# Local Linearizations of Rational Matrices with Application to Rational Approximations of Nonlinear Eigenvalue Problems

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## Abstract

This paper presents a definition for local linearizations of rational matrices and studies their properties. This definition allows to introduce matrix pencils associated to a rational matrix that preserve its structure of zeros and poles in subsets of any algebraically closed field and also at infinity. This new theory of local linearizations captures and explains rigorously the properties of all the different pencils that have been used from the 1970's until 2020 for computing zeros, poles and eigenvalues of rational matrices. Particular attention is paid to those pencils that have appeared recently in the numerical solution of nonlinear eigenvalue problems through rational approximation.

*Keywords:* rational matrix, rational eigenvalue problem, nonlinear eigenvalue problem, linearization, polynomial system matrix, rational approximation, block full rank pencils

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## 1. Introduction

Matrices whose entries are rational functions of a scalar variable received a lot of attention since the 1950s because of the fundamental role their structural properties play in linear systems and control theory [24] and their engineering applications [20, 27]. The most relevant structural data of a rational matrix are its poles and zeros, together with their partial multiplicities or structural indices, and its minimal indices, which exist only

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when the matrix is singular, i.e., rectangular or square with identically zero determinant. These structural data are very important in the applications mentioned above, which motivated in the 1970s a considerable research activity on the development of numerical algorithms for computing them, see [32] and the references therein. Among the different algorithms developed for this purpose in the 1970-80s, the most reliable ones were based on constructing a matrix pencil, i.e., a matrix polynomial of degree 1, containing exactly *all* the information about the structural data of the considered rational matrix [32, 36], and then applying to this matrix pencil backward stable algorithms, developed also in the 1970s, for computing the eigenvalues and/or other structural data of general pencils [26, 31]. These pencils are among the first examples of linearizations of rational matrices, but are in fact particular instances of polynomial system matrices, introduced by Rosenbrock [27] to include simultaneously all the information about the poles and zeros of a rational matrix into a polynomial matrix.

Recently, rational matrices received a renewed attention in what are called rational eigenvalue problems (REPs), which arise directly from applications [29], from rational approximations of nonlinear eigenvalue problems (NLEPs) [25, 19, 22, 28, 30, 18] and, besides, from rational approximations of polynomial eigenvalue problems to take advantage of low-rank structures [23]. Since this literature originated from a different perspective as the system and control theory literature, it is useful to point out here connections and differences between how rational matrices are viewed in the classic areas of linear systems and control theory and in the modern one of NLEPs.

In the REP literature, a scalar $\lambda_0$ is an eigenvalue of a rational matrix $G(\lambda)$ if there exists a nonzero constant vector $x$ such that $G(\lambda_0)x = 0$, with $G(\lambda_0)$ bounded (i.e., $\lambda_0$ is not a pole of $G(\lambda)$ and, therefore, $G(\lambda_0)$ has finite entries). Moreover, for this definition it is required that $G(\lambda)$ is regular (i.e., square and with non identically zero determinant). In the system and control theory, such a point $\lambda_0$ is called a zero of $G(\lambda)$, but zeros can also be defined when $G(\lambda_0)$ is unbounded or when $G(\lambda)$ is not regular. Thus, eigenvalues in REPs can be viewed as special cases of zeros in system and control theory.

In the NLEP literature, rational matrices appear as approximations in a certain target set. Moreover, the matrix defining the NLEP is often assumed to be analytic in the target region [19, 22, 28, 30], and this region does not contain the poles of the rational matrix defining the approximating REP, nor does it contain the point at infinity. In particular, in the REPs coming from a NLEP the poles are already known from the approximation process, and only the eigenvalues (i.e., those zeros that are not poles) in the target set have to be computed. This is in contrast with rational matrices arising in linear systems and control theory, which are transfer functions of time invariant linear systems and where all the finite and infinite structure of poles and zeros are of interest [20, 21, 32, 35, 36].

It is clear that the literature on rational approximations used for NLEPs (see e.g. [19]) was not aware of the earlier results in system and control theory. But since the emphasis is on approximations in a particular target set, this is not so surprising. In that area, pencils are called "linearizations" when they allow to recover correctly eigenvalues of a regular rational approximation in a target set. But nothing is claimed about the pole structure, or the zero structure outside this set. Moreover, nothing is claimed about the partial multiplicities of the eigenvalues of the linearization, except that their algebraic and geometric multiplicities are preserved. This is in contrast with the standard definition of (strong) linearization of polynomial matrices [17, 9], which guarantees that linearizations contain *all* the information about the eigenvalues of polynomial matrices (including at infinity in the strong case), as well as with the linear minimal polynomial system matrices used as linearizations of rational matrices in [32, 36], which contain *all* the information about poles and zeros of the rational matrices.

In the recent literature, references about (strong) linearizations of arbitrary rational matrices are [5], and, with some restrictions, also [1, 7, 8, 11]. These restrictions come from considering only square rational matrices and/or from imposing that some of the blocks of the linearization are constant matrices. However, the definition of (strong) linearization in [5] does not always apply to the pencils defined in [19, 22, 28, 29]. In particular, these pencils do not always satisfy the minimality requirements of [5] and, then, may not contain all the information about the poles of the rational matrix, and a zero of the linearization could be a pole of the rational matrix but not a zero. But, this is not a drawback in the setting of [19, 22, 28, 29] because there the poles are known, and only the eigenvalues in a certain target set have to be computed. This motivates us to develop in this paper a theory of what we call local linearizations of rational matrices, where the word local means that the linearization is only guaranteed to contain all the information about those zeros and poles of the rational matrix which are located in a certain set.

The theory of local linearizations of rational matrices captures all the pencils that have been used (as far as we know) in the literature for solving REPs arising from approximating NLEPs. We will apply this theory to the pencils in [19, 28] in several different ways. In addition, the definition of local linearizations includes the definition of linearizations and strong linearizations of arbitrary rational matrices presented in [5], just by considering as set the whole underlying field and including infinity in the strong case. As a consequence, local linearizations also include the pencils originally used in [32, 36]. Thus, this new local theory is a flexible tool that generalizes and includes most of the previous results available in the literature in this area. This is in part possible due to a new treatment of polynomial system matrices at infinity.

The theory of local linearizations of rational matrices is based on the extension of Rosenbrock's fundamental concept of minimal polynomial system matrix to a local perspective and the use of local equivalences. Such theory is developed in a very simple and applicable manner that avoids as much as possible the use of abstract algebraic concepts. This is in contrast with related local approaches as the one in [6] and the references therein, which, in addition, are focused on the underlying local equivalence relationships rather than on the properties of polynomial system matrices. The local linearization approach connects the concept of linearization with classical results as the local Smith form of polynomial matrices (see [17, Section S1.5]) and the local Smith–McMillan form of rational matrices (see [33]).

The paper is organized as follows. Section 2 summarizes some basic results that will be used in the rest of the paper. Locally minimal polynomial system matrices are defined and studied in Section 3. Section 4 presents the main definitions and properties of local linearizations of rational matrices. Section 5 introduces the so-called block full rank pencils, which are linearizations of rational matrices that do not contain any information about the poles, and are closely related to the block minimal bases linearizations of polynomial matrices recently presented in [10]. The application of the local theory to the pencils in [19] is analyzed in depth and from two perspectives in Section 6. Finally, Section 7 discusses the conclusions and some lines of future research.

## 2. Preliminaries

We assume throughout this paper that $\mathbb{F}$ is an algebraically closed field that does not include infinity. As usual, $\mathbb{F}[\lambda]$ denotes the ring of polynomials with coefficients in $\mathbb{F}$ and $\mathbb{F}(\lambda)$ the field of rational functions or, equivalently, the field of fractions of $\mathbb{F}[\lambda]$. A rational function $r(\lambda) = \frac{n(\lambda)}{d(\lambda)}$ is said to be proper if $\deg(n(\lambda)) \leq \deg(d(\lambda))$, and strictly proper if $\deg(n(\lambda)) < \deg(d(\lambda))$, where $\deg(\cdot)$ stands for "degree of". $\mathbb{F}^{p\times m}$, $\mathbb{F}[\lambda]^{p\times m}$ and $\mathbb{F}(\lambda)^{p\times m}$ denote the sets of $p \times m$ matrices with elements in $\mathbb{F}$, $\mathbb{F}[\lambda]$ and $\mathbb{F}(\lambda)$, respectively. The elements of $\mathbb{F}[\lambda]^{p\times m}$ are called polynomial matrices or matrix polynomials. In the sequel we will use both terms. Moreover, the elements of $\mathbb{F}(\lambda)^{p\times m}$ are called rational matrices. A (strictly) proper rational matrix is a rational matrix whose entries are (strictly) proper rational functions. The normal rank of a polynomial or rational matrix $G(\lambda)$ is the size of its largest nonidentically zero minor and is denoted by $\operatorname{nrank} G(\lambda)$. See [20] and [34] for more information on these and other concepts related to polynomial and rational matrices.

As a first step to define local linearizations of rational matrices, we present local notions and results about rational matrices. Given a rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and a nonempty set $\Sigma \subseteq \mathbb{F}$, $R(\lambda)$ is said to be *defined or bounded in* $\Sigma$ if $R(\lambda_0) \in \mathbb{F}^{p\times m}$ for all $\lambda_0 \in \Sigma$. If $\Sigma = \{\lambda_0\}$ then $R(\lambda)$ is said to be *defined or bounded at* $\lambda_0$. For the point at infinity, denoted by $\infty$, $R(\lambda)$ is said to be *defined or bounded at* $\infty$ if $R(1/\lambda)$ is defined at 0. Notice that a rational matrix is defined at $\infty$ if and only if it is proper. Moreover, a square rational matrix $R(\lambda)$ is said to be *invertible in* $\Sigma$ if it is defined in $\Sigma$ and $\det R(\lambda_0) \neq 0$ for all $\lambda_0 \in \Sigma$; and $R(\lambda)$ is said to be *regular* if it is invertible for some $\lambda_0 \in \mathbb{F}$, that is, if $\det R(\lambda) \not\equiv 0$. Unimodular matrices are those rational matrices that are invertible in $\mathbb{F}$. In addition, $R(\lambda)$ is said to be *invertible at* $\infty$ or *biproper* if $R(1/\lambda)$ is invertible at 0. In regard to the previous definitions, we introduce some equivalence relations defined in the set of rational matrices [3, 4, 16].

**Definition 2.1.** *Two rational matrices $G_1(\lambda), G_2(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ are said to be equivalent in a nonempty set $\Sigma \subseteq \mathbb{F}$ if there exist rational matrices $R_1(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$ and $R_2(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ both invertible in $\Sigma$ such that $R_1(\lambda)G_1(\lambda)R_2(\lambda) = G_2(\lambda)$. This is denoted by $G_1(\lambda) \sim_\Sigma G_2(\lambda)$. When $\Sigma = \{\lambda_0\}$, we have the local equivalence at $\lambda_0$ and is denoted by $G_1(\lambda) \sim_{\lambda_0} G_2(\lambda)$. If $R_1(\lambda)$ and $R_2(\lambda)$ are biproper then $G_1(\lambda)$ and $G_2(\lambda)$ are said to be equivalent at $\infty$. This is denoted by $G_1(\lambda) \sim_\infty G_2(\lambda)$.*

Note that if $\Sigma = \mathbb{F}$ is considered in Definition 2.1, then $R_1(\lambda)$ and $R_2(\lambda)$ are both unimodular, and the standard definition of unimodular equivalence is recovered.

We now introduce the definition of the local Smith–McMillan form of a rational matrix at a point (finite and infinite). The notion of the Smith–McMillan form of a rational matrix was first studied by McMillan in [24] and, then, in other works as [20, 27, 34]. The local Smith–McMillan form for rational matrices over the complex field can be found in [33], and a complete and rigorous modern treatment in [4]. Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be any rational matrix of normal rank $r$. Let $\lambda_0 \in \mathbb{F}$. Then

$$G(\lambda) \sim_{\lambda_0} \begin{bmatrix} \operatorname{diag}\left((\lambda-\lambda_0)^{\nu_1}, \ldots, (\lambda-\lambda_0)^{\nu_r}\right) & 0 \\ 0 & 0_{(p-r)\times(m-r)} \end{bmatrix}, \tag{1}$$

where $\nu_1 \leq \cdots \leq \nu_r$ are integers. The integers $\nu_1, \ldots, \nu_r$ are uniquely determined by $G(\lambda)$ and $\lambda_0$, and are called the invariant orders at $\lambda_0$ of $G(\lambda)$. The matrix on the right hand side in (1) is called the local Smith–McMillan form of $G(\lambda)$ at $\lambda_0$. For defining the Smith–McMillan form of $G(\lambda)$ at $\infty$, we replace the factor $(\lambda - \lambda_0)$ in (1) by $\left(\frac{1}{\lambda}\right)$.

In order to define zeros and poles we need to distinguish between positive and negative invariant orders [20, 34]. When we say that a rational matrix has $\nu_1 \leq \cdots \leq \nu_k < 0 = \nu_{k+1} = \cdots = \nu_{u-1} < \nu_u \leq \cdots \leq \nu_r$ as invariant orders at $\lambda_0$ (infinity) we mean that $k$ may take values from 0 to $r$ and $u$ from 1 to $r+1$. For instance, if $k = 0$ all the invariant orders are nonnegative; if, in addition, $u = 1$ then they are all positive, but if $k = 0$ and $u = r + 1$ they are all 0.

**Definition 2.2.** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and $\lambda_0 \in \mathbb{F}$. Let $\nu_1 \leq \cdots \leq \nu_k < 0 = \nu_{k+1} = \cdots = \nu_{u-1} < \nu_u \leq \cdots \leq \nu_r$ be the invariant orders at $\lambda_0$ (resp. at $\infty$) of $G(\lambda)$. Then $\lambda_0$ (resp. $\infty$) is said to be a pole of $G(\lambda)$ with partial multiplicities $-\nu_k, \ldots, -\nu_1$, and a zero of $G(\lambda)$ with partial multiplicities $\nu_u, \ldots, \nu_r$. In particular, the positive integers $-\nu_k, \ldots, -\nu_1$ and $\nu_u, \ldots, \nu_r$ are called the pole and zero partial multiplicities of $G(\lambda)$ at $\lambda_0$ (resp. at $\infty$), respectively. Moreover, $(\lambda - \lambda_0)^{-\nu_i}$ for $i = 1, \ldots, k$ are called the pole elementary divisors of $G(\lambda)$ at $\lambda_0$, while $(\lambda - \lambda_0)^{\nu_i}$ for $i = u, \ldots, r$ are called the zero elementary divisors of $G(\lambda)$ at $\lambda_0$. Finally, the pole (zero) algebraic multiplicity of $\lambda_0$ is the sum of its pole (zero) partial multiplicities, and the pole (zero) geometric multiplicity of $\lambda_0$ is the number of its pole (zero) partial multiplicities.*

If $G(\lambda)$ is a polynomial matrix then the polynomials $(\lambda - \lambda_0)^{\nu_i}$ with $\nu_i \neq 0$ are simply called elementary divisors of $G(\lambda)$ at $\lambda_0$, and the nonzero integers $\nu_i \neq 0$ are all positive and are called partial multiplicities of $G(\lambda)$ at $\lambda_0$. Some modern references, see for instance [1, 19, 29], also consider (finite) eigenvalues of rational matrices, a concept that is not mentioned at all in classical references of rational matrices. According to these modern references, we introduce the following definition.

