# MODEL REDUCTION OF LARGE-SCALE SYSTEMS RATIONAL KRYLOV VERSUS BALANCING TECHNIQUES

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**Abstract.** In this paper, we describe some recent developments in the use of projection methods to produce a reduced-order model for a linear time-invariant dynamical system which approximates its frequency response. We give an overview of the family of Rational Krylov methods and compare them with "near-optimal" approximation methods based on balancing transformations.

## 1. Introduction

Physical phenomena are often modeled with linear, time-invariant (LTI) dynamical systems because of the simplicity and low complexity of the approach (both in terms of the complexity of the approximation problem and its subsequent use for simulation). Such linear models can frequently be acquired through discretizations of partial or ordinary differential equations describing the physical system. However, such physical models are becoming more complex due to either increased system size or an increased desire for detail. Large scale problems (such as the North American power grid system) and fine grid discretizations (needed in high-speed circuit designs) require models of increasing complexity. Although such models tend to accurately describe the behavior of the underlying physical system, their complexity leads to high analysis and simulation costs which are too high

when one uses traditional numerical techniques. Methods which exploit the structure in the models such as sparsity have become critical. This lead to an increased interest in iterative methods for solving large sparse linear systems and/or eigenvalue problems for the simulation of such large dynamical systems.

In some cases, however, there is a need to go even further. Despite the use of efficient computational kernels, the model may still require an unacceptable amount of time to evaluate. It is then necessary to create a second model that is significantly smaller while preserving important aspects of the original system. This is the model reduction problem for linear time-invariant dynamical systems. It is assumed that the original system is described by the generalized state-space equations

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t). \end{cases} \tag{1}$$

The vectors $u(t) \in \mathcal{R}^m$, $y(t) \in \mathcal{R}^p$ and $x(t) \in \mathcal{R}^N$ are the vectors of input variables, output variables and state variables, respectively. For nearly all large-scale problems, it is assumed that the state transition matrices $A \in \mathcal{R}^{N\times N}$ and $E \in \mathcal{R}^{N\times N}$ are large and sparse or structured. Moreover, the input dimension $m$ and output dimension $p$ are assumed much smaller that the state dimension $N$. We point out that this system has a well-defined solution provided the pencil $(\lambda E - A)$ is *regular*, i.e. $\det(\lambda E - A) \neq 0$. A reduced-order approximation to (1) takes the corresponding form

$$\begin{cases} \hat{E}\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\ \hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}u(t). \end{cases} \tag{2}$$

The dimension $n$ of the reduced-order is supposed to be much smaller than $N$. Ideally, the reduced order model would produce an output $\hat{y}(t)$ approximating well the true output $y(t)$ for all inputs $u(t)$. It is more realistic to try to match the response $y(t)$ of some "representative" input $u(t)$ and typically one chooses the response of the zero initial state impulse response $(x(0) = 0, u_j(t) = \delta(t)e_j$, where $\delta(t)$ is the Dirac impulse and $e_j$ is the $j$-th column of the identity matrix). The reason for this is that the system response to an arbitrary input (with zero initial condition) can be represented as a convolution with the impulse response. Provided $E$ is invertible, this response equals (for each input $\delta(t)e_j$) :

$$y_j(t) = \{Ce^{E^{-1}At}E^{-1}B + D\delta(t)\}e_j, \quad j = 1, \ldots, m. \tag{3}$$

In the Laplace transform domain one derives the equivalent formula

$$\mathcal{L}y_j = \{C(sE - A)^{-1}B + D\}\mathcal{L}u_j, \tag{4}$$

which involves the *transfer function* $H(s) \doteq C(sE - A)^{-1}B + D$ of the system. It plays a key role in the description of the system behavior by describing the response of the system to a periodic input signal $u_j(t) \doteq e^{\jmath\omega t}e_j$ since the corresponding output equals $y_j(t) = H(\jmath\omega)u_j(t)$. The model reduction problem therefore reduces to an approximation of the frequency response $H(s)$ by another rational matrix of lower degree $\hat{H}(s) \doteq \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} + \hat{D}$. Since $D$ and $\hat{D}$ are of dimensions $p \times m$ where $p, m << N$, one typically chooses $D = \hat{D}$. Without a loss of generality, the feed-through term $D$ of the original model can therefore be assumed to be zero since the approximation problem clearly involves only the part that depends on $s$ and involve the large scale matrices.

