A different point of view: Linearizations of matrix polynomials as Rosenbrock's system matrices

Froilán M. Dopico[a,1], Silvia Marcaida[b,2], María C. Quintana[c,3], Paul Van Dooren.[d,4]

[a]*Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain.*

[b]*Departamento de Matemáticas, Universidad del País Vasco UPV/EHU, Apdo. Correos 644, Bilbao 48080, Spain.*

[c]*Aalto University, Department of Mathematics and Systems Analysis, P.O. Box 11100, FI-00076, Aalto, Finland.*

[d]*Department of Mathematical Engineering, Université catholique de Louvain, Avenue Georges Lemaître 4, B-1348 Louvain-la-Neuve, Belgium.*

Abstract

A well known method to solve the Polynomial Eigenvalue Problem (PEP) is via linearization. That is, transforming the PEP into a generalized linear eigenvalue problem with the same spectral information. Linearizations of matrix polynomials are defined using unimodular transformations. In this paper we establish a connection between the standard definition of linearization for matrix polynomials introduced by Gohberg, Lancaster and Rodman and the notion of polynomial system matrix introduced by Rosenbrock. This connection gives new techniques to show that a matrix pencil is a linearization of the corresponding matrix polynomial arising in a PEP.

Keywords: polynomial matrix, polynomial eigenvalue problem, linearization, polynomial system matrix

AMS subject classifications: 15A22, 93B18, 93C05

1. Introduction

Let $\mathbb{F}$ be an arbitrary field, and let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. $\mathbb{F}[\lambda]$ denotes the ring of polynomials with coefficients in $\mathbb{F}$, and $\mathbb{F}(\lambda)$ the field of rational functions over $\mathbb{F}[\lambda]$. The sets of $p \times m$ matrices with elements in $\mathbb{F}$, $\mathbb{F}[\lambda]$ and $\mathbb{F}(\lambda)$ are denoted by $\mathbb{F}^{p\times m}$, $\mathbb{F}[\lambda]^{p\times m}$ and $\mathbb{F}(\lambda)^{p\times m}$, respectively. The elements of $\mathbb{F}[\lambda]^{p\times m}$ are called matrix polynomials or polynomial matrices, and the elements of $\mathbb{F}(\lambda)^{p\times m}$ are called rational matrices.

Email addresses: `dopico@math.uc3m.es` (Froilán M. Dopico), `silvia.marcaida@ehu.eus` (Silvia Marcaida), `maria.quintanaponce@aalto.fi` (María C. Quintana), `paul.vandooren@uclouvain.be` (Paul Van Dooren.)

[1]Supported by

[2]Supported by

[3]Supported by an Academy of Finland grant (Suomen Akatemian päätös 331230) and by the Agencia Estatal de Investigación of Spain through grant PID2019-106362GB-I00 MCIN/AEI/10.13039/501100011033/.

[4]Supported by an Aalto Science Institute Visitor Programme

A polynomial matrix $P(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ can always be written in the form

$$P(\lambda) = P_k\lambda^k + P_{k-1}\lambda^{k-1} + \cdots + P_1\lambda + P_0, \tag{1}$$

where $P_k, \ldots, P_1, P_0 \in \mathbb{F}^{p\times m}$ with $P_k \neq 0$. The scalar k is then called the degree of $P(\lambda)$, and it is denoted by $\deg P(\lambda)$. Polynomial matrices of degree 1 or 0, i.e., linear polynomial matrices, are called pencils.

The (finite) eigenvalues of a polynomial matrix $P(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ are defined as the scalars $\lambda_0 \in \overline{\mathbb{F}}$ such that

$$\operatorname{rank} P(\lambda_0) < \max_{\mu\in\overline{\mathbb{F}}} \operatorname{rank} P(\mu).$$

The Polynomial Eigenvalue Problem (PEP) consists of finding the eigenvalues of $P(\lambda)$. If $P(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ is regular (i.e., square with $\det P(\lambda) \not\equiv 0$), the PEP is equivalent to the problem of finding scalars $\lambda_0 \in \overline{\mathbb{F}}$ such that there exist nonzero constant vectors $x \in \overline{\mathbb{F}}^{m\times 1}$ and $y \in \overline{\mathbb{F}}^{m\times 1}$ satisfying

$$P(\lambda_0)x = 0 \quad \text{and} \quad y^T P(\lambda_0) = 0,$$

respectively. The vectors x are called right eigenvectors associated with λ_0, and the vectors y are called left eigenvectors associated with λ_0.

To solve the PEP, in the eighties Gohberg, Lancaster and Rodman [13] introduced the notion of *linearization* of a matrix polynomial. Given a matrix polynomial $P(\lambda)$ of degree $k > 1$, a linearization of $P(\lambda)$ is a pencil $L(\lambda) := \lambda L_1 + L_0$ such that there exist unimodular matrices (i.e., square polynomial matrices with nonzero constant determinant) $U_1(\lambda)$ and $V_1(\lambda)$ satisfying

$$U_1(\lambda)L(\lambda)V_1(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_s \end{bmatrix}, \tag{2}$$

where I_s denotes the identity matrix of size an integer $s \geq 0$. It is known that $L(\lambda)$ has the same finite eigenvalues with the same partial multiplicities as $P(\lambda)$. $L(\lambda)$ is said to be a strong linearization of $P(\lambda)$ if, in addition, there exist unimodular matrices $U_2(\lambda)$ and $V_2(\lambda)$ satisfying

$$U_2(\lambda) \operatorname{rev}_1 L(\lambda)V_2(\lambda) = \begin{bmatrix} \operatorname{rev}_\ell P(\lambda) & 0 \\ 0 & I_s \end{bmatrix}, \tag{3}$$

where $\operatorname{rev}_1 L(\lambda) := \lambda L_0 + L_1$ and $\operatorname{rev}_\ell P(\lambda) := \lambda^\ell P(1/\lambda)$ for some $\ell \geq \deg P(\lambda)$ [12]. Here we remark that to have an equivalent definition of strong linearization, it is enough if the matrices $U_2(\lambda)$ and $V_2(\lambda)$ in (3) are invertible at 0 (i.e., $U_2(0)$ and $V_2(0)$ are invertible) instead of unimodular (see the proof of Proposition 3.3). In that case we say that $\operatorname{rev}_1 L(\lambda)$ and $\operatorname{diag}(\operatorname{rev}_\ell P(\lambda), I_s)$ are equivalent at 0.

However, already in the seventies Rosenbrock [16] introduced the notion of *polynomial system matrix* $S(\lambda)$ of a rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$. That is, a matrix polynomial of the form

$$S(\lambda) := \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)} \tag{4}$$

with $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ regular, such that its Schur complement with respect to $A(\lambda)$ is $G(\lambda)$, i.e,

$$G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda).$$

The rational matrix $G(\lambda)$ is called the transfer function matrix of $S(\lambda)$ and the matrix polynomial $A(\lambda)$ is called the state matrix of $S(\lambda)$. Although the state matrix $A(\lambda)$ will appear in the $(1,1)$-block of $S(\lambda)$ in the theory, it can be at any place in $S(\lambda)$. The important property is that $G(\lambda)$ is the Schur complement of $A(\lambda)$ in $S(\lambda)$.

Under minimality conditions, Rosenbrock showed that polynomial system matrices contain the pole and zero information of their transfer function matrices [16]. Poles and zeros of rational matrices are defined through the notion of the Smith–McMillan form, that we state in what follows (see [15], or [16] for a more recent reference).

Definition 1.1 (Smith–McMillan form)**.** *For any rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ there exist unimodular matrices $U_1(\lambda) \in \mathbb{F}[\lambda]^{p\times p}$ and $U_2(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ such that*

$$U_1(\lambda)G(\lambda)U_2(\lambda) = \begin{bmatrix} \operatorname{diag}\left(\dfrac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \dfrac{\epsilon_r(\lambda)}{\psi_r(\lambda)}\right) & 0 \\ 0 & 0_{(p-r)\times(m-r)} \end{bmatrix} \tag{5}$$

where r is the normal rank of $G(\lambda)$ and, for $i = 1, \dots, r$, $\dfrac{\epsilon_i(\lambda)}{\psi_i(\lambda)}$ are nonzero irreducible rational functions with $\epsilon_i(\lambda)$ and $\psi_i(\lambda)$ monic polynomials (i.e., with leading coefficient equal to 1) that satisfy the divisibility chains $\epsilon_1(\lambda) \mid \cdots \mid \epsilon_r(\lambda)$ and $\psi_r(\lambda) \mid \cdots \mid \psi_1(\lambda)$. The diagonal matrix in (5) is called the Smith–McMillan form of $G(\lambda)$.