**Definition 2.3.** *A finite eigenvalue of a rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ is any $\lambda_0 \in \mathbb{F}$ such that $\operatorname{rank} G(\lambda_0) < \operatorname{nrank} G(\lambda)$, with $G(\lambda_0) \in \mathbb{F}^{p\times m}$. That is, $\lambda_0$ is a finite zero of $G(\lambda)$ but not a pole.*

Observe that if $G(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$ is regular, an eigenvalue of $G(\lambda)$ is any $\lambda_0 \in \mathbb{F}$ such that there exists a nonzero vector $x \in \mathbb{F}^p$ satisfying $G(\lambda_0)x = 0$ with $G(\lambda_0) \in \mathbb{F}^{p\times p}$, which is the standard definition of REP (Rational Eigenvalue Problem).

As a consequence of [4, Theorem 2.3] (see [3, Section 2] for more details) we can also present the Smith–McMillan form of a rational matrix in a nonempty subset of $\mathbb{F}$, say $\Sigma$. Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ with normal rank $r$. Then

$$G(\lambda) \sim_\Sigma \begin{bmatrix} \operatorname{diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}\right) & 0 \\ 0 & 0_{(p-r)\times(m-r)} \end{bmatrix} \tag{2}$$

where, for $i = 1, \ldots, r$, $\frac{\epsilon_i(\lambda)}{\psi_i(\lambda)}$ are nonzero irreducible rational functions, $\epsilon_i(\lambda)$ and $\psi_i(\lambda)$ are monic (leading coefficient equal to 1) polynomials which are either constants or whose roots are in $\Sigma$ and $\epsilon_1(\lambda) \mid \cdots \mid \epsilon_r(\lambda)$ while $\psi_r(\lambda) \mid \cdots \mid \psi_1(\lambda)$, where $\mid$ stands for divisibility. We refer to the matrix on the right hand side in (2) as the Smith–McMillan form in $\Sigma$ of $G(\lambda)$. When we take $\Sigma = \mathbb{F}$, we obtain the (finite) Smith–McMillan form of $G(\lambda)$, i.e., the classical Smith–McMillan form of $G(\lambda)$. In this case, if $G(\lambda)$ is polynomial then

$\psi_1(\lambda) = \cdots = \psi_r(\lambda) = 1$, $\epsilon_1(\lambda), \ldots, \epsilon_r(\lambda)$ are the invariant polynomials of $G(\lambda)$, and (2) is called the Smith normal form of $G(\lambda)$.

The Smith–McMillan form of a rational matrix in a nonempty set $\Sigma \subseteq \mathbb{F}$ is invariant under equivalence in $\Sigma$. The next result shows that the equivalence of rational matrices in nonempty sets is a local property.

**Proposition 2.4.** *Let $\Sigma \subseteq \mathbb{F}$ be nonempty. Two rational matrices of the same size are equivalent in $\Sigma$ if and only if they are equivalent at each $\lambda_0 \in \Sigma$.*

*Proof.* If two rational matrices are equivalent in $\Sigma$ then, by Definition 2.1, it is straightforward that they are equivalent at each $\lambda_0 \in \Sigma$. For the converse, suppose that $G(\lambda) \sim_{\lambda_0} H(\lambda)$ for all $\lambda_0 \in \Sigma$. Then, $G(\lambda)$ and $H(\lambda)$ have the same local Smith–McMillan forms at each $\lambda_0 \in \Sigma$. Let us consider $M_G(\lambda)$ and $M_H(\lambda)$ as the global Smith–McMillan forms of $G(\lambda)$ and $H(\lambda)$, respectively. Thus, there exist unimodular matrices $U_i^G(\lambda)$, $U_i^H(\lambda)$ for $i = 1, 2$, such that $G(\lambda) = U_1^G(\lambda)M_G(\lambda)U_2^G(\lambda)$, $H(\lambda) = U_1^H(\lambda)M_H(\lambda)U_2^H(\lambda)$, and we can write

$$\begin{aligned} M_G(\lambda) &= \operatorname{diag}\left(f_1(\lambda)g_1(\lambda), \ldots, f_r(\lambda)g_r(\lambda), 0_{(p-r)\times(m-r)}\right), \text{ and} \\ M_H(\lambda) &= \operatorname{diag}\left(f_1(\lambda)h_1(\lambda), \ldots, f_r(\lambda)h_r(\lambda), 0_{(p-r)\times(m-r)}\right), \end{aligned}$$

where $f_i(\lambda)$ are rational functions which are either equal to one or have poles or zeros in $\Sigma$, while $g_i(\lambda)$ and $h_i(\lambda)$ are rational functions that do not have any poles or zeros in $\Sigma$. Let us define $R(\lambda) := \operatorname{diag}\left(\dfrac{h_1(\lambda)}{g_1(\lambda)}, \ldots, \dfrac{h_r(\lambda)}{g_r(\lambda)}, I_{m-r}\right)$. Hence, $M_H(\lambda) = M_G(\lambda)R(\lambda)$. Therefore, we deduce that $H(\lambda) = U_1^H(\lambda)U_1^G(\lambda)^{-1}G(\lambda)U_2^G(\lambda)^{-1}R(\lambda)U_2^H(\lambda)$, and $G(\lambda) \sim_\Sigma H(\lambda)$ since the matrices $U_1^H(\lambda)U_1^G(\lambda)^{-1}$ and $U_2^G(\lambda)^{-1}R(\lambda)U_2^H(\lambda)$ are invertible in $\Sigma$. $\square$

## 3. Polynomial system matrices minimal in subsets of $\mathbb{F}$ and at infinity

Polynomial system matrices are a classical tool for studying rational matrices. They were introduced by Rosenbrock and are analyzed in detail in [27]. Among them, minimal polynomial system matrices have been used in many problems dealing with rational matrices because they allow to extract all the information about finite poles and zeros. Recently, they have played a fundamental role in developing a rigorous theory of linearizations and strong linearizations of rational matrices [5]. In this section, we extend the concept of minimal polynomial system matrices from the classical global scenario to a local one. Some of the definitions in this section can also be found in [6] expressed in an abstract algebraic language.

### *3.1. Polynomial system matrices minimal in subsets of* $\mathbb{F}$

Consider the fact that any rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ can be written as $G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$ for some polynomial matrices $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ with $A(\lambda)$ nonsingular if $n > 0$ (see [27]). Then the matrix polynomial

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \tag{3}$$

is called a polynomial system matrix of $G(\lambda)$ [27]. That is, $G(\lambda)$ is the Schur complement of $A(\lambda)$ in $P(\lambda)$ and is called the transfer function matrix of $P(\lambda)$. We will refer to $A(\lambda)$

as the state matrix of $P(\lambda)$. If $n = 0$, we assume that the matrices $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are empty, and $P(\lambda) = G(\lambda) = D(\lambda)$ is a polynomial matrix. We emphasize that the definition of polynomial system matrix of a rational matrix includes a specific partition. Sometimes in this paper a certain polynomial matrix is partitioned in different ways giving rise to different polynomial system matrices of (possibly) different rational matrices. In such cases, we often use expressions as "$P(\lambda)$ is a polynomial system matrix of $G(\lambda)$ with state matrix $A(\lambda)$" in order to avoid ambiguities, where the words "of $G(\lambda)$" may be omitted because $P(\lambda)$ and $A(\lambda)$ determine $G(\lambda)$. We stress that although in (3) the state matrix is in the $(1,1)$-block, it might be a different submatrix of $P(\lambda)$. In the case $n = 0$, we will use "$P(\lambda)$ is a polynomial system matrix with empty state matrix".

We remark that the relation between the normal ranks of $P(\lambda)$ and its transfer function matrix $G(\lambda)$ is

$$\operatorname{nrank} P(\lambda) = n + \operatorname{nrank} G(\lambda), \tag{4}$$

since $P(\lambda) = \begin{bmatrix} I_n & 0 \\ -C(\lambda)A(\lambda)^{-1} & I_p \end{bmatrix} \begin{bmatrix} A(\lambda) & 0 \\ 0 & G(\lambda) \end{bmatrix} \begin{bmatrix} I_n & A(\lambda)^{-1}B(\lambda) \\ 0 & I_m \end{bmatrix}$. Next, we introduce one of the main definitions of this work.

**Definition 3.1** (Polynomial system matrix minimal in a subset of $\mathbb{F}$)**.** *Let $\Sigma \subseteq \mathbb{F}$ be nonempty. The polynomial system matrix $P(\lambda)$ in* (3)*, with $n > 0$, is said to be minimal in $\Sigma$ if, for each $\lambda_0 \in \Sigma$, the following condition holds:*

$$\operatorname{rank} \begin{bmatrix} A(\lambda_0) \\ C(\lambda_0) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A(\lambda_0) & B(\lambda_0) \end{bmatrix} = n. \tag{5}$$

Definition 3.1 extends to subsets of $\mathbb{F}$ the classical definition of minimal, or with least order, polynomial system matrices introduced in [27]. Rosenbrock's definition coincides with Definition 3.1 when $\Sigma = \mathbb{F}$.

**Remark 3.2.** Notice that $\operatorname{nrank} \begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix} = \operatorname{nrank} \begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} = n$ since $A(\lambda)$ is nonsingular. Thus, the rank condition (5) holds if and only if $\lambda_0$ is neither an eigenvalue of $\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}$ nor of $\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix}$.

**Remark 3.3.** For convenience, if $n = 0$ in (3), we adopt the agreement that $P(\lambda)$ is minimal at every point $\lambda_0 \in \mathbb{F}$.

In the next example, we illustrate Definition 3.1 with a rational matrix and a polynomial system matrix taken from the recent reference [28] dealing with numerical algorithms for solving NLEPs via rational approximation. We advance that we will use Example 3.4 several times for illustrating different concepts. In this respect, we emphasize that [28] does not mention polynomial system matrices at all, and neither do [19, 29].

**Example 3.4.** Let $G(\lambda)$ be a rational matrix of the form

$$G(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p \times p}, \tag{6}$$

with $A_0, B_0, \ldots, B_s \in \mathbb{C}^{p\times p}$, $\sigma_1, \ldots, \sigma_s \in \mathbb{C}$, and $\sigma_i \neq \sigma_j$ if $i \neq j$. Let us consider the linear polynomial matrix

$$P(\lambda) = \left[\begin{array}{cccc|c} (\lambda-\sigma_1)I & & & & I \\ & & \ddots & & \vdots \\ & & & (\lambda-\sigma_s)I & I \\ \hline -B_1 & & \cdots & -B_s & \lambda A_0 - B_0 \end{array}\right].$$

These matrices are introduced in [28] to tackle a NLEP $T(\lambda)v = 0$, in a certain region $\Omega \subseteq \mathbb{C}$, where the matrix $T(\lambda)$ is of the form $T(\lambda) = -B_0 + \lambda A_0 + f_1(\lambda)A_1 + \cdots + f_q(\lambda)A_q$, with $A_0, A_1, \ldots, A_q \in \mathbb{C}^{p\times p}$ and $f_i : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$, $i = 1, \ldots, q$, being scalar functions nonlinear in the variable $\lambda$ and holomorphic in $\Omega$. For solving a NLEP of this form, the nonlinear matrix $T(\lambda)$ is approximated in $\Omega$ by a rational matrix $G(\lambda)$ as in (6), and $P(\lambda)$ is considered to linearize $G(\lambda)$. It is easy to see that $P(\lambda)$ is, in fact, a linear polynomial system matrix of $G(\lambda)$, by setting the matrix $\operatorname{diag}((\lambda-\sigma_1)I, \ldots, (\lambda-\sigma_s)I)$ as state matrix $A(\lambda)$ in (3). Moreover, without any assumption, $P(\lambda)$ is minimal in $\Sigma := \mathbb{C} \setminus \{\sigma_1, \ldots, \sigma_s\}$. In particular, and according to [28], $\Omega$ is a subset of $\Sigma$. Therefore, $P(\lambda)$ is minimal in the target set $\Omega$. For completeness, notice that a polynomial system matrix as $P(\lambda)$ is minimal in $\mathbb{C}$ if and only if all the matrices $B_1, \ldots, B_s$ are nonsingular.

The next result provides the pole and zero elementary divisors of a rational matrix $G(\lambda)$ in a subset $\Sigma$ from any polynomial system matrix of $G(\lambda)$ minimal in $\Sigma$. This result is the counterpart of [27, Chapter 3, Theorem 4.1] for polynomial system matrices minimal in a particular subset instead of polynomial system matrices of least order.

