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MODEL REDUCTION OF MIMO SYSTEMS VIA TANGENTIAL INTERPOLATION*

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Abstract. In this paper, we address the problem of constructing a reduced order system of minimal McMillan degree that satisfies a set of tangential interpolation conditions with respect to the original system under some mild conditions. The resulting reduced order transfer function appears to be generically unique and we present a simple and efficient technique to construct this interpolating reduced order system. This is a generalization of the *multipoint Padé* technique which is particularly suited to handle multiinput multioutput systems.

Key words. linear time invariant, multivariable system, model reduction, tangential interpolation, Krylov method, multipoint Padé

AMS subject classifications. 41A21, 65F99

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1. Introduction.

Model reduction of large-scale dynamical systems has received a lot of attention during the last decade: it is a crucial tool in reducing the computational complexity of, e.g., analysis and design of micro-electro-mechanical systems (MEMS) [13], in simulation of electronic devices [5], in weather prediction [6], and in control of partial differential equations [12].

The construction of the reduced order model typically passes via the derivation of one or two projective subspaces of the state space in which the original system is modelled. There are several approaches to find such projective subspaces. In this paper, we focus on an approach related to tangential interpolation of the rational transfer function, which therefore only works for linear time invariant systems. Tangential interpolation of given input/output data has already been treated in the literature [3], [4]. Here, we address the case where these data are themselves obtained from tangential information of a given (large-scale) transfer function, which to our knowledge has not been considered.

In this paper, we consider $p \times m$ *strictly proper* transfer functions $T(s)$, i.e., where $\lim_{s\to\infty} T(s) = 0$. This implies that the point at infinity is a zero of $T(s)$. For this reason, a separate treatment of the point at infinity is required.

We begin with some definitions which will allow us to formalize the problem of tangential interpolation. We say that a rational matrix function $R(s)$ is $O(\lambda - s)^k$ in s with $k \in \mathbb{Z}$ if its Taylor expansion about the point λ can be written as follows:

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$$R(s) = O(\lambda - s)^k \Longleftrightarrow R(s) = \sum_{i=k}^{+\infty} R_i(\lambda - s)^i, \tag{1.1}$$

where the coefficients R_i are constant matrices. If $R_k \neq 0$, then we say that $R(s) = \Theta(\lambda - s)^k$. As a consequence, if $R(s) = \Theta(\lambda - s)^k$ and k is strictly negative, then λ is a pole of $R(s)$, and if k is strictly positive, then λ is a zero of $R(s)$. Analogously, we say that $R(s)$ is $O(s^{-1})^k$ if the following condition is satisfied:

$$R(s) = O(s^{-1})^k \Longleftrightarrow R(s) = \sum_{i=k}^{+\infty} R_i s^{-i}, \tag{1.2}$$

where the coefficients R_i are constant matrices. It should be stressed that, in general, $R(s)$ being $O(s)^{-k}$ is not equivalent to $R(s)$ being $O(s^{-1})^k$.

We must also use the well-established concept of a zero of a system (see, e.g., [14]) and the following related definition.

DEFINITION 1.1. *Suppose that $T(s)$ is a $p \times m$ rational function. The zeros of the numerator polynomials not equal to zero in the Smith–McMillan form of the transfer function $T(s)$ are called the zeros of $T(s)$. An $m \times 1$ polynomial vector $y(s)$ is a right zero direction of order k at λ if $y(\lambda) \neq 0$ and*

$$T(s)y(s) = O(\lambda - s)^k. \tag{1.3}$$

Analogously, a $1 \times p$ polynomial vector $x(s)$ is a left zero direction of $T(s)$ when $x^*(s)$ is a right zero of $T^*(s)$. The order of a zero is defined as the maximum order of the zero directions at this point.

For MIMO systems, a zero can also be a pole. If λ is not a pole of $T(s)$, only the k first Taylor coefficients of $y(s)$ about λ are important. If λ is a pole of $T(s)$, the situation is more complicated. Indeed, assume that λ is a pole of order p of $T(s)$ and that $y(s)$ has an expansion about λ; then

$$T(s)y(s) = \left(\sum_{i=-p}^{+\infty} T_i(\lambda - s)^i \right) \left(\sum_{j=0}^{\infty} y_j(\lambda - s)^j \right). \tag{1.4}$$

We see that the first $k + p$ terms in the Taylor expansion of $y(s)$ are important to ensure that the product (1.4) has a zero of order k. This case will not be discussed in this paper, but a few remarks will be made to indicate how it complicates the problem.

We now present the concept of tangential interpolation that will be considered in this paper. Three concepts are defined, namely left, right, and two-sided tangential interpolation. Interpolation at the point at infinity is considered as a special case.

Let z be a finite point in the complex plane. Let $T(s)$ and $\hat{T}(s)$ be two $p \times m$ strictly proper transfer functions that do not have a pole at $s = z$.

Left tangential interpolation. Let $x(s)$ be a $1 \times p$ polynomial vector of degree $\beta - 1$ and not equal to zero at $s = z$. We say that $\hat{T}(s)$ interpolates $T(s)$ at $(z, x(s))$ if

$$x(s)(T(s) - \hat{T}(s)) = O(z - s)^\beta. \tag{1.5}$$

Let $x(s)$ be a $1 \times p$ polynomial vector in s^{-1}, of degree $\beta - 1$ in s^{-1} and not equal to zero at $s = \infty$. We say that $\hat{T}(s)$ interpolates $T(s)$ at $(\infty, x(s))$ if

$$x(s)(T(s) - \hat{T}(s)) = O(s^{-1})^{\beta+1}. \tag{1.6}$$

Right tangential interpolation. Let $y(s)$ be a $m \times 1$ polynomial vector of degree $\delta - 1$ and not equal to zero at $s = z$. We say that $\hat{T}(s)$ interpolates $T(s)$ at $(z, y(s))$ if

$$(T(s) - \hat{T}(s))y(s) = O(z - s)^{\delta}. \tag{1.7}$$

Let $y(s)$ be a $m \times 1$ polynomial vector in s^{-1}, of degree $\delta - 1$ in s^{-1} and not equal to zero at $s = \infty$. We say that $\hat{T}(s)$ interpolates $T(s)$ at $(\infty, y(s))$ if the following condition is satisfied:

$$(T(s) - \hat{T}(s))y(s) = O(s^{-1})^{\delta+1}. \tag{1.8}$$

Two-sided tangential interpolation. Let $x(s)$ be a $1 \times p$ polynomial vector of degree $\beta - 1$ and not equal to zero at $s = z$. Let $y(s)$ be a $m \times 1$ polynomial vector of degree $\delta - 1$ and not equal to zero at $s = z$. We say that $\hat{T}(s)$ interpolates $T(s)$ at $(z, x(s), y(s))$ if the following condition is satisfied:

