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Approximate Solves in Krylov-based Modeling Methods

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Abstract

Rational Krylov methods are potentially one of the most robust and efficient algorithms to compute lower order approximations to linear time invariant dynamical systems that match a specified number of moments of the transfer function at multiple points in the complex plane. The characterization of multipoint rational interpolation in terms of bases of multiple Krylov spaces was developed recently. In this paper, we summarize some results concerning the use of approximate solutions to the linear systems of equations that arise on each step of the method. Such approximations are used to reduce the space and time required to produced the reduced order system.

1 Introduction

Model reduction, computing a low-order approximation to a linear, time-invariant dynamic system, is well-studied in system theory, see [3] and its references. The need to efficiently compute accurate low-order approximations continues to grow as descriptions of physical systems become more complex and potentially involve hundreds of thousands of discrete variables. The substitution of an accurate reduced-order model in place of the original description may significantly reduce the time for a simulation of the system or the construction of a controller for the original plant.

Recently, a large amount of work has appeared in the literature which utilizes Krylov subspace projection to generate the reduced-order model, see [5] and its references. Requiring only matrix-vector multiplications and/or matrix factorizations, these methods are reasonably well-suited for large, sparse problems. In fact, the methods yield reduced-order models which are partial realizations, Padé approximations or rational interpolants [4, 10]. The more powerful Padé approximations and rational interpolations require the solution of sparse linear systems during the construction of the Krylov subspaces. When this issue is discussed in the literature of model reduction it is typically assumed that matrix factorizations are used.

Although the matrices to be factored may be sparse, direct factorizations may not be feasible—especially for three-dimensional discretizations. As an alternative, a large body of literature exists on the topic of iteratively solving large-scale systems of equations. These iterative solvers, frequently based on Krylov methods themselves, may provide low-cost but approximate solutions. In the following, we consider the use of approximate solves in Krylov-based model reduction methods. Approaches for both implementing the approximate solves within the model reduction and choosing the solution techniques themselves are addressed.

2 Exact Solution Spaces

An approach to computing a Krylov-based reduced-order model of dimension M

$$\hat{E}\dot{\hat{x}} = \hat{A}\hat{x} + \hat{b}u \quad \text{and} \quad \hat{y} = \hat{c}\hat{x} + du \tag{1}$$

for the order-N dynamic system

$$E\dot{x} = Ax + bu \quad \text{and} \quad y = cx + du \tag{2}$$

follows from the following result which is proven in [6] in a slightly more general form:

Theorem 1 *If $\hat{A} = Z^T AV$, $\hat{E} = Z^T EV$, $\hat{b} = Z^T b$ and $\hat{c} = cV$ where $V, Z \in \Re^{N\times M}$ and if the spaces*

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{b_k}}\left((A-\sigma^{(k)}E)^{-1}(A-\zeta_k E), (A-\sigma^{(k)}E)^{-1}b\right) \tag{3}$$

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and

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{c_k}}\left((A-\sigma^{(k)}E)^{-T}(A-\zeta_k E)^T, (A-\sigma^{(k)}E)^{-T}c^T\right) \tag{4}$$

are in $Im(V)$ and $Im(Z)$ respectively then the moments of (1) and (2) satisfy

$$c\left\{(A-\sigma^{(k)}E)^{-1}E\right\}^{j_k-1}(A-\sigma^{(k)}E)^{-1}b = \hat{c}\left\{(\hat{A}-\sigma^{(k)}\hat{E})^{-1}\hat{E}\right\}^{j_k-1}(\hat{A}-\sigma^{(k)}\hat{E})^{-1}\hat{b}$$

for $j_k = 1, 2, \ldots, J_{b_k} + J_{c_k}$ and $k = 1, 2, \ldots K$, independently of the ζ_k values.