The rational functions $\epsilon_i(\lambda)/\psi_i(\lambda)$ in (5) are called the invariant rational functions of $G(\lambda)$ and the finite poles and zeros of $G(\lambda)$ are the roots in $\overline{\mathbb{F}}$ of the denominators and numerators of the invariant rational functions, respectively. If $G(\lambda)$ is a polynomial matrix then $\psi_i(\lambda) = 1$, for $i = 1, \dots, r$, and the diagonal matrix in (5) is just called the Smith form of $G(\lambda)$. In addition, for any $\lambda_0 \in \overline{\mathbb{F}}$ each polynomial $\epsilon_i(\lambda)$ can be factored as $\epsilon_i(\lambda) = (\lambda - \lambda_0)^{\alpha_i} p_i(\lambda)$ with $p_i(0) \neq 0$ and $\alpha_i \geq 0$. The factors $(\lambda - \lambda_0)^{\alpha_i}$ with $\alpha_i \neq 0$ are called the elementary divisors for the eigenvalue λ_0.

We now introduce the notion of minimality of a polynomial system matrix, in order to relate its zeros with the poles and zeros of its transfer function matrix. A polynomial system matrix $S(\lambda)$ as in (4) is said to be minimal if

$$\operatorname{rank} \begin{bmatrix} A(\lambda_0) & -B(\lambda_0) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A(\lambda_0) \\ C(\lambda_0) \end{bmatrix} = n \tag{6}$$

for all $\lambda_0 \in \overline{\mathbb{F}}$. Then, Rosenbrock proved the following result [16], about the recovery of the pole and zero information of a rational matrix from a minimal polynomial system matrix.

Theorem 1.2. *Let $S(\lambda)$ as in (4) be a minimal polynomial system matrix, with state matrix $A(\lambda)$, whose transfer function matrix is $G(\lambda)$. Then the finite eigenvalues of $A(\lambda)$ are the finite poles of $G(\lambda)$, and the finite eigenvalues of $S(\lambda)$ are the finite zeros of $G(\lambda)$.*

In the particular case of $A(\lambda)$ in (4) being unimodular $G(\lambda)$, has no finite poles, i.e., $G(\lambda)$ is a polynomial matrix, and condition (6) is satisfied. In addition, we have the following unimodular equivalence:

$$\underbrace{\begin{bmatrix} C(\lambda)A^{-1}(\lambda) & I_p \\ A^{-1}(\lambda) & 0 \end{bmatrix}}_{\text{unimodular}} \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \underbrace{\begin{bmatrix} -A^{-1}(\lambda)B(\lambda) & I_n \\ I_m & 0 \end{bmatrix}}_{\text{unimodular}} = \begin{bmatrix} G(\lambda) & \\ & I_n \end{bmatrix}. \tag{7}$$

Therefore, if $S(\lambda)$ is a pencil and $A(\lambda)$ is unimodular then $S(\lambda)$ is a linearization for the matrix polynomial $G(\lambda)$.

The purpose of this paper is to show that many of the linearizations for matrix polynomials in the literature are actually linear polynomial system matrices of the corresponding matrix polynomial. Moreover, this property establishes new tools to determine if a pencil $L(\lambda)$ is a linearization of a matrix polynomial. Namely, by computing the transfer function matrix of $L(\lambda)$ (i.e., the Schur complement) with respect to a unimodular submatrix $A(\lambda)$.

Before giving some auxiliary results in Section 3, we first study the Frobenius companion form or the (first) companion form [12, 13] in the next Section 2, that is one the most classic linearizations. Then, we consider in Section 4 the family of "comrade" linearizations [11], that are particular cases of CORK linearizations [17] studied in Section 5. We also analyze the family of (extended) block Kronecker linearizations [7] in Sections 6 and 7. Finally, we give a note in Section 8 on how to use these ideas to construct linearizations for rational matrices.

2. Frobenius companion form

Given a matrix polynomial $P(\lambda)$ written in terms of the monomial basis as in (1), the Frobenius companion form is the following pencil

$$C_1(\lambda) := \begin{bmatrix} \lambda P_k + P_{k-1} & P_{k-2} & \cdots & P_1 & P_0 \\ -I_m & \lambda I_m & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_m & \\ & & & -I_m & \lambda I_m \end{bmatrix}.$$

It is known that $C_1(\lambda)$ is a strong linearization of $P(\lambda)$ [12]. We now show that $C_1(\lambda)$ can be seen as a polynomial system matrix of $P(\lambda)$ with unimodular state matrix, which in turn implies that $C_1(\lambda)$ is a linearization of $P(\lambda)$.

2.1. Frobenius companion form as a Rosenbrock's system matrix

We consider the following partition:

$$C_1(\lambda) = \left[\begin{array}{cccc|c} \lambda P_k + P_{k-1} & P_{k-2} & \cdots & P_1 & P_0 \\ \hline -I_m & \lambda I_m & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_m & \\ & & & -I_m & \lambda I_m \end{array}\right] =: \begin{bmatrix} -C(\lambda) & D(\lambda) \\ A(\lambda) & B(\lambda) \end{bmatrix}, \tag{8}$$

as a Rosenbrock's system matrix with state matrix $A(\lambda)$. Then, $A(\lambda)$ is clearly unimodular and the transfer function matrix is

$$D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda) = P_0 - \begin{bmatrix} \lambda P_k + P_{k-1} & P_{k-2} & \cdots & P_1 \end{bmatrix} A(\lambda)^{-1}B(\lambda). \tag{9}$$

To compute (9), we consider the polynomial vector containing the elements of the monomial basis. Namely,

$$\Lambda_{k-1}(\lambda) := \begin{bmatrix} \lambda^{k-1} & \lambda^{k-2} & \cdots & \lambda & 1 \end{bmatrix}^T.$$

Then, observe that

$$\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} (\Lambda_{k-1}(\lambda) \otimes I_m)^T = 0.$$

Therefore, $A(\lambda) \begin{bmatrix} \lambda^{k-1} I_m & \lambda^{k-2} I_m & \cdots & \lambda I_m \end{bmatrix}^T + B(\lambda) = 0$ and

$$A(\lambda)^{-1} B(\lambda) = - \begin{bmatrix} \lambda^{k-1} I_m & \lambda^{k-2} I_m & \cdots & \lambda I_m \end{bmatrix}^T. \tag{10}$$

Finally, by (9) and (10), we obtain that the transfer function matrix of (8) is $P(\lambda)$.

2.2. Reversal of the Frobenius companion form as a Rosenbrock's system matrix

Now, we consider the reversal $\operatorname{rev}_1 C_1(\lambda)$ and the following partition:

$$\operatorname{rev}_1 C_1(\lambda) = \left[\begin{array}{c|cccc} P_k + \lambda P_{k-1} & \lambda P_{k-2} & \cdots & \lambda P_1 & \lambda P_0 \\ \hline -\lambda I_m & I_m & & & \\ & \ddots & \ddots & & \\ & & \ddots & I_m & \\ & & & -\lambda I_m & I_m \end{array}\right] =: \begin{bmatrix} D_r(\lambda) & -C_r(\lambda) \\ B_r(\lambda) & A_r(\lambda) \end{bmatrix}. \tag{11}$$

Then, we have that $A_r(\lambda)$ is unimodular, and the transfer function matrix of $\operatorname{rev}_1 C_1(\lambda)$ with the partition in (11) is

$$D_r(\lambda) + C_r(\lambda) A_r(\lambda)^{-1} B_r(\lambda) = P_k + \lambda P_{k-1} - \begin{bmatrix} \lambda P_{k-2} & \cdots & \lambda P_1 & \lambda P_0 \end{bmatrix} A_r(\lambda)^{-1} B_r(\lambda).$$

Taking into account that

$$\begin{bmatrix} B_r(\lambda) & A_r(\lambda) \end{bmatrix} (\operatorname{rev}_{k-1} \Lambda_{k-1}(\lambda) \otimes I_m)^T = 0,$$

we obtain that

$$A_r(\lambda)^{-1} B_r(\lambda) = - \begin{bmatrix} \lambda I_m & \cdots & \lambda^{k-2} I_m & \lambda^{k-1} I_m \end{bmatrix}^T.$$

Therefore, the transfer function matrix is

$$D_r(\lambda) + C_r(\lambda) A_r(\lambda)^{-1} B_r(\lambda) = P_k + \lambda P_{k-1} + \lambda^2 P_{k-2} + \cdots + \lambda^k P_0 = \operatorname{rev}_k P(\lambda).$$

It then follows that $C_1(\lambda)$ is a strong linearization of $P(\lambda)$.

