ON THE GENERALITY OF MULTIPOINT PADÉ APPROXIMATIONS

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Abstract: Multipoint Padé interpolation methods were shown to be very efficient for the construction of reduced order models of large-scale dynamical systems. The objective of this paper is to analyse the generality of this approach. In the SISO case, we show that the reduced order interpolating system is unique if and only if it can be constructed via Multipoint Padé. In the MIMO case, an extension of Multipoint Padé method to create reduced order models that tangentially interpolate the original model is developed. The generality of this approach for MIMO model reduction will be discussed in a later paper.

Keywords:
Multipoint Padé, interpolation, tangential interpolation, model reduction

1. INTRODUCTION

For the sake of brevity, we will not give all proofs in this extended abstract. In the first section, we recall some basic results about Krylov subspaces and Multipoint Padé approximations, which will be used in the sequel. In section 2, we show the connection between Multipoint Padé and uniqueness of the interpolating transfer fonction. In section 3, we extend for MIMO systems the method of Multipoint Padé to tangential interpolation, and discuss the generality of this new method.

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1.1 *Introduction to Model Reduction*

Although most of the theory presented in this paper holds for both continuous-time and discrete-time systems, we only cover here the continuous-time case. Extensions of the theory for discrete-time systems are straightforward. Every linear time-invariant continuous-time system can be represented by a generalized state-space model :

$$\left\{ \begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \right. \tag{1}$$

with input $u(t) \in \mathbb{R}^m$, state $x(t) \in \mathbb{R}^N$ and output $y(t) \in \mathbb{R}^p$. Without loss of generality, we can assume that the system is controllable and observable since otherwise we can always find a smaller dimensional model that is controllable and observable, and that has exactly the same transfer function. In addition to this, we will assume that the system is stable, i.e. the generalized eigenvalues of the pencil $sE - A$ lie in the open left half plane (this also implies that E is nonsingular).

When the system order N is too large for solving various control problems within a reasonable computing time, it is natural to consider approximating it by a reduced order system

$$\begin{cases} \hat{E}\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\ \hat{y} = \hat{C}\hat{x} + \hat{D}u \end{cases} \tag{2}$$

driven with the same input $u(t) \in \mathbb{R}^m$, but having different output $\hat{y}(t) \in \mathbb{R}^p$ and state $\hat{x}(t) \in \mathbb{R}^n$. For the same reasons as above, we will assume that the reduced order model is minimal. The degree n of the reduced order system is also assumed to be much smaller than the degree N of the original system.

The objective of the reduced order model is to project the state space (of dimension N) of the system onto a space of lower dimension n in such a way that the behavior of the reduced order model is sufficiently close that of the full order system. For a same input $u(t)$, we thus want $\hat{y}(t)$ to be close to $y(t)$. This also implies that the reduced order system will have to be stable since otherwise both system responses can not be close to each other.

One shows that in the frequency domain, this is equivalent to imposing conditions on the frequency responses of both systems (Zhou *et al.*, 1995), (Van Dooren, 2000) : we want to find a reduced order model such that the transfer functions of both models, i.e.

$$T(s) = C(sE - A)^{-1}B \tag{3}$$
$$\hat{T}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}, \tag{4}$$

are such that the error $\|T(.) - \hat{T}(.)\|$ is minimal for the H_∞ norm.

The reduced order models we will consider in this paper are built as follows. To construct a n-th order reduced system, we project the matrices of the original system using $(N \times n)$ matrices Z and V as follows :

$$\{\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D}\} = \{Z^T EV, Z^T AV, Z^T B, CV, D\}. \tag{5}$$

The matrices Z and V can therefore be viewed as (respectively left and right) *projectors*. Below we present a general theorem linking the image of the projectors Z and V with interpolation.

1.2 *Multipoint Padé*

We review first some results about *moment matching* methods. Let us consider an expansion of $T(s)$ about a point σ that is *not* a pole of $T(s)$. It then follows that $\sigma E - A$ is invertible and one obtains the following formal series expansion :

$$T(s) = C\left(\sigma E - A - (\sigma - s)E\right)^{-1} B$$
$$= C\left(I - (\sigma E - A)^{-1}E(\sigma - s)\right)^{-1}(\sigma E - A)^{-1}B$$
$$= \sum_{j=0}^{+\infty} C\left((\sigma E - A)^{-1}E\right)^j (\sigma E - A)^{-1}B \cdot (\sigma - s)^j$$
$$\doteq \sum_{j=0}^{+\infty} T_\sigma^{(j)} \cdot (\sigma - s)^j \tag{6}$$