By Theorem 1, the reduced-order model (1) is a rational interpolant of the original system if V and Z are constructed according to (3) and (4). The individual subspaces $\mathcal{K}_j$ in (3) and (4) are known as Krylov subspaces,

$$\mathcal{K}_j(G, g) = \text{span}\{g, Gg, G^2g, \ldots, G^{j-1}g\}.$$

The difference between the reduced-order model and the original system follows from a result which is also proven in [6]:

Theorem 2 *The difference between the frequency responses of the original and reduced-order systems is $r_c^T(sE-A)^{-1}r_b$ where r_b and r_c are the residuals,*

$$\begin{aligned} r_b(s) &= b - (sE-A)V(s\hat{E}-\hat{A})^{-1}\hat{b} \\ &\text{and} \\ r_c(s) &= c^T - (sE-A)^T Z(s\hat{E}-\hat{A})^{-T}\hat{c}^T. \end{aligned} \tag{5}$$

Furthermore, the residuals satisfy the Petrov-Galerkin conditions, $Z^T r_b(s) = 0$ and $W^T r_c(s) = 0$ for all values of s.

Note the residuals correspond to the approximate solutions $\hat{x}_b(s) = V(s\hat{E}-\hat{A})^{-1}\hat{b}$ and $\hat{x}_c(s) = Z(s\hat{E}-\hat{A})^{-T}\hat{c}^T$ for the dual systems of linear equations

$$(sE-A)x_b = b \quad \text{and} \quad (sE-A)^T x_c = c^T. \tag{6}$$

Theorems 1 and 2 can also be extended to handle multiple-input-multiple-output systems. It is also possible to show Galerkin conditions when solving matrix equations. For example, if an RK method is used to create a reduced order Lyapunov equation and its solution is projected from the low order space back to the original space of dimension n, then the projected solution satisfies a Galerkin condition.

Attempting to compute an exact rational interpolant as in Theorem 1 requires (at least implicitly) the knowledge of the exact inverse $(A-\sigma^{(k)}E)^{-1}$ in V and Z. For this reason, the V and Z in (3) and (4) are denoted exact subspaces. However, the ability to solve systems of large-scale equations involving $(A-\sigma^{(k)}E)$ may not be practical due to work or memory constraints. An alternative point of view follows by attempting to minimize the residual expressions $r_b(s)$ and $r_c(s)$ in the error expression of Theorem 2.

3 Preconditioned Solution Subspaces

Theorem 1 characterizes the subspaces $\mathcal{K}_j$ for which bases must be produced and combined to form the matrices V and Z. It does not specify how they should be constructed or what constraints should be imposed in order to determine which bases are chosen. The Rational Krylov (RK) family of methods for model reduction (both multipoint moment-matched and those based on approximate solutions) can be represented by Algorithm 1. (The subscript p_m is the index of the iteration on which $\sigma^{(k_m)}$ was last used as an interpolation point.) By choosing the parameters appropriately various RK family members can be produced. Common to all that match moments is the selection of $\Phi_m = (A-\sigma^{(k_m)}E)^{-1}$ where $\sigma^{(k_m)} \in \{\sigma^{(1)}, \ldots, \sigma^{(K)}\}$, i.e., it is the interpolation point chosen from the set of K possible points for use on the m–th iteration. The various moment-matching members are then distinguished by the constraints placed on the relationships between the matrices V_m, Z_m, Q_m and W_m. The multipoint Rational Arnoldi (RA), Rational Lanczos (RL), Dual Rational Arnoldi (DRA), and Rational Power (RPM) methods are explored in detail in [6].