Several measures of the accuracy of the reduced-order model are possible. Formally, one wants to bound the difference between the actual and low-order outputs, $y(t) - \hat{y}(t)$, given a selected input $u(t)$, and this can be characterized by a system norm. The popular $H_\infty$ error norm measures, in the time domain, the worst ratio of output error energy to input energy. In the frequency domain, this represents the largest magnitude of the frequency-response error. A second measure of the accuracy of the approximation is to assess which properties of the original model are preserved in the reduced-order one. A common choice is modal approximation [1, 5] which preserves the dominant system's poles (eigenvalues) $\lambda_n$ and corresponding residues which arise in the partial fraction expansion of the transfer function. A reduced-order model that matches (or approximates) dominant modal components of the original model is expected to approximate well its response. Unfortunately, it can be difficult to identify a priori which modes are the truly dominant modal components of the original system [23]. Alternative invariant properties that may be retained in model reduction are the Hankel singular values. Hankel singular values are related to the controllability and observability properties of a system [18]. Constructing a reduced-order model preserving the largest Hankel singular values is known as balanced truncation. A variant of this is known as optimal Hankel norm approximation [11]. These last two approximations are nearly optimal in terms of the $H_\infty$ norm of the error and are constructed from balanced realizations. There exist sparse matrix techniques that compute these "near-optimal approximations" in an approximate fashion as well.

## 2. Moment matching

In this paper we focus on approximations defined from a power series expansion of the rational matrix function $H(s)$. Let $\sigma$ be a point in the complex

plane, then $H(s)$ has an expansion

$$H(s) = H_{-\ell}(s-\sigma)^{-\ell} + \cdots + H_{-1}(s-\sigma)^{-1} + H_0 + H_1(s-\sigma)^1 + H_2(s-\sigma)^2 + \cdots, \tag{5}$$

where $\ell$ is the order of its pole at $\sigma$. Typically, one chooses interpolation points that are *not* a pole of $H(s)$ and then $\ell = 0$. The coefficients $H_i$ are then easy to construct from the system model via a Neumann expansion of $H(s) = C\{(s-\sigma)E - (A-\sigma E)\}^{-1}B$ since $(A-\sigma E)$ is non-singular :

$$H_i = C(\sigma E - A)^{-1}\{E(\sigma E - A)^{-1}\}^i B = C\{(\sigma E - A)^{-1}E\}^i(\sigma E - A)^{-1}B. \tag{6}$$

The solution techniques proposed determine a reduced-order model that accurately matches the leading coefficients $H_i$ – also called *moments* – arising in this power series. In general, one can produce a reduced-order model that interpolates the frequency response and its derivatives at multiple points $\{\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(K)}\}$. Since

$$\left.\frac{\partial^i H(s)}{\partial s^i}\right|_{s=\sigma} \doteq i! H_i \ , \tag{7}$$

we seek to match the moments $H_i^{(j)}$ at each interpolation point $\sigma^{(j)}$, $j = 1, \ldots, K$. The first $2J_1$ moments are matched at $\sigma^{(1)}$, the next $2J_2$ moments are matched at $\sigma^{(2)}$, etc., where $J_1 + J_2 + \ldots + J_K = M$. A model meeting these constraints is denoted a multipoint Padé approximation or a rational interpolant [2, 3]. By varying the location and number of interpolation points utilized with the underlying problem in mind, one can construct accurate reduced-order models in a variety of situations.

Moment matching methods are relatively old and are based on Padé approximations [6]. For the more general rational interpolation problem described here, one has to solve a system of equations involving a Loewner matrix [2]. It is important to note that the system matrices only enter the modeling problem through its moments explicitly. Unfortunately, these explicit moment-matching methods exhibit numerical instabilities, which was first pointed out in [8] and later on in [7]. The reader is referred to those papers and to [14] for a detailed discussion. The numerical difficulties come from the construction of the Hankel and Loewner matrices involved and the ill-conditioning of the associated linear systems. Both [8] and [7] point out that moment-matching via Krylov projection methods is a preferred numerical implementation.