which defines the so-called *moments*

$$T_\sigma^{(j)} \doteq C\left((\sigma E - A)^{-1}E\right)^j (\sigma E - A)^{-1}B \tag{7}$$

about an expansion point σ. These moments exist for every σ for which $(\sigma E - A)$ is non-singular. The same expansion can be made of a lower order model $\hat{T}(s)$ about the same point σ provided $(\sigma\hat{E} - \hat{A})$ is invertible. Define the ensemble

$$P = \{(\sigma_i, \nu_i) : i = 1 \dots r,\ \nu_i \in \mathbb{N}_0,\ \sum_{i=1}^{r} \nu_i = 2n\}, \tag{8}$$

where all the points σ_i are different and are not eigenvalues of A. We say that a reduced order transfer function $\hat{T}(s)$ matches the interpolating conditions given by the ensemble P when

$$\hat{T}_{\sigma_i}^{(j)} = T_{\sigma_i}^{(j)} \quad \forall\ 1 \le j \le \nu_i,\ \forall\ 1 \le i \le r, \tag{9}$$

where $(\sigma_i, \nu_i) \in P$. The objective of a *moment matching* method is to construct a reduced order model $\hat{T}(s)$ such that $\hat{T}(s)$ verifies the interpolating conditions (9). We now show that such expansions must match for a certain number of moments when the lower order model is constructed via a projection on particular Krylov spaces. For matrices $G \in \mathbb{R}^{N\times N}$, $H \in \mathbb{R}^{N\times n}$, we define the Krylov space of index j as follows:

$$\mathcal{K}_j(G, H) = \mathrm{Im}\{H, GH, G^2H, \dots, G^{j-1}H\}. \tag{10}$$

The following lemma related to such subspaces will prove to be useful.

Lemma 1.1. Let $V \in \mathbb{R}^{N\times n}$ be a full rank matrix such that

$$\mathcal{K}_j(G, H) \subseteq \mathcal{V} \doteq \mathrm{Im}V$$

and let W be an arbitrary $n \times N$ matrix such that $W^T V = I_n$. Then the *projected* matrices

$$\hat{G} \doteq W^T GV, \quad \hat{H} \doteq W^T H,$$

satisfy the equalities

$$G^i H = V\hat{G}^i\hat{H}, \quad i = 0, \dots, j-1.$$

Proof :
Since V is full rank, there exists a matrix W^T such that $W^T V = I_n$. Since the image of each $G^i H$ is spanned by the columns of V there exists for all $i = 0, \dots, j-1$ a matrix Y_i such that

$$G^i H = VY_i, \quad \text{and hence} \quad Y_i = W^T G^i H.$$

The proof now goes by induction. For $i = 0$ clearly $Y_0 = W^T H = \hat{H}$. If $Y_i = \hat{G}^i\hat{H}$ then also $Y_{i+1} = W^T G.G^i H = W^T G V \hat{G}^i \hat{H} = \hat{G}^{i+1}\hat{H}$, which proves the result. $\square$

The following theorem is shown in (Grimme, 1992) and (Gallivan *et al.*, 1999) for a SISO system, and is rewritten here for the MIMO case.

Theorem 1.1. If the spaces $\mathcal{V} \doteq Im(V)$ and $\mathcal{Z} \doteq Im(Z)$ satisfy

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{b_k}}((\sigma_k E - A)^{-1}E, (\sigma_k E - A)^{-1}B) \subseteq \mathcal{V}$$

and

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{c_k}}((\sigma_k E - A)^{-T}E^T, (\sigma_k E - A)^{-T}C^T) \subseteq \mathcal{Z}$$

where the interpolation points points σ_k are chosen such that the matrices $\sigma_k E - A$ are invertible $\forall k \in \{1, \ldots, K\}$, then the moments of the systems (1) and (2) around the points σ_k satisfy

$$T_{\sigma_k}^{(j_k)} = \hat{T}_{\sigma_k}^{(j_k)} \qquad (11)$$

for $j_k = 1, 2, \ldots, J_{b_k} + J_{c_k}$ and $k = 1, 2, \ldots, K$, provided these moments exist, i.e. provided the matrices $\hat{A} - \sigma_k \hat{E}$ are invertible.