Table 1: Rational Krylov Algorithm (Approximate General Version)

Initialize: $q_1 = (\gamma_1^q)^{-1}b$ and $w_1 = (\beta_1^w)^{-1}c^T$;

For $m = 1$ to M,

(S1.1) Input: $\sigma^{(k_m)}$, the interpolation point for m^{th} iteration;

(S1.2) $\tilde{v}_m = \Phi_m q_{p_m+1}$ and $\tilde{z}_m = \Phi_m^T w_{p_m+1}$;

(S1.3) $\gamma_m^v v_m = \tilde{v}_m - V_{m-1}\bar{v}_m$ and $\beta_m^z z_m = \tilde{z}_m - Z_{m-1}\bar{z}_m$;

(S1.4) $\tilde{q}_{m+1} = (A-\zeta_m E)v_m$ and $\tilde{w}_{m+1} = (A-\zeta_m E)^T z_m$;

(S1.5) $\gamma_{m+1}^q q_{m+1} = \tilde{q}_{m+1} - Q_m\bar{q}_{m+1}$ and $\beta_{m+1}^w w_{m+1} = \tilde{w}_{m+1} - W_m\bar{w}_{m+1}$;

end

The approximate form of the algorithms allow the choice of the two parameters, Φ_m and ζ_m. The operator Φ_m approximates the action of $(A-\sigma^{(k_m)}E)^{-1}$ on a vector. Numerous possibilities exist for finding preconditioner that approximates $(A-\sigma^{(k_m)}E)^{-1}$ [7]. Alternatively, and more generally, one can think of Φ_m as an operation that takes in the vectors q_{p_m+1}, w_{p_m+1} and outputs the vectors $\tilde{v}_m$, $\tilde{z}_m$. Hence, Φ_m can represent an iterative system solver that computes approximate solutions to the equations. However, in this case, Φ_m represents a nonlinear operation which is no longer associated with a fixed matrix P_k. This is due to the fact that a single interpolation point $\sigma^{(k)}$ may be used in more than one step of the iteration with different right-hand side vectors and therefore there would be more than one approximation to $(A-\sigma^{(k)}E)^{-1}$. If the methods are always iterated to near working precision then this can be analyze as roundoff and covered by the appropriate error bounds. If, however, to reduce the time required the iterations are stopped early then the theory must be generalized. As can be seen from the experiments in [2] and [6], in practice it is not an important distinction.

If (and only if) Φ_m is chosen so that moment matching results then the choice of ζ_m in (S1.4) does not contribute to the specification of the V and Z column spaces and ζ_k can be replaced in Theorem 1 by a variable s and removed from the algorithm, i.e., $\zeta_m=\infty$. When $P_m \approx (A-\sigma^{(k_m)}E)^{-1}$ it is possible to tune ζ_m to improve the results. For example, $\zeta=\infty$ can still be used, or more commonly, the setting $\zeta_m=\sigma^{(k_m)}$ is used. This can be motivated by subspace and/or eigenvalue mapping considerations [6]. As the experiments below indicate, the choice of ζ can lead to significant, but often unpredictable differences in the convergence of the reduced-order model. In practice, ζ_m can be tuned between ∞ and $\sigma^{(k_m)}$ by using available information on the preconditioner quality and/or the convergence behavior of previous solves. Various choices for ζ_m appear in related approaches to problems in the eigenvalue literature [8, 9]. Alternatively, more sophisticated approaches for implementing the approximate solvers can be used to reduce signifcantly the importance of ζ_m.

When $P_m \approx (A-\sigma^{(k_m)}E)^{-1}$, the V and Z generated by any version of the approximate RK algorithm no longer form bases for the unions of Krylov subspaces required for rational interpolation. However, the residuals of the reduced order model still satisfies the Petrov-Galerkin conditions, $Z^T r_b(s)=0$ and $W^T r_c(s)=0$ for all values of s. As long as reasonable approximations $V\hat{\mathbf{x}}_b$ and $Z\hat{\mathbf{x}}_c$ to $\mathbf{x}_b$ and $\mathbf{x}_c$ are acquired, a good reduced-order model is achievable. It is interesting to note that even the exactly preconditioned V and Z do not necessarily lead to optimal approximations to $\mathbf{x}_b$ and $\mathbf{x}_c$ at all s. Rational interpolation leads to exact preconditioners, which are only optimally suited for a few discrete points, i.e., the interpolation points. For frequencies away from the interpolation points, it is uncertain as to whether $(A-\sigma^{(k)}E)^{-1}$ is necessarily a better preconditioner than some $P_k \approx (A-\sigma^{(k)}E)^{-1}$. We are therefore interested in finding approximate solutions to the equations in (6) which lie, respectively, in the dual solution subspaces