Notice that we computed the transfer function matrices of $C_1(\lambda)$ and $\operatorname{rev}_1 C_1(\lambda)$ without computing the inverse of the state matrix $A(\lambda)$. In general, although the computation of the transfer function matrix involves the inverse of $A(\lambda)$, such computation is simple in most of the cases if there is a linear relation satisfied by the considered polynomial basis as in the example above. We will see more such examples in what follows.

3. Auxiliary results

The discussion in the introduction can be summarized in the following Proposition 3.1. The proof follows from (7).

Proposition 3.1. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let*

$$\mathcal{L}(\lambda) = \left[\begin{array}{cc} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{array} \right] \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

be a linear polynomial system matrix of $G(\lambda)$, with state matrix $A(\lambda) := A_1\lambda + A_0$. If $A(\lambda)$ is unimodular then $G(\lambda)$ is a matrix polynomial and $\mathcal{L}(\lambda)$ is a linearization of $G(\lambda)$.

In the next Proposition 3.2 we give a necessary and sufficient condition for the state matrix $A(\lambda)$ of a polynomial system matrix to be unimodular. This result can be useful in problems where the transfer function matrix $G(\lambda)$ is polynomial and computing $G(\lambda)$ is easier than showing that $A(\lambda)$ is unimodular.

Proposition 3.2. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let*

$$S(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

be a polynomial system matrix of $G(\lambda)$, with state matrix $A(\lambda)$. Then $A(\lambda)$ is unimodular if and only if $S(\lambda)$ is minimal and $G(\lambda)$ is a matrix polynomial.

Proof. It is clear that if $A(\lambda)$ is unimodular then $G(\lambda)$ is a matrix polynomial and $S(\lambda)$ is minimal. That is, condition (6) holds. Conversely, if $G(\lambda)$ is a matrix polynomial and $S(\lambda)$ is minimal then $A(\lambda)$ has no finite eigenvalues, since the finite eigenvalues of $A(\lambda)$ would be the finite poles of $G(\lambda)$ by Theorem 1.2. But $G(\lambda)$ has no finite poles since it is a matrix polynomial. Therefore, $\det A(\lambda)$ is constant, which implies that $A(\lambda)$ is unimodular. □

We could apply Proposition 3.1 to the reversal $\operatorname{rev}_1 \mathcal{L}(\lambda)$ to see that a linearization $\mathcal{L}(\lambda)$ is, in addition, a strong linearization. However, we may need to select another submatrix of the system pencil as appropriate state matrix and that is not always possible. In that case, we can also try to apply the following Proposition 3.3, which requires milder conditions.

Proposition 3.3. *Let $\mathcal{L}(\lambda)$ be a linearization of a polynomial matrix $P(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$. Assume that we can write*

$$\operatorname{rev}_1 \mathcal{L}(\lambda) = \left[\begin{array}{cc} \widetilde{A}_1\lambda + \widetilde{A}_0 & \widetilde{B}_1\lambda + \widetilde{B}_0 \\ -(\widetilde{C}_1\lambda + \widetilde{C}_0) & \widetilde{D}_1\lambda + \widetilde{D}_0 \end{array} \right] \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

as a linear polynomial system matrix with state matrix $\widetilde{A}(\lambda) := \widetilde{A}_1\lambda + \widetilde{A}_0$. If $\widetilde{A}(\lambda)$ is invertible at 0 and the transfer function matrix of $\operatorname{rev}_1 \mathcal{L}(\lambda)$ is equivalent at 0 to $\operatorname{rev}_\ell P(\lambda)$, for some $\ell \geq \deg P(\lambda)$, then $\mathcal{L}(\lambda)$ is a strong linearization of $P(\lambda)$.

Proof. Since $\mathcal{L}(\lambda)$ is a linearization of $P(\lambda)$, by [14] we know that

(a) the elementary divisors for those $\lambda_0 \neq 0$ of $\operatorname{rev}_1 \mathcal{L}(\lambda)$ and $\operatorname{rev}_\ell P(\lambda)$ are the same.

(b) In addition, $\dim \mathcal{N}_r(P) = \dim \mathcal{N}_r(\mathcal{L})$ and $\dim \mathcal{N}_\ell(P) = \dim \mathcal{N}_\ell(\mathcal{L})$.

Now, by assuming that $\widetilde{A}(\lambda)$ is invertible at 0 and that the transfer function matrix of $\operatorname{rev}_1 \mathcal{L}(\lambda)$ is equivalent at 0 to $\operatorname{rev}_\ell P(\lambda)$, we have by [9, Theorem 4.4] that

(c) the elementary divisors at 0 of $\operatorname{rev}_1 \mathcal{L}(\lambda)$ and $\operatorname{rev}_\ell P(\lambda)$ are also the same.

Statements (a), (b) and (c) imply that $\operatorname{rev}_1 \mathcal{L}(\lambda)$ and $\operatorname{diag}(\operatorname{rev}_\ell P(\lambda), I_n)$ are unimodularly equivalent [6, Theorem 4.1]. Thus, $\mathcal{L}(\lambda)$ is a strong linearization of $P(\lambda)$.

□

3.1. Recovery of eigenvectors

An advantage of considering linearizations of matrix polynomials as Rosenbrock's system matrices with unimodular state matrix is that the eigenvectors associated with an eigenvalue λ_0 can be recovered always in the same way. Given $\lambda_0 \in \overline{\mathbb{F}}$ and a polynomial matrix $P(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$, we consider the following vector spaces over $\overline{\mathbb{F}}$:

$$\begin{aligned}\mathcal{N}_r(P(\lambda_0)) &= \{x \in \overline{\mathbb{F}}^{m\times 1} : P(\lambda_0)x = 0\}, \text{ and}\\ \mathcal{N}_\ell(P(\lambda_0)) &= \{y^T \in \overline{\mathbb{F}}^{1\times p} : y^T P(\lambda_0) = 0\},\end{aligned}$$

which are called, respectively, the right and left nullspaces over $\overline{\mathbb{F}}$ of $P(\lambda_0)$. If λ_0 is an eigenvalue of a regular $P(\lambda)$, then $\mathcal{N}_r(P(\lambda_0))$ and $\mathcal{N}_\ell(P(\lambda_0))$ are non trivial and contain, respectively, the right and left eigenvectors of $P(\lambda)$ associated with λ_0.

In the following Proposition 3.4 we state the relation between the right and left nullspaces of a polynomial system matrix with unimodular state matrix and those of its polynomial transfer function matrix $P(\lambda)$. This is a particular case of the results from [8, Proposition 5.1] and [8, Proposition 5.2]. Then, in the particular case of $P(\lambda)$ being regular, Proposition 3.4 can be used to recover the right and left eigenvectors of $P(\lambda)$ from those of a polynomial system matrix of it with unimodular state matrix.

Proposition 3.4. *Let $S(\lambda)$ be a polynomial system matrix as in Proposition 3.2 with $A(\lambda)$ unimodular, and let $P(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ be its transfer function matrix. Let $\lambda_0 \in \overline{\mathbb{F}}$. Then, the following statements hold:*

(a) *The linear map*

$$\begin{aligned}E_r : \mathcal{N}_r(P(\lambda_0)) &\longrightarrow \mathcal{N}_r(S(\lambda_0))\\ x &\longmapsto \begin{bmatrix} -A(\lambda_0)^{-1}B(\lambda_0) \\ I_m \end{bmatrix} x\end{aligned}$$

is a bijection between the right nullspaces over $\overline{\mathbb{F}}$ of $P(\lambda_0)$ and $S(\lambda_0)$.

(b) *The linear map*

$$\begin{aligned}E_\ell : \mathcal{N}_\ell(P(\lambda_0)) &\longrightarrow \mathcal{N}_\ell(S(\lambda_0))\\ y^T &\longmapsto y^T \begin{bmatrix} C(\lambda_0)A(\lambda_0)^{-1} & I_p \end{bmatrix}\end{aligned}$$

is a bijection between the left nullspaces over $\overline{\mathbb{F}}$ of $P(\lambda_0)$ and $S(\lambda_0)$.