**Theorem 3.5.** *Let $\Sigma \subseteq \mathbb{F}$ be nonempty. Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

*be a polynomial system matrix minimal in $\Sigma$ whose transfer function matrix is $G(\lambda)$. Then the elementary divisors of $A(\lambda)$ in $\Sigma$ are the pole elementary divisors of $G(\lambda)$ in $\Sigma$, and the elementary divisors of $P(\lambda)$ in $\Sigma$ are the zero elementary divisors of $G(\lambda)$ in $\Sigma$.*

*Proof.* We give the proof for a finite point $\lambda_0 \in \Sigma$. Then, the result can be extended to $\Sigma$ in a natural way. Let us consider the Smith normal form of $\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix}$. Namely, $U(\lambda) \begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} V(\lambda) = \begin{bmatrix} S(\lambda) & 0 \end{bmatrix}$ with $U(\lambda)$ and $V(\lambda)$ unimodular matrices. Observe that $S(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ is regular since $\operatorname{nrank} \begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} = n$. We set $H_1(\lambda) := S(\lambda)^{-1}U(\lambda)$. Since $P(\lambda)$ is minimal at $\lambda_0$, $S(\lambda)$ has no zeros at $\lambda_0$. Therefore, $H_1(\lambda)$ is invertible at $\lambda_0$. Moreover, $\begin{bmatrix} H_1(\lambda)A(\lambda) & H_1(\lambda)B(\lambda) \end{bmatrix}$ is a polynomial matrix, as it is equal to $\begin{bmatrix} I_n & 0 \end{bmatrix} V(\lambda)^{-1}$, has full row normal rank, and has no zeros in $\mathbb{F}$. Now, let us consider the Smith normal form of the polynomial matrix $\begin{bmatrix} H_1(\lambda)A(\lambda) \\ -C(\lambda) \end{bmatrix}$. Namely, $\widetilde{U}(\lambda) \begin{bmatrix} H_1(\lambda)A(\lambda) \\ -C(\lambda) \end{bmatrix} \widetilde{V}(\lambda) = \begin{bmatrix} \widetilde{S}(\lambda) \\ 0 \end{bmatrix}$ with $\widetilde{U}(\lambda)$ and $\widetilde{V}(\lambda)$ unimodular matrices. Observe that $\widetilde{S}(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ is regular since $H_1(\lambda)$ is regular and $\operatorname{nrank} \begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix} = n$. We set $H_2(\lambda) := \widetilde{V}(\lambda)\widetilde{S}(\lambda)^{-1}$. Moreover, the matrix $\begin{bmatrix} H_1(\lambda)A(\lambda)H_2(\lambda) \\ -C(\lambda)H_2(\lambda) \end{bmatrix}$ is also polynomial, as it is equal to $\widetilde{U}(\lambda)^{-1}\begin{bmatrix} I_n \\ 0 \end{bmatrix}$, has full column normal rank, and has no zeros in $\mathbb{F}$. Since $P(\lambda)$ is minimal at $\lambda_0$ and $H_1(\lambda)$ is invertible at $\lambda_0$, $\widetilde{S}(\lambda)$ has no zeros at $\lambda_0$. Therefore, $H_2(\lambda)$ is invertible at $\lambda_0$. Let us define now the polynomial system matrix

$$\widetilde{P}(\lambda) := \begin{bmatrix} H_1(\lambda) & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \begin{bmatrix} H_2(\lambda) & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} H_1(\lambda)A(\lambda)H_2(\lambda) & H_1(\lambda)B(\lambda) \\ -C(\lambda)H_2(\lambda) & D(\lambda) \end{bmatrix}.$$

We claim that $\widetilde{P}(\lambda)$ is a minimal polynomial system matrix in $\mathbb{F}$ or in the classical sense of [27]. For that, it remains to prove that the matrix $Z(\lambda) := \begin{bmatrix} H_1(\lambda)A(\lambda)H_2(\lambda) & H_1(\lambda)B(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{F}$. Let us suppose that there exists $\lambda_1 \in \mathbb{F}$ such that $\operatorname{rank} Z(\lambda_1) < n$. On the one hand, $\operatorname{rank}\begin{bmatrix} H_1(\lambda_1)A(\lambda_1)\widetilde{V}(\lambda_1) & H_1(\lambda_1)B(\lambda_1) \end{bmatrix} = n$ since the Smith normal form of $\begin{bmatrix} H_1(\lambda)A(\lambda) & H_1(\lambda)B(\lambda) \end{bmatrix}$ is equal to $\begin{bmatrix} I_n & 0 \end{bmatrix}$ and $\widetilde{V}(\lambda)$ is unimodular. On the other hand, we have that $\operatorname{rank}\begin{bmatrix} H_1(\lambda_1)A(\lambda_1)\widetilde{V}(\lambda_1) & H_1(\lambda_1)B(\lambda_1) \end{bmatrix} = \operatorname{rank}\left( Z(\lambda_1)\begin{bmatrix} \widetilde{S}(\lambda_1) & 0 \\ 0 & I_m \end{bmatrix} \right) \leq \operatorname{rank} Z(\lambda_1) < n$, which is a contradiction. Therefore, $\widetilde{P}(\lambda)$ is a minimal polynomial system matrix. Its transfer function matrix is $G(\lambda)$. Then, by [27, Chapter 3, Theorem 4.1], we know that the zero elementary divisors of $G(\lambda)$ are the elementary divisors of $\widetilde{P}(\lambda)$, and that the pole elementary divisors of $G(\lambda)$ are the elementary divisors of $H_1(\lambda)A(\lambda)H_2(\lambda)$. Finally, the result follows by taking into account that $P(\lambda) \sim_{\lambda_0} \widetilde{P}(\lambda)$ and $A(\lambda) \sim_{\lambda_0} H_1(\lambda)A(\lambda)H_2(\lambda)$, since $H_1(\lambda)$ and $H_2(\lambda)$ are both invertible at $\lambda_0$. $\square$

**Example 3.6.** If Theorem 3.5 is applied in Example 3.4, we obtain that (without any hypothesis) the eigenvalues of $P(\lambda)$ in $\Sigma$ coincide exactly with the zeros of $G(\lambda)$ in $\Sigma$, with exactly the same multiplicities (geometric, algebraic and partial). In addition, all the zeros of $G(\lambda)$ in $\Sigma$ are, in fact, eigenvalues of $G(\lambda)$ because the only potential poles of $G(\lambda)$ are $\sigma_1, \ldots, \sigma_s$. This result is stronger than Lemma 3.1 and Corollary 3.2 in [28] from two perspectives: [28] deals with determinants and, so, only gives information on algebraic multiplicities, and the requests in [28] impose the additional hypothesis that $A_0$ is nonsingular. Note that, under the assumption that all the matrices $B_1, \ldots, B_s$ are nonsingular, $P(\lambda)$ and $A(\lambda)$ allow us to obtain the complete information on finite zeros and poles (including all the multiplicities) of $G(\lambda)$ in $\mathbb{C}$.

*3.2. Polynomial system matrices minimal at infinity*

Theorem 3.5 characterizes polynomial system matrices that contain the information of the invariant orders at finite points of their transfer functions. The extension of these results for including the information at infinity is an old problem that has been considered in classical papers as, for instance, in [35, 36]. However, a satisfactory solution has been found, so far, only for polynomial system matrices with state matrix $A(\lambda)$ being a linear polynomial matrix (also called a matrix pencil) and the other blocks $B(\lambda)$, $C(\lambda)$, $D(\lambda)$ being constant matrices. In other cases, recovering the information at infinity requires to embed the polynomial system matrix into a larger matrix. In this section, we propose a new approach for obtaining a counterpart of Theorem 3.5 at infinity.

First, we introduce the notion of $g$-reversal of a rational matrix in Definition 3.7, where $g$ is any integer. In this definition we will use, for a particular value of $g$, the well-known fact that any rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ can be uniquely written as

$$G(\lambda) = Q(\lambda) + G_{sp}(\lambda) \tag{7}$$

where $Q(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ is a polynomial matrix and $G_{sp}(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ is a strictly proper rational matrix. The matrices $Q(\lambda)$ and $G_{sp}(\lambda)$ are called the polynomial part and the strictly proper part of $G(\lambda)$, respectively. A polynomial matrix $Q(\lambda)$ is said to have degree $d$ if $d$ is the largest exponent of the variable $\lambda$ of its entries with nonzero coefficient. In such a case, $d$ is denoted by $\deg(Q(\lambda))$.

**Definition 3.7** ($g$-reversal of a rational matrix)**.** *Let $G(\lambda)$ be a rational matrix, and let $g$ be an integer. We define the $g$-reversal of $G(\lambda)$ as the rational matrix*

$$\operatorname{rev}_g G(\lambda) := \lambda^g G\left(\frac{1}{\lambda}\right).$$

*Let $G(\lambda)$ be expressed as in* (7)*. If $g = \deg(Q(\lambda))$ whenever $G(\lambda)$ is not strictly proper, or $g = 0$ if $G(\lambda)$ is strictly proper, then the $g$-reversal is called the reversal of $G(\lambda)$ and is often denoted by just* $\operatorname{rev} G(\lambda)$.

Definition 3.7 extends the definition of $g$-reversal for polynomial matrices (see, for instance, [9, Definition 2.12]). However, in the definition of $g$-reversal of a polynomial matrix considered previously in the literature, $g$ is always taken larger than or equal to the degree of the polynomial matrix, while in Definition 3.7 we only ask for $g$ to be an integer.

Given a polynomial system matrix $P(\lambda)$ as in (3) of degree $d$, we have that $\operatorname{rev} P(\lambda) = \begin{bmatrix} \operatorname{rev}_d A(\lambda) & \operatorname{rev}_d B(\lambda) \\ -\operatorname{rev}_d C(\lambda) & \operatorname{rev}_d D(\lambda) \end{bmatrix}$ is also a polynomial matrix. Moreover, $\operatorname{rev}_d A(\lambda)$ is nonsingular since $A(\lambda)$ is nonsingular. Therefore, $\operatorname{rev} P(\lambda)$ is also a polynomial system matrix. We now introduce the concept of minimality at infinity of a polynomial system matrix.

**Definition 3.8** (Polynomial system matrix minimal at infinity)**.** *The polynomial system matrix $P(\lambda)$ in* (3) *is minimal at $\infty$ if* $\operatorname{rev} P(\lambda)$ *is minimal at* 0.

**Example 3.9.** The polynomial system matrix in Example 3.4 is minimal at $\infty$ since $\operatorname{rev}P(\lambda)$ is, obviously, minimal at 0.

**Remark 3.10.** A polynomial system matrix $P(\lambda)$ as in (3), with $\deg(P(\lambda)) = d$ and $n > 0$, is minimal at $\infty$ if and only if $\operatorname{rank} \begin{bmatrix} \operatorname{rev}_d A(0) \\ \operatorname{rev}_d C(0) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \operatorname{rev}_d A(0) & \operatorname{rev}_d B(0) \end{bmatrix} = n$. More precisely, let $A_d$, $B_d$ and $C_d$ be the matrix coefficients of $\lambda^d$ in $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$, respectively. Then $P(\lambda)$ is minimal at $\infty$ if and only if $\operatorname{rank} \begin{bmatrix} A_d \\ C_d \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A_d & B_d \end{bmatrix} = n$. Notice that if $d = 0$ then $P(\lambda)$ is a constant polynomial system matrix, and $A_0$ must be invertible. Therefore, in this case, the rank condition above is automatically satisfied, and $P(\lambda)$ is minimal at $\infty$.

Theorem 3.11 is essentially the counterpart of Theorem 3.5 at infinity. We state it in terms of reversals and their elementary divisors at 0 as we only have defined elementary divisors for finite points. The implications of Theorem 3.11 on the structure at infinity are made explicit in Theorem 3.13.

**Theorem 3.11.** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

*be a polynomial system matrix of degree $d$ minimal at $\infty$ whose transfer function matrix is $G(\lambda)$. Then the elementary divisors of $\operatorname{rev}_d A(\lambda)$ at $0$ are the pole elementary divisors of $\operatorname{rev}_d G(\lambda)$ at $0$, and the elementary divisors of $\operatorname{rev} P(\lambda)$ at $0$ are the zero elementary divisors of $\operatorname{rev}_d G(\lambda)$ at $0$.*

*Proof.* It can be easily proved that the transfer function matrix of $\operatorname{rev} P(\lambda)$ is $\operatorname{rev}_d G(\lambda)$. The theorem then follows by applying Theorem 3.5, since $\operatorname{rev} P(\lambda)$ is minimal at 0. □

Once we have obtained the elementary divisors of the $d$-reversal of a rational matrix at 0, from one of its polynomial system matrices of degree $d$ minimal at $\infty$, we can then obtain its invariant orders at infinity as we state in Theorem 3.13. For proving that, we use Lemma 3.12.

**Lemma 3.12.** Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ with $\operatorname{nrank} G(\lambda) = r$, and let $g$ be an integer. Let $e_1, \ldots, e_r$ be the invariant orders of $\operatorname{rev}_g G(\lambda)$ at 0, and let $q_1, \ldots, q_r$ be the invariant orders at infinity of $G(\lambda)$. Then

$$e_i = q_i + g \quad i = 1, \ldots, r. \tag{8}$$

*Proof.* Note that $G(\lambda) \sim_\infty \operatorname{diag}\left((1/\lambda)^{q_1}, \ldots, (1/\lambda)^{q_r}, 0_{(p-r)\times(m-r)}\right)$. By the transformation $\lambda \mapsto 1/\lambda$, $G(1/\lambda) \sim_0 \operatorname{diag}\left(\lambda^{q_1}, \ldots, \lambda^{q_r}, 0_{(p-r)\times(m-r)}\right)$. Then multiply by $\lambda^g$. □

**Theorem 3.13.** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ with $\operatorname{nrank} G(\lambda) = r$ and let*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

*be a polynomial system matrix of degree $d$ minimal at $\infty$ whose transfer function matrix is $G(\lambda)$. Let $e_1 \leq \cdots \leq e_s$ be the partial multiplicities of $\operatorname{rev}_d A(\lambda)$ at $0$ and let $\widetilde{e}_1 \leq \cdots \leq \widetilde{e}_u$ be the partial multiplicities of $\operatorname{rev} P(\lambda)$ at $0$. Then the invariant orders at infinity $q_1 \leq \cdots \leq q_r$ of $G(\lambda)$ are $(q_1, q_2, \ldots, q_r) = (-e_s, -e_{s-1}, \ldots, -e_1, \underbrace{0, \ldots, 0}_{r-s-u}, \widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_u) - (d, d, \ldots, d)$.*

*Proof.* By Theorem 3.11, we know that $e_i$ and $\widetilde{e}_j$ with $i = 1, \ldots, s$ and $j = 1, \ldots, u$ are the pole and zero partial multiplicities of $\operatorname{rev}_d G(\lambda)$ at 0, respectively. Thus, the invariant orders of $\operatorname{rev}_d G(\lambda)$ at 0 are $-e_s \leq -e_{s-1} \leq \cdots \leq -e_1 < \underbrace{0 = \cdots = 0}_{r-s-u} < \widetilde{e}_1 \leq \cdots \leq \widetilde{e}_u$. Then the use of Lemma 3.12 completes the proof. □

**Example 3.14.** By combining Theorem 3.13 and Example 3.9, we see that $P(\lambda)$ contains the complete information about the invariant orders at $\infty$ of $G(\lambda)$ (without imposing any hypothesis). Note that, in this case, $d = 1$ and that the 1-reversal of the state matrix, i.e., $\operatorname{rev}_1 A(\lambda) = \operatorname{diag}((1 - \lambda\sigma_1)I, \ldots, (1 - \lambda\sigma_s)I)$, has no partial multiplicities at 0. This result on the relationship between the infinite structure of $G(\lambda)$ and the reversal of $P(\lambda)$ is not mentioned in [28].

Polynomial system matrices that are minimal at infinity and, also, at every finite point are called strongly minimal in [13, Definition 3.3]. However, in [13] the definition is given in terms of eigenvalues instead of minimality at every point, but both definitions are equivalent. We emphasize that, as a consequence of Theorems 3.5 and 3.13, strongly minimal polynomial system matrices contain all the information about the invariant orders of their transfer function matrices, both at finite points and at infinity.

## 4. Local linearizations of rational matrices

In practice, one is often interested in studying the pole and zero structure of rational matrices not in the whole space $\mathbb{F} \cup \{\infty\}$ but in a particular region (see [18, 19, 22, 28]). For instance, this happens when a REP arises from approximating a NLEP, since the approximation is usually reliable only in a target region not containing poles. As a consequence, the eigenvalues (those zeros that are not poles) of the approximating REP need to be computed only in that region. In this scenario, one can use local linearizations of the corresponding rational matrix which contain the information about the poles and zeros in the target region, but possibly not in the whole space $\mathbb{F} \cup \{\infty\}$. In addition, they do not satisfy, in general, the conditions of the strong linearizations of rational matrices introduced in [5]. Thereby local linearizations provide extra flexibility in solving NLEPs.

