{aligned}$$

From Theorem 6.3, we find:

$$\begin{aligned} s_4^1 &= r_4^1 \\ s_4^2 &= r_4^2 \end{aligned}$$

and

$$r_{(1)|(2)(3)(4)} = r_1 + s_4^2$$

so that

$$r_4^2 = r_{1|(2)(3)(4)} - r_1$$

Finally, since $r_4 = r_4^1 + r_4^2 + r_4^3 + r_4^4$, we find

$$r_4^1 = r_1 + r_{(2)(3)(4)} - r_{1|(2)(3)(4)}$$

Observe that this last relation can be interpreted geometrically as the dimension of the intersection between the row spaces of A_1 and $A_2A_3A_4$:

$$\begin{aligned} r_4^1 &= dim\ span_{row}(A_1) + dim\ span_{row}(A_2A_3A_4) \\ & \quad -dim\ span_{row}\begin{pmatrix} A_1 \\ A_2A_3A_3 \end{pmatrix} \end{aligned}$$

Example:
Consider the determination of $r_5^1, r_5^2, r_5^3, r_5^4, r_5^5$ in a PQ^3**-SVD** of 5 matrices A_1, A_2, A_3, A_4, A_5 with Theorem 6.3, which will coincide with the structure of a **UULUL-URV** or a **LLULU-URV**:

	s_5^i
$r_{4\|5}$	$s_5^1 = r_5^1 + r_5^2 + r_5^3 + r_5^4$ $s_5^2 = r_5^5$
$r_{3\|4\|5}$	$s_5^1 = r_5^1 + r_5^2 + r_5^3$ $s_5^2 = r_5^4$ $s_5^3 = r_5^5$
$r_{2\|3\|4\|5}$	$s_5^1 = r_5^1 + r_5^2$ $s_5^2 = r_5^3$ $s_5^3 = r_5^4$ $s_5^4 = r_5^5$
$r_{(1)(2)\|3\|4\|5}$	$s_5^1 = r_5^1$ $s_5^2 = r_5^2 + r_5^3$ $s_5^3 = r_5^4$ $s_5^4 = r_5^5$

These relations can be used to set up a set of equations for the unknowns $r_5^1, r_5^2, r_5^3, r_5^4, r_5^5$, using Theorem 6.3 as:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_5^1 \\ r_5^2 \\ r_5^3 \\ r_5^4 \\ r_5^5 \end{pmatrix} = \begin{pmatrix} r_5 \\ r_{4|5} - r_4 \\ r_{3|4|5} - r_{3|4} \\ r_{2|3|4|5} - r_{2|3|4} \\ r_{(1)(2)|3|4|5} - r_{(1)(2)|3|4} \end{pmatrix}$$

the solution of which is

$$\begin{aligned} r_5^1 &= r_{3|4|5} - r_{3|4} + r_{(1)(2)|3|4} - r_{(1)(2)|3|4|5} \\ r_5^2 &= r_{(1)(2)|3|4|5} - r_{(1)(2)|3|4} - r_{2|3|4|5} + r_{2|3|4} \\ r_5^3 &= r_{2|3|4|5} - r_{2|3|4} - r_{4|5} + r_4 \\ r_5^4 &= r_5 - r_{3|4|5} + r_{3|4} \\ r_5^5 &= r_{4|5} - r_4 \end{aligned}$$

7. A further block diagonalization of the GQRD . In this section, we point out that a further block diagonalization of a **GQRD** can be interpreted as a preliminary step towards the corresponding **GSVD**. We will proceed in two stages. First we observe that each upper- or lower-triangular matrix in the generalized *URV*-decomposition of Theorem 5.3 can be block diagonalized. Next we show how these block diagonalizations can be propagated backward through the **GQRD**. The first step is the factorization of the upper- and lower-triangular matrices $\tilde{T}_i$ of Theorem 5.3 into an upper- or lower triangular matrix and a block diagonal matrix. For lower triangular matrices, $\tilde{T}_i = \tilde{T}_i^l$, we can obtain a factorization in the form:

$$\tilde{T}_i^l = L_i \tilde{D}_i^l$$

where

$$L_i = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{ccccc} r_{i-1}^1 & r_{i-1}^2 & \dots & r_{i-1}^{i-1} & r_{i-1}^i \end{array} \\ \begin{pmatrix} I & 0 & \dots & 0 & 0 \\ * & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * & I \end{pmatrix} \end{array}$$

$$\tilde{D}_i^l = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{ccccc} r_i^1 & r_i^2 & \dots & r_i^i & r_i^{i+1} \end{array} \\ \begin{pmatrix} R_{i,1} & 0 & \dots & 0 & 0 \\ 0 & R_{i,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_{i,i} & 0 \end{pmatrix} \end{array}$$