We can see that, in particular, right and left eigenvectors of a polynomial matrix $P(\lambda)$ can be directly recovered from the last block of the right and left eigenvectors of its polynomial system matrix $S(\lambda)$. With extra information, we can also recover the right and left eigenvectors of $P(\lambda)$ from any block, as in the following example.

Example 3.5. *Recall the Frobenius companion form $C_1(\lambda)$ in Section 2 and the partition as a polynomial system matrix in (8). By (10), we have that, for any $\lambda_0 \in \overline{\mathbb{F}}$,*

$$-A(\lambda_0)^{-1}B(\lambda_0) = \begin{bmatrix} \lambda_0^{k-1} I_m & \lambda_0^{k-2} I_m & \cdots & \lambda_0 I_m \end{bmatrix}^T.$$

Therefore, by Proposition 3.4, the linear map

$$\begin{aligned}F_r : \mathcal{N}_r(P(\lambda_0)) &\longrightarrow \mathcal{N}_r(C_1(\lambda_0))\\ x &\longmapsto \begin{bmatrix} \lambda_0^{k-1} I_m & \lambda_0^{k-2} I_m & \cdots & \lambda_0 I_m & I_m \end{bmatrix}^T x\end{aligned}$$

is a bijection between the right nullspaces over $\overline{\mathbb{F}}$ of $P(\lambda_0)$ and $C_1(\lambda_0)$.

4. Comrade linearizations

Consider a polynomial matrix

$$P(\lambda) = P_k\phi_k(\lambda) + P_{k-1}\phi_{k-1}(\lambda) + \cdots + P_1\phi_1(\lambda) + P_0\phi_0(\lambda) \in \mathbb{F}[\lambda]^{p\times m},$$

written in terms of a polynomial basis satisfying a three-term recurrence relation of he form:

$$\alpha_j\phi_{j+1}(\lambda) = (\lambda - \beta_j)\phi_j(\lambda) - \gamma_j\phi_{j-1}(\lambda) \quad j \geq 0$$

where $\alpha_j, \beta_j, \gamma_j \in \mathbb{F}$, $\alpha_j \neq 0$, $\phi_{-1}(\lambda) = 0$, and $\phi_0(\lambda) = 1$. It is "well-known" that the following "comrade" companion matrix introduced in [3, Chapter 5] is a strong linearization of $P(\lambda)$ [1, 8, 11]:

$$C_\phi(\lambda) = \begin{bmatrix} \frac{(\lambda-\beta_{k-1})}{\alpha_{k-1}}P_k + P_{k-1} & P_{k-2} - \frac{\gamma_{k-1}}{\alpha_{k-1}}P_k & P_{k-3} & \cdots & P_1 & P_0 \\ -\alpha_{k-2}I & (\lambda-\beta_{k-2})I & -\gamma_{k-2}I & & & \\ & -\alpha_{k-3}I & (\lambda-\beta_{k-3})I & -\gamma_{k-3}I & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\alpha_1 I & (\lambda-\beta_1)I & -\gamma_1 I \\ & & & & -\alpha_0 I & (\lambda-\beta_0)I \end{bmatrix}.$$

This can be proved also via Rosenbrock's system matrices.

4.1. Comrade linearizations as Rosenbrock's system matrices

With the following partition:

$$C_\phi(\lambda) = \left[\begin{array}{ccccc|c} \frac{(\lambda-\beta_{k-1})}{\alpha_{k-1}}P_k + P_{k-1} & P_{k-2} - \frac{\gamma_{k-1}}{\alpha_{k-1}}P_k & P_{k-3} & \cdots & P_1 & P_0 \\ \hline -\alpha_{k-2}I & (\lambda-\beta_{k-2})I & -\gamma_{k-2}I & & & \\ & -\alpha_{k-3}I & (\lambda-\beta_{k-3})I & -\gamma_{k-3}I & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\alpha_1 I & (\lambda-\beta_1)I & -\gamma_1 I \\ & & & & -\alpha_0 I & (\lambda-\beta_0)I \end{array}\right]$$

$$=: \begin{bmatrix} -C(\lambda) & D(\lambda) \\ A(\lambda) & B(\lambda) \end{bmatrix},$$

we get that $C_\phi(\lambda)$ is a linear polynomial system matrix with unimodular state matrix $A(\lambda)$ and transfer function matrix $P(\lambda)$. Then $C_\phi(\lambda)$ is a linearization of $P(\lambda)$ by Proposition 3.1. Notice that comrade linearizations are constructed by considering the recurrence relation satisfied by the polynomial basis. They are particular cases of the more general notion of CORK linearizations, described in Section 5. How to compute the transfer function matrix of $C_\phi(\lambda)$ is a particular case of the computation in the proof of Theorem 5.1.

4.2. Reversal of comrade linearizations as Rosenbrock's system matrices

To see that $C_\phi(\lambda)$ is a strong linearization, it is not possible to identify a unimodular submatrix of $\operatorname{rev}_1 C_\phi(\lambda)$ such that the transfer function matrix is $\operatorname{rev}_k P(\lambda)$. However, we can use Proposition 3.3. For that, we consider the following partition:

$$\operatorname{rev}_1 C_\phi(\lambda) = \left[\begin{array}{c|cccccc} \frac{(1-\lambda\beta_{k-1})}{\alpha_{k-1}}P_k + \lambda P_{k-1} & \lambda P_{k-2} - \lambda\frac{\gamma_{k-1}}{\alpha_{k-1}}P_k & \lambda P_{k-3} & \cdots & \lambda P_1 & \lambda P_0 \\ \hline -\lambda\alpha_{k-2}I & (1-\lambda\beta_{k-2})I & -\lambda\gamma_{k-2}I & & & \\ & -\lambda\alpha_{k-3}I & (1-\lambda\beta_{k-3})I & -\lambda\gamma_{k-3}I & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\lambda\alpha_1 I & (1-\lambda\beta_1)I & -\lambda\gamma_1 I \\ & & & & -\lambda\alpha_0 I & (1-\lambda\beta_0)I \end{array}\right]$$
$$=: \begin{bmatrix} \widetilde{D}(\lambda) & -\widetilde{C}(\lambda) \\ \widetilde{B}(\lambda) & \widetilde{A}(\lambda) \end{bmatrix},$$

so that $\operatorname{rev}_1 C_\phi(\lambda)$ is a linear polynomial system matrix of $\dfrac{1}{f(\lambda)}\operatorname{rev}_k P(\lambda)$, with $f(\lambda) := \lambda^{k-1}\phi_{k-1}(1/\lambda)$, and state matrix $\widetilde{A}(\lambda)$. In addition, $\widetilde{A}(\lambda)$ is invertible at 0 and $f(0) \neq 0$ since $\deg \phi_{k-1}(\lambda) = k-1$. How to compute the transfer function matrix of $\operatorname{rev}_1 C_\phi(\lambda)$ is a particular case of the computation given in the proof of Theorem 5.2.

5. CORK linearizations

In this section we consider polynomial matrices $P(\lambda)$ written as

$$P(\lambda) = \sum_{i=0}^{k-1}(A_i - \lambda B_i)p_i(\lambda) \in \mathbb{F}[\lambda]^{p\times m}, \tag{12}$$

where $p_i(\lambda)$ are scalar polynomials with $p_0(\lambda) \equiv 1$ and A_i, $B_i \in \mathbb{F}^{p\times m}$. Define the polynomial vector

$$p(\lambda) := [p_{k-1}(\lambda) \quad \cdots \quad p_0(\lambda)]^T,$$

and assume that the polynomials $p_i(\lambda)$ satisfy a linear relation

$$(X - \lambda Y)p(\lambda) = 0, \tag{13}$$

where $\operatorname{rank}(X - \lambda_0 Y) = k-1$ for all $\lambda_0 \in \overline{\mathbb{F}}$, and $X - \lambda Y$ has size $(k-1)\times k$. Then the matrix pencil

$$C(\lambda) = \left[\begin{array}{c} A_{k-1} - \lambda B_{k-1} \quad \cdots \quad A_0 - \lambda B_0 \\ \hdashline (X - \lambda Y)\otimes I_m \end{array}\right] \tag{14}$$

is called a CORK linearization of $P(\lambda)$ [17]. We show in the following result that $C(\lambda)$ can be seen as a linear polynomial system matrix of $P(\lambda)$ with unimodular state matrix.

