Model reduction via tangential interpolation

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May 14, 2002

1 Introduction

Although most of the theory presented in this paper holds for both continuous-time and discrete-time systems, we only cover here the continuous-time case. Extensions of the theory for discrete-time systems are straightforward. Every linear time-invariant continuous-time system can be represented by a generalized state-space model :

$$\left\{ \begin{array}{rcl} E\dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{array} \right. \tag{1}$$

with input $u(t) \in \mathbb{R}^m$, state $x(t) \in \mathbb{R}^N$ and output $y(t) \in \mathbb{R}^p$. Without loss of generality, we can assume that the system is controllable and observable since otherwise we can always find a smaller dimensional model that is controllable and observable, and that has exactly the same transfer function. In addition to this, we will assume that the system is stable, i.e. the generalized eigenvalues of the pencil $sE - A$ lie in the open left half plane (this also implies that E is non-singular).

When the system order N is too large for solving various control problems within a reasonable computing time, it is natural to consider approximating it by a reduced order system

$$\left\{ \begin{array}{rcl} \hat{E}\dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u \\ \hat{y} & = & \hat{C}\hat{x} + \hat{D}u \end{array} \right. \tag{2}$$

driven with the same input $u(t) \in \mathbb{R}^m$, but having different output $\hat{y}(t) \in \mathbb{R}^p$ and state $\hat{x}(t) \in \mathbb{R}^n$. For the same reasons as above, we will assume that the reduced order model is minimal. The degree n of the reduced order system is also assumed to be much smaller than the degree N of the original system.

The objective of the reduced order model is to project the state-space (of dimension N) of the system onto a space of lower dimension n in such a way that the behavior of the reduced order model is sufficiently close to that of the full order system. For a same input $u(t)$, we thus want $\hat{y}(t)$ to be close to $y(t)$. This also implies that the reduced order system will have to be stable since otherwise both system responses can not be close to each other. One shows that in the frequency domain, this is equivalent to imposing conditions on the

*A research fellowship from the Belgian National Fund for Scientific Research is gratefully acknowledged by the second author. This paper presents research supported by the Belgian Programme on Inter-university Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. This work was also supported by the National Science Foundation under Grant No. CCR-9912415.

frequency responses of both systems [7] : we want to find a reduced order model such that the transfer functions of both models, i.e.

$$\begin{aligned} T(s) &= C(sE-A)^{-1}B+D & (3)\\ \hat{T}(s) &= \hat{C}(s\hat{E}-\hat{A})^{-1}\hat{B}+\hat{D}, & (4)\end{aligned}$$

are such that the error $\|T(.)-\hat{T}(.)\|$ is minimal for the H_∞ norm.

2 Tangential interpolation

The reduced order models we will consider in this paper are built as follows. To construct a n-th order reduced system, we project the matrices of the original system using $(N\times n)$ matrices Z and V as follows :

$$\{\hat{E},\hat{A},\hat{B},\hat{C},\hat{D}\}=\{Z^TEV,Z^TAV,Z^TB,CV,D\}. \tag{5}$$

The matrices Z and V can therefore be viewed as (respectively left and right) *projectors*. Below we present a general theorem linking the image of the projectors Z and V with the following tangential interpolation (TI) problem :

Definition 2.1 *Let there be given distinct points $z_1,\ldots,z_r$ in the complex plane. For every z_α, a collection of vector polynomials $x_{\alpha,1}(s),\ldots,x_{\alpha,k_\alpha}(s)$ of size $1\times p$ is given. Let there be given distinct points $w_1,\ldots,w_t$ in the complex plane, and for every w_γ, a collection of vector polynomials $u_{\gamma,1}(s),\ldots,u_{\gamma,l_\gamma}(s)$ of size $m\times 1$ be given. Finally, if some z_α is equal to some w_γ, say $\xi_{\alpha,\gamma}=z_\alpha=w_\gamma$ then define*

$$x^{(g)}_{\alpha,j}(s)=\sum_{f=0}^{g}x^{[f]}_{\alpha,j}\Big|_{s=z_\alpha}(s-z_\alpha)^f \tag{6}$$