## 3. Krylov projection methods

In projection methods, the $M$-th order reduced system is produced by applying two $N \times n$ matrices $Z$ and $V$ to the system matrices of the original

system: $\hat{A} = Z^T AV$, $\hat{E} = Z^T EV$, $\hat{B} = Z^T B$, $\hat{C} = V^T C$. The matrices $Z$ and $V$ in fact define projections onto Krylov spaces

$$\mathcal{K}_j(G, g) = \text{Im}\,\{g, Gg, G^2 g, \ldots, G^{j-1} g\} \tag{8}$$

for specific choices of $G$ and $g$. The first connection between the Lanczos algorithm, a Krylov-based technique, and Padè approximations was given in [12]. Later work proposed related Krylov space techniques for model reduction of dynamical systems in various application areas [19, 16, 26, 24, 4, 25]. New results in the area included stability retention of the reduced order model [13] and multipoint rational Lanczos methods [10], i.e., starting from the Lanczos procedure and modifying it to produce a reduced system that matched multiple moments at multiple frequency values.

We now give a basic theorem describing the relationships between Krylov subspaces, the iterative algorithms for constructing these subspaces, and model reduction via rational interpolation. It was proven in [14] and [15] and extends [26] to multipoint approximations.

**Theorem 1**

If

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{b_k}} \left( (\sigma^{(k)} E - A)^{-1} E, (\sigma^{(k)} E - A)^{-1} B \right) \subseteq \mathcal{V} \tag{9}$$

and

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{c_k}} \left( (\sigma^{(k)} E - A)^{-T} E^T, (\sigma^{(k)} E - A)^{-T} C^T \right) \subseteq \mathcal{Z} \tag{10}$$

then the moments of (1) and (2) satisfy

$$H_k^{(j_k)} = C \left\{ (\sigma^{(k)} E - A)^{-1} E \right\}^{j_k - 1} (\sigma^{(k)} E - A)^{-1} B = \tag{11}$$

$$\hat{H}_k^{(j_k)} = \hat{C} \left\{ (\sigma^{(k)} \hat{E} - \hat{A})^{-1} \hat{E} \right\}^{j_k - 1} (\sigma^{(k)} \hat{E} - \hat{A})^{-1} \hat{B} \tag{12}$$

for $j_k = 1, 2, \ldots, J_{b_k} + J_{c_k}$ and $k = 1, 2, \ldots K$, provided these moments exist.

**Proof:** This is a trivial extension of the proof given in [14] for the case $m = p = 1$. ■

The moments of the original system exist if one chooses interpolation points that are not poles of the system (one chooses e.g. points in the right half plane, where a stable system has no poles). But the non-singularity of the pencils $(\sigma^{(k)} \hat{E} - \hat{A})$ is not automatically guaranteed (see [14] for details on how to handle this case). Any pair of projection bases satisfying (9) and (10) achieve the desired rational interpolant. Restrictions on $V$ or $Z$, such as biorthogonality or orthogonality, are implementation specific choices and

lead to different variants. The Dual Rational Arnoldi method referred to later on corresponds $V$ and $Z$ having orthogonal columns. For algorithmic implementations and further details on the different variants we refer to [14].

## 4. Modal approximation

A classical method for model reduction is modal approximation. Let us assume that the system (1) has $t$ different poles, then there exist invertible matrices $Z$ and $V$ transforming $\lambda E - A$ to a block diagonal form. In this new coordinate system, the model matrices $\{E, A, B, C\}$ in (1) become :

$$\left[\begin{array}{c|c} Z^T(\lambda E - A)V & Z^T B \\ \hline CV & 0 \end{array}\right] \doteq \left[\begin{array}{ccc|c} \lambda E_1 - A_1 & & & B_1 \\ & \ddots & & \vdots \\ & & \lambda E_t - A_t & B_t \\ \hline C_1 & \dots & C_t & 0 \end{array}\right], \quad (13)$$

where each subpencil $\lambda E_i - A_i$ has only one eigenvalue $\lambda_i$. In general each eigenvalue $\lambda_i$ can be repeated and its multiplicity $k_i$ is the dimension of the block $\lambda E_i - A_i$. The expansion

$$H(s) = \sum_1^t C_i(\lambda E_i - A_i)^{-1} B_i \quad (14)$$

is essentially the partial fraction expansion of $H(s)$ since each term has only one pole (but of degree $k_i$ which is possibly higher than 1). From a comparison of this expansion and the expansion (5) around the pole $\lambda_i$ we obtain the identity

$$C_i(\lambda E_i - A_i)^{-1} B_i = H_{-\ell}^{(i)}(\lambda - \lambda_i)^{-\ell} + \ldots + H_{-1}^{(i)}(\lambda - \lambda_i)^{-1} \quad (15)$$

which shows indeed that (14) is the partial fraction expansion of $H(s)$.