Proof : This is based on Lemma 1.1 and is omitted here for the sake of brevity. $\square$

If we want to find an order n reduced order transfer function $\hat{T}(s)$ that verifies the interpolating conditions (9), we proceed as follows. Choose an ensemble P as defined in (8). Partition P into 2 ensembles P_1 and P_2 of cardinality n in the following way :

$$P_i \doteq \{\sigma_1, \sigma_1, \ldots, \sigma_r\}, \qquad (12)$$

where σ_j appears $\nu_{i,j}$ times and $\nu_{1,j} + \nu_{2,j} = \nu_j$. The elements of P_i are ordered and denoted by $p_{i,j}$, $j = 1, \ldots, n$. Then, we construct the projector V such that

$$\bigcup_{j=1}^{r} \mathcal{K}_{\nu_{1,j}}((\sigma_j E - A)^{-1}E, (\sigma_j E - A)^{-1}B) \subseteq \mathcal{V}$$

and Z such that

$$\bigcup_{j=1}^{r} \mathcal{K}_{\nu_{2,j}}((\sigma_j E - A)^{-T}E^T, (\sigma_j E - A)^{-T}C^T) \subseteq \mathcal{Z}.$$

By Theorem 1.1, if all the moments are well defined, such a transfer function verifies (9).

Remark 1.1. Suppose that we choose a particular ensemble P. In the proof of Theorem 1.1, all the matrices $\sigma_k E - A$ of the original model have to be invertible. If this is not the case, the Krylov subspaces defined in the Thm 1.1 do not make sense anymore. All the matrices $\sigma_k \hat{E} - \hat{A}$ of the reduced order model have to be invertible as well. The singularity of $\sigma\hat{E} - \hat{A} = Z^T(\sigma E - A)V$ may be due to different reasons.
(a) Either Z (or V) is not full rank. In such a case, $\hat{E}$ and $\hat{A}$ are singular, and the reduced system is not controllable or observable anymore. In the next section, we show that, in the SISO case, this may only appear when the original transfer function was not minimal.
(b) It is possible that Z, V and $\sigma E - A$ are full rank matrices but that their product is singular. We show in the next section, again in the SISO case, that it occurs iff the minimal reduced order transfer function that matches (9) is not of order n for this particular choice of P.

1.3 *Some facts about Krylov subspaces*

In this section, we give some preleminary results about Krylov subspaces that will be used in the sequel of this paper. Most of the results of this section are very close to those developed in (Li, 2000). We focus our attention on Krylov subspaces $\mathcal{K}_n(A, B)$ of the same order as the dimension of A. The following Lemma is obvious.

Lemma 1.2. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$. Consider n polynomials of degree at most $n-1$,

$$\phi_i(x) = \sum_{j=0}^{n-1} \alpha_{i,j} x^j, \qquad 1 \leq i \leq n. \qquad (13)$$

Define the matrix $M \in \mathbb{R}^{n\times n}$ such that

$$M(i,j) = \alpha_{i,j-1}, \qquad \forall 1 \leq i, j \leq n. \qquad (14)$$

If M is invertible (i.e. if the polynomials are independent), then

$$Im\{\phi_1(A)B, \ldots, \phi_n(A)B\} = \mathcal{K}_n(A, B) \qquad (15)$$

Proof :
This follows from

$$\begin{pmatrix} \phi_1(A)B & \ldots & \phi_n(A)B \end{pmatrix} = \begin{pmatrix} B & \ldots & A^{n-1}B \end{pmatrix} M, \qquad (16)$$

and the fact that M is invertible. $\square$

Lemma 1.3. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$. Let ϕ be any invertible polynomial function of A of degree at most $n-1$. Then

$$\phi(A)\mathcal{K}_n(A, B) = \mathcal{K}_n(A, B) \qquad (17)$$

Proof :
By Caley-Hamilton,

$$\phi(A)\mathcal{K}_n(A, B) = r(A)\mathcal{K}_n(A, B) \subseteq \mathcal{K}_n(A, B), \qquad (18)$$

where $r(A)$ is the interpolating polynomial of $\phi(A)$. Since $\phi(A)$ is invertible,

$dim(\phi(A)\mathcal{K}_n(A,B)) = dim(\mathcal{K}_n(A,B))$ and equality of the 2 subspaces follows. $\square$

Now, we focus our attention on particular cases. Consider the set of pairs of points and corresponding orders

$$R = \{(s_i, \nu_i) : i = 1 \dots r,\ \nu_i \in \mathbb{N}_0,\ \sum_{i=1}^{r} \nu_i = n\}, \tag{19}$$

where all the points s_i are different and are not eigenvalues of A. Here, we rewrite a result already found in (Li, 2000).