$$x(s)(T(s) - \hat{T}(s))y(s) = O(z - s)^{\beta+\delta}. \tag{1.9}$$

Let $x(s)$ be a $1 \times p$ polynomial vector in s^{-1}, of degree $\beta - 1$ in s^{-1} and not equal to zero at $s = \infty$. Let $y(s)$ be a $m \times 1$ polynomial vector in s^{-1}, of degree $\delta - 1$ in s^{-1} and not equal to zero at $s^{-1} = 0$. We say that $\hat{T}(s)$ interpolates $T(s)$ at $(\infty, x(s), y(s))$ if the following condition is satisfied:

$$x(s)(T(s) - \hat{T}(s))y(s) = O(s^{-1})^{\beta+\delta+1}. \tag{1.10}$$

The objective of this paper is the following. We are given a transfer function $T(s)$ and a set of tangential interpolation conditions of the type (1.5) to (1.10) in a number of points of the complex plane, and we want to construct the transfer function of minimal McMillan degree that satisfies these interpolation conditions. In order to make the problem more precise, we need to introduce the following concepts.

DEFINITION 1.2. *Let $z_1, \ldots, z_{k_{left}}$ be points in the complex plane, not necessarily distinct or finite. For each finite z_α, a $1 \times p$ polynomial vector $x_\alpha(s)$ of degree $\beta_\alpha - 1$ and not equal to zero at $s = z_\alpha$ is given:*

$$x_\alpha(s) = \sum_{j=0}^{\beta_\alpha - 1} x_\alpha^{[j]} (z_\alpha - s)^j, \quad x_\alpha^{[0]} \neq 0. \tag{1.11}$$

If $z_\alpha = \infty$, then a $1 \times p$ polynomial vector in s^{-1}, $x_\alpha(s)$ of degree $\beta_\alpha - 1$ in s^{-1} and not equal to zero at $s = \infty$ is given:

$$x_\alpha(s) = \sum_{j=0}^{\beta_\alpha - 1} x_\alpha^{[j]} s^{-j}, \quad x_\alpha^{[0]} \neq 0. \tag{1.12}$$

The left interpolation set I_{left} is defined as follows:

$$I_{left} \doteq \left\{ (z_1, x_1(s)), \ldots, \left(z_{k_{left}}, x_{k_{left}}(s)\right) \right\}. \tag{1.13}$$

The size of I_{left}, written $s(I_{left})$, is defined as follows:

$$s(I_{left}) \doteq \sum_{i=1}^{k_{left}} \beta_i. \tag{1.14}$$

Finally, the set of interpolation points of I_{left}, written $p(I_{left})$ is defined as follows:

$$p(I_{left}) = \{z_1, \ldots, z_{k_{left}}\}. \tag{1.15}$$

Analogously, a right tangential interpolation set

$$I_{right} \doteq \left\{(w_1, y_1(s)), \ldots, \left(w_{k_{right}}, y_{k_{right}}(s)\right)\right\}, \tag{1.16}$$

with the points $w_1, \ldots, w_{k_{right}}$ arbitrarily chosen in $\mathbb{C} \cup \infty$ and each $m \times 1$ polynomial vector $y_\alpha(s)$, $1 \leq \alpha \leq k_{right}$ of degree $\delta_\alpha - 1$ in s if w_α is finite (of degree $\delta_\alpha - 1$ in s^{-1} otherwise) defined with the same conventions as above.

Let I_l be a left tangential interpolation set. Let I_r be a right tangential interpolation set. The set

$$I = \{I_l, I_r\} \tag{1.17}$$

is called a tangential interpolation set. The set of interpolation points of I, written $p(I)$, is defined by

$$p(I) \doteq p(I_l) \cup p(I_r). \tag{1.18}$$

Let $T(s)$ be a transfer function, then we say that the tangential interpolation set I is $T(s)$-admissible if $T(s)$ has m inputs and p outputs and no point belonging to $p(I)$ is a pole of $T(s)$, i.e., no interpolation point is a pole of $T(s)$.

Let the tangential interpolation set $I = \{I_l, I_r\}$ be defined as above. If some $z_\alpha \in I_l$ is equal to some $w_\gamma \in I_r$, say $\xi_{\alpha,\gamma} = z_\alpha = w_\gamma$, then define $x_\alpha^{(f)}(s)$ to be the polynomial vector of size $1 \times p$ of degree f obtained by keeping the first f terms in the Taylor expansion of $x_\alpha(s)$ about z_α, and analogously for $y_\gamma^{(g)}(s)$:

$$x_\alpha^{(f)}(s) \doteq \sum_{j=0}^{f-1} x_\alpha^{[j]}(z_\alpha - s)^j, \quad y_\gamma^{(g)}(s) \doteq \sum_{j=0}^{g-1} y_\gamma^{[j]}(w_\gamma - s)^j. \tag{1.19}$$

Use the same notation if z_α or w_γ is equal to ∞:

$$x_\alpha^{(f)}(s) \doteq \sum_{j=0}^{f-1} x_\alpha^{[j]} s^{-j}, \quad y_\gamma^{(g)}(s) \doteq \sum_{j=0}^{g-1} y_\gamma^{[j]} s^{-j}. \tag{1.20}$$

We are now able to define the tangential interpolation problem.

DEFINITION 1.3. *Let $T(s)$ and $\hat{T}(s)$ be two* strictly proper *$p \times m$ transfer functions. $\hat{T}(s)$ interpolates $T(s)$ at I if the three following conditions are satisfied:*

1. *$\hat{T}(s)$ interpolates $T(s)$ at any couple $(z_\alpha, x_\alpha(s))$ belonging to I_l,*
2. *$\hat{T}(s)$ interpolates $T(s)$ at any couple $(w_\gamma, y_\gamma(s))$ belonging to I_r,*
3. *Finally, for every $z_\alpha = w_\gamma \doteq \xi_{\alpha,\gamma}$, we impose in addition that for all $f = 1, \ldots, \beta_\alpha$; $g = 1, \ldots, \delta_\gamma$, $\hat{T}(s)$ interpolates $T(s)$* at *$(\xi_{\alpha,\gamma}, x_\alpha^{(f)}(s), y_\gamma^{(g)}(s))$.*

Two remarks are in order. In this paper, we consider only the simple case when the interpolation set I is $T(s)$-admissible and $\hat{T}(s)$-admissible. Second, the tangential interpolation problem has been studied in a slightly different form in the literature, e.g., in [4], and the reader is directed there for general results about the theory of interpolation of rational matrix functions. At first sight, one could think that our definition of the two-sided tangential interpolation problem is not the same as the one treated in [4]. A lemma showing the equivalence between the two formulations is proved in the appendix.