5.1. CORK linearizations as Rosenbrock's system matrices

Theorem 5.1. *Let $P(\lambda)$ be a matrix polynomial as in (12) and consider the matrix pencil $C(\lambda)$ in (14). Consider the following partition*

$$C(\lambda) = \left[\begin{array}{ccc|c} A_{k-1} - \lambda B_{k-1} & \cdots & A_1 - \lambda B_1 & A_0 - \lambda B_0 \\ \hline & X_1(\lambda) & & X_2(\lambda) \end{array}\right],$$

where $(X - \lambda Y) \otimes I_m =: \begin{bmatrix} X_1(\lambda) & X_2(\lambda) \end{bmatrix}$ and $X_1(\lambda)$ has size $(k-1)m \times (k-1)m$. Then, $C(\lambda)$ is a linear polynomial system matrix with state matrix $X_1(\lambda)$ and transfer function matrix $P(\lambda)$. In addition, $X_1(\lambda)$ is unimodular.

Proof. By (13), we have that $\begin{bmatrix} X_1(\lambda) & X_2(\lambda) \end{bmatrix} (p(\lambda) \otimes I_m) = 0$ and, thus,

$$X_1(\lambda)[p_{k-1}(\lambda)I_m \quad \cdots \quad p_1(\lambda)I_m]^T + X_2(\lambda) = 0, \tag{15}$$

taking into account that $p_0(\lambda) = 1$. From (15) follows that $X_1(\lambda)$ is regular. By contradiction, if $X_1(\lambda)$ is singular there exists a nonzero polynomial vector $w(\lambda)$ such that $w(\lambda)^T X_1(\lambda) = 0$ and, therefore, $w(\lambda)^T X_2(\lambda) = 0$ by (15). Thus, $w(\lambda)^T \begin{bmatrix} X_1(\lambda) & X_2(\lambda) \end{bmatrix} = 0$. But this is a contradiction since $\begin{bmatrix} X_1(\lambda) & X_2(\lambda) \end{bmatrix}$ has full row normal rank. Then $C(\lambda)$ is a linear polynomial system matrix with state matrix $X_1(\lambda)$ and its transfer function matrix is

$$A_0 - \lambda B_0 - \begin{bmatrix} A_{k-1} - \lambda B_{k-1} & \cdots & A_1 - \lambda B_1 \end{bmatrix} X_1(\lambda)^{-1} X_2(\lambda). \tag{16}$$

By (15), we have that

$$X_1(\lambda)^{-1} X_2(\lambda) = -[p_{k-1}(\lambda)I_m \quad \cdots \quad p_1(\lambda)I_m]^T, \tag{17}$$

and, by (16) and (17), we obtain that the transfer function matrix is

$$A_0 - \lambda B_0 + \begin{bmatrix} A_{k-1} - \lambda B_{k-1} & \cdots & A_1 + \lambda B_1 \end{bmatrix} \begin{bmatrix} p_{k-1}(\lambda)I_m \\ \vdots \\ p_1(\lambda)I_m \end{bmatrix} = P(\lambda).$$

In addition, the state matrix $X_1(\lambda)$ is unimodular. To see this, we consider the following pencil

$$X(\lambda) := \begin{bmatrix} X_1(\lambda) & X_2(\lambda) \\ I_{(k-1)m} & 0 \end{bmatrix}$$

as a polynomial system matrix with state matrix $X_1(\lambda)$. Then we have that $X(\lambda)$ is minimal, since $\operatorname{rank}(X - \lambda_0 Y) = k - 1$ for all $\lambda_0 \in \overline{\mathbb{F}}$, and the transfer function matrix (i.e., $-X_1(\lambda)^{-1} X_2(\lambda)$) is a polynomial matrix by (17). Then, by Proposition 3.2, $X_1(\lambda)$ is unimodular. $\square$

Theorem 5.1 together with Proposition 3.1 implies that $C(\lambda)$ is a linearization of $P(\lambda)$.

5.2. Reversal of CORK linearizations as Rosenbrock's system matrices

By assuming extra conditions in (13), it follows from Proposition 3.3 and the next Theorem 5.2 that $C(\lambda)$ is, in addition, a strong linearization by considering $\operatorname{rev}_1 C(\lambda)$ as a Rosenbrock's system matrix.

Theorem 5.2. *Let $P(\lambda)$ be a matrix polynomial as in (12) and consider the matrix pencil $C(\lambda)$ in (14). Assume that Y in (14) is invertible and that $\deg p_{k-1}(\lambda) = k-1$. Consider the following partition for $\operatorname{rev}_1 C(\lambda)$:*

$$\operatorname{rev}_1 C(\lambda) = \left[\begin{array}{c|ccc} \lambda A_{k-1} - B_{k-1} & \lambda A_{k-2} - B_{k-2} & \cdots & \lambda A_0 - B_0 \\ \hline \operatorname{rev}_1 Y_1(\lambda) & & \operatorname{rev}_1 Y_2(\lambda) & \end{array}\right].$$

Then $\operatorname{rev}_1 C(\lambda)$ is a linear polynomial system matrix with state matrix $\operatorname{rev}_1 Y_2(\lambda)$ of size $(k-1)m \times (k-1)m$ and transfer function matrix $\dfrac{1}{q(\lambda)} \operatorname{rev}_k P(\lambda)$, where $q(\lambda) := \operatorname{rev}_{k-1} p_{k-1}(\lambda)$ and $q(0) \neq 0$. In addition, $\operatorname{rev}_1 Y_2(\lambda)$ is invertible at 0.

Proof. First, taking into account that $\begin{bmatrix} Y_1(\lambda) & Y_2(\lambda) \end{bmatrix} (p(\lambda) \otimes I_m) = 0$, we have that $\begin{bmatrix} \operatorname{rev}_1 Y_1(\lambda) & \operatorname{rev}_1 Y_2(\lambda) \end{bmatrix} (\lambda^{k-1} p(1/\lambda) \otimes I_m) = 0$ and, thus,

$$q(\lambda) \operatorname{rev}_1 Y_1(\lambda) + \operatorname{rev}_1 Y_2(\lambda) [\lambda^{k-1} p_{k-2}(1/\lambda) I_m \quad \cdots \quad \lambda^{k-1} p_0(1/\lambda) I_m]^T = 0, \tag{18}$$

where $q(\lambda) := \operatorname{rev}_{k-1} p_{k-1}(\lambda) = \lambda^{k-1} p_{k-1}(1/\lambda)$. From (18), and the fact that the matrix $\begin{bmatrix} \operatorname{rev}_1 Y_1(0) & \operatorname{rev}_1 Y_2(0) \end{bmatrix}$ has full row rank since Y is invertible, follows that $\operatorname{rev}_1 Y_2(\lambda)$ is invertible at 0, i.e., that $\operatorname{rev}_1 Y_2(0)$ is invertible. By contradiction, if $\operatorname{rev}_1 Y_2(0)$ is not invertible, there exists a constant vector w such that $w^T \operatorname{rev}_1 Y_2(0) = 0$ and, by (18), $w^T \operatorname{rev}_1 Y_1(0) = 0$ since $q(0) \neq 0$. Therefore, $w^T \begin{bmatrix} \operatorname{rev}_1 Y_1(0) & \operatorname{rev}_1 Y_2(0) \end{bmatrix} = 0$ and this is a contradiction since $\begin{bmatrix} \operatorname{rev}_1 Y_1(0) & \operatorname{rev}_1 Y_2(0) \end{bmatrix}$ has full row rank.

We now compute the transfer function matrix of $\operatorname{rev}_1 C(\lambda)$ as a linear polynomial system matrix with state matrix $\operatorname{rev}_1 Y_2(\lambda)$. That is,

$$T(\lambda) := \lambda A_{k-1} - B_{k-1} - \begin{bmatrix} \lambda A_{k-2} - B_{k-2} & \cdots & \lambda A_0 - B_0 \end{bmatrix} (\operatorname{rev}_1 Y_2(\lambda))^{-1} \operatorname{rev}_1 Y_1(\lambda). \tag{19}$$

By (18), we know that

$$(\operatorname{rev}_1 Y_2(\lambda))^{-1} \operatorname{rev}_1 Y_1(\lambda) = -\frac{1}{p_{k-1}(1/\lambda)} [p_{k-2}(1/\lambda) I_m \quad \cdots \quad p_0(1/\lambda) I_m]^T = 0. \tag{20}$$

Combining (19) and (20), we obtain

$$T(\lambda) := \lambda A_{k-1} - B_{k-1} + \frac{1}{p_{k-1}(1/\lambda)} \begin{bmatrix} \lambda A_{k-2} - B_{k-2} & \cdots & \lambda A_0 - B_0 \end{bmatrix} \begin{bmatrix} p_{k-2}(1/\lambda) I_m \\ \vdots \\ p_0(1/\lambda) I_m \end{bmatrix}. \tag{21}$$

Multiplying $T(\lambda)$ by $q(\lambda)$ we obtain

$$q(\lambda) T(\lambda) = \sum_{i=0}^{k-1} (\lambda A_i - B_i)(\lambda^{k-1} p_i(1/\lambda)) = \operatorname{rev}_k P(\lambda).$$

□

6. Block Kronecker linearizations

In this section, we consider the block Kronecker pencils introduced in [7] and show that they can also be seen as Rosenbrock's system matrices with unimodular state matrix.