Lemma 1.4. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $(s_i,\nu_i) \in R$, then

$$\begin{aligned} &Im\{(s_1 I - A)^{-1}B, \dots, (s_1 I - A)^{-\nu_1}B,\\ &\quad \dots, (s_r I - A)^{-1}B, \dots, (s_r I - A)^{-\nu_r}B\}\\ &= Im\{(s_1 I - A)^{-1}B, \dots, (s_1 I - A)^{-\nu_1}B,\\ &\quad (s_1 I - A)^{-\nu_1}(s_2 I - A)^{-1}B, \dots, \prod_{i=1}^{r}(s_i I - A)^{-\nu_i}B\}\\ &= Im\{B, AB, \dots, A^{n-1}B\} \end{aligned} \tag{20}$$

The proof is rather technical and is omitted here.

2. UNIQUENESS OF SOLUTIONS

In this section, we give necessary and sufficient conditions to construct a minimal order n transfer function that matches the $2n$ interpolating conditions (9). Several results shown here are taken from (Antoulas and Anderson, 1986) and (Anderson and Antoulas, 1990). In this paper, we apply these results to Multipoint Padé techniques. Choose an ensemble P and construct P_1 and P_2 as explained in (8) and (12). In order to define the generalized Loewner matrix, $L \in \mathbb{R}^{n\times n}$, we need to define $m_{i,j}$ as the number of times that $p_{i,j}$ occurs in the subset $\{p_{i,1}, \dots, p_{i,j}\}$. We then define the Loewner matrix via its elements $(k,l), 1 \le k,l \le n$:

$$L_T(k,l) = C(p_{1,k}I - A)^{-m_{1,k}}(p_{2,l}I - A)^{-m_{2,l}}B, \tag{21}$$

where the subscript T refers to the transfer function $T(s) = C(sI-A)^{-1}B$. A first property of this matrix is the following

Lemma 2.1.

$$L_T(k,l) = L_{\hat{T}}(k,l), \quad \forall\ 1 \le k,l \le n. \tag{22}$$

Equation (22) is a straightforward consequence of the interpolating conditions (9). We can suppose without loss of generality that the ensembles P_i are ordered in such a way that we can write

$$L_T = \begin{pmatrix} C(\sigma_1 I - A)^{-1}\\ \vdots\\ C(\sigma_1 I - A)^{-\nu_{1,1}}\\ C(\sigma_2 I - A)^{-1}\\ \vdots\\ C(\sigma_r I - A)^{-\nu_{1,r}} \end{pmatrix} \tag{23}$$

$$\begin{aligned} &\left((\sigma_1 I - A)^{-1}B \ \dots\ (\sigma_r I - A)^{-\nu_{2,r}}B \right)\\ &\doteq \mathcal{O}_T\mathcal{C}_T, \end{aligned} \tag{24}$$

where we delete the row associated with $\nu_{1,i}$ if $\nu_{1,i} = 0$, and the column associated with $\nu_{2,j} = 0$ if $\nu_{2,j} = 0$. It follows from Lemma 2.1 that

$$\mathcal{O}_T\mathcal{C}_T = \mathcal{O}_{\hat{T}}\mathcal{C}_{\hat{T}}. \tag{25}$$

Finally, we define the following matrix that will be used in the next subsection.

$$\mathcal{D}_T \doteq [B|\mathcal{C}_T]. \tag{26}$$

2.1 *The regular case*

This is the case when it is possible to construct an order n reduced-order transfer function $\tilde{T}(s)$ via Multipoint Padé that verifies the interpolating conditions (9). The main result of this sub-section is the following Proposition

Proposition 1. Suppose that there exists an order n transfer function $\hat{T}(s)$ that verifies the interpolating conditions (9). If $\hat{T}(s)$ is minimal, then
(1) $\hat{T}(s)$ is the unique transfer function of order n that verifies the interpolating conditions (9)
(2) $\hat{T}(s)$ can be constructed via Multipoint Padé.

The proof of this new result is rather technical and is omitted here. A complementary result is the following Corollary.

Corollary 2.1. There exists a unique transfer function $\hat{T}(s)$ of order n that matches the interpolating conditions (9) iff the transfer function $\tilde{T}(s)$ obtained via Multipoint Padé verifies (9), i.e. the interpolation points are not poles of $\tilde{T}(s)$.

The "only if" part of Corollary 2.1 is a straightforward consequence of Proposition 1. The "if" part is not treated here.