(i.e., $x^{(g)}_{\alpha,j}(s)$ is obtained from $x_{\alpha,j}(s)$ by keeping its first $g+1$ terms in the Taylor expansion around z_α) and analogously

$$u^{(g)}_{\gamma,j}(s)=\sum_{f=0}^{g}u^{[f]}_{\gamma,j}\Big|_{s=z_\gamma}(s-w_\gamma)^f. \tag{7}$$

Our TI problem can be stated as follows : Find a reduced order transfer function $\hat{T}(s)$ of Mc Millan degree n such that the three following types of interpolation conditions are satisfied :
Left Interpolation Conditions

$$\frac{d^{i-1}}{ds^{i-1}}\{x_{\alpha,j}(s)T(s)\}\Big|_{s=z_\alpha}=\frac{d^{i-1}}{ds^{i-1}}\{x_{\alpha,j}(s)\hat{T}(s)\}\Big|_{s=z_\alpha}, \tag{8}$$

$i=1,\ldots,\beta_{\alpha,j}$; $j=1,\ldots,k_\alpha$; $\alpha=1,\ldots,r$ (here $\beta_{\alpha,j}$ are given positive integers depending on α and j).
Right Interpolation Conditions

$$\frac{d^{i-1}}{ds^{i-1}}\{T(s)u_{\gamma,j}(s)\}\Big|_{s=w_\gamma}=\frac{d^{i-1}}{ds^{i-1}}\{\hat{T}(s)u_{\gamma,j}(s)\}\Big|_{s=w_\gamma}, \tag{9}$$

$i = 1, \ldots, \delta_{\gamma,j}$; $j = 1, \ldots, l_\gamma$; $\gamma = 1, \ldots, s$ (here $\delta_{\gamma,j}$ are given positive integers depending on γ and j). Finally, when $z_\alpha = w_\gamma \doteq \xi_{\alpha,\gamma}$, we impose the following

Two Sided Interpolation Conditions

$$\frac{d^{f+g-1}}{ds^{f+g-1}} \left\{x_{\alpha,i}^{(f)}(s) T(s) u_{\gamma,j}^{(g)}(s)\right\}\Big|_{s=\xi_{\alpha,\gamma}} = \frac{d^{f+g-1}}{ds^{f+g-1}} \left\{x_{\alpha,i}^{(f)}(s) \hat{T}(s) u_{\gamma,j}^{(g)}(s)\right\}\Big|_{s=\xi_{\alpha,\gamma}}, \tag{10}$$

where $S_{i,j,\alpha,\gamma}^{(fg)}$ are prescribed numbers, $f = 0, \ldots, \beta_{\alpha,i}$; $g = 0, \ldots, \delta_{\gamma,j}$; $i = 1, \ldots, k_\alpha$; $j = 1, \ldots, l_\gamma$. We emphasize that conditions (10) are imposed for every pair of indices α, γ such that $z_\alpha = w_\gamma$.

This formulation is a particular case of the general TI problem as stated in [2] and [3]. Notice also that we can assume w.l.o.g. that $\hat{D} = D = 0$ since these matrices are independent of the state space dimension.

To make the ideas simpler, we illustrate now the solution of the TI problem in a particular case, namely when

$$E = I_n, \quad r = k_1 = s = l_1 = 1, \quad n = \beta_{1,1} = \delta_{1,1}. \tag{11}$$

Moreover, we assume $z_1 \neq w_1$ and redefine $\alpha \doteq z_1$ and $\gamma \doteq w_1$ so that we can delete most subscripts. In other words, we want to construct a reduced order transfer function $\hat{T}(s)$ of Mc Millan degree n such that

$$x(s)T(s) = x(s)\hat{T}(s) + O(s-\alpha)^n, \tag{12}$$

$$T(s)u(s) = \hat{T}(s)u(s) + O(s-\gamma)^n. \tag{13}$$

For every vector polynomial $v(s)$, for any $\alpha \in \mathbb{C}$ that is not a pole of $v(s)$, we can write

$$v(s) = \sum_{k=0}^{\infty} v^{[k]}(\alpha - s)^k. \tag{14}$$

For every transfer function $T(s) = C(sI - A)^{-1}B$, for any $\alpha \in \mathbb{C}$ that is not a pole of $T(s)$, we can write