Definition 6.1. *Let $\lambda M_1 + M_0$ be an arbitrary pencil. Any pencil of the form*

$$C_K(\lambda) = \left[\begin{array}{c|c} \lambda M_1 + M_0 & L_\eta(\lambda)^T \otimes I_p \\ \hline L_\epsilon(\lambda) \otimes I_m & 0 \end{array}\right],$$

is called a block Kronecker pencil, where

$$L_k(\lambda) := \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{k\times(k+1)}.$$

They are a linearization of the polynomial matrix

$$P(\lambda) := (\Lambda_\eta(\lambda)^T \otimes I_p)(\lambda M_1 + M_0)(\Lambda_\epsilon(\lambda) \otimes I_m).$$

The one-block row and one-block column cases are included, i.e., the second block row or the second block column can be empty.

6.1. Block Kronecker linearizations as Rosenbrock's system matrices

Observe that we can write

$$L_\epsilon(\lambda) \otimes I_m = \left[\begin{array}{cccc|c} -I_m & \lambda I_m & & & \\ & -I_m & \lambda I_m & & \\ & & \ddots & \ddots & \\ & & & -I_m & \lambda I_m \end{array}\right] =: \left[\begin{array}{c|c} A_{\epsilon,m}(\lambda) & B_{\epsilon,m}(\lambda) \end{array}\right], \tag{22}$$

and $A_{\epsilon,m}(\lambda)$ is unimodular. Analogously,

$$L_\eta(\lambda) \otimes I_p = \left[\begin{array}{cccc|c} -I_p & \lambda I_p & & & \\ & -I_p & \lambda I_p & & \\ & & \ddots & \ddots & \\ & & & -I_p & \lambda I_p \end{array}\right] =: \left[\begin{array}{c|c} A_{\eta,p}(\lambda) & B_{\eta,p}(\lambda) \end{array}\right], \tag{23}$$

and $A_{\eta,p}(\lambda)$ is unimodular. Then, $C_K(\lambda)$ can be partitioned as:

$$C_K(\lambda) = \left[\begin{array}{cc|c} M_{11}(\lambda) & M_{12}(\lambda) & A_{\eta,p}(\lambda)^T \\ M_{21}(\lambda) & M_{22}(\lambda) & B_{\eta,p}(\lambda)^T \\ \hline A_{\epsilon,m}(\lambda) & B_{\epsilon,m}(\lambda) & 0 \end{array}\right],$$

and we set

$$A(\lambda) := \begin{bmatrix} M_{11}(\lambda) & A_{\eta,p}(\lambda)^T \\ A_{\epsilon,m}(\lambda) & 0 \end{bmatrix}, \quad B(\lambda) := \begin{bmatrix} M_{12}(\lambda) \\ B_{\epsilon,m} \end{bmatrix}, \tag{24}$$

$$C(\lambda) := -\begin{bmatrix} M_{21}(\lambda) & B_{\eta,p}(\lambda)^T \end{bmatrix}, \quad \text{and} \quad D(\lambda) := M_{22}(\lambda). \tag{25}$$

Notice that $A(\lambda)$ is unimodular for any $M_{11}(\lambda)$. With the partition above, we have that $C_K(\lambda)$ is a linear polynomial system matrix with unimodular state matrix $A(\lambda)$, and the transfer function matrix is the matrix polynomial $P(\lambda)$ in Theorem 6.2.

Theorem 6.2. *Let $C_K(\lambda)$ be a block Kronecker pencil as in Definition 6.1. Then, the following statements hold:*

(a) *The submatrix $A(\lambda)$ of $C_K(\lambda)$ as in (24) is unimodular.*

(b) *The Schur complement of $A(\lambda)$ in $C_K(\lambda)$ is the polynomial matrix*

$$P(\lambda) := (\Lambda_\eta(\lambda)^T \otimes I_p)(\lambda M_1 + M_0)(\Lambda_\epsilon(\lambda) \otimes I_m) \in \mathbb{F}[\lambda]^{p\times m},$$

where

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{1\times(k+1)}.$$

(c) *$C_K(\lambda)$ is a linearization of $P(\lambda)$.*

Proof. Statement (c) follows from (a) and (b), and we only remain to prove (b). First, we write

$$\begin{aligned} A(\lambda)^{-1} &:= \begin{bmatrix} 0 & A_{\epsilon,m}(\lambda)^{-1} \\ A_{\eta,p}(\lambda)^{-T} & -A_{\eta,p}(\lambda)^{-T} M_{11}(\lambda) A_{\epsilon,m}(\lambda)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I_{\epsilon,m} & 0 \\ 0 & A_{\eta,p}(\lambda)^{-T} \end{bmatrix} \begin{bmatrix} 0 & I_{\epsilon m} \\ I_{\eta p} & -M_{11}(\lambda) \end{bmatrix} \begin{bmatrix} I_{\eta p} & 0 \\ 0 & A_{\epsilon,m}(\lambda)^{-1} \end{bmatrix}. \end{aligned}$$

Now, observe that $\begin{bmatrix} A_{\epsilon,m}(\lambda) & B_{\epsilon,m}(\lambda) \end{bmatrix} (\Lambda_\epsilon(\lambda) \otimes I_m) = 0$ and, thus,

$$A_{\epsilon,m}(\lambda)(\lambda\Lambda_{\epsilon-1}(\lambda) \otimes I_m) + B_{\epsilon,m}(\lambda) = 0.$$

Therefore,

$$A_{\epsilon,m}(\lambda)^{-1} B_{\epsilon,m}(\lambda) = -(\lambda\Lambda_{\epsilon-1}(\lambda) \otimes I_m).$$

Analogously,

$$B_{\eta,p}(\lambda)^T A_{\eta,p}(\lambda)^{-T} = -(\lambda\Lambda_{\eta-1}(\lambda)^T \otimes I_p).$$

Thus, the transfer function matrix is

$$\begin{aligned} &D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda) = \\ &M_{22}(\lambda) - \begin{bmatrix} M_{21}(\lambda) & B_{\eta,p}(\lambda)^T \end{bmatrix} \begin{bmatrix} I_{\epsilon,m} & 0 \\ 0 & A_{\eta,p}(\lambda)^{-T} \end{bmatrix} \begin{bmatrix} 0 & I_{\epsilon m} \\ I_{\eta p} & -M_{11}(\lambda) \end{bmatrix} \begin{bmatrix} I_{\eta p} & 0 \\ 0 & A_{\epsilon,m}(\lambda)^{-1} \end{bmatrix} \begin{bmatrix} M_{12}(\lambda) \\ B_{\epsilon,m} \end{bmatrix} = \\ &M_{22}(\lambda) - \begin{bmatrix} M_{21}(\lambda) & -(\lambda\Lambda_{\eta-1}(\lambda)^T \otimes I_p) \end{bmatrix} \begin{bmatrix} 0 & I_{\epsilon m} \\ I_{\eta p} & -M_{11}(\lambda) \end{bmatrix} \begin{bmatrix} M_{12}(\lambda) \\ -(\lambda\Lambda_{\epsilon-1}(\lambda) \otimes I_m) \end{bmatrix} = \\ &M_{22}(\lambda) + (\lambda\Lambda_{\eta-1}(\lambda)^T \otimes I_p)M_{12}(\lambda) + M_{21}(\lambda)(\lambda\Lambda_{\epsilon-1}(\lambda) \otimes I_m) + (\lambda\Lambda_{\eta-1}(\lambda)^T \otimes I_p)M_{11}(\lambda)(\lambda\Lambda_{\epsilon-1}(\lambda) \otimes I_m) = \\ &(\Lambda_\eta(\lambda)^T \otimes I_p) \begin{bmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{bmatrix} (\Lambda_\epsilon(\lambda) \otimes I_m) = P(\lambda). \end{aligned}$$