$$\begin{aligned}\mathrm{Im}(V) &= \bigcup_{k=1}^{K} \mathcal{K}_{J_k}(\Phi_k(A-\zeta_k E), \Phi_k b)\\ \mathrm{Im}(Z) &= \bigcup_{k=1}^{K} \mathcal{K}_{J_k}(\Phi_k^T(A-\zeta_k E)^T, \Phi_k^T c^T)\end{aligned} \tag{7}$$

The loss of moment matching is not the only consequence of the use of an approximate RK method. Care must be taken to assess exactly how the constraints that relate V_m, Z_m, Q_m and W_m in the rational interpolation case are effected. More than other RK variants seen in [6], the rational Lanczos (RL) algorithm relies on the properties of the exact preconditioner. This reliance allowed the development of efficient algorithm to produce a sparse reduced order model (hence, short recurrences and less storage); it also makes the approach a questionable one for inexact preconditioning. If approximations to $(A-\sigma^{(k)}E)^{-1}$ are used in the RL algorithm, the reduced-order systems matrices become dense. Either these now nonzero off-diagonal terms must be computed or an error is incurred. Computing all of the elements in $\hat{A}$ or $\hat{E}$ corresponds to full-length biorthogonality recursions which increases the cost of an approximate RL algorithm to levels comparable with the DRA approach which is is numerically more reliable; an $O(M^2N)$ version of the RL algorithm is of little value. However, an option in dealing with an approximate RL algorithm is to simply ignore the problem, i.e., the error between Φ_m and $(A-\sigma^{(k_m)}E)^{-1}$). In this case, simply edit the RL algorithm to use Φ_m and use the algorithm as if all was well. Under appropriate assumptions about the point ordering, sparse $\hat{A}$ and $\hat{E}$ are produced by this algorithm; they are still formed implicitly by using coefficients of short recursions. The error in this theoretically unsupported approach is characterized by Theorem 3.

Theorem 3 *The output residual expression for the approximate RL algorithm is*

$$\begin{aligned}\mathbf{r}_c(s) &= \beta_{M+1,M+1} w_{M+1} e_m^T (s\hat{E}-\hat{A})^{-T}\hat{c}(\sigma^{(k_M)}-s)\\ &\quad -\tilde{R}(s\hat{E}-\hat{A})^{-T}\hat{c},\end{aligned} \tag{8}$$

where the m^{th} column of $\tilde{R}$ is

$$\tilde{r}_m = w_m - (A-\sigma^{(k_m)}E)^T(\Phi_m^T w_m), \tag{9}$$

the residual associated with the approximate system solve in the m^{th} iteration.

In the exact case, the output residual resulting from the RL algorithm is a scaled version of w_{M+1}. This

vector w_{M+1} stays biorthogonal with at least the recent directions of V. Moreover, the scaling of w_{m+1} drops to zero in regions around the interpolation point. When approximations are employed, the output residual is corrupted by the error in Φ_m. The residual $\tilde{r}_m$, associated with the approximation for the vector $(A-\sigma^{(k_m)}E)^{-T}w_m$, appears in $\mathbf{r}_c$. Several facts should be noted concerning this corruption:

1. The corruption of $\mathbf{r}_c(s)$ is proportional to the error in Φ_m.
2. Unlike the exact case, the residual $\mathbf{r}_c$ in approximate RL is not generally forced to zero in the regions about the interpolation point.
3. Errors in the computation of $(A-\sigma^{(k_m)}E)^{-1}$ in the m^{th} iteration continue to appear in the reduced-order models of later iterations (the entire matrix $\tilde{R}$ appears in (8)).
4. The total corruption behaves as the sum (rather than product) of previous errors in the computation of $(A-\sigma^{(k_m)}E)^{-1}$, because the matrix $\tilde{R}$ appears in a matrix-vector product in (8). This fact is good news for moment matching versions of the RL algorithm that are implemented in finite precision. Machine-level precision errors in the linear system solvers are not blown up in later steps.