In this section, we give separately the definitions of linearizations of rational matrices in subsets of $\mathbb{F}$ and at infinity. These linearizations will be useful in order to study the pole and zero structure of rational matrices in different sets containing infinity or not. In particular, and as an application of these definitions, we will study in Section 6 the structure of the linearizations that appear in [19].

### *4.1. Linearizations in subsets of* $\mathbb{F}$

In this subsection we introduce the definition of linearization of a rational matrix in a set not containing infinity and study some of its properties.

**Definition 4.1** (Linearization in a subset of $\mathbb{F}$)**.** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $\Sigma \subseteq \mathbb{F}$ be nonempty. Let*

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)} \tag{9}$$

*be a linear polynomial system matrix with state matrix $A_1\lambda + A_0$ and let*

$$\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0) \in \mathbb{F}(\lambda)^{q\times r}$$

*be its transfer function matrix. $\mathcal{L}(\lambda)$ is a linearization of $G(\lambda)$ in $\Sigma$ if the following conditions hold:*

(a) *$\mathcal{L}(\lambda)$ is minimal in $\Sigma$, and*

(b) *there exist nonnegative integers $s_1, s_2$ satisfying $s_1 - s_2 = q - p = r - m$, such that*

$$\operatorname{diag}(G(\lambda), I_{s_1}) \sim_{\Sigma} \operatorname{diag}(\widehat{G}(\lambda), I_{s_2}). \tag{10}$$

Linearizations of rational matrices are polynomial system matrices and their definition includes a specific partition. Thus, a fixed linear polynomial matrix may be partitioned in different ways giving rise to different linearizations of the same or of different rational matrices, or in different subsets. To deal with different partitions, we will use expressions as "$\mathcal{L}(\lambda)$ is a linearization of $G(\lambda)$ in $\Sigma$ with state matrix $A_1\lambda+A_0$" when it is necessary for avoiding any ambiguity. The expression "$\mathcal{L}(\lambda)$ is a linearization of $G(\lambda)$ in $\Sigma$ with empty state matrix" will cover the case $n = 0$ in (9), which does not give us pole information (see Remark 4.5).

**Remark 4.2.** We remark the following extreme cases since they are important in applications and make Definition 4.1 very general:

1. $\widehat{G}(\lambda) = G(\lambda)$. Then we just have to check condition (a). It follows that any linear polynomial system matrix $\mathcal{L}(\lambda)$ is a linearization of its transfer function matrix in the sets where $\mathcal{L}(\lambda)$ is minimal.

2. $n = 0$. Then it is not necessary to take into account condition (a) (it is automatically satisfied by the agreement in Remark 3.3) and, therefore, we just have to check condition (b) with $\widehat{G}(\lambda) = D_1\lambda + D_0 = \mathcal{L}(\lambda)$. That is, $\operatorname{diag}(G(\lambda), I_{s_1}) \sim_\Sigma \operatorname{diag}(\mathcal{L}(\lambda), I_{s_2})$. Notice that if we want a linearization of $G(\lambda)$ in $\Sigma = \mathbb{F}$ we can not consider the case $n = 0$ unless $G(\lambda)$ is polynomial.

In condition (10), one can always take $s_1 = 0$ or $s_2 = 0$, according to $p \geq q$ and $m \geq r$ or $q \geq p$ and $r \geq m$, respectively. This is a consequence of the local Smith–McMillan forms of $\operatorname{diag}(G(\lambda), I_{s_1})$ and $\operatorname{diag}(\widehat{G}(\lambda), I_{s_2})$ being equivalent to each other in $\Sigma$. In the rest of the results of this subsection, we will consider $s := s_1 \geq 0$ and $s_2 = 0$, since it corresponds to the most interesting situation in applications. The next result gives the relation between the Smith–McMillan forms at a finite point of the rational matrices $G(\lambda)$ and $\operatorname{diag}(G(\lambda), I_s)$, with $s > 0$. It is motivated by (10) and can be easily proved.

**Lemma 4.3.** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $\operatorname{diag}\left((\lambda-\lambda_0)^{\nu_1}, \ldots, (\lambda-\lambda_0)^{\nu_k}, (\lambda-\lambda_0)^{\nu_{k+1}}, \ldots, (\lambda-\lambda_0)^{\nu_r}, 0_{(p-r)\times(m-r)}\right)$ be the Smith–McMillan form at $\lambda_0 \in \mathbb{F}$ of $G(\lambda)$, with $\nu_i \leq 0$ for $i = 1, \ldots, k$ and $\nu_i > 0$ for $i = k+1, \ldots, r$. Then the Smith–McMillan form at $\lambda_0$ of $\operatorname{diag}(G(\lambda), I_s)$ is $\operatorname{diag}\left((\lambda-\lambda_0)^{\nu_1}, \ldots, (\lambda-\lambda_0)^{\nu_k}, I_s, (\lambda-\lambda_0)^{\nu_{k+1}}, \ldots, (\lambda-\lambda_0)^{\nu_r}, 0_{(p-r)\times(m-r)}\right)$.*

Theorem 4.4 states the spectral information that one can obtain from local linearizations in the spirit of [5, Theorem 3.10].

**Theorem 4.4** (Spectral characterization of linearizations in a subset of $\mathbb{F}$)**.** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$, $\Sigma \subseteq \mathbb{F}$ nonempty and let*

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

*be a linear polynomial system matrix, with state matrix $A_1\lambda + A_0$, minimal in $\Sigma$. Then $\mathcal{L}(\lambda)$ is a linearization of $G(\lambda)$ in $\Sigma$ if and only if the following conditions hold:*

(a) $\operatorname{nrank} \mathcal{L}(\lambda) = \operatorname{nrank} G(\lambda) + n + s$,

(b) *the pole elementary divisors of $G(\lambda)$ in $\Sigma$ are the elementary divisors of $A_1\lambda + A_0$ in $\Sigma$, and the zero elementary divisors of $G(\lambda)$ in $\Sigma$ are the elementary divisors of $\mathcal{L}(\lambda)$ in $\Sigma$.*

*Proof.* We give the proof for a finite point $\lambda_0 \in \Sigma$. Then, the result can be extended to $\Sigma$ in a natural way. Let $\widehat{G}(\lambda)$ be the transfer function matrix of $\mathcal{L}(\lambda)$. First, assume that $\mathcal{L}(\lambda)$ is a linearization of $G(\lambda)$ at $\lambda_0$. By (4), $\operatorname{nrank} \widehat{G}(\lambda) = \operatorname{nrank} \mathcal{L}(\lambda) - n$. And, by Lemma 4.3, $\operatorname{nrank} \widehat{G}(\lambda) = \operatorname{nrank} G(\lambda) + s$. Then, $\operatorname{nrank} \mathcal{L}(\lambda) = \operatorname{nrank} G(\lambda) + n + s$. By Lemma 4.3, we also have that $G(\lambda)$ and $\widehat{G}(\lambda)$ have the same pole and zero elementary divisors at $\lambda_0$. Then (b) follows from Theorem 3.5, since the pole elementary divisors of $\widehat{G}(\lambda)$ at $\lambda_0$ are the elementary divisors of $A_1\lambda + A_0$ at $\lambda_0$, and the zero elementary divisors of $\widehat{G}(\lambda)$ at $\lambda_0$ are the elementary divisors of $\mathcal{L}(\lambda)$ at $\lambda_0$. For the converse, suppose that $\operatorname{diag}((\lambda - \lambda_0)^{\nu_1}, \ldots, (\lambda - \lambda_0)^{\nu_k}, (\lambda - \lambda_0)^{\nu_{k+1}}, \ldots, (\lambda - \lambda_0)^{\nu_r}, 0_{(p-r)\times(m-r)})$ is the Smith–McMillan form at $\lambda_0$ of $G(\lambda)$, with $\nu_i \le 0$ for $i = 1, \ldots, k$ and $\nu_i > 0$ for $i = k+1, \ldots, r$. From (b) and Theorem 3.5, the pole and zero elementary divisors of $G(\lambda)$ and $\widehat{G}(\lambda)$ are the same. Moreover, by (4) and (a), $\operatorname{nrank} \widehat{G}(\lambda) = \operatorname{nrank} G(\lambda) + s$. Therefore, the Smith–McMillan form at $\lambda_0$ of $\widehat{G}(\lambda)$ must be $\operatorname{diag}\big((\lambda - \lambda_0)^{\nu_1}, \ldots, (\lambda - \lambda_0)^{\nu_k}, I_s, (\lambda - \lambda_0)^{\nu_{k+1}}, \ldots, (\lambda - \lambda_0)^{\nu_r}, 0_{(p-r)\times(m-r)}\big)$. This is also the Smith–McMillan form at $\lambda_0$ of $\operatorname{diag}(G(\lambda), I_s)$, as stated in the previous lemma. Thus, $\operatorname{diag}(G(\lambda), I_s) \sim_{\lambda_0} \widehat{G}(\lambda)$. □

**Remark 4.5.** Notice that if $n = 0$ in Theorem 4.4 then we can not obtain pole information in $\Sigma$ from the linearization $\mathcal{L}(\lambda)$ since the state matrix is empty.

**Example 4.6.** Consider Example 3.4. By combining the discussion in that example with Remark 4.2(case 1), we immediately obtain that $P(\lambda)$ is a linearization of $G(\lambda)$ in $\Sigma$. With a bit more effort, it is also easy to obtain the following stronger result: $P(\lambda)$ is a linearization of $G(\lambda)$ in $\mathbb{C} \setminus \Pi$ where $\Pi := \{\sigma_i : B_i \text{ is singular for } 1 \le i \le s\}$.

### *4.2. Linearizations at infinity and in sets containing infinity*

Our definition of linearization of a rational matrix at infinity is based on the notion of $g$-reversal of a rational matrix introduced in Definition 3.7.

**Definition 4.7** (Linearization at infinity of grade $g$)**.** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$. Let*

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)} \tag{11}$$

*be a linear polynomial system matrix with state matrix $A_1\lambda + A_0$ and let*

$$\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0) \in \mathbb{F}(\lambda)^{q\times r}$$

*be its transfer function matrix. Let $g$ be an integer. $\mathcal{L}(\lambda)$ is a linearization of $G(\lambda)$ at $\infty$ of grade $g$ if the following conditions hold:*

(a) $\operatorname{rev} \mathcal{L}(\lambda)$ *is minimal at* $0$, *and*

(b) *there exist nonnegative integers $s_1, s_2$, with $s_1 - s_2 = q - p = r - m$, such that*

$$\operatorname{diag}(\operatorname{rev}_g G(\lambda), I_{s_1}) \sim_0 \operatorname{diag}(\operatorname{rev}_\ell \widehat{G}(\lambda), I_{s_2}), \tag{12}$$

*where $\ell = \deg(\mathcal{L}(\lambda))$.*

Observe that Definition 4.7 allows, for completeness, the possibility of $\ell = \deg(\mathcal{L}(\lambda))$ being equal to 0. We admit that this case has a very limited interest in applications, since it corresponds to $\mathcal{L}(\lambda)$ and $\operatorname{rev}_\ell \widehat{G}(\lambda) = \widehat{G}(\lambda)$ being constant matrices. However, it includes linearizations at $\infty$ of rational matrices $G(\lambda)$ such that, for some integer $g$, $\operatorname{rev}_g G(\lambda)$ has all its invariant orders at 0 equal to zero. Moreover, notice that, in any case, $\operatorname{rev} \mathcal{L}(\lambda)$ is also a linear polynomial system matrix since $\operatorname{rev}_\ell(A_1\lambda + A_0)$ is nonsingular. We then have the following characterization of linearizations at infinity, which follows from Definition 4.1 and the fact that $\operatorname{rev}_\ell \widehat{G}(\lambda)$ with $\ell = \deg(\mathcal{L}(\lambda))$ is the transfer function matrix of $\operatorname{rev} \mathcal{L}(\lambda)$.

**Proposition 4.8.** *A linear polynomial system matrix $\mathcal{L}(\lambda)$ as in* (11) *is a linearization of a rational matrix $G(\lambda)$ at $\infty$ of grade $g$ if and only if $\operatorname{rev} \mathcal{L}(\lambda)$ is a linearization of $\operatorname{rev}_g G(\lambda)$ at* 0.

Conditions (a) and (b) in Definition 4.7 can be stated in a different way as we show in Remarks 4.9 and 4.11, respectively.

**Remark 4.9.** As a particular case of what is discussed in Remark 3.10, condition ($a$) in Definition 4.7 is equivalent to

$$\operatorname{rank} \begin{bmatrix} A_1 \\ C_1 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A_1 & B_1 \end{bmatrix} = n, \tag{13}$$

if $\mathcal{L}(\lambda)$ is nonconstant, i.e., if $\ell = 1$. If $\mathcal{L}(\lambda)$ is constant, i.e., $\ell = 0$, condition ($a$) is automatically satisfied since $\mathcal{L}(\lambda)$ is a polynomial system matrix and, therefore, $A_0$ is invertible. We emphasize that when a nonconstant linear polynomial system matrix $\mathcal{L}(\lambda)$ as in (11) satisfies condition (13) then $\mathcal{L}(\lambda)$ is a linearization of its transfer function matrix $\widehat{G}(\lambda)$ at $\infty$ of grade 1. If $\mathcal{L}(\lambda)$ is constant then $\mathcal{L}(\lambda)$ is a linearization of $\widehat{G}(\lambda)$ at $\infty$ of grade 0.

**Example 4.10.** Consider the matrices in Example 3.4. By Remark 4.9, the linear polynomial system matrix $P(\lambda)$ is a linearization of $G(\lambda)$ at $\infty$ of grade 1.

**Remark 4.11.** By performing the transformation $\lambda \mapsto 1/\lambda$, condition ($b$) in Definition 4.7 is equivalent to $\operatorname{diag}((1/\lambda)^g G(\lambda), I_{s_1}) \sim_\infty \operatorname{diag}((1/\lambda)^\ell \widehat{G}(\lambda), I_{s_2})$.

We state in Theorem 4.12 a characterization of linearizations at $\infty$ analogous to the one in Theorem 4.4 for linearizations at finite points. In this characterization, we consider the most usual situation $s_1 := s \geq 0$ and $s_2 = 0$, assuming $q \geq p$ and $r \geq m$. Its proof is omitted since it follows immediately from Theorem 4.4 and Proposition 4.8.

**Theorem 4.12** (Spectral characterization of linearizations at infinity)**.** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let*

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

*be a linear polynomial system matrix, with state matrix $A_1\lambda + A_0$, such that $\operatorname{rev} \mathcal{L}(\lambda)$ is minimal at 0. Let $\ell = \deg(\mathcal{L}(\lambda))$. Then $\mathcal{L}(\lambda)$ is a linearization of $G(\lambda)$ at $\infty$ of grade $g$ if and only if the following conditions hold:*

(a) $\operatorname{nrank} \mathcal{L}(\lambda) = \operatorname{nrank} G(\lambda) + n + s$,

(b) *the pole elementary divisors of* $\operatorname{rev}_g G(\lambda)$ *at* 0 *are the elementary divisors of* $\operatorname{rev}_\ell(A_1\lambda + A_0)$ *at* 0, *and the zero elementary divisors of* $\operatorname{rev}_g G(\lambda)$ *at* 0 *are the elementary divisors of* $\operatorname{rev} \mathcal{L}(\lambda)$ *at* 0.