The problem solved in this paper can be stated as follows.

PROBLEM 1.1. *We are given a strictly proper $p \times m$ transfer function $T(s)$ of McMillan degree N, and a corresponding minimal state space realization (C, A, B), such that*

$$T(s) = C(sI_N - A)^{-1}B,$$

with $C \in \mathbb{C}^{p\times N}$, $A \in \mathbb{C}^{N\times N}$, and $B \in \mathbb{C}^{N\times m}$. We are also given a $T(s)$-admissible tangential interpolation set I. We want to construct a $p \times m$ reduced order transfer function $\hat{T}(s)$ of minimal McMillan degree n,

$$\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}, \tag{1.21}$$

with $\hat{C} \in \mathbb{C}^{p\times n}$, $\hat{A} \in \mathbb{C}^{n\times n}$, $\hat{B} \in \mathbb{C}^{n\times m}$ such that I is $\hat{T}(s)$-admissible and $\hat{T}(s)$ tangentially interpolates $T(s)$ at I.

The remainder of this paper is organized as follows. In section 2, the tangential interpolation problem is solved for two simple sets of interpolation conditions. In section 3, the background necessary to solve the general problem, Problem 1.1, is introduced. In section 4, the multipoint Padé approximation is constructed and its main properties are analyzed. Concluding remarks are given in section 5.

2. Preliminary results.

In this section, we present the solution of Problem 1.1 for two particular interpolation sets. The general results are given in sections 3 and 4.

2.1. One set of n distinct right interpolation conditions.

The first simpler problem solved in this section is the following.

PROBLEM 2.1. *Let $T(s)$ be a $p \times m$ transfer function of McMillan degree N. Let $\{\lambda_1, \ldots, \lambda_n\}$ be n (where $n < N$) distinct finite points in the complex plane that are not poles of $T(s)$. Let $\{y_1, \ldots, y_n\}$ be n $m \times 1$ nonzero vectors. We want to construct a $p \times m$ transfer function $\hat{T}(s)$ of McMillan degree n such that for all $1 \le i \le n$,*

$$T(\lambda)y_i = \hat{T}(\lambda_i)y_i. \tag{2.1}$$

Let C, A, B be a minimal state space realization of the $p \times m$ transfer function $T(s)$. In order to solve the problem, we construct the $N \times n$ matrix $V \doteq \begin{bmatrix} v_1 \ldots v_n \end{bmatrix}$ that satisfies the following Sylvester equation:

$$A\begin{bmatrix} v_1 \ldots v_n \end{bmatrix} - \begin{bmatrix} v_1 \ldots v_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} + B\begin{bmatrix} y_1 \ldots y_n \end{bmatrix} = 0. \tag{2.2}$$

Assume that V has full column rank n. Construct $Z \in \mathbb{C}^{N\times n}$ such that

$$Z^T V = I_n.$$

Construct $\hat{C} \in \mathbb{C}^{p\times n}$, $\hat{A} \in \mathbb{C}^{n\times n}$, and $\hat{B} \in \mathbb{C}^{n\times m}$ as follows:

$$\hat{C} \doteq CV, \quad \hat{A} \doteq Z^T AV, \quad \hat{B} \doteq Z^T B.$$

To verify that the transfer function

$$\hat{T}(s) \doteq \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$$

solves Problem 2.1, first note that for any $1 \leq i \leq k$ the columns of V can be computed as follows:

$$v_i = (\lambda_i I_N - A)^{-1} B y_i.$$

We will also use the following well-known result.

LEMMA 2.1. *Let $V \in \mathbb{C}^{N \times n}$. If the vector v belongs to the column span of the matrix V. Then, for any matrix $W \in \mathbb{C}^{N \times n}$ such that $W^T V = I_k$,*

$$v = VW^T v.$$

Proof. Because v belongs to the linear span of the columns of V, there exists a vector $\hat{v} \in \mathbb{C}^n$ such that $v = V\hat{v}$. For any W^T satisfying $W^T V = I_n$, we have $\hat{v} = W^T v$. This in turn implies that $v = VW^T v$. ∎

Defining W by

$$W^T \doteq \left(Z^T(\lambda_1 I_N - A)V\right)^{-1} Z^T(\lambda_1 I_N - A).$$

clearly yields $W^T V = I_n$ and applying the preceding lemma, we obtain the following equalities:

$$\begin{aligned}
T(\lambda_1)y_1 &= C(\lambda_1 I_N - A)^{-1} B y_1 && (2.3)\\
&= CVW^T(\lambda_1 I_N - A)^{-1} B y_1 && (2.4)\\
&= CV(\lambda_1 I_k - Z^T AV)^{-1} Z^T B y_1 && (2.5)\\
&= \hat{T}(\lambda_1) y_1. && (2.6)
\end{aligned}$$

This proves that $\hat{T}(s)$ solves Problem 2.1.

REMARK 2.1.

1. *This reasoning is very similar to the technique used in the SISO case in* [7] *and* [11] *. These papers develop techniques to construct a SISO transfer function of McMillan degree n that satisfies a set of (scalar) interpolation conditions with respect to an original transfer function.*
2. *It should be pointed out that the transfer function $\hat{T}(s)$ of McMillan degree n that solves Problem* 2.1 *is not unique. This is due to the fact that there exist infinitely many matrices $Z \in \mathbb{C}^{N \times n}$ such that $Z^T V = I_n$, where V satisfies* (2.2) *and is generically unique. We will see in what follows that, by imposing n additional left interpolation conditions, one generically determines a unique reduced order transfer function $\hat{T}(s)$ of McMillan degree n.*

2.2. One unique two-sided interpolation condition. We next consider the case where the interpolation set consists of only one finite interpolation point $\alpha \in \mathbb{C}$, i.e., in terms of the parameters of Problem 1.1,

$$k_{left} = k_{right} = 1, \quad \beta_1 = \delta_1 = n, \quad z_1 = w_1 = \alpha. \tag{2.7}$$

Moreover, we assume that α is not a pole of $T(s)$. Deleting the subscripts not required due to the simpler conditions to clarify the notation allows the problem to be stated as follows.