Modal approximation now consists of keeping those terms in this expansion that correspond to "dominant" poles. The reduced order model is then obtained from keeping only the columns of $Z$ and $V$ corresponding to the blocks containing the "selected" dominant poles. One can describe this more formally by using the concept of deflating subspace.

**Definition 1**
The column spaces of the full rank matrices $V_i$ and $Z_i$ are called right, respectively left deflating subspaces of the regular pencil $(\lambda E - A)$ if and only if $(\lambda E - A)V_i = \hat{V}_i(\lambda E_i - A_i)$, respectively $Z_i^T(\lambda E - A) = (\lambda E_i - A_i)\hat{Z}_i^T$, and $(\lambda E_i - A_i)$ is also a regular pencil. ■

The spectrum of a deflating subspace is that of the pencil $(\lambda E_i - A_i)$. If that spectrum is only one point (say $\lambda_i$) then $\mathrm{Im}V_i$ and $\mathrm{Im}Z_i$ are called deflating subspace of the eigenvalue $\lambda_i$. The largest dimension of a deflating subspace with spectrum $\lambda_i$ equals the multiplicity $k_i$ of that eigenvalue [17].

**Theorem 2**
Let $\mathrm{Im}V_i$ and $\mathrm{Im}Z_i$ be left and right invariant subspaces of the regular pencil $(\lambda E - A)$ with a given spectrum, and consider the regular reduced order system $(\lambda \hat{E} - \hat{A}) \doteq Z^T(\lambda E - A)V$. If

$$\mathrm{Im}V_i \subseteq \mathrm{Im}V \doteq \mathcal{V} \tag{16}$$

then $(\lambda \hat{E} - \hat{A})$ has a right invariant subspace with the same spectrum. If

$$\mathrm{Im}Z_i \subseteq \mathrm{Im}Z \doteq \mathcal{Z} \tag{17}$$

then $(\lambda \hat{E} - \hat{A})$ has a left invariant subspace with the same spectrum. In both cases the corresponding poles of the original system are retained in the reduced order system.

**Proof:** We only prove the result for the right deflating subspaces since both cases are dual. Since $\mathrm{Im}V_i \subseteq \mathrm{Im}V$ we have $V_i = VX_i$ for some full rank matrix $X_i$. Since $\mathrm{Im}V_i$ is a deflating subspace of $(\lambda E - A)$, we have $(\lambda E - A)V_i = \hat{V}_i(\lambda E_i - A_i)$. It follows that $(\lambda \hat{E} - \hat{A})X_i = Z^T(\lambda E - A)VX_i = Z^T\hat{V}_i(\lambda E_i - A_i) = Y_i(\lambda E_i - A_i)$, whence $X_i$ is a right deflating subspace of the reduced order pencil. ■

Together with Theorem 1, this allows to combine moment matching with pole (or modal) matching. A simple example of this is when one has a system of differential algebraic equations (DAE's) and one wants to retain these algebraic equations in the reduced order system. For the original system this implies that $E$ is singular with a kernel $V_\infty$ of a particular dimension $k_\infty$. The subscript $\infty$ is intentional since this kernel is in fact a deflating subspace corresponding to the eigenvalue $\lambda = \infty$. By imposing $\mathrm{Im}V_\infty \subseteq \mathrm{Im}V$, the reduced order system will have an $\hat{E}$ matrix with a kernel of the same dimension and hence will retain these algebraic equations. If no such condition is imposed, the reduced order system built via moment matching typically is not a DAE anymore.

We end this section by pointing out that modal matching can also be interpreted as moment matching of the transfer function $(\lambda - \lambda_i)^\ell H(s)$, which has no poles anymore at $\lambda_i$. Its first $\ell$ moments are indeed the matrices $H_{-\ell}^{(i)}, \ldots, H_{-1}^{(i)}$ of (15) and will be retained in the modified reduced order model $(\lambda - \lambda_i)^\ell \hat{H}(s)$, provided both $V$ and $Z$ contain $V_i$ and $Z_i$ as submatrices.