In summary, significant errors between $(A-\sigma^{(k_m)}E)^{-1}$ and Φ_m do not appear to be acceptable in any iteration of the approximate RL algorithm. Limited numerical experience supports this result.

While the approximate RL algorithm produces the reduced model implicitly, it is possible to create versions of the approximate RK algorithm that are theoretically well-founded by producing the bases V and Z explicitly and forming the reduced system explictly by applying them to A and E. The approximate DRA algorithm produces orthogonal V and Z matrices by effectively running two preconditioned Arnoldi iterations simultaneously. This method is costly but also very reliable in both its exact and approximate form. The explicit production of the reduced model has important convergence consequences. Approximate forms of other methods are possible, e.g. for RPM, and are discussed along with some unconventional ways of reducing storage and computation in [6].

4 Experiments

In this section, we present some experiments that indicate the trends that occur when using the approximate RK model reduction algorithms. The full sets of experiments can be seen in [2] and [6]. We first explore the effect of altering degrees of precision used in solving the linear systems on each iteration. The PDE $x_t(v,z) = x_{zz} + x_{vv} + 20x_z + 180x + f(v,z)u(t)$ was discretized using central differences. The input vector f was taken to be random and the output vector is taken to be the same as the input vector for simplicity. The results for approximate DRA and approximate RL for 2 and 10 digits accuracy in the solves using purely imaginary interpolation points are shown in Table 2 as a function of model size. A small problem illustrates the salient points so a 7×12 grid is used. The trend predicted by the analysis is clearly shown. The approximate RL algorithm converges until it reaches a stagnation point reflecting the solve accuracy. It is inferior in accuracy to the approximate RA even when a significant number of digits are found in the solves. (Of course, it requires less work to produce a given order model but, doing more work does not improve the accuracy.) The approximate RA steadily improves as the outer iteration helps damp the inner errors.

Table 2: Effect of linear solution accuracy

m	Modeling Error			
	10 digits		2 digits	
	RL	DRA	RL	DRA
10	7.6e-02	7.6e-02	4.8e-01	3.7e-01
20	1.0e-06	5.7e-08	2.1e-01	8.9e-02
30	1.9e-08	2.2e-13	1.6e-01	1.2e-01
40	1.9e-08	2.4e-13	5.8e-01	4.8e-02
50	1.9e-08	6.6e-13	1.1e-01	7.1e-02
60	1.9e-08	1.3e-12	1.2e-01	1.4e-03
70	1.9e-08	2.9e-12	1.0e-01	4.1e-04
80	1.9e-08	3.9e-13	1.5e-01	1.8e-04

The effects of using an iterative method with different numbers of iterations and varying the value of ζ_m are considered using the PDE on a 40×60 grid. This grid leads to an A matrix of dimension $N = 2400$ with 11800 nonzero elements. The approximate dual RA algorithm was applied to this problem with a real shift of 2π. The systems of linear equations are approximately solved with k iterations the GMRES method preconditioned with $ILUT(4,0)$ [7]. (We refer to this as the inner preconditioner.) Therefore, the operator Φ_m that preconditions the model reduction problem, combines both a fixed preconditioner of the linear system and an iterative solver.

The results of 100 dual RA iterations with $k = 5$ and $k = 20$ presented in Table 3. A total of 500 GMRES iterations take place in the first case and 2000 in the second (these iterations only involve matrix-vector prod-

ucts).