Since the diagonal blocks $R_{i,j}$ are of full column rank, such a factorization is always possible. In a similar way, for upper triangular matrices, $\tilde{T}_i = \tilde{T}_i^u$, we find a factorization in the form

$$\tilde{T}_i^u = U_i \tilde{D}_i^u$$

with U_i an upper triangular block matrix with identity matrices on the block diagonal:

$$U_i = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{ccccc} r_{i-1}^1 & r_{i-1}^2 & \dots & r_{i-1}^{i-1} & r_{i-1}^i \end{array} \\ \begin{pmatrix} I & * & \dots & * & * \\ 0 & I & \dots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I \end{pmatrix} \end{array}$$

$$\tilde{D}_i^u = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{ccccc} r_i^1 & r_i^2 & \dots & r_i^i & r_i^{i+1} \end{array} \\ \begin{pmatrix} R_{i,1} & 0 & \dots & 0 & 0 \\ 0 & R_{i,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_{i,i} & 0 \end{pmatrix} \end{array}$$

Suppose now that we have done this for all matrices $\tilde{T}_i, i = 1, \dots, k$ in a **GQRD** of

Theorem 5.3. We will now show how we can propagate a further block diagonalization backward through the **GQRD**, in a way which is completely consistent with the corresponding **GSVD** of Theorem 4.1. To simplify the notation we will simply replace in the following $\tilde{T}_i$ by T_i and $\tilde{D}_i$ by D_i.
First, assume that T_k be block triangular. It then follows from the previous section that we can factorize T_k as

$$T_k^l = L_k D_k$$

Depending on the fact whether T_{k-1} is upper or lower triangular, we have two cases:

- $T_{k-1} = T_{k-1}^l$ lower triangular.
  In this case, the product $T_{k-1}^l L_k$ will be lower triangular as well and we can obtain a similar decomposition:

  $$T_{k-1}^l L_k = L_{k-1} D_{k-1}$$

  where L_{k-1} is again lower triangular and D_{k-1} has the same diagonal blocks $R_{i,j}$ as T_{k-1}^l.
- $T_{k-1} = T_{k-1}^u$ upper triangular:
  In this case, the product $T_{k-1}^u L_k^{-*}$ will be upper triangular and we can obtain a factorization:

  $$T_{k-1}^u L_k^{-*} = U_{k-1} D_{k-1}$$

  where U_{k-1} is upper triangular and D_{k-1} has the same diagonal blocks $R_{i,j}$ as T_{k-1}^u.

It is easily verified that when T_k is upper triangular, similar conclusions can be obtained.

In general, let T_i be lower triangular and assume that it is factorized as

$$T_i^l = L_i D_i Z_i$$

Assume that T_{i-1} is lower triangular. Then T_{i-1} can be factored as

$$T_{i-1}^l = L_{i-1} D_{i-1} L_i^{-1}$$

If T_{i-1} is upper triangular, it can be factored as

$$T_{i-1}^u = U_{i-1} D_{i-1} L_i^*$$

where U_{i-1} is upper triangular. The cases with T_i upper triangular are similar. The following table summarizes all possibilities:

T_i Lower Triangular	T_{i-1} Lower triangular	$T_{i-1} = L_{i-1} D_{i-1} L_i^{-1}$
$T_i = L_i D_i Z_i$	T_{i-1} Upper triangular	$T_{i-1} = U_{i-1} D_{i-1} L_i^*$
T_i Upper triangular	T_{i-1} Lower triangular	$T_{i-1} = L_{i-1} D_{i-1} U_i^*$
$T_i = U_i D_i Z_i$	T_{i-1} Upper triangular	$T_{i-1} = U_{i-1} D_{i-1} U_i^{-1}$