Next, we study in Proposition 4.13 how to recover the invariant orders at infinity of rational matrices from linearizations at infinity of grade $g$. Its proof is analogous to the one for Theorem 3.13. It follows from combining Theorem 4.12 and Lemma 3.12.

**Proposition 4.13.** *Let* $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *with* $\operatorname{nrank} G(\lambda) = r$, *and let*

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

*be a linearization at infinity of grade* $g$ *of* $G(\lambda)$ *with* $\ell = \deg(\mathcal{L}(\lambda))$. *Let* $e_1 \le \cdots \le e_t$ *be the partial multiplicities of* $\operatorname{rev}_\ell(A_1\lambda + A_0)$ *at* 0, *and let* $\widetilde{e}_1 \le \cdots \le \widetilde{e}_u$ *be the partial multiplicities of* $\operatorname{rev} \mathcal{L}(\lambda)$ *at* 0. *Then the invariant orders at infinity* $q_1 \le q_2 \le \cdots \le q_r$ *of* $G(\lambda)$ *are* $(q_1, q_2, \ldots, q_r) = (-e_t, -e_{t-1}, \ldots, -e_1, \underbrace{0, \ldots, 0}_{r-t-u}, \widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_u) - (g, g, \ldots, g)$.

A linear polynomial system matrix that satisfies Definition 4.1 in $\mathbb{F}$ and Definition 4.7 allows us to recover the complete information about the poles and zeros of the corresponding rational matrix, finite and at infinity. In Example 4.14 we consider a linear polynomial system matrix $\mathcal{L}(\lambda)$ that is a linearization of a rational matrix $G(\lambda)$ in $\mathbb{F}\cup\{\infty\}$. However, it is not a strong linearization in the sense of [5, Definition 3.4]. In particular, the grade of $\mathcal{L}(\lambda)$ as linearization at $\infty$ is not equal to the degree of the polynomial part of $G(\lambda)$. Actually, the grade is less than the degree of the polynomial part.

**Example 4.14.** Let us consider the rational matrix

$$G(\lambda) = \begin{bmatrix} \dfrac{\lambda^2+\lambda-1}{\lambda} & -\dfrac{1}{\lambda} \\ -1 & -\lambda^2+\lambda-2 \end{bmatrix}.$$

It can be easily proved that

$$\mathcal{L}(\lambda) = \left[\begin{array}{cc|cc} \lambda & 0 & 1 & 1 \\ 0 & 1 & 0 & \lambda \\ \hline 1 & 0 & \lambda+1 & 0 \\ \lambda & \lambda & 0 & \lambda-1 \end{array}\right] := \left[\begin{array}{c|c} A_1\lambda + A_0 & B_1\lambda + B_0 \\ \hline -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{array}\right]$$

is a linear polynomial system matrix of $G(\lambda)$. Moreover, note that $\mathcal{L}(\lambda)$ is minimal for all $\lambda_0 \in \mathbb{F}$. Therefore, by Remark 4.2(case 1), $\mathcal{L}(\lambda)$ is a linearization of $G(\lambda)$ in $\mathbb{F}$. By Remark 4.9, $\mathcal{L}(\lambda)$ is also a linearization of $G(\lambda)$ at $\infty$ of grade 1 since $\operatorname{rank}\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_1 & B_1 \end{bmatrix} = 2$. However, $\mathcal{L}(\lambda)$ is not a strong linearization according to [5, Definition 3.4] since $A_1$ is singular. Nevertheless, we can recover easily the invariant orders at $\infty$ from $\mathcal{L}(\lambda)$ by applying Proposition 4.13 with $g = 1$. For this purpose, note that $\operatorname{rev} \mathcal{L}(\lambda)$ does not have elementary divisors at 0, since $\operatorname{rev} \mathcal{L}(\lambda)$ is invertible at 0. Moreover, the only elementary divisor at 0 of $A_1 + A_0\lambda$ is $\lambda$. Therefore, the invariant orders at infinity of $G(\lambda)$ are $-2$ and $-1$ by Proposition 4.13. The invariant orders of $G(\lambda)$ at any finite point can be recovered from $\mathcal{L}(\lambda)$ by using Theorem 4.4. It is worthwhile to emphasize that the grade of $\mathcal{L}(\lambda)$ as linearization at $\infty$ of $G(\lambda)$ is different from the degree of the polynomial part of $G(\lambda)$.

## 5. Block full rank pencils

In this section, we introduce a wide family of pencils that give us the information about the zeros of rational matrices locally. More precisely, they are linearizations with empty state matrix of rational matrices in some subsets of $\mathbb{F}$, as well as at $\infty$ under some conditions. These pencils will be called block full rank pencils, since they generalize the block minimal bases pencils introduced in [10, Definition 3.1]. The definition of block full rank pencils is motivated by the fact that most of the linearizations for rational approximations of NLEPs that have been constructed so far are pencils of this type. The key results in this section are Theorems 5.3 and 5.5, which will be applied in the following section to establish rigorously and very easily the properties of the linearizations used in [19]. Note that, according to Theorem 4.4, the results in this section are not useful for studying, or computing, the finite poles of rational matrices because the considered linearizations have empty state matrix. This may be a drawback in certain situations, but we emphasize again that it is not in the development of algorithms for solving large-scale NLEPs via rational approximations [18, 19, 22, 28]. This is due to the fact that, in those cases, the poles of the rational matrix are known, since they are chosen for constructing the approximation, and/or are located outside the target set.

**Definition 5.1.** (Block full rank pencil) *A block full rank pencil is a linear polynomial matrix over $\mathbb{F}$ with the following structure*

$$L(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \tag{14}$$

*where $K_1(\lambda)$ and $K_2(\lambda)$ are pencils with full row normal rank.*

Note that Definition 5.1 includes the cases when $K_1(\lambda)$ or $K_2(\lambda)$ are empty matrices, that is, when $L(\lambda)$ has only one block row or only one block column, respectively.

We introduce some auxiliary concepts and results before establishing the most important properties of block full rank pencils in Theorems 5.3 and 5.5. We will say that a rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ has full row rank in $\Sigma \subseteq \mathbb{F}$ if, for all $\lambda_0 \in \Sigma$, $R(\lambda_0) \in \mathbb{F}^{p\times m}$, i.e., $R(\lambda)$ is defined or bounded at $\lambda_0$, and $\operatorname{rank} R(\lambda_0) = p$. Observe that this implies that $R(\lambda)$ has no poles in $\Sigma$. A polynomial matrix $K(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ (with $p < m$) is a minimal basis if its rows form a minimal basis of the rational subspace they span. One of the most useful characterizations of minimal bases (see [14, Main Theorem] or [10, Theorem 2.2]) is that $K(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ is a minimal basis if and only if $K(\lambda_0)$ has full row rank for all $\lambda_0 \in \mathbb{F}$ and $K(\lambda)$ is row reduced, i.e., its highest row degree coefficient matrix has full row rank (see [10, Definition 2.1]). Moreover, a minimal basis $N(\lambda) \in \mathbb{F}[\lambda]^{q\times m}$ is said to be dual to $K(\lambda)$ if $p + q = m$ and $K(\lambda)N(\lambda)^T = 0$ [10, Definition 2.5]. The following lemma connects rational matrices with full row rank in $\Sigma$ with minimal bases, and establishes other properties that will be used later.

**Lemma 5.2.** *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix with full row normal rank and let $T(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ be a minimal basis of the row space of $R(\lambda)$. Then the following statements hold:*

(a) *There exists a unique regular rational matrix $S(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$ such that $R(\lambda) = S(\lambda)T(\lambda)$.*

(b) *$R(\lambda)$ has full row rank in $\Sigma \subseteq \mathbb{F}$ if and only if $S(\lambda)$ in* (a) *is invertible in $\Sigma$.*

(c) *$R(\lambda)$ is a polynomial matrix if and only if $S(\lambda)$ in* (a) *is a polynomial matrix.*

(d) *If $R(\lambda)$ is a matrix pencil, then $S(\lambda)$ in* (a) *and $T(\lambda)$ are both matrix pencils.*

*Proof.* For part (a), consider the entries of the rows of $S(\lambda)$ to be the unique rational coefficients that allow us to express the corresponding row of $R(\lambda)$ as a unique linear combination of the rows of $T(\lambda)$. Part (b) follows from the Smith form of $T(\lambda)$ and the fact that any minimal basis has a polynomial right inverse. Parts (c) and (d) follow from [14, Main Theorem, part 4]. □

A rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ (with $p < m$) is said to be a rational basis if it is a basis of the rational subspace spanned by its rows, i.e., if it has full row normal rank. Two rational bases $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and $H(\lambda) \in \mathbb{F}(\lambda)^{q\times m}$ are said to be dual if $p+q = m$, and $G(\lambda)\, H(\lambda)^T = 0$.

**Theorem 5.3.** *Let $L(\lambda)$ be a block full rank pencil as in* (14) *and let $N_1(\lambda)$ and $N_2(\lambda)$ be any rational bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively. Let $\Omega \subseteq \mathbb{F}$ be nonempty. If $K_i(\lambda)$ and $N_i(\lambda)$ have full row rank in $\Omega$, for $i = 1, 2$, then $L(\lambda)$ is a linearization of the rational matrix $G(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$ in $\Omega$ with empty state matrix.*

*Proof.* In order to simplify the notation, throughout this proof we do not specify the sizes of different identity matrices and all of them are denoted by $I$. Let $\widetilde{K}_1(\lambda), \widetilde{K}_2(\lambda), \widetilde{N}_1(\lambda)$ and $\widetilde{N}_2(\lambda)$ be minimal bases of the row spaces of $K_1(\lambda)$, $K_2(\lambda)$, $N_1(\lambda)$ and $N_2(\lambda)$, respectively. Then, Lemma 5.2 implies that there exist regular rational matrices $S_1(\lambda)$, $S_2(\lambda)$, $W_1(\lambda)$ and $W_2(\lambda)$ such that

$$\begin{aligned} K_i(\lambda) &= S_i(\lambda)\widetilde{K}_i(\lambda), \quad \text{and } S_i(\lambda) \text{ is invertible in } \Omega, \text{ for } i = 1, 2. \\ N_i(\lambda) &= W_i(\lambda)\widetilde{N}_i(\lambda), \quad \text{and } W_i(\lambda) \text{ is invertible in } \Omega, \text{ for } i = 1, 2. \end{aligned}$$

Moreover, $\widetilde{K}_1(\lambda), \widetilde{K}_2(\lambda), S_1(\lambda)$ and $S_2(\lambda)$ are all matrix pencils. Then, $L(\lambda)$ can be factorized as follows,

$$L(\lambda) = \begin{bmatrix} I & 0 \\ 0 & S_1(\lambda) \end{bmatrix} \begin{bmatrix} M(\lambda) & \widetilde{K}_2(\lambda)^T \\ \widetilde{K}_1(\lambda) & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S_2(\lambda)^T \end{bmatrix}, \tag{15}$$

where the first and third factors are invertible in $\Omega$. Note that the factor in the middle is a block minimal bases pencil (see [10, Definition 3.1]) associated with the polynomial matrix $\widetilde{N}_2(\lambda)M(\lambda)\widetilde{N}_1(\lambda)^T$, since the regularity of $S_i(\lambda)$ and $W_i(\lambda)$ implies that $\widetilde{K}_i(\lambda)$ and $\widetilde{N}_i(\lambda)$ are dual minimal bases for $i = 1, 2$. Then, there exist unimodular matrices $U(\lambda)$ and $V(\lambda)$ such that

$$\begin{aligned} \begin{bmatrix} M(\lambda) & \widetilde{K}_2(\lambda)^T \\ \widetilde{K}_1(\lambda) & 0 \end{bmatrix} &= U(\lambda) \begin{bmatrix} \widetilde{N}_2(\lambda)M(\lambda)\widetilde{N}_1(\lambda)^T & 0 \\ 0 & I \end{bmatrix} V(\lambda) \\ &= U(\lambda) \begin{bmatrix} W_2(\lambda)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G(\lambda) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} W_1(\lambda)^{-T} & 0 \\ 0 & I \end{bmatrix} V(\lambda), \end{aligned} \tag{16}$$

where $U(\lambda)\,\mathrm{diag}(W_2(\lambda)^{-1}, I)$ and $\mathrm{diag}(W_1(\lambda)^{-T}, I)V(\lambda)$ are invertible in $\Omega$. From combining (15) and (16), we obtain that $L(\lambda) \sim_\Omega \mathrm{diag}(G(\lambda), I)$. □

**Remark 5.4.** Under the conditions of Theorem 5.3, we will say for brevity that "$L(\lambda)$ is a block full rank pencil associated with $G(\lambda)$ in $\Omega$". We emphasize that this "association" is not one-to-one because there are infinitely many rational bases $N_i(\lambda)$ dual to $K_i(\lambda)$. If $K_1(\lambda)$ (resp. $K_2(\lambda)$) is an empty matrix, we can take any rational matrix $N_1(\lambda) \in \mathbb{F}(\lambda)^{s_1 \times s_1}$ (resp. $N_2(\lambda) \in \mathbb{F}(\lambda)^{s_2 \times s_2}$) invertible in $\Omega$, where $s_1$ (resp. $s_2$) is the number of colums (resp. rows) of $M(\lambda)$. The standard choices are $N_1(\lambda) = I_{s_1}$ and $N_2(\lambda) = I_{s_2}$.

In the scenario of Theorem 5.3, Theorem 4.4 guarantees that the elementary divisors of $L(\lambda)$ in $\Omega$ coincide exactly with the zero elementary divisors of $G(\lambda)$ in $\Omega$. Moreover, it is clear from the expression $G(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$ that $G(\lambda)$ does not have poles in $\Omega$, since the matrices $N_i(\lambda)$ must be defined in $\Omega$ but they are not defined at the poles of $G(\lambda)$. Thus, $G(\lambda)$ has only eigenvalues in $\Omega$, and all the information about them, i.e., geometric, algebraic and partial multiplicities, is contained in $L(\lambda)$.

Next, we present sufficient conditions for a block full rank pencil to be a linearization of $G(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$ at $\infty$ of a certain grade $g$. In order to avoid cases with limited interest in applications, in Theorem 5.5 we assume $\deg(L(\lambda)) = 1$.