## 5. Near-optimal solutions

The approximation problem in the frequency domain can also be phrased in terms of the $H_\infty$ norm of the error function $\Delta H(s) \doteq H(s) - \hat{H}(s)$. There is a well developed theory for finding nearly optimal solutions of this problem when the transfer function is given in a state-space formulation. These are based on the so-called balanced realizations of a state-space system $\{A, B, C\}$. Such realizations have Gramians that are equal and diagonal [11], and hence a diagonal product as well. We develop here the equivalent formulas for a generalized state-space $\{E, A, B, C\}$, under the assumption that $E$ is non-singular (which can therefore be reduced to a standard state-space system).

The controllability Gramian $G_c$ and observability Gramian $G_o$ of a system (1) can be defined as follows :

$$G_c = \int_0^{+\infty} (e^{E^{-1}At}E^{-1}B)(e^{E^{-1}At}E^{-1}B)^T dt, \tag{18}$$

$$G_o = \int_0^{+\infty} (CE^{-1}e^{AE^{-1}t})^T (CE^{-1}e^{AE^{-1}t}) dt, \tag{19}$$

which by Parseval's theorem are also equal to

$$G_c = \frac{1}{2\pi} \int_0^{+\infty} (\jmath\omega E - A)^{-1} BB^T (\jmath\omega E - A)^{-*} d\omega, \tag{20}$$

$$G_o = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\jmath\omega E - A)^{-*} C^T C (\jmath\omega E - A)^{-1} d\omega. \tag{21}$$

These Gramians can be computed as the solution of the generalized state-space equations

$$AG_cE^T + EG_cA^T = -BB^T \text{ and } A^TG_oE + E^TG_oA = -C^TC. \tag{22}$$

The Gramians of the corresponding state space realization $\{E^{-1}A, E^{-1}B, C\}$ are in fact equal to $G_c$ and $E^TG_oE$, respectively. It is the product $E^TG_oEG_c$ of these two matrices that one diagonalizes via a state-space similarity transformation [11]. One then truncates the smallest diagonal elements (i.e. eigenvalues of $E^TG_oEG_c$), which yields the reduced order approximation. An $n$-th order approximation is thus obtained from the eigenspace corresponding to the $n$ largest eigenvalues of $E^TG_oEG_c$. An slightly better approximation to $H(s)$ is obtained from the optimal Hankel norm approximation which is also derived from the balanced realization and hence eigendecompostion of $E^TG_oEG_c$. In practice both approximations give nearly optimal approximations in the $H_\infty$ norm [11].

But (20,21) suggest than the rational Krylov approach tries to approximate the same objects, since the frequency response $C(\jmath\omega E - A)^{-1}B$ clearly

is retrieved in the integrals describing the Gramians. Let us define the approximations to these Gramians as

$$\tilde{G}_c \doteq V\hat{G}_c V^T, \text{ and } \tilde{G}_o \doteq Z\hat{G}_o Z^T, \tag{23}$$

where $\hat{G}_c$ and $\hat{G}_o$ satisfy the projected equations

$$\hat{A}\hat{G}_c\hat{E}^T + \hat{E}\hat{G}_c\hat{A}^T = -\hat{B}\hat{B}^T \text{ and } \hat{A}^T\hat{G}_o\hat{E} + \hat{E}^T\hat{G}_o\hat{A} = -\hat{C}^T\hat{C}. \tag{24}$$

Define also the residuals

$$A\tilde{G}_c E^T + E\tilde{G}_c A^T + BB^T \doteq \Delta_c \text{ and } A^T\tilde{G}_o E + E^T\tilde{G}_o A + C^T C \doteq \Delta_o. \tag{25}$$

Then the approximations clearly satisfy the Galerkin conditions

$$Z^T\Delta_c Z = 0, \text{ and } V^T\Delta_o V = 0. \tag{26}$$

As the spaces $\mathcal{V}$ and $\mathcal{Z}$ grow, these residuals decrease and we are thus trying to improve the approximation $\tilde{G}_c \approx G_c$ and $\tilde{G}_o \approx G_o$. By choosing $Z$ and $V$ such that the dominant features of the transfer function $C(\jmath\omega E - A)^{-1}B$ are captured, we look for the dominant spaces of $G_c$ and $G_o$, which in the balanced coordinate system are also the dominant eigenspaces of these two positive definite matrices.