PROBLEM 2.2. *Given $T(s) = C(sI_N - A)^{-1}B$, $\alpha \in \mathbb{C}$, $x(s) \doteq \sum_{i=0}^{n-1} x^{[i]}(\alpha - s)^i$ and $y(s) \doteq \sum_{i=0}^{n-1} y^{[i]}(\alpha - s)^i$, construct a reduced order transfer function $\hat{T}(s)$ of McMillan degree n such that*

$$x(s)T(s) = x(s)\hat{T}(s) + O(\alpha - s)^n, \tag{2.8}$$

$$T(s)y(s) = \hat{T}(s)y(s) + O(\alpha - s)^n, \tag{2.9}$$

and for all $f = 1, \ldots, n$, $g = 1, \ldots, n$,

$$x^{(f)}(s)(T(s) - \hat{T}(s))y^{(g)}(s) = O(\alpha - s)^{f+g}. \tag{2.10}$$

In order to solve the problem, we first rewrite (2.8)–(2.10) as matrix equations. Note that for any $\alpha \in \mathbb{C}$ is not a pole of $T(s)$, we can write

$$T(s) = C(sI_N - A)^{-1}B = C\left((s-\alpha)I_N + \alpha I - A\right)^{-1}B \tag{2.11}$$
$$= C(\alpha I_N - A)^{-1}\left(I - (\alpha - s)(\alpha I - A)^{-1}\right)^{-1}B \tag{2.12}$$
$$= \sum_{k=0}^{\infty} C(\alpha I - A)^{-k-1}B(\alpha - s)^k. \tag{2.13}$$

Let us consider the left interpolation conditions corresponding to equation (2.8). By imposing the n first coefficients of the Taylor expansion of the product $x(s)(T(s) - \hat{T}(s))$ to be zero, we find the following system of equations:

$$\begin{aligned} &x^{[0]}C(\alpha I - A)^{-1}B \\ &= x^{[0]}\hat{C}(\alpha I - \hat{A})^{-1}\hat{B} \end{aligned} \tag{2.14}$$
$$\begin{aligned} &x^{[1]}C(\alpha I - A)^{-1}B + x^{[0]}C(\alpha I - A)^{-2}B \\ &= x^{[1]}\hat{C}(\alpha I - \hat{A})^{-1}\hat{B} + x^{[0]}\hat{C}(\alpha I - \hat{A})^{-2}\hat{B} \end{aligned} \tag{2.15}$$
$$\vdots$$
$$\begin{aligned} &x^{[n-1]}C(\alpha I - A)^{-1}B + \cdots + x^{[0]}C(\alpha I - A)^{-n}B \\ &= x^{[n-1]}\hat{C}(\alpha I - \hat{A})^{-1}\hat{B} + x^{[0]}\hat{C}(\alpha I - \hat{A})^{-n}\hat{B}. \end{aligned} \tag{2.16}$$

Defining the matrix $X \in \mathbb{C}^{n \times np}$ and the generalized observability matrix $\mathcal{O}_{C,A} \in \mathbb{C}^{np \times N}$ as follows:

$$X \doteq \begin{bmatrix} x^{[0]} & & \\ \vdots & \ddots & \\ x^{[n-1]} & \ldots & x^{[0]} \end{bmatrix}; \quad \mathcal{O}_{C,A} \doteq \begin{bmatrix} C(\alpha I - A)^{-1} \\ \vdots \\ C(\alpha I - A)^{-n} \end{bmatrix} \tag{2.17}$$

and defining matrix $\mathcal{O}_{\hat{C},\hat{A}} \in \mathbb{C}^{np \times n}$ analogously by replacing the matrices C and A by $\hat{C}$ and $\hat{A}$ in (2.17), we are able to state the following lemma.

LEMMA 2.2. *A* $p \times m$ *transfer function* $\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$ *satisfies the interpolation conditions* (2.8) *if and only if*

$$X\mathcal{O}_{\hat{C},\hat{A}}\hat{B} = X\mathcal{O}_{C,A}B. \tag{2.18}$$

Proof. Equation (2.18) is simply a matrix form of the system (2.14)–(2.16). □

We can transpose the preceding reasoning to the right interpolation condition (2.9). Defining

$$Y = \begin{bmatrix} y^{[0]} & \ldots & y^{[n-1]} \\ & \ddots & \vdots \\ & & y^{[0]} \end{bmatrix}; \quad \mathcal{C}_{A,B} = \left[(\alpha I - A)^{-1}B \ldots (\alpha I - A)^{-n}B\right] \tag{2.19}$$

and following the same reasoning as before, we obtain the following lemma.

LEMMA 2.3. *A $p \times m$ transfer function $\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$ verifies the interpolation conditions* (2.9) *if and only if*

$$\hat{C}\mathcal{C}_{\hat{A},\hat{B}}Y = C\mathcal{C}_{A,B}Y. \tag{2.20}$$

At this point, all that we have done is to rewrite the left and right interpolation conditions into matrix equations. Next, we define the generalized Loewner matrix as

$$\mathcal{L}_{T(s)} = X\mathcal{O}_{C,A}\mathcal{C}_{A,B}Y. \tag{2.21}$$

The matrix $\mathcal{L}_{\hat{T}(s)}$ is defined as $\mathcal{L}_{T(s)}$ by replacing the matrices C, A, and B by $\hat{C}, \hat{A}$, and $\hat{B}$. By rewriting the two-sided interpolation conditions corresponding to (2.10), we obtain the following lemma.

LEMMA 2.4. *A $p \times m$ transfer function $\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$ verifies the interpolation conditions* (2.10) *if and only if*

$$\mathcal{L}_{\hat{T}(s)} = \mathcal{L}_{T(s)}. \tag{2.22}$$

The following result can be proven using partial fraction expansion and Lemmas 2.2 to 2.3.

PROPOSITION 2.5. *Every transfer function $\hat{T}(s)$ that verifies* (2.8), (2.9) *and* (2.10) *is such that*

$$X\mathcal{O}_{C,A}A\mathcal{C}_{A,B}Y = X\mathcal{O}_{\hat{C},\hat{A}}\hat{A}\mathcal{C}_{\hat{A},\hat{B}}Y. \tag{2.23}$$

The main result of this section can now be stated as follows.

PROPOSITION 2.6. *If the matrix $\mathcal{L}_{T(s)}$ is invertible, then every transfer function that verifies the interpolation conditions* (2.8)–(2.10) *has a McMillan degree greater than or equal to n. Moreover, the transfer function of degree n that satisfies the equations* (2.8)–(2.10) *is unique if it exists and it can be constructed by the projection matrices V and Z that satisfy*

$$\mathrm{Im}(V) = \mathrm{Im}\left(\mathcal{C}_{C,A}Y\right), \tag{2.24}$$

$$\mathrm{Ker}(Z^T) = \mathrm{Ker}\left(X\mathcal{O}_{A,B}\right), \tag{2.25}$$

$$Z^TV = I_n, \tag{2.26}$$

if α is not a pole of $\hat{A}$.