**Theorem 5.5.** *Let $L(\lambda)$ be a block full rank pencil as in* (14) *with* $\deg(L(\lambda)) = 1$ *and let $N_1(\lambda)$ and $N_2(\lambda)$ be rational bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively. If, for $i = 1, 2$, $\operatorname{rev}_1 K_i(\lambda)$ has full row rank at 0, and there exists an integer number $t_i$ such that $\operatorname{rev}_{t_i} N_i(\lambda)$ has full row rank at 0, then $L(\lambda)$ is a linearization of the rational matrix $G(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$ at $\infty$ of grade $1 + t_1 + t_2$ with empty state matrix.*

*Proof.* Note that $\operatorname{rev} L(\lambda) = \begin{bmatrix} \operatorname{rev}_1 M(\lambda) & \operatorname{rev}_1 K_2(\lambda)^T \\ \operatorname{rev}_1 K_1(\lambda) & 0 \end{bmatrix}$ is a block full rank pencil. Moreover, for $i = 1, 2$, $\operatorname{rev}_{t_i} N_i(\lambda)$ has full row normal rank, and $K_i(\lambda)\, N_i(\lambda)^T = 0$ implies $(\operatorname{rev}_1 K_i(\lambda))\, (\operatorname{rev}_{t_i} N_i(\lambda))^T = 0$. Therefore, $\operatorname{rev}_{t_i} N_i(\lambda)$ is a rational basis dual to $\operatorname{rev}_1 K_i(\lambda)$. Then, Theorem 5.3 applied to $\operatorname{rev} L(\lambda)$ proves that $\operatorname{rev} L(\lambda)$ is a linearization at 0 of $(\operatorname{rev}_{t_2} N_2(\lambda))\, (\operatorname{rev}_1 M(\lambda))\, (\operatorname{rev}_{t_1} N_1(\lambda)^T) = \operatorname{rev}_{1+t_1+t_2} G(\lambda)$, with empty state matrix, which combined with Proposition 4.8 proves the result. □

As a consequence of Theorems 5.3 and 5.5, we obtain Corollary 5.6. It generalizes the structure of most of the linearizations of rational approximations of NLEPs that appear in the literature in a constructive way. Moreover, it is very useful in order to characterize easily some pencils as linearizations of rational matrices locally and to obtain the information about the zeros of such rational matrices in subsets not containing poles.

**Corollary 5.6.** *Let $R(\lambda) = (A_0 - \lambda B_0)R_0(\lambda) + (A_1 - \lambda B_1)R_1(\lambda) + \cdots + (A_N - \lambda B_N)R_N(\lambda)$ be a $p \times m$ rational matrix written in terms of some matrix pencils $A_i - \lambda B_i \in \mathbb{F}[\lambda]^{p \times n_i}$ and rational matrices $R_i(\lambda) \in \mathbb{F}(\lambda)^{n_i \times m}$. Define*

$$\begin{aligned} M(\lambda) &:= [(A_0 - \lambda B_0) \quad (A_1 - \lambda B_1) \quad \cdots \quad (A_N - \lambda B_N)] \ \textit{and} \\ N_1(\lambda) &:= \begin{bmatrix} R_0(\lambda)^T & R_1(\lambda)^T & \cdots & R_N(\lambda)^T \end{bmatrix}, \end{aligned}$$

*and assume that $N_1(\lambda)$ has full row normal rank. Let $L(\lambda) = \begin{bmatrix} M(\lambda) \\ \hdashline K_1(\lambda) \end{bmatrix}$ be a block full rank pencil of degree* 1 *with only one block column and such that $K_1(\lambda)$ and $N_1(\lambda)$ are dual rational bases. Let $\Omega \subseteq \mathbb{F}$ be nonempty. Then the following statements hold:*

(a) *If* $K_1(\lambda)$ *and* $N_1(\lambda)$ *have full row rank in* $\Omega$ *then* $L(\lambda)$ *is a linearization of* $R(\lambda)$ *in* $\Omega$ *with empty state matrix.*

(b) *If* $\operatorname{rev}_1 K_1(\lambda)$ *has full row rank at* $0$*, and there exists an integer* $t$ *such that* $\operatorname{rev}_t N_1(\lambda)$ *has full row rank at* $0$*, then* $L(\lambda)$ *is a linearization of* $R(\lambda)$ *at* $\infty$ *of grade* $1+t$ *with empty state matrix.*

In the next example, we revisit the pencil introduced in Example 3.4 from the perspective of the block full rank pencils. This example illustrates how the theory of block full rank pencils may simplify the analysis of the properties of important linearizations of rational matrices when one is not interested on the information about the poles.

**Example 5.7.** Consider Example 3.4. We partition $P(\lambda)$ as follows:

$$P(\lambda) = \left[\begin{array}{cccc} (\lambda-\sigma_1)I & & & I \\ & \ddots & & \vdots \\ & & (\lambda-\sigma_s)I & I \\ \hdashline -B_1 & \cdots & -B_s & \lambda A_0 - B_0 \end{array}\right] =: \left[\begin{array}{c} K_1(\lambda) \\ \hdashline M(\lambda) \end{array}\right].$$

Observe that, in the above partition, we are considering a permuted version of the structure of the pencil $L(\lambda)$ in Corollary 5.6. Note now that $K_1(\lambda)$ has full row rank in $\mathbb{C}$, and

$$N_1(\lambda) := \left[\begin{array}{cccc} \frac{1}{\sigma_1-\lambda}I & \cdots & \frac{1}{\sigma_s-\lambda}I & I \end{array}\right]$$

is a rational basis dual to $K_1(\lambda)$ with full row rank in $\Sigma := \mathbb{C} \setminus \{\sigma_1, \ldots, \sigma_s\}$. Then, by Corollary 5.6(a), $P(\lambda)$ is a linearization of $G(\lambda)$ in $\Sigma$ with empty state matrix. Moreover, note that $\operatorname{rev}_1 K_1(\lambda)$ and $\operatorname{rev}_0 N_1(\lambda) = \left[\begin{array}{cccc} \frac{\lambda}{\lambda\sigma_1-1}I & \cdots & \frac{\lambda}{\lambda\sigma_s-1}I & I \end{array}\right]$ both have full row rank at 0. Thus, by Corollary 5.6(b), $P(\lambda)$ is a linearization of $G(\lambda)$ at $\infty$ of grade 1 with empty state matrix.

## 6. Application of the local linearization theory to NLEIGS pencils

In this section we study in depth the pencils introduced in the influential reference [19] for linearizing rational matrices obtained from approximating NLEPs. This reference presents one of the first systematic approaches for solving large scale NLEPs. For brevity of exposition, and also for recognizing the key contribution of [19], we will call *NLEIGS pencils* to the pencils introduced in [19]. The main goal of this section is to replace the vague usage of the word "linearization" in [19] by a number of rigorous results on NLEIGS pencils which, combined with the results in Sections 4 and 5, establish the precise properties enjoyed with respect to eigenvalues (and poles) of the NLEIGS pencils. We remark that NLEIGS pencils were the initial motivation for developing the results of this paper, since they are not linearizations of the corresponding rational matrix according to the definitions in [1, 5].

As in the rest of the paper, the results in this section are valid and are stated in any algebraically closed field $\mathbb{F}$ that does not include infinity. Note, however, that reference [19] considers only the complex field and that this restriction is important in the approximation phase of the NLEP. Moreover, although [19] deals with regular rational matrices $Q_N(\lambda)$, we will not impose such condition initially in our developments.

Reference [19] uses two families of rational matrices, and corresponding pencils, depending on whether or not a certain low rank structure is present in the original NLEP. We will refer to them as the *NLEIGS basic problem* and the *NLEIGS low rank structured problem*, respectively. The NLEIGS pencils corresponding to each of these two cases will be studied from two perspectives giving rise to the four subsections included in this section. These two perspectives are considering NLEIGS pencils as block full rank pencils and, thus, as linearizations with empty state matrices, and considering them as polynomial system matrices with transfer function matrices equivalent to $Q_N(\lambda)$ everywhere except at a point $\xi_N$. Both perspectives allow us to state in a rigorous sense that NLEIGS pencils are linearizations of $Q_N(\lambda)$, but the one based on block full rank pencils is much simpler, does not require any hypothesis and covers fully the applications of interest in [19]. In contrast, the polynomial system matrix perspective provides more information on $Q_N(\lambda)$ but at the cost of extra hypotheses which are not imposed in [19] and that require considerable effort to check.

### *6.1. The NLEIGS basic problem from the point of view of block full rank pencils*

The families of rational matrices considered in [19] are defined in terms of the following parameters: a list of nodes $(\sigma_0, \sigma_1, \ldots, \sigma_{N-1})$ in $\mathbb{F}$, a list of nonzero poles $(\xi_1, \xi_2, \ldots, \xi_N)$ in $\mathbb{F} \cup \{\infty\}$, and a list of nonzero scaling parameters $(\beta_0, \beta_1, \ldots, \beta_N)$ in $\mathbb{F}$. It is important to bear in mind that [19] assumes that the poles are all distinct from the nodes. However, we do not assume such property, except in a few results where it will be explicitly stated. With these parameters, the following sequence of rational scalar functions is defined:

$$b_0(\lambda) = \frac{1}{\beta_0}, \quad b_i(\lambda) = \frac{1}{\beta_0} \prod_{k=1}^{i} \frac{\lambda - \sigma_{k-1}}{\beta_k(1 - \lambda/\xi_k)}, \quad i = 1, \ldots, N. \tag{17}$$

Let us now define the linear scalar functions

$$g_i(\lambda) := \beta_i \left(1 - \lambda/\xi_i\right), \qquad \text{and} \qquad h_j(\lambda) := \lambda - \sigma_j, \tag{18}$$

for $i = 1, \ldots, N$, and $j = 0, \ldots, N-1$. Then, the rational functions $b_i(\lambda)$ satisfy the simple recursion

$$g_{j+1}(\lambda)\, b_{j+1}(\lambda) = h_j(\lambda)\, b_j(\lambda), \quad j = 0, 1, \ldots, N-1\,,$$

which will be useful in the sequel. Note that the rational functions $b_i(\lambda)$ could not be proper, since for any infinite pole $\xi_i = \infty$ the corresponding factor $1 - \lambda/\xi_i$ is just equal to 1, and, therefore, $b_i(\lambda)$ has a nonconstant polynomial part.

With all this information, we are in the position of introducing the first family of rational matrices considered in [19], whose elements are defined as

$$Q_N(\lambda) = b_0(\lambda)D_0 + b_1(\lambda)D_1 + \cdots + b_N(\lambda)D_N \in \mathbb{F}(\lambda)^{m\times m}, \tag{19}$$

where $D_0, \ldots, D_N \in \mathbb{F}^{m\times m}$ are constant matrices. In this section, all the parameters that allow us to define the considered family of rational matrices are arbitrary. However, in [19] these parameters are carefully chosen in such a way that $Q_N(\lambda)$ approximates satisfactorily the matrix defining the NLEP to be solved in a target set $\Sigma \subset \mathbb{F}$. In this scenario, it is important to stress that the poles $(\xi_1, \ldots, \xi_N)$ are always chosen outside $\Sigma$ [19, p. A2852], which implies that all the zeros of $Q_N(\lambda)$ located in $\Sigma$ are eigenvalues of $Q_N(\lambda)$. Thus, the REP associated with $Q_N(\lambda)$ is an explicit example of a problem with a property that has been mentioned before in this paper, i.e., the poles are located outside the region of interest and, then, it is not needed to compute them. Note, however, the following subtlety: though it is clear that the finite poles of $Q_N(\lambda)$ are included in the list $(\xi_1, \ldots, \xi_N)$, we can construct examples of matrices as in (19) for which some of the finite numbers in $(\xi_1, \ldots, \xi_N)$ are not poles due to some cancellations. Thus, all the finite numbers in $(\xi_1, \ldots, \xi_N)$ are not necessarily finite poles of $Q_N(\lambda)$ and, even more, the partial multiplicities of such poles are not immediately visible from (19). Despite these comments, we will call the numbers $(\xi_1, \ldots, \xi_N)$ poles, following the usage in [19].

In order to solve the REP $Q_N(\lambda)\, y = 0$, the authors of [19] solve the generalized eigenvalue problem corresponding to the pencil

$$L_N(\lambda) = \begin{bmatrix} M_N(\lambda) \\ K_N(\lambda) \end{bmatrix}, \tag{20}$$

where

$$M_N(\lambda) := \begin{bmatrix} \frac{g_N(\lambda)}{\beta_N} D_0 & \frac{g_N(\lambda)}{\beta_N} D_1 & \cdots & \frac{g_N(\lambda)}{\beta_N} D_{N-2} & \frac{g_N(\lambda)}{\beta_N} D_{N-1} + \frac{h_{N-1}(\lambda)}{\beta_N} D_N \end{bmatrix},$$

$$K_N(\lambda) := \begin{bmatrix} -h_0(\lambda) & g_1(\lambda) & & & \\ & -h_1(\lambda) & g_2(\lambda) & & \\ & & \ddots & \ddots & \\ & & & -h_{N-2}(\lambda) & g_{N-1}(\lambda) \end{bmatrix} \otimes I_m.$$

In [19] the use of $L_N(\lambda)$ for solving the REP associated to $Q_N(\lambda)$ is supported by [19, Theorem 3.2], which states that $L_N(\lambda)$ is a strong linearization of the rational matrix $Q_N(\lambda)$ without specifying the exact meaning of "strong linearization" in this rational context. Moreover, the proof of [19, Theorem 3.2] consists of a reference to [2, Theorem 3.1], which is a paper dealing with strong linearizations of *polynomial* matrices in the classical sense of [17]. However, as a consequence of the results in Section 5, it is very easy to prove that $L_N(\lambda)$ is always a linearization of $Q_N(\lambda)$ in a set including the region of interest in [19], as well as at infinity. This is proved in Theorem 6.1, where the nomenclature introduced in Remark 5.4 is used.

**Theorem 6.1.** *Let $Q_N(\lambda)$ be the rational matrix in (19) and $L_N(\lambda)$ be the pencil in (20). Let $\mathcal{P}_N$ and $i_N$ be, respectively, the set of finite poles and the number of infinite poles in the list $(\xi_1, \xi_2, \ldots, \xi_N)$. Then, the following statements hold:*

(a) *$L_N(\lambda)$ partitioned as in (20) is a block full rank pencil with only one block column associated with $Q_N(\lambda)$ in $\mathbb{F} \setminus \mathcal{P}_N$ and, then, $L_N(\lambda)$ is a linearization of $Q_N(\lambda)$ in $\mathbb{F} \setminus \mathcal{P}_N$ with empty state matrix.*

(b) *$L_N(\lambda)$ is a linearization of $Q_N(\lambda)$ at $\infty$ of grade $i_N$ with empty state matrix.*

*Proof.* It is immediate to check that

$$N_N(\lambda) := \frac{1}{1 - \frac{\lambda}{\xi_N}} \begin{bmatrix} b_0(\lambda) & b_1(\lambda) & \cdots & b_{N-1}(\lambda) \end{bmatrix} \otimes I_m \tag{21}$$

is a rational basis dual to $K_N(\lambda)$. Note also that $K_N(\lambda)$ and $N_N(\lambda)$ have both full row rank in $\mathbb{F}\setminus\mathcal{P}_N$. In addition, an easy direct computation proves $M_N(\lambda)N_N(\lambda)^T = Q_N(\lambda)$. Thus, part (a) follows from Theorem 5.3. Observe that (a) can also be proved from Corollary 5.6, since the structures of $Q_N(\lambda)$, $L_N(\lambda)$ and $N_N(\lambda)$ are particular cases of those described in that corollary.