Example:
Let us apply this result to a sequence of 4 matrices A_1, A_2, A_3, A_4 with compatible

dimensions. If the required sequence is **ULUU**, then :

$$\begin{array}{lllllll} A_1 & = & Q_0 T_1^u Q_1^* & = & Q_0(U_1 D_1 L_2^*)Q_1^* & = & (Q_0U_1)D_1(Q_1L_2)^* \\ A_2 & = & Q_1 T_2^l Q_2^* & = & Q_1(L_2 D_2 U_3^*)Q_2^* & = & (Q_1L_2)D_2(Q_2U_3)^* \\ A_3 & = & Q_2 T_3^u Q_3^* & = & Q_2(U_3 D_3 U_4^{-1})Q_3^* & = & (Q_2U_3)D_3(Q_3U_4)^{-1} \\ A_4 & = & Q_3 T_4^u Q_4^* & = & Q_3(U_4 D_4 \ \ )Q_4^* & = & (Q_3U_4)D_4Q_4^* \end{array}$$

Notice that $U_1 = I_{n_0}$. This follows immediately from the block structure of U_i for $i = 1$. Observe that the relationships between the common factors in the left hand side of these expressions are conform with the requirements for a **QQP-SVD**. Only the middle factors $D_i, i = 1, 2, 3, 4$ are not *quasi-diagonal.*

8. Conclusions . In this paper, a constructive proof was given of a multi-matrix generalization of the concept of rank factorization. The connection of this new decomposition with the analogous generalized singular value decomposition was also shown. The block structure of both generalizations and the ranks of the individual diagonal blocks in both decompositions were indeed shown to be identical. As is shown in a forthcoming paper, the spaces spanned by certain block columns of the orthogonal transformation matrices Q_i are in fact identical to those of the **GSVD**. The difference lies only in a particular choice of basis vectors for these spaces. The consequences of these connections are still under investigation. We already mention here the following results.

- Updating the above decomposition to yield the **GSVD** requires non orthogonal transformation. These updating transformations can be chosen block triangular with diagonal block sizes compatible with the index sets derived in Theorem 4.1.
- A modified orthogonal decomposition can be defined where the individual matrices are *not* diagonalized but triangularized. This new factorization is a variant of the above decomposition where now a special coordinate system is chosen for each of the individual orthogonal transformations Q_i. The result is an orthogonal decomposition of the type of Theorem 5.3 where now the generalized singular values can be extracted from the diagonal elements of some triangular blocks. The orthogonal updating needed to get to this new decomposition can be done with techniques described in [2].
- A geometric interpretation can be given of the bases obtained from the transformation matrices Q_i in 5.1. As particular examples of theses spaces we retrieve the following well known concepts
  a) For the case $A_i = (A - \alpha I)$ the **GQRD** in fact reconstructs the nested null spaces of the matrices $(A - \alpha I)^i$, which reveal the Jordan structure of the matrix A at the eigenvalue α (see also the example in section 2).
  b) For the case $A_{2i} = (A - \alpha B)$ and $A_{2i+1} = B$ the decomposition reconstructs the nested null spaces of the sequences $[B^{-1}(A - \alpha B)]^i$ and $[(A - \alpha B)B^{-1}]^i$, which reveal the Kronecker structure of the pencil $\lambda B - A$ at the generalized eigenvalue α (see [30] [31]).
  c) For the case $A_1 = D$ and $A_i = C \cdot A^{i-1} \cdot B, i = 1, \ldots$
  the decomposition reconstructs the invertibility subspaces of the discrete time system

$$\left\{ \begin{array}{lll} x_{k+1} & = & Ax_k + Bu_k \\ y_k & = & Cx_k + Du_k \end{array} \right.$$

These are in fact also the spaces constructed by the structure algorithm of Silverman [29] and they play a role in several key problems of geometrical systems theory [34].

Other applications of generalized singular value decompositions have been described in [7] [8] [11] [13] [35] while applications of the generalized QR-decompositions are described in [25] [36].

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