Sketch of the proof. Suppose that there exists a transfer function of McMillan degree n such that (2.8)–(2.10) are satisfied. It follows that

$$X\mathcal{O}_{\hat{C},\hat{A}}\hat{B} = X\mathcal{O}_{C,A}B, \tag{2.27}$$

$$\hat{C}\mathcal{C}_{\hat{C},\hat{A}}U = C\mathcal{C}_{C,A}Y, \tag{2.28}$$

$$X\mathcal{O}_{\hat{C},\hat{A}}\hat{A}\mathcal{C}_{\hat{A},\hat{B}}Y = X\mathcal{O}_{C,A}A\mathcal{C}_{A,B}Y. \tag{2.29}$$

Because of the invertibility of $\mathcal{L}_{T(s)}$, the matrices $X\mathcal{O}_{\hat{C},\hat{A}} \in \mathbb{C}^{n\times n}$ and $\mathcal{C}_{\hat{A},\hat{B}}Y \in \mathbb{C}^{n\times n}$ are invertible. If we define

$$M = (X\mathcal{O}_{\hat{C},\hat{A}})^{-1}, \tag{2.30}$$

$$N = (\mathcal{C}_{\hat{A},\hat{B}}Y)^{-1}, \tag{2.31}$$

$$Z^T = MX\mathcal{O}_{C,A}, \tag{2.32}$$

$$V = \mathcal{C}_{A,B}YN, \tag{2.33}$$

it is straightforward to show that

$$\hat{A} = Z^TAV, \quad \hat{B} = Z^TB, \quad \hat{C} = CV, \quad Z^TV = I_n. \tag{2.34}$$

3. Auxiliary results. In this section, we define a generalized Loewner matrix that will allow us to construct explicitly the solution of the interpolation problem (1.1) under some mild conditions. This generalized Loewner matrix is inspired by the discussion in [2]. For the SISO case previous results based on [1], [8], and [10] may be found in [9].

In this section, we are given a strictly proper transfer function $T(s)$ and a $T(s)$-admissible interpolation set $I = \{I_l, I_r\}$ as defined in section 1. The objective of this section is to find a way to characterize the set of strictly proper transfer functions $\hat{T}(s)$ such that I is $\hat{T}(s)$-admissible (the interpolation points are not poles of $\hat{T}(s)$) and $\hat{T}(s)$ tangentially interpolates $T(s)$ at I.

We define first several matrices that will be used in the development. Consider the set I_l and associate with the pair $(z_\alpha, x_\alpha(s)) \in I_l$ defined in (1.11)–(1.12) the matrix $X_\alpha \in \mathbb{C}^{\beta_\alpha \times p\beta_\alpha}$

$$X_\alpha \doteq \begin{bmatrix} x_\alpha^{[0]} & & \\ \vdots & \ddots & \\ x_\alpha^{[\beta_\alpha - 1]} & \dots & x_\alpha^{[0]} \end{bmatrix}, \tag{3.1}$$

and define the matrix $X(I_l) \in \mathbb{C}^{s(I_l) \times ps(I_l)}$ by

$$X(I_l) \doteq \operatorname{diag}\{X_\alpha\}_{\alpha=1}^{k_{left}}. \tag{3.2}$$

Analogously, with the pair $(w_\alpha, y_\alpha(s)) \in I_r$, we associate the matrix

$$Y_\alpha \doteq \begin{bmatrix} y_\alpha^{[0]} & \dots & y_\alpha^{[\delta_\alpha - 1]} \\ & \ddots & \vdots \\ & & y_\alpha^{[0]} \end{bmatrix} \tag{3.3}$$

and define

$$Y(I_r) \doteq \operatorname{diag}\{Y_\alpha\}_{\alpha=1}^{k_{right}} \tag{3.4}$$

related to, respectively, the left and right interpolation sets I_l and I_r.

The Jordan matrices will play an important role in this paper, and we therefore introduce the following compact notation.

DEFINITION 3.1. *The matrix $J_{w,\delta,k} \in \mathbb{C}^{k\delta \times k\delta}$ is defined to be*

$$J_{w,\delta,k} \doteq \begin{bmatrix} wI_k & -I_k & & \\ & \ddots & \ddots & \\ & & \ddots & -I_k \\ & & & wI_k \end{bmatrix}. \tag{3.5}$$

When $k = 1$, $J_{w,\delta,1}$ is simply a Jordan matrix of size $\delta \times \delta$ at eigenvalue w and is written $J_{w,\delta}$.

With this definition, we easily obtain the following lemma.

LEMMA 3.2.

$$J_{w,\delta,m} Y_\alpha = Y_\alpha J_{w,\delta}, \quad J_{w,\beta}^T X_\alpha = X_\alpha J_{w,\beta,p}^T. \tag{3.6}$$

Proof. The case $w = 0$ is nothing but the shift invariance property of block Toeplitz matrices. It then also follows for $J_{w,\delta,m} = wI + J_{0,\delta,m}$ since we add the same term on both sides of (3.6). $\square$

Two matrices associated to the $p \times m$ transfer function $T(s) = C(sI_N - A)^{-1}B$ with $A \in \mathbb{C}^{N\times N}$ are the controllability matrix $Contr(A,B) \in \mathbb{C}^{pN\times N}$ and the observability matrix $Obs(C,A) \in \mathbb{C}^{N\times mN}$ defined by

$$Contr(A,B) \doteq \left[B \dots A^{N-1}B\right], \quad Obs(C,A) = \begin{bmatrix} C \\ \vdots \\ CA^{N-1} \end{bmatrix}. \tag{3.7}$$

The quantities occurring in $Contr(A,B)$ and $Obs(C,A)$,

$$\mu_{A,B}(\infty,k) \doteq A^{k-1}B \quad \nu_{C,A}(\infty,k) \doteq CA^{k-1}, \tag{3.8}$$

can be seen as "moments" of $(sI-A)^{-1}B$ and $C(sI-A)^{-1}$ about infinity. Similarly, from the dyadic expansion about a point $\lambda \notin \Lambda(A)$

$$(sI-A)^{-1} = \sum_{k=0}^{+\infty}(\lambda I - A)^{-k-1}(\lambda - s)^k, \tag{3.9}$$

we define the moments about a finite expansion point $\lambda \in \mathbb{C}$

$$\mu_{A,B}(\lambda,k) \doteq (\lambda I - A)^{-k}B, \quad \nu_{C,A}(\lambda,k) \doteq C(\lambda I - A)^{-k}. \tag{3.10}$$