2.2 *The singular case*

We know that the reduced order transfer function constructed via Multipoint Padé verifies the interpolating conditions (9) iff there exists a unique interpolating transfer function of order n that matches (9). Moreover, we know that in this case the reduced order transfer function is minimal.

Now, we analyse the case where at least one matrix $Z^T(\sigma_i I_N - A)V$ is singular, i.e. one moment of the reduced order transfer function constructed via Multipoint Padé does not exist. We will show that there are two cases to consider : either there is a unique transfer function of order less than n that verifies (9), or there exists a class of interpolating transfer functions of minimal order greater than n that verifies (9). We refer to (Antoulas and Anderson, 1986) for the following Theorem :

Theorem 2.1. Suppose that the generalized Loewner matrix $\mathcal{O}_T\mathcal{D}_T$ is of rank q. Then
(1) If every $q \times q$ Loewner submatrix of $\mathcal{O}_T\mathcal{D}_T$ is invertible, then there exists an unique minimal transfer function of order q that verifies (9). Moreover, there is no transfer function of degree less than q that verifies 9.
(2) If some $q \times q$ Loewner submatrices of $\mathcal{O}_T\mathcal{D}_T$ are singular, all the transfer functions that verify (9) are of degree greater than or equal to $2n - q$. The family of all interpolating transfer function of degree $2n - q$ is parametrized in terms of one parameter.

3. THE MIMO SYLVESTER EQUATION

Suppose we want to solve

$$AV_+ + V_+\hat{A}^T + B\hat{B}^T = 0 \tag{27}$$

where $A \in \mathbb{R}^{N\times N}, B \in \mathbb{R}^{N\times p}, \hat{A} \in \mathbb{R}^{n\times n}$. For simplicity we assume $\Lambda(A) \cap \Lambda(-\hat{A}) = \emptyset$, which makes the solution unique and well-defined. By linearity, we can write

$$V_+ = \sum_{i=1}^{p} V_i \tag{28}$$

$$AV_i + V_i\hat{A}^T + b_i^c\hat{b_i^c}^T = 0, \tag{29}$$

where x_i^c is the i^{th} column of a matrix X. Moreover, it is easy to check that the image of V_+ does not change when we are solving (27) with the Jordan canonical form $\hat{A}_J$ of $\hat{A}$ instead of $\hat{A}$. Hence, we can write

$$\hat{A} = T\hat{A}_J T^{-1} \tag{30}$$

$$T^{-1}\hat{B} = \tilde{B} \tag{31}$$

$$\tilde{V_i} = V_i T^{-T} \tag{32}$$

Because of the bloc diagonal structure of the Jordan canonical form, we can decompose the system (27) into smaller systems with only one Jordan bloc of $\hat{A}$. So, we restrict our analysis in the case

$$\hat{A}_J = \begin{pmatrix} \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & \lambda \end{pmatrix}. \tag{33}$$

By solving (29) with $\hat{A} = \hat{A}_J$, it is possible to show that

$$\tilde{V_i} = \left((A+\lambda I)^{-1}b_i^c, \ \ldots, \ (A+\lambda I)^{-n}b_i^c \right) \begin{pmatrix} \tilde{b_i}^c(1) & \ldots & \tilde{b_i}^c(n) \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ \tilde{b_i}^c(n) & & \end{pmatrix}. \tag{34}$$

This last equation allows us to write

$$\tilde{V}_+ = \left((A+\lambda I)^{-1}B, \ \ldots, \ (-1)^{n+1}(A+\lambda I)^{-n}B \right) \begin{pmatrix} \tilde{b_1^r}^T & \ldots & \tilde{b_n^r}^T \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ \tilde{b_n^r}^T & & \end{pmatrix}, \tag{35}$$

where we write x_i^r for the i^{th} row of a matrix X. This last result allows us to show the following.

Proposition 2. With the notations and conventions stated above, the reduced-order system constructed with V_+ as right-projector, $\forall 1 \leq p \leq n$,

$$\begin{aligned} &\sum_{i=1}^{p} T^{i-1}(-\lambda)\tilde{b^r}_{n-p+1}^T \\ &= \sum_{i=1}^{p} \hat{T}^{i-1}(-\lambda)\tilde{b^r}_{n-p+1}^T. \end{aligned} \tag{36}$$

Sketch of the proof :
By induction,
- For $p = 1$,

$$(A+\lambda I)^{-1}B\tilde{b^r}_n^T \in Im(V_+) \tag{37}$$

Hence,

$$\begin{aligned} &(A+\lambda I)^{-1}B\tilde{b^r}_n^T \\ &= V_+(\hat{A}^+ + \lambda I)^{-1}\hat{B}^+\tilde{b^r}_n^T \end{aligned} \tag{38}$$