□

6.2. Reversal of block Kronecker linearizations as Rosenbrock's system matrices

We can consider the following partition for $\operatorname{rev}_1 C_K(\lambda)$:

$$\operatorname{rev}_1 C_K(\lambda) = \left[\begin{array}{c|cc} \widehat{M}_{11}(\lambda) & \widehat{M}_{12}(\lambda) & \widehat{B}_{\eta,p}(\lambda)^T \\ \hline \widehat{M}_{21}(\lambda) & \widehat{M}_{22}(\lambda) & \widehat{A}_{\eta,p}(\lambda)^T \\ \widehat{B}_{\epsilon,m}(\lambda) & \widehat{A}_{\epsilon,m}(\lambda) & 0 \end{array}\right] := \begin{bmatrix} D_r(\lambda) & -C_r(\lambda) \\ B_r(\lambda) & A_r(\lambda) \end{bmatrix}, \tag{26}$$

as a linear polynomial system matrix with state matrix $A_r(\lambda)$, where

$$\operatorname{rev}_1 L_\epsilon(\lambda) \otimes I_m = \left[\begin{array}{c|cccc} -\lambda I_m & I_m & & & \\ & -\lambda I_m & I_m & & \\ & & \ddots & \ddots & \\ & & & -\lambda I_m & I_m \end{array}\right] =: \left[\begin{array}{cc} \widehat{B}_{\epsilon,m}(\lambda) & \widehat{A}_{\epsilon,m}(\lambda) \end{array}\right], \tag{27}$$

and

$$\operatorname{rev}_1 L_\eta(\lambda) \otimes I_p = \left[\begin{array}{c|cccc} -\lambda I_p & I_p & & & \\ & -\lambda I_p & I_p & & \\ & & \ddots & \ddots & \\ & & & -\lambda I_p & I_p \end{array}\right] =: \left[\begin{array}{cc} \widehat{B}_{\eta,p}(\lambda) & \widehat{A}_{\eta,p}(\lambda) \end{array}\right]. \tag{28}$$

Then, we have the following result.

Theorem 6.3. *Let $C_K(\lambda)$ be a block Kronecker pencil as in Definition 6.1. Then, the following statements hold:*

(a) *The submatrix $A_r(\lambda)$ of $\operatorname{rev}_1 C_K(\lambda)$ as in (26) is unimodular.*

(b) *The Schur complement of $A_r(\lambda)$ in $\operatorname{rev}_1 C_K(\lambda)$ is $\operatorname{rev}_{\eta+\epsilon+1} P(\lambda)$.*

(c) *$C_K(\lambda)$ is a strong linearization of $P(\lambda)$.*

7. Extended block Kronecker linearizations

In this section we consider a more general version of the notion of block Kronecker linearization.

Definition 7.1. *Let $\lambda M_1 + M_0$ be an arbitrary pencil and $Y \in \mathbb{F}^{\varepsilon m \times \varepsilon m}$ and $Z \in \mathbb{F}^{\eta p \times \eta p}$ be arbitrary constant matrices. Then any pencil of the form*

$$C_{EK}(\lambda) = \left[\begin{array}{c|c} \lambda M_1 + M_0 & (Z(L_\eta(\lambda) \otimes I_p))^T \\ \hline Y(L_\epsilon(\lambda) \otimes I_m) & 0 \end{array}\right],$$

is called an extended block Kronecker pencil. The one-block row and one-block column cases are also included, i.e., the second block row or the second block column can be empty. Note that if $Z = I_{\eta p}$ and $Y = I_{\varepsilon m}$ then $C_{EK}(\lambda)$ is just a block Kronecker pencil.

We can also write $C_{EK}(\lambda)$ as a polynomial system matrix with unimodular state matrix.

7.1. Extended block Kronecker linearizations as Rosenbrock's system matrices

Recall (22) and (23), and observe that

$$Y(L_\epsilon(\lambda) \otimes I_m) = \left[\begin{array}{c|c} YA_{\epsilon,m}(\lambda) & YB_{\epsilon,m}(\lambda) \end{array}\right],$$

and $YA_{\epsilon,m}(\lambda)$ is unimodular if Y is invertible. Analogously,

$$Z(L_\eta(\lambda) \otimes I_p) = \left[\begin{array}{c|c} ZA_{\eta,p}(\lambda) & ZB_{\eta,p}(\lambda) \end{array}\right],$$

and $ZA_{\eta,p}(\lambda)$ is unimodular if Z is invertible. Then, $C_{EK}(\lambda)$ can be partitioned as:

$$C_{EK}(\lambda) = \left[\begin{array}{c:c|c} M_{11}(\lambda) & M_{12}(\lambda) & A_{\eta,p}(\lambda)^T Z^T \\ \hdashline M_{21}(\lambda) & M_{22}(\lambda) & B_{\eta,p}(\lambda)^T Z^T \\ \hline YA_{\epsilon,m}(\lambda) & YB_{\epsilon,m}(\lambda) & 0 \end{array}\right],$$

and we set

$$\widetilde{A}(\lambda) := \begin{bmatrix} M_{11}(\lambda) & A_{\eta,p}(\lambda)^T Z^T \\ YA_{\epsilon,m}(\lambda) & 0 \end{bmatrix}, \quad \widetilde{B}(\lambda) := \begin{bmatrix} M_{12}(\lambda) \\ YB_{\epsilon,m} \end{bmatrix}, \tag{29}$$

$$\widetilde{C}(\lambda) := -\begin{bmatrix} M_{21}(\lambda) & B_{\eta,p}(\lambda)^T Z^T \end{bmatrix}, \quad \text{and} \quad \widetilde{D}(\lambda) := M_{22}(\lambda). \tag{30}$$

Notice that $\widetilde{A}(\lambda)$ is unimodular if Y and Z are invertible, for any $M_{11}(\lambda)$. With the partition above, we have that $C_{EK}(\lambda)$ is a linear polynomial system matrix with unimodular state matrix $\widetilde{A}(\lambda)$, and the transfer function matrix is $P(\lambda)$ in Theorem 7.2.

Theorem 7.2. *Let $C_{EK}(\lambda)$ be an extended Block Kronecker pencil as in Definition 7.1. Assume that Y and Z are invertible. Then, the following statements hold:*

(a) *The submatrix $\widetilde{A}(\lambda)$ of $C_{EK}(\lambda)$ as in* (29) *is unimodular.*

(b) *The Schur complement of $\widetilde{A}(\lambda)$ in $C_{EK}(\lambda)$ is the polynomial matrix*

$$P(\lambda) := (\Lambda_\eta(\lambda)^T \otimes I_p)(\lambda M_1 + M_0)(\Lambda_\epsilon(\lambda) \otimes I_m) \in \mathbb{F}[\lambda]^{p\times m}.$$

(c) *$C_{EK}(\lambda)$ is a linearization of $P(\lambda)$.*

Proof. Statement (c) follows from (a) and (b), and we only remain to prove (b). For that, we write the matrices in (29) as follows:

$$\widetilde{A}(\lambda) := \begin{bmatrix} I_{\eta p} & 0 \\ 0 & Y \end{bmatrix} A(\lambda) \begin{bmatrix} I_{\epsilon m} & 0 \\ 0 & Z^T \end{bmatrix}, \quad \widetilde{B}(\lambda) := \begin{bmatrix} I_{\eta p} & 0 \\ 0 & Y \end{bmatrix} B(\lambda),$$

$$\widetilde{C}(\lambda) := C(\lambda) \begin{bmatrix} I_{\epsilon m} & 0 \\ 0 & Z^T \end{bmatrix}, \quad \text{and} \quad \widetilde{D}(\lambda) := D(\lambda),$$

where $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ are as in (24) and (25). Then, the transfer function matrix of $C_{EK}(\lambda)$ is:

$$\begin{aligned}
&\widetilde{D}(\lambda) + \widetilde{C}(\lambda)\widetilde{A}(\lambda)^{-1}\widetilde{B}(\lambda) = \\
&D(\lambda) + C(\lambda) \begin{bmatrix} I_{\epsilon m} & 0 \\ 0 & Z^T \end{bmatrix} \begin{bmatrix} I_{\epsilon m} & 0 \\ 0 & Z^{-T} \end{bmatrix} A(\lambda)^{-1} \begin{bmatrix} I_{\eta p} & 0 \\ 0 & Y^{-1} \end{bmatrix} \begin{bmatrix} I_{\eta p} & 0 \\ 0 & Y \end{bmatrix} B(\lambda) = \\
&D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda) = P(\lambda),
\end{aligned}$$