## 6. A numerical comparison

We now compare moment matching techniques with the near-optimal approaches in terms of numerical accuracy and complexity. For the complexity, we only need to consider the most time consuming steps of each approach. We assume that the system is given in generalized state-space system. For the solution of the generalized Lyapunov equations one needs first to put the pencil $\lambda E - A$ in generalized Schur form. For a dense $N \times N$ system this requires approximately $70N^3$ flops (floating point operations). The subsequent eigendecomposition of $E^T G_o E G_c$ needed for balanced truncation takes another $30N^3$ flops, while the additional work for constructing an optimal Hankel norm approximation requires about $60N^3$ flops. Both near-optimal solutions therefore require over $100N^3$ flops. The rational Krylov approach on the other hand requires the $LU$ factorizations of each matrix $(\sigma^{(i)}E - A)$, which is a total of $\frac{2}{3}KN^3$ flops. All other steps are less important and so this approach is of much lower complexity since the number of interpolation points $K$ is typically small.

For sparse large-scale systems, the comparison is still in favor of rational Krylov methods. The near-optimal solutions still need the solution of the generalized Lyapunov equations but here one can also use sparse matrix

techniques. Efficient methods are based on the Smith iteration and multi-point accelerations of it [20] and compute a low rank approximation of the true solutions. This still has a complexity of the order of $c_1 k\alpha N$ flops where $\alpha$ is the average number of non-zero elements in each row of $A$ and $E$, and $k$ is the number of interpolation points used in this method. All other parts of the algorithm involve $n \times n$ matrices where $n$ is the approximate rank of the Gramians. Notice that for this reason, the near-optimality is lost and that these approximations become much less accurate. For the rational Krylov approach one uses iterative solvers for the solution of the systems involving $(\sigma^{(i)}E - A)$, and this requires $c_2 K\alpha N$ flops, where $K$ is the number of interpolation points. Both approaches are therefore comparable in complexity, but the near-optimal methods tend to loose their accuracy. Moreover, the rational Krylov methods rely on independent matrix solves which can be implemented efficiently in parallel.

Now we look at the accuracy of both approaches. In order not to disfavor the near-optimal schemes we use the full Lyapunov solvers and start from standard state-space models so that no accuracy can get lost from the inversion of $E$. All computations were performed on a machine with IEEE standard arithmetic and using MATLAB (which is also why all models are reasonably small). In Figures 1, 2 and 3 we compare three 15th order approximations of a single-input/single-output 120th transfer function of degree 120, used for the Compact Disc regulator [27]. The solid lines represent the frequency response of the full system, the approximations are dotted for the Hankel norm approximation, dash-dotted for the balanced truncation and dashed for the rational Krylov method.