DEFINITION 3.3. *Let I be a $T(s)$-admissible interpolation set. For any state-space realization (A,B,C) of $T(s)$, we associate with the right tangential interpolation set I_r the generalized controllability matrix $\mathcal{C}_{A,B}(I_r)$ by the following equations:*

$$\mathcal{C}_{A,B}(z_\alpha,\beta_\alpha) \doteq \left[\mu(z_\alpha,1)\dots\mu(z_\alpha,\beta_\alpha)\right], \tag{3.11}$$

$$\mathcal{C}_{A,B}(I_r) \doteq \left[\mathcal{C}_{A,B}(z_1,\beta_1)\dots\mathcal{C}_{A,B}(z_{k_{left}},\beta_{k_{left}})\right]. \tag{3.12}$$

Similarly, we define a generalized observability matrix $\mathcal{O}_{C,A}$ with the left tangential interpolation set I_l:

$$\mathcal{O}_{C,A}(w_\alpha,\delta_\alpha) \doteq \begin{bmatrix} \nu(w_\alpha,1) \\ \vdots \\ \nu(w_\alpha,\delta_\alpha) \end{bmatrix}, \mathcal{O}_{C,A}(I_l) \doteq \begin{bmatrix} \mathcal{O}_{C,A}(w_1,\delta_1) \\ \vdots \\ \mathcal{O}_{C,A}(w_{k_{right}},\delta_{k_{right}}) \end{bmatrix}. \tag{3.13}$$

We associate with the tangential interpolation set I the generalized Loewner matrix $\mathcal{L}_{T(s)}(I) \in \mathbb{C}^{s(I_l)\times s(I_r)}$ defined by

$$\mathcal{L}_{T(s)}(I) \doteq X(I_l)\mathcal{O}_{C,A}(I_l)\mathcal{C}_{A,B}(I_r)Y(I_r), \tag{3.14}$$

where (A,B,C) is a minimal realization of $T(s)$.

It is straightforward to verify then that $\mathcal{L}_{T(s)}(I)$ does not depend on the particular state space realization of $T(s)$. Next, we derive a series of lemmas that are needed for our main result in Theorem 3.10.

LEMMA 3.4. *If $z_\alpha \neq w_\gamma$ and both interpolation points are finite,*

$$\begin{aligned}
&\mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)\\
&\quad = \frac{1}{w_\gamma - z_\alpha}\mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\left(\begin{bmatrix} B & 0\ldots 0\end{bmatrix} - \mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)J_{0,\delta_\gamma,m}\right)\\
&\qquad + \frac{1}{z_\alpha - w_\gamma}\left(\begin{bmatrix} C\\ 0\\ \vdots\\ 0\end{bmatrix} - J^T_{0,\beta_\alpha,p}\mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\right)\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma).
\end{aligned} \tag{3.15}$$

If $z_\alpha \neq w_\gamma$ and z_α is infinite, then

$$\begin{aligned}
&\mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)\\
&\quad = \begin{bmatrix} C\\ 0\\ \vdots\\ 0\end{bmatrix}\mathcal{C}_{A,B}(z_\alpha,\delta_\alpha) - J^T_{0,\beta_\alpha}\mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)J_{0,\delta_\gamma,m}\\
&\qquad - w_\gamma J^T_{0,\beta}\mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma) + J^T_{0,\beta_\alpha}\mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\begin{bmatrix} B & 0\ldots 0\end{bmatrix}.
\end{aligned} \tag{3.16}$$

Proof. We first prove (3.15). Recall that if $\alpha \neq \beta \in \mathbb{C}$, then

$$(\alpha I - A)^{-1}(\beta I - A)^{-1} = \frac{1}{\beta - \alpha}(\alpha I - A)^{-1} + \frac{1}{\alpha - \beta}(\beta I - A)^{-1}. \tag{3.17}$$

This permits us to write that

$$\begin{aligned}
&\mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)\\
&= \begin{bmatrix} C(z_\alpha I - A)^{-1}\\ \vdots\\ C(z_\alpha I - A)^{-\beta_\alpha}\end{bmatrix}\begin{bmatrix}(w_\gamma I - A)^{-1}B\ldots(w_\gamma I - A)^{-\delta_\gamma}B\end{bmatrix} &(3.18)\\
&= \frac{1}{w_\gamma - z_\alpha}\begin{bmatrix} C(z_\alpha I - A)^{-1}\\ \vdots\\ C(z_\alpha I - A)^{-\beta_\alpha}\end{bmatrix}\begin{bmatrix}(B\ldots(w_\gamma I - A)^{-\delta_\gamma+1}B\end{bmatrix} &(3.19)\\
&\quad + \frac{1}{z_\alpha - w_\gamma}\begin{bmatrix} C\\ \vdots\\ C(z_\alpha I - A)^{-\beta_\alpha+1}\end{bmatrix}\begin{bmatrix}(w_\gamma I - A)^{-1}B\ldots(w_\gamma I - A)^{-\delta_\gamma}B\end{bmatrix}.
\end{aligned}$$

This last equation is equal to (3.15). This concludes the proof for the finite case.

Next, consider the case $z_\alpha = \infty$. The proof is similar but uses the following equality:

$$A(\lambda I - A)^{-1} = -I + \lambda(\lambda I - A)^{-1}. \tag{3.20}$$

This permits us to write that

$$\mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)$$

$$= \begin{bmatrix} C \\ \vdots \\ CA^{\beta_\alpha-1} \end{bmatrix} \begin{bmatrix}(w_\gamma I - A)^{-1}B \dots (w_\gamma I - A)^{-\delta_\gamma} B\end{bmatrix} \tag{3.21}$$

$$= \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mathcal{C}_{A,B}(w_\gamma,\delta_\gamma) - J^T_{0,\beta_\alpha}\mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)(A - w_\gamma I + w_\gamma I)\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma) \tag{3.22}$$

$$= \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mathcal{C}_{A,B}(w_\gamma,\delta_\gamma) - w_\gamma J^T_{0,\beta_\alpha}\mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma) \tag{3.23}$$
$$+ J^T_{0,\beta_\alpha}\mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\left(\begin{bmatrix} B & 0\dots 0\end{bmatrix} - \mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)J_{0,\delta}\right).$$

This last term is equal to the right-hand side of (3.16). □

To prove Theorem 3.10, we need the important result that the matrix $\mathcal{L}_{\hat{T}(s)}(I)$ is invariant for any matrix $\hat{T}(s)$ interpolating $T(s)$ at I (for which I is $\hat{T}(s)$-admissible). However, to show this result, we need the following lemmas.