In order to prove part (b), note first that $\operatorname{rev}_1 K_N(\lambda)$ has full row rank at 0. We now consider the rational matrix $\operatorname{rev}_{i_N-1} N_N(\lambda) = \lambda^{i_N-1}N_N\left(\frac{1}{\lambda}\right)$, which is of the form $\lambda^{i_N-1}N_N\left(\frac{1}{\lambda}\right) = \left[\begin{array}{cccc} * & \cdots & * & \frac{\lambda}{\lambda-1/\xi_N}\lambda^{i_N-1}b_{N-1}\left(\frac{1}{\lambda}\right)I_m\end{array}\right]$, where the entries $*$ are defined at 0. Denote by $i_{N-1}$ the number of infinite poles in the list $(\xi_1,\xi_2,\ldots,\xi_{N-1})$. Then, $b_{N-1}\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda^{i_{N-1}}}c(\lambda)$, for a certain rational function $c(\lambda)$ with $c(0)\neq 0$. Thus, we obtain that $\operatorname{rev}_{i_N-1} N_N(\lambda)$ has full row rank at 0, taking into account that $i_{N-1} = i_N$ if $\xi_N\neq\infty$, and $i_{N-1} = i_N - 1$ if $\xi_N = \infty$. Then, part (b) follows from Theorem 5.5. □

Combining Theorems 6.1 and 4.4, we get that $L_N(\lambda)$ contains all the information about the finite eigenvalues of $Q_N(\lambda)$ in $\mathbb{F}\setminus\mathcal{P}_N$, including all type of multiplicities (algebraic, geometric and partial). Moreover, Proposition 4.13 allows us to recover the complete pole-zero structure of $Q_N(\lambda)$ at $\infty$ from the eigenvalue structure at 0 of $\operatorname{rev} L_N(\lambda)$, just by noting that, in this case, $t = 0$ in Proposition 4.13 since we are taking an empty state matrix. We stress that all these results hold for *any* rational matrix $Q_N(\lambda)$ either regular or singular. However, no information is provided on the finite poles of $Q_N(\lambda)$, and some of them could also be zeros. As explained above, this is not an issue in [19], since $\mathcal{P}_N$ is outside the target set $\Sigma$. Nevertheless, at the cost of imposing extra hypotheses, we will solve this problem in Section 6.3 for completeness and also because it is of interest for the theory of REPs.

### *6.2. The NLEIGS low rank problem from the point of view of block full rank pencils*

The second family of rational matrices considered in [19] comes from approximating NLEPs, $A(\lambda)x = 0$, such that the associated matrix $A(\lambda)$ is the sum of a polynomial matrix plus a matrix of the form $\sum_{i=1}^{n} C_i f_i(\lambda)$, where the constant matrices $C_i$ have much smaller rank than the size of $A(\lambda)$ and $f_i(\lambda)$ are scalar nonlinear functions of $\lambda$. This type of NLEPs arise in several applications [18] and are approximated in [19, eq. (6.2)] by a family of rational matrices of the form

$$\widetilde{Q}_N(\lambda) = \sum_{i=0}^{p} b_i(\lambda)\,\widetilde{D}_i + \sum_{i=p+1}^{N} b_i(\lambda)\,\widetilde{L}_i\,\widetilde{U}^T \in \mathbb{F}(\lambda)^{m\times m}, \tag{22}$$

where $b_0(\lambda),\ldots,b_N(\lambda)$ are the scalar rational functions in (17), $\widetilde{D}_0,\ldots,\widetilde{D}_p\in\mathbb{F}^{m\times m}$, $\widetilde{L}_{p+1},\ldots,\widetilde{L}_N\in\mathbb{F}^{m\times r}$ and $\widetilde{U}\in\mathbb{F}^{m\times r}$ are constant matrices, and $r\ll m$. For the functions in (18), let us consider the simpler notation $h_i := h_i(\lambda)$ and $g_i := g_i(\lambda)$. Then, in order to solve the REP $\widetilde{Q}_N(\lambda)y = 0$ efficiently by taking advantage of the low rank structure of $\widetilde{Q}_N(\lambda)$, the following pencil is introduced in [19, Sec. 6.4]:

$$\widetilde{L}_N(\lambda) = \begin{bmatrix} \widetilde{M}_N(\lambda) \\ \widetilde{K}_N(\lambda) \end{bmatrix}, \tag{23}$$

where

$$\widetilde{M}_N(\lambda) := \begin{bmatrix} \frac{g_N}{\beta_N}\widetilde{D}_0 & \frac{g_N}{\beta_N}\widetilde{D}_1 & \cdots & \frac{g_N}{\beta_N}\widetilde{D}_p & \frac{g_N}{\beta_N}\widetilde{L}_{p+1} & \cdots & \frac{g_N}{\beta_N}\widetilde{L}_{N-2} & \frac{g_N}{\beta_N}\widetilde{L}_{N-1} + \frac{h_{N-1}}{\beta_N}\widetilde{L}_N \end{bmatrix},$$

$$\widetilde{K}_N(\lambda) := \begin{bmatrix} -h_0 I_m & g_1 I_m & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -h_{p-1} I_m & g_p I_m & & & & \\ & & & -h_p \widetilde{U}^T & g_{p+1} I_r & & & \\ & & & & -h_{p+1} I_r & g_{p+2} I_r & & \\ & & & & & \ddots & \ddots & \\ & & & & & & -h_{N-2} I_r & g_{N-1} I_r \end{bmatrix}.$$

A result analogous to Theorem 6.1 can be proved for the pencil $\widetilde{L}_N(\lambda)$ and the matrix $\widetilde{Q}_N(\lambda)$. This is accomplished in Theorem 6.2. We remark, nevertheless, that the result concerning the linearizations at $\infty$ is weaker in Theorem 6.2 than in Theorem 6.1. This is an unavoidable consequence of the used approach and the low rank structure of $\widetilde{Q}_N(\lambda)$.

**Theorem 6.2.** *Let $\widetilde{Q}_N(\lambda)$ be the rational matrix in (22) and $\widetilde{L}_N(\lambda)$ be the pencil in (23). Let $\mathcal{P}_N$ and $i_N$ be, respectively, the set of finite poles and the number of infinite poles in the list $(\xi_1, \xi_2, \ldots, \xi_N)$. Then, the following statements hold:*

(a) *$\widetilde{L}_N(\lambda)$ partitioned as in (23) is a block full rank pencil with only one block column associated with $\widetilde{Q}_N(\lambda)$ in $\mathbb{F} \setminus \mathcal{P}_N$ and, then, $\widetilde{L}_N(\lambda)$ is a linearization of $\widetilde{Q}_N(\lambda)$ in $\mathbb{F} \setminus \mathcal{P}_N$ with empty state matrix.*

(b) *If, in addition, the poles $\xi_{p+1}, \xi_{p+2}, \ldots, \xi_{N-1}$ are all finite, then $\widetilde{L}_N(\lambda)$ is a linearization of $\widetilde{Q}_N(\lambda)$ at $\infty$ of grade $i_N$ with empty state matrix.*

*Proof.* The proof is similar to that of Theorem 6.1 with some differences coming from the presence of the low rank term in $\widetilde{Q}_N(\lambda)$. It is immediate to check that

$$\widetilde{N}_N(\lambda) = \frac{1}{1 - \frac{\lambda}{\xi_N}} \begin{bmatrix} b_0(\lambda) I_m & \cdots & b_p(\lambda) I_m & b_{p+1}(\lambda)\widetilde{U} & \cdots & b_{N-1}(\lambda)\widetilde{U} \end{bmatrix} \tag{24}$$

is a rational basis dual to $\widetilde{K}_N(\lambda)$, that $\widetilde{K}_N(\lambda)$ and $\widetilde{N}_N(\lambda)$ have both full row rank in $\mathbb{F}\setminus\mathcal{P}_N$ and that $\widetilde{M}_N(\lambda)\widetilde{N}_N(\lambda)^T = \widetilde{Q}_N(\lambda)$. Thus, part (a) follows from Theorem 5.3.

In order to prove part (b), note first that $\operatorname{rev}_1 \widetilde{K}_N(\lambda)$ has full row rank at 0 as a consequence of the fact that the poles $\xi_{p+1}, \xi_{p+2}, \ldots, \xi_{N-1}$ are all finite. We now consider the rational matrix $\operatorname{rev}_{i_N - 1} \widetilde{N}_N(\lambda) = \lambda^{i_N - 1}\widetilde{N}_N\left(\frac{1}{\lambda}\right)$, which is of the form

$$\lambda^{i_N-1}\widetilde{N}_N\left(\frac{1}{\lambda}\right) = \begin{bmatrix} * & \cdots & * & \frac{\lambda}{\lambda - 1/\xi_N}\lambda^{i_N-1} b_p\left(\frac{1}{\lambda}\right) I_m & * & \cdots & * \end{bmatrix},$$

where the entries $*$ are defined at 0. Denote by $i_p$ the number of infinite poles in the list $(\xi_1, \xi_2, \ldots, \xi_p)$. Then, $b_p\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda^{i_p}}\tilde{c}(\lambda)$ for a certain rational function $\tilde{c}(\lambda)$ with $\tilde{c}(0) \neq 0$. Taking into account that the poles $\xi_{p+1}, \xi_{p+2}, \ldots, \xi_{N-1}$ are all finite, we have that $i_p = i_N$ if $\xi_N \neq \infty$, and $i_p = i_N - 1$ if $\xi_N = \infty$. Therefore, $\operatorname{rev}_{i_N-1} \widetilde{N}_N(\lambda)$ has full row rank at 0 because $\tilde{c}(0) \neq 0$. Thus, part (b) follows from Theorem 5.5. $\square$

A discussion similar to the one in the last paragraph of Section 6.1 can be developed on the basis of Theorem 6.2. The details are omitted for brevity. The open problem corresponding to the information of the finite poles will be solved in Section 6.4.

*6.3. The NLEIGS basic problem from the point of view of polynomial system matrices*

As discussed previously, the approach presented in Section 6.1 to the NLEIGS pencil $L_N(\lambda)$ in (20) considers $L_N(\lambda)$ as a linearization with empty state matrix and, thus, it does not provide any information on the finite poles of $Q_N(\lambda)$. In order to get this information, we need to identify a convenient square regular submatrix $A_N(\lambda)$ of $L_N(\lambda)$ that may be used as state matrix. The block structure of $L_N(\lambda)$ makes it not possible to find such a matrix $A_N(\lambda)$ in a way that it includes the information of all the potential poles $(\xi_1, \dots, \xi_N)$. This is related with the comment included in [19, p. A2849] on the fact that $\xi_N$ plays a special role and that it is convenient to choose $\xi_N = \infty$. In what follows we will *not* assume that $\xi_N = \infty$, though the obtained results are simpler and stronger under such assumption, but we will focus on getting information on the finite poles in $(\xi_1, \dots, \xi_{N-1})$. With this spirit, we consider the following partition of $L_N(\lambda)$ in (20), where $A_N(\lambda)$ will play the role of the state matrix,

$$L_N(\lambda) =: \left[\begin{array}{c|c} D_N(\lambda) & -C_N(\lambda) \\ \hline B_N(\lambda) & A_N(\lambda) \end{array}\right], \quad \text{where } D_N(\lambda) := \left(1 - \tfrac{\lambda}{\xi_N}\right) D_0, \tag{25}$$

and the rest of the blocks are easily described from the blocks in (20). With this partition, the next technical lemma reveals the transfer function matrix of $L_N(\lambda)$ and establishes necessary and sufficient conditions for $L_N(\lambda)$ to be minimal in the whole field $\mathbb{F}$. By definition, $L_N(\lambda)$ is minimal in $\mathbb{F}$ if $\begin{bmatrix} B_N(\lambda_0) & A_N(\lambda_0) \end{bmatrix} \in \mathbb{F}^{m(N-1)\times mN}$ and $\begin{bmatrix} -C_N(\lambda_0)^T & A_N(\lambda_0)^T \end{bmatrix}^T \in \mathbb{F}^{mN\times m(N-1)}$ have, respectively, full row and column rank for all $\lambda_0 \in \mathbb{F}$. The conditions in Lemma 6.3(b) require to evaluate the rational matrix $R_N(\lambda)$ of size $m \times m$, which for practical problems is much smaller than $m(N-1) \times mN$.

**Lemma 6.3.** *Let us consider the pencil $L_N(\lambda)$ in* (20) *as a polynomial system matrix with state matrix $A_N(\lambda)$, where $A_N(\lambda)$ is defined through the partition* (25)*, and let $Q_N(\lambda)$ be the rational matrix in* (19)*. Then the following statements hold:*

(a) *The transfer function matrix of $L_N(\lambda)$ is $\beta_0 \left(1 - \frac{\lambda}{\xi_N}\right) Q_N(\lambda)$.*

(b) *Let us define the rational matrix $R_N(\lambda) := (Q_N(\lambda) - b_0(\lambda) D_0)/b_N(\lambda)$, whose explicit expression is*

$$R_N(\lambda) = \sum_{j=1}^{N-1} \left( \prod_{k=j+1}^{N} \frac{g_k(\lambda)}{h_{k-1}(\lambda)} \right) D_j \,+\, D_N \in \mathbb{F}(\lambda)^{m\times m}, \tag{26}$$

*let $\mathcal{P}_{N-1}$ be the set of finite poles in the list $(\xi_1, \xi_2, \dots, \xi_{N-1})$, and assume $\xi_i \neq \sigma_j$, $1 \le i \le N$, $0 \le j \le N-1$. Then, $L_N(\lambda)$ is minimal in $\mathbb{F}$ if and only if the matrix $R_N(\xi_k) \in \mathbb{F}^{m\times m}$ is nonsingular for all $\xi_k \in \mathcal{P}_{N-1}$.*

*Proof.* The computation of the transfer function matrix of $L_N(\lambda)$ is easy because $B_N(\lambda) = \begin{bmatrix} -h_0(\lambda)I_m & 0 & \cdots & 0 \end{bmatrix}^T$, which implies that only the first block column of $A_N(\lambda)^{-1}$ is needed. It is immediate to check that $\frac{1}{b_1(\lambda)g_1(\lambda)} [b_1(\lambda) \quad \cdots \quad b_{N-1}(\lambda)]^T \otimes I_m$ is that first block column. The rest of the proof of part (a) is just an elementary algebraic manipulation. The proof of part (b) is elementary but long. It is in [12, Appendix A]. $\square$

The constant matrix $A_N(\lambda_0)$ is invertible for any $\lambda_0 \in \mathbb{F} \setminus \mathcal{P}_{N-1}$ and, so, $L_N(\lambda)$ is minimal in $\mathbb{F} \setminus \mathcal{P}_{N-1}$. Combining this with the fact that $Q_N(\lambda)$ and $\beta_0 \left(1 - \frac{\lambda}{\xi_N}\right) Q_N(\lambda)$ are equivalent in $\mathbb{F}$ if $\xi_N = \infty$ or in $\mathbb{F} \setminus \{\xi_N\}$ if $\xi_N$ is finite, we immediately obtain from Definition 4.1 that $L_N(\lambda)$ is a linearization of $Q_N(\lambda)$ with state matrix $A_N(\lambda)$ in $\mathbb{F} \setminus \mathcal{P}_N$, which is a result analogous to Theorem 6.1(a). This approach, of course, does not give any information on the finite poles of $Q_N(\lambda)$, because the finite eigenvalues of $A_N(\lambda)$ coincide with $\mathcal{P}_{N-1}$. Such information is obtained from the next result, which is the main result of this section and is a corollary of Lemma 6.3.