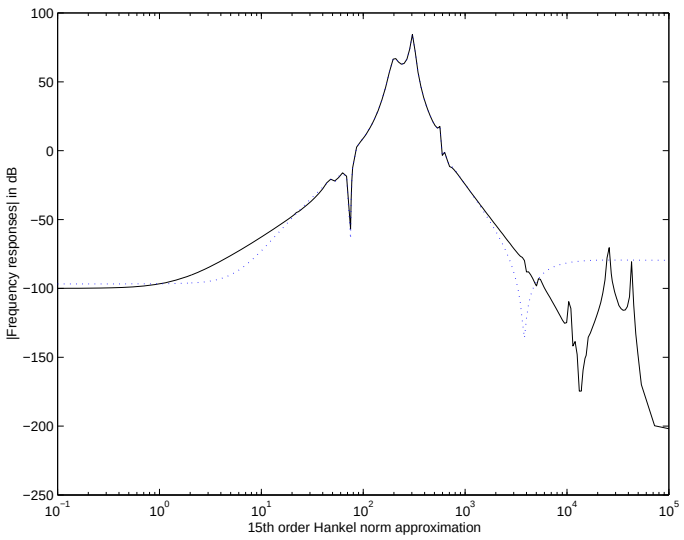


Fig. 1. Optimal Hankel norm approximation

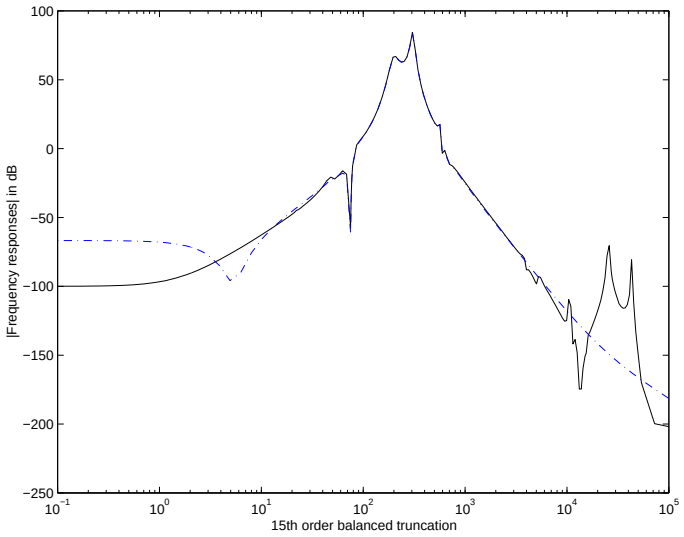


Fig. 2. Balanced truncation

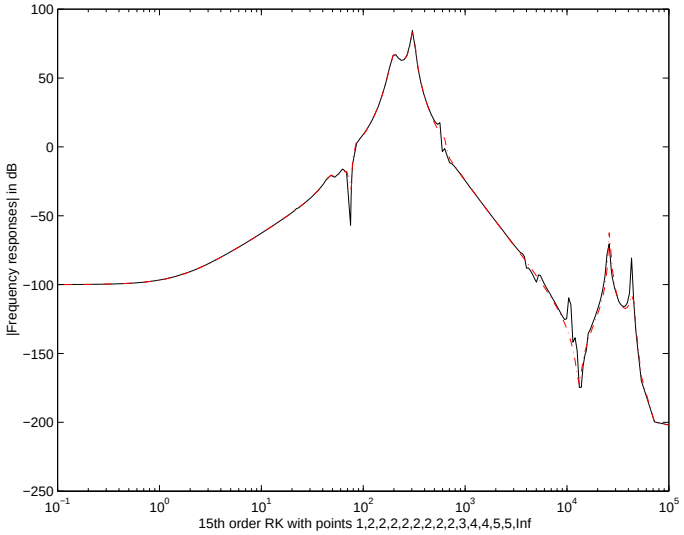


Fig. 3. Rational Krylov with 6 interpolation points

Although the near-optimal schemes should behave much better that the rational Krylov approximations, the pictures seem to indicate the opposite. This impression is due to the fact that a logarithmic scale was used. The *absolute error* of the near-optimal schemes is indeed much better that that of the Rational Krylov approach as indicated in Table 1, but for the logarithmic errors this is just the opposite. Notice that for the rational Krylov method we used 6 different interpolation points. We observed the

same phenomenon on random 100th order models for various orders of approximation. As expected, the absolute errors for the near-optimal scheme are systematically smaller than for the rational Krylov technique, but the logarithmic errors of both approaches are comparable. In practice it is important to take into account the large range of scales of the frequency response, since a very small value of the frequency response corresponds to "blocking" frequency and it is often desirable to maintain this in the reduced order model. This is precisely what is obtained by a logarithmic fit of the frequency response.

| Errors | $\|\Delta ln(h)\|$ | $\|\Delta h\|$ |
|---|---|---|
| Hankel norm approximation | 6.1 | 0.02 |
| Balanced truncation | 4.1 | 0.04 |
| Rational Krylov approximation | 1.5 | 4.02 |

Table 1 : Logarithmic versus absolute errors of the approximations

## 7. Concluding remarks

In this paper we showed the advantages of the rational Krylov approach for constructing reduced order models of generalized state-space systems. These methods rely on sparse matrix solves which can be implemented efficiently using iterative or direct methods and are easy to parallelize. They can be mixed with partial modal matching and also work for systems with singular $E$ matrices. Finally, their accuracy is comparable to near-optimal schemes provided the point selection is handled appropriately. Issues that were not handled in this paper are the point selection mechanism and the stopping criterion both for the model approximation and the iterative solves. These additional features are handled elsewhere but are equally important in order to obtain an efficient general purpose scheme.

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