Table 3: Effects of shift and GMRES iterations

m	Modeling Error w/ k GMRES steps			
	$\zeta_m = \infty$		$\zeta_m = \sigma^{(k_m)}$	
	$k=5$	$k=20$	$k=5$	$k=20$
10	9.9-01	1.3e-03	5.9e+00	1.6e+00
20	1.1+00	1.9e-04	9.0e-02	1.0e+00
30	2.1+00	4.5e-05	3.7e-01	1.1e+00
40	1.2+00	2.7e-04	9.2e-03	1.2e+00
50	3.4+00	3.4e-04	3.2e-02	1.6e+00
60	3.4+00	8.7e-05	2.7e-02	3.2e-01
70	1.2+00	8.2e-05	8.1e-02	1.2e-01
80	2.0+00	5.0e-05	1.1e-01	1.2e-02
90	1.0+00	2.7e-04	4.0e-01	7.1e-02
100	2.5e+00	2.0e-05	4.5e-02	3.3e-01

Note that the $\zeta_m = \sigma^{(k_m)}$ case performs well when the model reduction preconditioner, Φ_m is poorer (fewer GMRES steps), but is unacceptable when the Φ_m is more accurate. This is similar to behavior observed in the eigenvalue literature. The opposite behavior occurs when ζ_m is ∞, i.e., $(A - \zeta_m E)$ is replaced with E. Although the best results are obtained in the second column, the results with a well-chosen ζ_m and only five GMRES iterations in column three are reasonably good and are achieved at lower cost.

The data in Tables 2 and 3 demonstrate the fact that accuracy in solving the linear systems cannot be sacrificed for the sake of efficiency. While the explicit model production of the approximate DRA can damp the inner errors it does so at a cost of increased order for the reduced model. The choice of ζ_m can also be difficult and suffers from the same unpredicability of effect seen in the eigenvalue literature. Of course, the details of the performance are intimately related with choice of iterative method and the inner preconditioner when computing reasonably accurate solutions to the dual system of equations [2].

The discussion of setting Φ_m and ζ_m has ignored the fact that at step m a reduced order model of size $m-1$ and its projectors V_{m-1} and Z_{m-1} are available. So the vector

$$h = V_m(\hat{A}_m - \sigma^{(k_{m+1})}\hat{E}_m)^{-1} Z_m^T (A - \zeta_{p_m} E) v_{p_m} \quad (10)$$
$$\approx (A - \sigma^{(k_{m+1})}E)^{-1}(A - \zeta_{p_m}E)v_{p_m}.$$

can be used as an initial guess, $\tilde{v}^0_{m+1}$, for the iterative solver.

This initial guess finds an approximate solution in the column space of V_m that satisfies a Petrov-Galerkin constraint with respect to Z_m. Using the projection associated with model reduction to generate an initial guess $\tilde{v}^0_{m+1}$ can lead to better inner-solver results than simply choosing $\tilde{v}^0_{m+1}$ to be a random vector or a vector of zeros.

Unfortunately, (10) is not enough by itself. Because the purpose of solving the inner system is to compute a new direction $\tilde{v}_{m+1}$, an approximate solution that lies entirely in the existing directions of V_m is not acceptable. The initial guess in (10) must be improved upon. This improvement can be accomplished by incorporating the initial guess into (10) which yields a correction to $\tilde{v}^0_{m+1}$ by approximately solving

$$(A-\sigma^{(k_{m+1})}E)\tilde{u}_{m+1} = (A-\zeta_{p_m}E)v_{p_m} - (A-\sigma^{(k_{m+1})}E)\tilde{v}^0_{m+1}$$

and adding $\tilde{v}^0_{m+1}$ to the result. This can be reworked to get

$$\tilde{u}_{m+1} = (\sigma^{(k_{m+1})} - \zeta_{p_m})\Phi_{m+1}\{Ev_{p_m} - u\} \quad (11)$$

where

$$u = -(A - \sigma^{(k_{m+1})}E)V_m(\hat{A}_m - \sigma^{(k_{m+1})}\hat{E})^{-1}Z_m^T E v_{p_m}.$$