- For $p = 2$,

$$\begin{aligned} &(A+\lambda I)^{-1}B\tilde{b^r}_{n-1}^T - (A+\lambda I)^{-2}B\tilde{b^r}_n^T \in Im(V_+) \\ &= V_+\left((\hat{A}^+ + \lambda I)^{-1}\hat{B}^+\tilde{b^r}_{n-1}^T - (\hat{A}^+ + \lambda I)^{-2}\hat{B}^+\tilde{b^r}_n^T \right) \end{aligned}$$

- For $p > 2$ ...
The result follows from similar arguments as the ones used to prove Theorem 1.1 and will not be developped here.
General results about tangential interpolation may be found in (Antoulas *et al.*, 1990) and (Ball

et al., 1990). Interpolation of MIMO systems via a block *Multipoint Pade* technique is already discussed in (Gallivan *et al.*, 2001).

4. CONCLUSIONS AND FUTURE WORK

4.1 *Conclusions*

In this paper, we have shown the generality of *Multipoint Padé* technique to construct interpolating reduced-order transfer functions. When the reduced-order transfer function constructed via *Multipoint Padé* does not check the interpolating conditions (9), we have shown that this is not due to the fact that we are using *Multipoint Padé* to construct our reduced-order model but this is due to the fact that the minimal reduced-order transfer function that verify the interpolating conditions (9) is not of rank n. In such a case, either we have to add a new interpolating condition until the reduced-order transfer function constructed via *Multipoint Padé* verifies all the interpolating conditions; or it is possible to construct a transfer function of order less than n that matches the interpolating conditions (9). Another possibility is to perturb the interpolating data (i.e. the points $\in P$) in such a way that the new ensemble P determines one unique interpolating transfer function. We should point out here that, from a practical point of vue, almost all interpolating conditions (i.e. almost every ensemble P) determine one unique interpolating transfer function that can be constructed via *Multipoint Padé*.

In the MIMO case, we have shown here a generalization of *Multipoint Padé* that allows us to construct a reduced-order transfer function that tangentially interpolate the original transfer function.

A big advantage of *Multipoint Padé* compared to others model reduction technique is its low computational cost. Hence, it can be applied to large-scale linear systems. A weakness of *Multipoint Padé* is that there exists no global error bound between the original and the reduced-order model.

4.2 *Future work*

The generality of Multipoint Padé in the MIMO case is still under investigation and will appear in a later paper.

Finding interpolating conditions such that there exists a global bound between the original and the reduced-order transfer function is an open question. For instance, we could look at well-known model reduction techniques such as balanced truncation or optimal Hankel norm approximation and try to characterize the interpolation points of reduced-order transfer function constructed via these techniques.

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6. REFERENCES

Anderson, B. D. O. and A. C. Antoulas (1990). Rational interpolation and state-variable realizations. *Linear Algebra and its Applications.*

Antoulas, A. C., J. A. Ball, J. Kang and J. C. Willems (1990). On the solution of the minimal rational interpolation problem. *Linear Algebra and its Applications.*

Antoulas, A.C. and B. D. O. Anderson (1986). On the scalar rational interpolation problem. *IMA J. Math. Control Inform.* **3**(2,3), 61–88.

Ball, Joseph A., Israel Gohberg and Leiba Rodman (1990). *Interpolation of Rational Matrix Functions.*

Gallivan, K., A. Vandendorpe and P. Van Dooren (2001). Sylvester equations and projection-based model reduction. Florida State University, Universite catholique de Louvain.

Gallivan, K., E. Grimme and P. Van Dooren (1999). Model reduction of large-scale systems, rational Krylov versus balancing techniques. In: *Error control and adaptivity in scientific computing (Antalya, 1998).* pp. 177–190. Kluwer Acad. Publ.. Dordrecht.

Grimme, Eric James (1992). Krylov projection methods for model reduction. PhD thesis. Ohio State University.

Li, Jing-Rebecca (2000). Model Reduction of Large Linear Systems via Low Rank System Gramians. PhD thesis. Massachusetts Institute of Technology.

Van Dooren, P. (2000). Gramian based model reduction of large-scale dynamical systems. *in Numerical Analysis 1999* pp. 231–247.

Zhou, K., J.C. Doyle and K. Glover (1995). Robust and optimal control. *Prentice Hall.*