as in the proof of Theorem 6.2. □

7.2. Reversal of extended block Kronecker linearizations as Rosenbrock's system matrices

We can consider the following partition for $\mathrm{rev}_1\, C_{EK}(\lambda)$:

$$\mathrm{rev}_1\, C_{EK}(\lambda) = \left[\begin{array}{c|cc} \widehat{M}_{11}(\lambda) & \widehat{M}_{12}(\lambda) & \widehat{B}_{\eta,p}(\lambda)^T Z^T \\ \hline \widehat{M}_{21}(\lambda) & \widehat{M}_{22}(\lambda) & \widehat{A}_{\eta,p}(\lambda)^T Z^T \\ Y\widehat{B}_{\epsilon,m}(\lambda) & Y\widehat{A}_{\epsilon,m}(\lambda) & 0 \end{array}\right] := \begin{bmatrix} \widetilde{D}_r(\lambda) & -\widetilde{C}_r(\lambda) \\ \widetilde{B}_r(\lambda) & \widetilde{A}_r(\lambda) \end{bmatrix}, \tag{31}$$

as a linear polynomial system matrix with state matrix $\widetilde{A}_r(\lambda)$, where

$$Y(\mathrm{rev}_1\, L_\epsilon(\lambda) \otimes I_m) = \begin{bmatrix} Y\widehat{B}_{\epsilon,m}(\lambda) & Y\widehat{A}_{\epsilon,m}(\lambda) \end{bmatrix},$$

and

$$Z(\mathrm{rev}_1\, L_\eta(\lambda) \otimes I_p) = \begin{bmatrix} Z\widehat{B}_{\eta,p}(\lambda) & Z\widehat{A}_{\eta,p}(\lambda) \end{bmatrix},$$

by using the notation in (27) and (28), respectively. Then, we have the following result.

Theorem 7.3. *Let $C_{EK}(\lambda)$ be an extended block Kronecker pencil as in Definition 7.1. Assume that Y and Z are invertible. Then, the following statements hold:*

(a) *The submatrix $\widetilde{A}_r(\lambda)$ of $\mathrm{rev}_1\, C_{EK}(\lambda)$ as in (31) is unimodular.*

(b) *The Schur complement of $\widetilde{A}_r(\lambda)$ in $\mathrm{rev}_1\, C_{EK}(\lambda)$ is $\mathrm{rev}_{\eta+\epsilon+1}\, P(\lambda)$, where*

$$P(\lambda) := (\Lambda_\eta(\lambda)^T \otimes I_p)(\lambda M_1 + M_0)(\Lambda_\epsilon(\lambda) \otimes I_m) \in \mathbb{F}[\lambda]^{p\times m}.$$

(c) *$C_{EK}(\lambda)$ is a strong linearization of $P(\lambda)$.*

Remark 7.4. Modulo permutations, extended block Kronecker linearizations include: Fiedler pencils (FP), Fiedler pencils with repetitions (FPR), generalized Fiedler pencils (GFP), generalized Fiedler pencils with repetitions (GFPR) [5] and all the pencils in the canonical basis of $\mathbb{DL}(P)$ since they are FPR [4].

8. A note on construction of linearizations for rational matrices from linear system matrices of their polynomial parts

By the division algorithm for polynomials, any rational function $r(\lambda)$ can be uniquely written as $r(\lambda) = p(\lambda) + r_{sp}(\lambda)$, where $p(\lambda)$ is a polynomial and $r_{sp}(\lambda)$ is a strictly proper rational function. That is, $\lim_{\lambda\to\infty} r_{sp}(\lambda) = 0$. Therefore, any rational matrix $R(\lambda)$ can be expressed uniquely as

$$R(\lambda) = P(\lambda) + R_{sp}(\lambda),$$

where $P(\lambda)$ is a polynomial matrix and $R_{sp}(\lambda)$ is a strictly proper rational matrix. That is, the entries of $R_{sp}(\lambda)$ are strictly proper rational functions. If we consider a linearization of $P(\lambda)$ that is a Rosenbrock's system matrix

$$L(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix},$$

with $A(\lambda)$ unimodular, and a minimal state-space realization

$$R_{sp}(\lambda) = C_s(\lambda I_s - A_s)^{-1} B_s$$

of the strictly proper part $R_{sp}(\lambda)$. Then, we obtain that

$$\mathcal{L}(\lambda) = \left[\begin{array}{cc|c} (\lambda I_s - A_s) & 0 & B_s \\ 0 & A(\lambda) & B(\lambda) \\ \hline -C_s & -C(\lambda) & D(\lambda) \end{array}\right]$$

is a linear minimal polynomial system matrix of $R(\lambda)$ and, thus, $\mathcal{L}(\lambda)$ contains the information about finite poles and zeros of $R(\lambda)$ by Theorem 1.2. Therefore, $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$. More information about linearizations of rational matrices and how to construct linear polynomial system matrices that also preserve the pole and zero information at infinity, i.e., the pole and zero information at 0 of $R(1/\lambda)$ can be found, for instance, in [2, 9, 10].

References

[1] A. Amiraslani, R.M. Corless, P. Lancaster, *Linearization of matrix polynomials expressed in polynomial bases*, IMA J. Numer. Anal. 29 (2009) 141–157.

[2] A. Amparan, F. M. Dopico, S. Marcaida, I. Zaballa, *Strong linearizations of rational matrices*, SIAM J. Matrix Anal. Appl. 39(4) (2018) 1670–1700.

[3] S. Barnett, *Polynomials and Linear Control Systems*, Marcel Dekker Inc., New York (1983).

[4] M. I. Bueno, K. Curlett, S. Furtado, *Structured strong linearizations from Fiedler pencils with repetition I*, Linear Algebra Appl. 460 (2014) 51–80.

[5] M. I. Bueno, F. M. Dopico, J. Pérez, R. Saavedra, B. Zykoski, *A simplified approach to Fiedler-like pencils via block minimal bases pencils*, Linear Algebra Appl. 547 (2018) 45–104.

[6] F. De Terán, F. M. Dopico, D. S. Mackey, *Spectral equivalence of matrix polynomials and the index sum theorem,* Linear Algebra Appl. 459 (2014) 264–333.

[7] F. M. Dopico, P. W. Lawrence, J. Pérez, P. Van Dooren, *Block Kronecker linearizations of matrix polynomials and their backward errors*, Numer. Math. 140 (2018) 373–426. Extended version available as MIMS EPrint 2016.34, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2016.

[8] F. M. Dopico, S. Marcaida, M. C. Quintana, *Strong linearizations of rational matrices with polynomial part expressed in an orthogonal basis*, Linear Algebra Appl. 570 (2019) 1–45.

[9] F. M. Dopico, S. Marcaida, M. C. Quintana, P. Van Dooren, *Local linearizations of rational matrices with application to rational approximations of nonlinear eigenvalue problems*, Linear Algebra Appl. 604 (2020) 441–475.

[10] F. M. Dopico, M. C. Quintana, P. Van Dooren, *Linear system matrices of rational transfer functions*, "Realization and Model Reduction of Dynamical Systems, A Festschrift in honor of the 70th birthday of Thanos Antoulas", Springer (2022) 95–113.

[11] H. Faßbender, P. Saltenberger, *On vector spaces of linearizations for matrix polynomials in orthogonal bases*, Linear Algebra Appl. 525 (2017) 59–83.

[12] I. Gohberg, M. A. Kaashoek, P. Lancaster, *General theory of regular matrix polynomials and band Toeplitz operators*, Integral Equations Operator Theory 11(6) (1988) 776–882.

[13] I. Gohberg, P. Lancaster, L. Rodman, *Matrix Polynomials*, SIAM Publications, 2009. Originally published: Academic Press, New York, 1982.

[14] D. S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, *Möbius transformations of matrix polynomials,* Linear Algebra Appl. 470 (2015) 120–184.

[15] B. McMillan, *Introduction to formal realizability theory II,* Bell System Tech. J. 31 (1952) 541–600.

[16] H. H. Rosenbrock, *State-space and Multivariable Theory*, Thomas Nelson and Sons, London, 1970.

[17] R. Van Beeumen, K. Meerbergen, W. Michiels, *Compact rational Krylov methods for nonlinear eigenvalue problems,* SIAM J Matrix Anal Appl. 36(2) (2015) 820–838.