There are two important things to note about (11). First, the update $\tilde{u}_{m+1}$ consists of the vector $\Phi_{m+1}Ev_{p_m}$, which is perhaps the most naive approximation to the ideal new direction $(A - \sigma^{(k_{m+1})}E)^{-1}Ev_{p_m}$, and a correction vector

$$\Phi_{m+1}(A - \sigma^{(k_{m+1})}E)V_m(\hat{A}_m - \sigma^{(k_{m+1})}\hat{E})^{-1}Z_m^T E v_{p_m},$$

which incorporates information from the existing reduced-order model. When Φ_{m+1} nears $(A - \sigma^{(k_{m+1})}E)^{-1}$, the vector $\Phi_{m+1}Ev_{p_m}$ nears the ideal direction, while the correction $V_m(\hat{A}_m - \sigma^{(k_{m+1})}\hat{E})^{-1}Z_m^T E v_{p_m}$ is contained in $\mathrm{Im}(V_m)$ and is, hence, irrelevant. On the other hand, if the reduced-order model is accurate (a fortunate event), then $V_m(\hat{A} - \sigma^{(k_{m+1})}\hat{E})^{-1}Z_m^T$ nears $(A - \sigma^{(k_{m+1})}E)^{-1}$ and the update $\tilde{u}_{m+1}$ becomes small. A second important feature of (11) is that the parameter ζ_{p_m} only comes into play as a scaling. However, scalings do not matter in constructing subspaces, it can be removed from the vector, i.e., set $\zeta_{p_m} = 0$. Even though Φ_m is an approximation, the choice of $\tilde{v}^0_{m+1}$ as a starting guess leads to a new direction that is independent of ζ. We have thus found a second way (Φ_m taken as exact was the first) for generating V directions that are independent of the evaluation point ζ. As a side note, it is claimed that the derivation above provides an alternative path for obtaining Davidson's method for the eigenvalue problem.

If the discretized PDE on the 40×60 grid is solved with the improved initial guess for 100 modeling iterations,

the data in Table 4 are the result. The use of the improved starting vector leads to a convergence which is slightly improved or comparable to the best cases in Table 3. However, the best results in Table 3 required careful choices for ζ, an issue that is no longer a concern here.

Table 4: Effects of improved starting vector in PDE

m	Modeling Error k GMRES steps	
	$k = 5$	$k = 20$
10	1.1e+00	1.0e-02
20	4.7e-01	4.3e-04
30	6.1e-02	1.0e-03
40	2.7e-02	1.8e-04
50	1.4e-02	3.2e-04
60	4.1e-02	2.3e-04
70	4.6e-02	7.1e-05
80	3.1e-03	5.2e-04
90	9.5e-03	6.1e-05
100	4.5e-03	1.8e-04

5 Current Work

Work currently in progress focuses on determining desirable approaches for constructing Φ_m and developing more efficient variations of the ADRA method. One may for example employ a variety of iterative methods to implicitly apply some Φ_m to $(A - \zeta_m E)v_{m-1}$. Although the ADRA is robust, one might hope to find algorithms based on shorter orthogonalization recursions in the spirit of the Lanczos method. The search is based on the results of using approximate solves via GMRES, GMRESR, GCRO, QMR and GMRES(k) with and without preconditioning via SSOR and ILU in the approximate RL algorithm contained in [2]. They show that it is possible to use the approximate RL algorithm to achieve reasonably accurate reduced order models if the linear systems are solved to reasonable accuracy. The choice of interpolation points is also shown to be very important for the approximate solves. If one takes points far from the spectrum (say, *large* real interpolation points) then the spectrum of $A - \sigma^{(k)}E$ appears more "clustered" and the number of iterations needed will drop. Notice that this corresponds to a "global" approximation of the frequency response and is also discussed in detail relative to point selection and placement algorithms in [6].

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