LEMMA 3.5. *Let $T(s) = C(sI - A)^{-1}B$ and $\hat{T}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ be two $p \times m$ strictly proper transfer functions. Let I_l be a left interpolation set that is $T(s)$- and $\hat{T}(s)$-admissible. Then, $\hat{T}(s)$ interpolates $T(s)$ at I_l if and only if*

$$X(I_l)\mathcal{O}_{\hat{C},\hat{A}}(I_l)\hat{B} = X(I_l)\mathcal{O}_{C,A}(I_l)B. \tag{3.24}$$

Proof. Because of the diagonal structure of X, if we prove (3.24) for one diagonal block of X, say for instance X_α, we prove it for the entire equation (3.24). So we consider the block associated with X_α, and we drop I_l from $x_\alpha(s), X_\alpha, \mathcal{O}_{C,A}(I_l), \mathcal{O}_{\hat{C},\hat{A}}(I_l)$ to make the notation simpler. In other words, we consider the case where there is only one vector $x(s)$ of degree $\beta - 1$ associated with one interpolation point z in the left interpolation set I_l. We assume that z is finite (appropriate change must be made for the case $z = \infty$). We have to show that (1.5) is satisfied if and only if

$$X\mathcal{O}_{\hat{C},\hat{A}}\hat{B} = X\mathcal{O}_{C,A}B. \tag{3.25}$$

We can write that

$$T(s) = \sum_{i=0}^{+\infty} C(zI - A)^{-i-1}B(z-s)^i, \quad \hat{T}(s) = \sum_{i=0}^{+\infty} \hat{C}(zI - \hat{A})^{-i-1}\hat{B}(z-s)^i. \tag{3.26}$$

Equation (1.5) says that $x(s)$ is a left zero of $T(s) - \hat{T}(s)$. This means that the first β Taylor coefficients of $x(s)(T(s) - \hat{T}(s))$ at $s = z$ are zero. In other words, for all $1 \le i \le \beta$, the following equation must be satisfied:

$$\sum_{k=0}^{i-1} x^{[k]}\hat{C}(zI - \hat{A})^{i-k}\hat{B} = \sum_{k=0}^{i-1} x^{[k]}C(zI - A)^{i-k}B, \tag{3.27}$$

and this equation turns out to be exactly the ith row of (3.25). □

Analogously, for the right interpolation conditions, we have the following lemma.

LEMMA 3.6. *Let $T(s) = C(sI - A)^{-1}B$ and $\hat{T}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ be two $p \times m$ strictly proper transfer functions. Let I_r be a right interpolation set that is $T(s)$- and $\hat{T}(s)$-admissible. Then, $\hat{T}(s)$ interpolates $T(s)$ at I_r if and only if*

$$\hat{C}\mathcal{C}_{\hat{A},\hat{B}}Y = C\mathcal{C}_{A,B}Y. \tag{3.28}$$

The proof is similar to the proof of Lemma 3.5.

LEMMA 3.7. *Let $T(s) = C(sI - A)^{-1}B$ and $\hat{T}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ be two $p \times m$ strictly proper transfer functions. Let $I = \{I_l, I_r\}$ be an interpolation set that is $T(s)$- and $\hat{T}(s)$-admissible. If $\hat{T}(s)$ interpolates $T(s)$ at I and then, for every pair of indices α, γ such that $z_\alpha = w_\gamma = \xi$, (where ξ is finite),*

$$X_\alpha \mathcal{O}_{\hat{C},\hat{A}}(z_\alpha, \beta_\alpha)\mathcal{C}_{\hat{A},\hat{B}}(w_\gamma, \delta_\gamma)Y_\gamma = X_\alpha \mathcal{O}_{C,A}(z_\alpha, \beta_\alpha)\mathcal{C}_{A,\hat{B}}(w_\gamma, \delta_\gamma)Y_\gamma; \tag{3.29}$$

and for every pair of indices α, γ such that $z_\alpha = w_\gamma = \xi$, (where $\xi = \infty$),

$$X_\alpha \mathcal{O}_{\hat{C},\hat{A}}(z_\alpha, \beta_\alpha)\hat{A}\mathcal{C}_{\hat{A},\hat{B}}(w_\gamma, \delta_\gamma)Y_\gamma = X_\alpha \mathcal{O}_{C,A}(z_\alpha, \beta_\alpha)A\mathcal{C}_{A,B}(w_\gamma, \delta_\gamma)Y_\gamma. \tag{3.30}$$

Proof. We consider the finite case. To simplify the notation, we drop the subscripts α, γ. Let us choose two integers f, g such that $1 \le f \le \beta$ and $1 \le g \le \delta$. Condition 3 of Definition 1.3 applied to $x(s) = x_\alpha^{(f)}(s)$ and $y(s) = y_\gamma^{(g)}(s)$ says that the $f + g$ first derivatives of $x^{(f)}(s)(T(s) - \hat{T}(s))y^{(g)}(s)$ at $s = \xi$ are zero. The condition corresponding to the derivative of highest order is

$$\frac{1}{(f+g-1)!}\frac{d^{f+g-1}}{ds^{f+g-1}}\left\{x^{(f)}(s)\hat{T}(s)y^{(g)}(s)\right\}\Big|_{s=\xi}$$

$$= \sum_{k=0}^{f-1}\sum_{l=0}^{g-1} x^{[k]}C(\xi I - A)^{k+l-f-g}By^{[l]} \tag{3.31}$$

$$= \sum_{k=0}^{f-1}\sum_{l=0}^{g-1}(x^{[k]}C(\xi I - A)^{k-f})((\xi I - A)^{l-g}Bu^{[l]}) \tag{3.32}$$

$$= \left(X\mathcal{O}_{C,A}\mathcal{C}_{A,B}Y\right)_{f,g}. \tag{3.33}$$

Thus, (3.29) is a consequence of the interpolation conditions. The proof is similar for the infinite interpolation point. ☐

Equations (3.25), (2.20), (3.29), and (3.30) are just a matrix version of the interpolation conditions of Definition 1.3. We now proceed to prove that (3.25) and (2.20) imply as well that $X\mathcal{O}_{\hat{C},\hat{A}}\mathcal{C}_{\hat{A},\hat{B}}Y = X\mathcal{O}_{C,A}\mathcal{C}_{A,B}Y$ and $X\mathcal{O}_{\hat{C},\hat{A}}\hat{A}\mathcal{C}_{\hat{A},\hat{B}}Y = X\mathcal{O}_{C,A}A\mathcal{C}_{A,B}Y$, provided the two-sided interpolation condition 3 of Definition 1.3 is added for every pair $z_\alpha = w_\gamma$. This may seem surprising but it is a simple consequence of Lemma 3.7 when $z_{\alpha \ne w_\gamma}$ and follows from the two-sided condition when $z_\alpha = w_\alpha$.