$$T(s) = \sum_{k=0}^{\infty} C(\alpha I - A)^{-k-1}B(\alpha - s)^k. \tag{15}$$

Let us first consider equation (12). By imposing the n first coefficients of the Taylor expansion of the product $x(s)(T(s) - \hat{T}(s))$ to be zero, we find the following system of equations : Define the following matrices :

$$X \doteq \begin{bmatrix} x^{[0]} & & \\ \vdots & \ddots & \\ x^{[n-1]} & \ldots & x^{[0]} \end{bmatrix}, \quad U \doteq \begin{bmatrix} u^{[0]} & \ldots & u^{[n-1]} \\ & \ddots & \vdots \\ & & u^{[0]} \end{bmatrix}; \tag{16}$$

and the *generalized observability* and *generalized controllability* matrices

$$\mathcal{O}_T \doteq \begin{bmatrix} C(\alpha I - A)^{-1} \\ \vdots \\ C(\alpha I - A)^{-n} \end{bmatrix}, \quad \mathcal{C}_T \doteq \begin{bmatrix} (\gamma I - A)^{-1}B & \ldots & (\gamma I - A)^{-n}B \end{bmatrix}, \tag{17}$$

where the subscript T refers to the transfer function $T(s) = C(sI - A)^{-1}B$. We are now able to write the following lemma

Lemma 2.1 *A $p \times m$ rational transfer function $\hat{T}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$, satisfies the interpolation conditions* (12)-(13) *iff*

$$\begin{aligned} X\mathcal{O}_{\hat{T}}\hat{B} &= X\mathcal{O}_T B, & (18) \\ \hat{C}\mathcal{C}_{\hat{T}}U &= C\mathcal{C}_T U. & (19) \end{aligned}$$

Proof :
Equations (18)-(19) are just a more compact way to write the system of equations (12)-(13). Indeed, each row of (18) corresponds to imposing that one of the n first terms of the Taylor expansion of $x(s)T(s)$ around α equals the corresponding term of the Taylor expansion of $x(s)\hat{T}(s)$. The same observation can be made for each column of (19) and the n first terms of the Taylor expansion in (13). □
We define now the generalized Loewner matrix to be

$$\mathcal{L}_T \doteq X\mathcal{O}_T\mathcal{C}_T U. \quad (20)$$

The following result will not be proved in this extended abstract.

Proposition 2.1 *Every transfer function $\hat{T}(s)$ that satisfies the equations (12) and (13) is such that*

$$\begin{aligned} \mathcal{L}_T &= \mathcal{L}_{\hat{T}} & (21) \\ X\mathcal{O}_T A\mathcal{C}_T U &= X\mathcal{O}_{\hat{T}}\hat{A}\mathcal{C}_{\hat{T}}U. & (22) \end{aligned}$$

The proof is based on partial fraction expansion and Lemma 2.1.
This then leads us to the main result of this paper :

Proposition 2.2 *If the matrix $\mathcal{L}_T$ is invertible, then every transfer function that satisfies the interpolation conditions (12)-(13) has a Mc Millan degree greater than or equal to n. Moreover, the transfer function of degree n that satisfies the equations (12)-(13), if it exists, is unique and can be constructed with the following projecting matrices :*

$$\begin{aligned} Im(V) &= Im\,(\mathcal{C}_T U) & (23) \\ Ker(Z^T) &= Ker\,(X\mathcal{O}_T) & (24) \\ Z^T V &= I_n & (25) \end{aligned}$$

Proof :
Firstly, consider a transfer function $\hat{T}(s)$ that satisfies the interpolation conditions (12)-(13). Then, from Proposition 2.1, $\mathcal{L}_T = \mathcal{L}_{\hat{T}}$. This implies that the generalized observability and controllability matrices $\mathcal{O}_{\hat{T}}$ and $\mathcal{C}_{\hat{T}}$ are at least of rank n. Hence, the Mc Millan degree of $\hat{T}(s)$ is greater than or equal to n. This proves the first part of the proposition.
Suppose now that there exists a transfer function $\hat{T}(s)$ of Mc Millan degree n such that the equations (12) to (13) are satisfied. Then,

$$\begin{aligned} X\mathcal{O}_{\hat{T}}\hat{B} &= X\mathcal{O}_T B & (26) \\ \hat{C}\mathcal{C}_{\hat{T}}U &= C\mathcal{C}_T U & (27) \\ X\mathcal{O}_T A\mathcal{C}_T U &= X\mathcal{O}_{\hat{T}}\hat{A}\mathcal{C}_{\hat{T}}U. & (28) \end{aligned}$$