**Theorem 6.4.** *Let $Q_N(\lambda)$ be the rational matrix in* (19)*, $L_N(\lambda)$ be the pencil in* (20)*, $A_N(\lambda)$ be the submatrix of $L_N(\lambda)$ in* (25)*, and $R_N(\lambda)$ be the rational matrix in* (26)*. Consider $\mathcal{P}_{N-1}$ the set of finite poles in the list $(\xi_1, \xi_2, \ldots, \xi_{N-1})$, and assume $\xi_i \neq \sigma_j$, $1 \leq i \leq N$, $0 \leq j \leq N-1$. If $R_N(\xi_k) \in \mathbb{F}^{m\times m}$ is nonsingular for every $\xi_k \in \mathcal{P}_{N-1}$, then $L_N(\lambda)$ is a linearization of $Q_N(\lambda)$ with state matrix $A_N(\lambda)$ in $\mathbb{F}$, if $\xi_N = \infty$, or in $\mathbb{F} \setminus \{\xi_N\}$, if $\xi_N$ is finite.*

*Proof.* Under the hypotheses of Theorem 6.4, $L_N(\lambda)$ is minimal in $\mathbb{F}$. Moreover, its transfer function matrix, i.e., $\beta_0 \left(1 - \frac{\lambda}{\xi_N}\right) Q_N(\lambda)$ is equivalent to $Q_N(\lambda)$ in $\mathbb{F}$, if $\xi_N = \infty$, or in $\mathbb{F} \setminus \{\xi_N\}$, if $\xi_N$ is finite. $\square$

We emphasize that the hypotheses that the constant matrices $R_N(\xi_k)$ in Theorem 6.4 are nonsingular are not mentioned at all in [19], but, fortunately, are generic, in the sense that they are satisfied by almost all regular rational matrices $Q_N(\lambda)$ expressed as in (19).

**Remark 6.5.** Under the conditions of Theorem 6.4, the pole elementary divisors of $Q_N(\lambda)$ in $\mathbb{F}$, if $\xi_N = \infty$, or in $\mathbb{F} \setminus \{\xi_N\}$, if $\xi_N$ is finite, are the elementary divisors of $A_N(\lambda)$, as a consequence of Theorem 4.4. These elementary divisors can be easily determined as follows: first express $A_N(\lambda) = \widehat{A}_N(\lambda) \otimes I_m$; second note that if $\widehat{S}_N(\lambda)$ is the Smith form of $\widehat{A}_N(\lambda)$, then $\widehat{S}_N(\lambda) \otimes I_m$ is the Smith form of $A_N(\lambda)$; third, use the fact that $\xi_i \neq \sigma_j$, $1 \leq i \leq N$, $0 \leq j \leq N-1$, to prove that the greatest common divisor of all $(N-2) \times (N-2)$ minors of $\widehat{A}_N(\lambda)$ is equal to 1, which implies, according to [15, Ch. VI], that there is only one invariant polynomial of $\widehat{S}_N(\lambda)$ different from 1 and that is equal to $p(\lambda) = c\,(1 - \lambda/\xi_1) \cdots (1 - \lambda/\xi_{N-1})$, where $c \in \mathbb{F}$ is a constant that makes $p(\lambda)$ monic. Finally, we get that $A_N(\lambda)$ has $m$ invariant polynomials different from 1 all equal to $p(\lambda)$. This allows us to obtain easily the finite elementary divisors of $A_N(\lambda)$ and, thus, the finite pole elementary divisors of $Q_N(\lambda)$ (in $\mathbb{F}$ if $\xi_N = \infty$, or in $\mathbb{F} \setminus \{\xi_N\}$ if $\xi_N$ is finite). In particular, they are of the form $(\lambda - \xi_i)^{\nu_i}$ and, in order to obtain the partial multiplicities $\nu_i$, we have to take into account possible repetitions in $(\xi_1, \ldots, \xi_{N-1})$. Observe that the infinite $\xi_i$ for $i = 1, \ldots, N-1$ do not contribute at all to the finite pole elementary divisors of $Q_N(\lambda)$. Moreover, if $\xi_N = \infty$, then we can state the compact and simple result that the $m$ denominators of the global Smith–McMillan form of $Q_N(\lambda)$ are all equal to $p(\lambda)$.

However, with this choice of state matrix, there is no way of obtaining information on the pole structure of $\xi_N$ when it is finite. This is the reason why, even if $L_N(\lambda)$ is minimal in $\mathbb{F}$, $L_N(\lambda)$ is not a linearization of $Q_N(\lambda)$ in $\mathbb{F}$.

*6.4. The NLEIGS low rank problem from the point of view of polynomial system matrices*

The results in this section are the counterpart for $\widetilde{Q}_N(\lambda)$ in (22) and $\widetilde{L}_N(\lambda)$ in (23) of those presented in Section 6.3 for $Q_N(\lambda)$ and $L_N(\lambda)$. For this purpose, we consider the following partition of $\widetilde{L}_N(\lambda)$ in (23), where $\widetilde{A}_N(\lambda)$ will play the role of the state matrix,

$$\widetilde{L}_N(\lambda) =: \left[\begin{array}{c|c} \widetilde{D}_N(\lambda) & -\widetilde{C}_N(\lambda) \\ \hline \widetilde{B}_N(\lambda) & \widetilde{A}_N(\lambda) \end{array}\right], \quad \text{where } \widetilde{D}_N(\lambda) = \left(1 - \tfrac{\lambda}{\xi_N}\right) \widetilde{D}_0, \tag{27}$$

and the rest of the blocks are easily described from the blocks in (23). The next lemma is the counterpart of Lemma 6.3. Note that the low rank structure in $\widetilde{Q}_N(\lambda)$ complicates the minimality conditions in part (b) of Lemma 6.6, which are expressed in terms of matrices of size $(2m+r) \times (m+r)$.

**Lemma 6.6.** *Let us consider the pencil $\widetilde{L}_N(\lambda)$ in (23) as a polynomial system matrix with state matrix $\widetilde{A}_N(\lambda)$, where $\widetilde{A}_N(\lambda)$ is defined through the partition (27), and let $\widetilde{Q}_N(\lambda)$ be the rational matrix in (22). Then the following statements hold:*

(a) *The transfer function matrix of $\widetilde{L}_N(\lambda)$ is $\beta_0 \left(1 - \tfrac{\lambda}{\xi_N}\right) \widetilde{Q}_N(\lambda)$.*

(b) *Let us define the rational matrices*

$$\widetilde{R}_N^{(1)}(\lambda) := \frac{g_N}{h_{N-1}} \left[ \sum_{j=1}^{p-1} \left( \prod_{k=j+1}^{p} \frac{g_k}{h_{k-1}} \right) \widetilde{D}_j \, + \, \widetilde{D}_p \right] \in \mathbb{F}(\lambda)^{m\times m},$$

$$\widetilde{R}_N^{(2)}(\lambda) := \sum_{j=p+1}^{N-1} \left( \prod_{k=j+1}^{N} \frac{g_k}{h_{k-1}} \right) \widetilde{L}_j \, + \, \widetilde{L}_N \in \mathbb{F}(\lambda)^{m\times r},$$

$$\widetilde{R}_N(\lambda) := \left[\begin{array}{c|c} \widetilde{R}_N^{(1)}(\lambda) & \widetilde{R}_N^{(2)}(\lambda) \\ \hline \left(\prod_{i=1}^{p-1} \frac{g_i}{h_i}\right) g_p \, I_m & 0 \\ \hline -\,h_p\, \widetilde{U}^T & \left(\prod_{i=p+1}^{N-2} \frac{g_i}{h_i}\right) g_{N-1}\, I_r \end{array}\right]. \tag{28}$$

*Let $\mathcal{P}_{N-1}$ be the set of finite poles in the list $(\xi_1, \xi_2, \ldots, \xi_{N-1})$, and assume that* $\operatorname{rank} \widetilde{U} = r$ *and that $\xi_i \neq \sigma_j$, $1 \le i \le N$, $0 \le j \le N-1$. Then, $\widetilde{L}_N(\lambda)$ is minimal in $\mathbb{F}$ if and only if the matrix $\widetilde{R}_N(\xi_k) \in \mathbb{F}^{(2m+r)\times(m+r)}$ has full column rank for all $\xi_k \in \mathcal{P}_{N-1}$.*

*Proof.* The proof of part (a) is similar to that of Lemma 6.3(a) with some differences coming from the presence of the low rank term in $\widetilde{Q}_N(\lambda)$. The computation of the transfer function matrix of $\widetilde{L}_N(\lambda)$ is easy because, again, $\widetilde{B}_N(\lambda) = \begin{bmatrix} -h_0 I_m & 0 & \cdots & 0 \end{bmatrix}^T$, and only the first block column of $\widetilde{A}_N(\lambda)^{-1}$ is needed, which, in this case, is equal to $\frac{1}{b_1(\lambda)g_1} \begin{bmatrix} b_1(\lambda)I_m & \cdots & b_p(\lambda)I_m & b_{p+1}(\lambda)\widetilde{U} & \cdots & b_{N-1}(\lambda)\widetilde{U} \end{bmatrix}^T$. The proof of part (b) is elementary but long. It is in [12, Appendix B]. □

**Remark 6.7.** If, in addition to $\operatorname{rank}\widetilde{U} = r$ and $\xi_i \neq \sigma_j$, $1 \leq i \leq N$, $0 \leq j \leq N-1$, we assume that $\xi_1 = \cdots = \xi_p = \infty$, then the necessary and sufficient conditions for minimality in Lemma 6.6(b) can be considerably simplified, since we get as an immediate corollary of Lemma 6.6(b) that "$\widetilde{L}_N(\lambda)$ is minimal in $\mathbb{F}$ if and only if the matrix $\widetilde{R}_N^{(2)}(\xi_k) \in \mathbb{F}^{m\times r}$ has full column rank for every $\xi_k \in \mathcal{P}_{N-1}$". Note that the hypothesis $\xi_1 = \cdots = \xi_p = \infty$ implies that the "no-low rank" term $\sum_{i=0}^{p} b_i(\lambda)\widetilde{D}_i$ of $\widetilde{Q}_N(\lambda)$ is a polynomial matrix, as often happens in NLEPs [19].

Observe also that if $\widehat{R}_N(\lambda)$ is the $(m+r)\times(m+r)$ matrix obtained from $\widetilde{R}_N(\lambda)$ in (28) by removing the second block row then, under the assumptions $\operatorname{rank}\widetilde{U} = r$ and $\xi_i \neq \sigma_j$, $1 \leq i \leq N$, $0 \leq j \leq N-1$, we get, as another immediate corollary of Lemma 6.6(b), the following sufficient condition for minimality: "if $\widehat{R}_N(\xi_k) \in \mathbb{F}^{(m+r)\times(m+r)}$ is invertible for every $\xi_k \in \mathcal{P}_{N-1}$ then $\widetilde{L}_N(\lambda)$ is minimal in $\mathbb{F}$".

Theorem 6.8 is the main result in this section and is an easy corollary of Lemma 6.6. Its proof is omitted because it is very similar to that of Theorem 6.4.

**Theorem 6.8.** *Let $\widetilde{Q}_N(\lambda)$ be the rational matrix in* (22)*, $\widetilde{L}_N(\lambda)$ be the pencil in* (23)*, $\widetilde{A}_N(\lambda)$ be the submatrix of $\widetilde{L}_N(\lambda)$ in* (27)*, and $\widetilde{R}_N(\lambda)$ be the rational matrix in* (28)*. Consider $\mathcal{P}_{N-1}$ the set of finite poles in the list $(\xi_1, \xi_2, \ldots, \xi_{N-1})$. If $\operatorname{rank}\widetilde{U} = r$, $\xi_i \neq \sigma_j$, $1 \leq i \leq N$, $0 \leq j \leq N-1$, and $\widetilde{R}_N(\xi_k) \in \mathbb{F}^{(2m+r)\times(m+r)}$ has full column rank for every $\xi_k \in \mathcal{P}_{N-1}$, then $\widetilde{L}_N(\lambda)$ is a linearization of $\widetilde{Q}_N(\lambda)$ with state matrix $\widetilde{A}_N(\lambda)$ in $\mathbb{F}$, if $\xi_N = \infty$, or in $\mathbb{F} \setminus \{\xi_N\}$, if $\xi_N$ is finite.*

Finally, note that the conditions in Theorem 6.8 on the full column rank of the matrices $\widetilde{R}_N(\xi_k)$ can be simplified as in Remark 6.7 under extra hypotheses.

## 7. Conclusions and future work

A theory of local linearizations of rational matrices has been carefully presented in this paper, by developing as starting point the extension of Rosenbrock's minimal polynomial system matrices to a local scenario. Moreover, this theory has been applied to a number of pencils that have appeared recently in some influential papers on solving numerically NLEPs by combining rational approximations, linearizations of the resulting rational matrices, and efficient numerical algorithms for generalized eigenvalue problems adapted to the structure of such linearizations. It has been emphasized throughout the paper that the theory of local linearizations allows us to view these pencils, and to explain their properties, from rather different perspectives, which depend on the particular choice of the submatrix of the pencil to be considered as state matrix. In particular, we have seen that the choice of an empty state matrix is simple and adequate for those rational matrices and pencils arising in NLEPs, when the poles are already known from the approximation process. This has led us to define and analyze the very general family of block full rank pencils, as a template that covers many of the pencils, available in the literature, that linearize the rational approximations in the corresponding target set. We plan to extend these ideas, and other ways to choose the state matrices will be explored. Finally, we also plan to study numerical properties of some of the linearizations analyzed in this work. In particular, given a linearization of the REP in a set, it is important to study the backward stability in terms of the structure of the rational matrix defining the REP when applying a numerical method to compute the eigenvalues of the linearization. In addition, we plan to investigate the conditioning of eigenvalues, that is, the sensitivity to perturbations, both in the original REP and its linearization, of a zero that is not a pole of the rational matrix.

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