Because of the invertibility of $\mathcal{L}_T$, the matrices $X\mathcal{O}_{\hat{T}} \in \mathbb{C}^{n\times n}$ and $\mathcal{C}_{\hat{T}}U \in \mathbb{C}^{n\times n}$ are invertible. Define

$$M \doteq \left(X\mathcal{O}_{\hat{T}}\right)^{-1}, \quad N \doteq \left(\mathcal{C}_{\hat{T}}U\right)^{-1}; \quad (29)$$

$$Z^T \doteq MX\mathcal{O}_T, \quad V \doteq \mathcal{C}_T UN. \tag{30}$$

It is easy to see that

$$\hat{A} = Z^T AV, \quad \hat{B} = Z^T B, \quad \hat{C} = CV, \quad Z^T V = I_n. \tag{31}$$

We call the transfer function constructed via equations (29) to (31) $\hat{T}_{MP}(s)$, where the subscript MP means *Multipoint Padé*. It is not difficult to check that $\hat{T}_{MP}(s)$ is well-defined because of the invertibility assumption of $\mathcal{L}_T$.
This result implies that any transfer function of Mc Millan degree n that satisfies the equations (12) to (13) is equal to $\hat{T}_{MP}(s)$, and uniqueness follows. This completes the proof. □
Until now, we have just proved that if there exists a transfer function of Mc Millan degree n that satisfies the interpolation conditions (12)-(13), then it is the unique transfer function of Mc Millan degree n that satisfies (12)-(13), and this transfer function is $\hat{T}_{MP}(s)$. It remains to characterize when $\hat{T}_{MP}(s)$ is a solution of (12)-(13). Namely, if the interpolation points α and γ are not poles of $\hat{T}_{MP}(s)$, then $\hat{T}_{MP}(s)$ is the unique solution of Mc Millan degree n.

3 Concluding remarks

Firstly, it is important to point out that all the results of section 2 can be extended to the problem stated in section 1. There are no additional difficulties, but the notation is heavier.
Results about rational interpolation in the SISO case may be found in [1], [5] and [6]. In the MIMO case, more general results about existence and uniqueness of the solutions of the TI problem may be found in [2]. Generically, imposing $2n$ interpolation conditions with respect to a transfer function $T(s)$ determines one unique interpolating reduced transfer function of Mc Millan degree n, and this transfer function is $\hat{T}_{MP}(s)$. Moreover, from Proposition 2.2, we see that the construction of the projecting matrices Z and V only requires the computation of Krylov subspaces that appear in the matrices $\mathcal{C}_T$ and $\mathcal{O}_T$ (cfr equation (17)). Hence, constructing $\hat{T}_{MP}(s)$ is cheap, and tangential interpolation can be applied to large scale systems.
A severe drawback of the interpolation technique for building reduced order transfer function is that, up to now, there exists no global error bound between the interpolating reduced order transfer function and the original one. There are several open questions. The most important could be : "How can we choose interpolation conditions that guarantee to have a global error bound?". A more general question is : "Can every reduced transfer function be built via tangential interpolation of a transfer function of greater Mc Millan degree?" Let us consider briefly the latter question. For simplicity, we consider the SISO case. Let $T(s)$ be a SISO transfer function of Mc Millan degree N and $\hat{T}(s)$ be a SISO transfer function of Mc Millan degree $k < N$. The following result has recently been obtained in [4].

Theorem 3.1 *Choose $T(s) = C(sI_N - A)^{-1}B$, an arbitrary strictly proper SISO transfer function of Mc Millan degree N. Choose $\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}B$, an arbitrary strictly proper SISO transfer function of Mc Millan degree $n < N$. Then $\hat{T}(s)$ can be constructed via truncation of $T(s)$. Moreover, if $T(s)$ and $\hat{T}(s)$ do not have common poles, then $\hat{T}(s)$ can be constructed via Multipoint Pade from $T(s)$.*

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