LEMMA 3.8. *If the strictly proper transfer function $\hat{T}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ interpolates $T(s)$ at $I = \{I_l, I_r\}$ (where the interpolation set I is $T(s)$- and $\hat{T}(s)$-admissible), then*

$$X\mathcal{O}_{\hat{C},\hat{A}}\mathcal{C}_{\hat{A},\hat{B}}Y = X\mathcal{O}_{C,A}\mathcal{C}_{A,B}Y. \tag{3.34}$$

Proof. The proof will be done block by block. If $z_\alpha = w_\gamma = \xi_{\alpha,\gamma}$ and $\xi_{\alpha,\gamma}$ is finite, the proof follows from Lemma 3.7. Let us consider the case $\xi_{\alpha,\gamma}$ infinite.

$$\begin{aligned} & X_\alpha \mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)Y_\gamma \\ &= X_\alpha \mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\left[B \dots A^{\delta_\gamma-1}\right]\begin{bmatrix} y_\gamma^{[0]} & \dots & y_\gamma^{\delta_\gamma-1} \\ & \ddots & \vdots \\ & & y_\gamma^{[0]} \end{bmatrix} \qquad (3.35)\\ &= X_\alpha \mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)B\left[y_\gamma^{[0]} \dots y_\gamma^{\delta_\gamma-1}\right] \qquad (3.36)\\ &\quad - X_\alpha \mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)A\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)Y_\gamma J_{0,\delta} \qquad (3.37)\\ &= X_\alpha \mathcal{O}_{\hat{C},\hat{A}}(z_\alpha,\beta_\alpha)\mathcal{C}_{\hat{A},\hat{B}}(w_\gamma,\delta_\gamma)Y_\gamma. \qquad (3.38)\end{aligned}$$

Second, we suppose that

$$z_\alpha \neq w_\gamma. \tag{3.39}$$

We assume that z_α and w_γ are finite. The idea is to recursively use (3.15). We want to show that

$$X_\alpha \mathcal{O}_{\hat{C},\hat{A}}(z_\alpha,\beta_\alpha)\hat{B} = X_\alpha \mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)B \tag{3.40}$$

and

$$\hat{C}\mathcal{C}_{\hat{A},\hat{B}}(w_\gamma,\delta_\gamma)Y_\gamma = C\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)Y_\gamma \tag{3.41}$$

imply

$$X_\alpha \mathcal{O}_{\hat{C},\hat{A}}(z_\alpha,\beta_\alpha)\mathcal{C}_{\hat{A},\hat{B}}(w_\gamma,\delta_\gamma)Y_\gamma = X_\alpha \mathcal{O}_{C,A}(z_\alpha,\beta_\alpha)\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)Y_\gamma. \tag{3.42}$$

We drop again $\alpha, \gamma, (z_\alpha,\beta_\alpha), (w_\gamma,\delta_\gamma)$ to simplify the notation.

$$\begin{aligned} X\mathcal{O}_{C,A}\mathcal{C}_{A,B}Y &= \frac{1}{w-z}X\mathcal{O}_{C,A}\left(\begin{bmatrix}B & 0\dots 0\end{bmatrix} - \mathcal{C}_{A,B}J_{0,\delta,m}\right)Y \\ &\quad + \frac{1}{z-w}X\left(\begin{bmatrix} C \\ 0 \\ \vdots \\ 0\end{bmatrix} - J_{0,\beta,p}^T\mathcal{O}_{C,A}\right)\mathcal{C}_{A,B}Y \qquad (3.43)\\ &= \frac{1}{w-z}\begin{bmatrix}X\mathcal{O}_{C,A}B & 0\dots 0\end{bmatrix}Y + \frac{1}{z-w}X\begin{bmatrix} C\mathcal{C}_{A,B}Y \\ 0 \\ \vdots \\ 0\end{bmatrix} \\ &\quad - \frac{1}{w-z}X\mathcal{O}_{C,A}\mathcal{C}_{A,B}YJ_{0,\delta} - \frac{1}{z-w}J_{0,\beta}X\mathcal{O}_{C,A}\mathcal{C}_{A,B}Y. \qquad (3.44)\end{aligned}$$

From Lemmas 3.5 and 3.6 we deduce

$$\frac{1}{w-z}\left[X\mathcal{O}_{\hat{C},\hat{A}}\hat{B}\ 0\dots 0\right]Y = \frac{1}{w-z}\left[X\mathcal{O}_{\hat{C},\hat{A}}\hat{B}\ 0\dots 0\right]Y, \tag{3.45}$$

$$\frac{1}{z-w}X\begin{bmatrix} C\mathcal{C}_{A,B}Y \\ 0 \\ \vdots \\ 0\end{bmatrix} = \frac{1}{z-w}X\begin{bmatrix} \hat{C}\mathcal{C}_{\hat{A},\hat{B}}Y \\ 0 \\ \vdots \\ 0\end{bmatrix}. \tag{3.46}$$

By using a recursive argument, it can be shown that

$$X\mathcal{O}_{C,A}\mathcal{C}_{A,B}YJ_{0,\delta} = X\mathcal{O}_{\hat{C},\hat{A}}\mathcal{C}_{\hat{A},\hat{B}}YJ_{0,\delta}, \tag{3.47}$$

$$J_{0,\beta}X\mathcal{O}_{C,A}\mathcal{C}_{A,B}Y = J_{0,\beta}X\mathcal{O}_{\hat{C},\hat{A}}\mathcal{C}_{\hat{A},\hat{B}}Y. \tag{3.48}$$

Finally, we have to consider the case with one infinite interpolation point, say for instance $z_\alpha = \infty$ and the other point w_γ finite. This can be treated similarly by recursively using (3.16). ∎

LEMMA 3.9. *If the strictly proper transfer function $\hat{T}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ interpolates $T(s)$ at $I = \{I_l, I_r\}$ and I is $T(s)$- and $\hat{T}(s)$-admissible, then*

$$X\mathcal{O}_{\hat{C},\hat{A}}\hat{A}\mathcal{C}_{\hat{A},\hat{B}}Y = X\mathcal{O}_{C,A}A\mathcal{C}_{A,B}Y. \tag{3.49}$$

Proof. We recall that

$$A\mathcal{C}_{A,B}Y = \left[A\mathcal{C}_{A,B}(w_1,\delta_1)Y_1 \ldots A\mathcal{C}_{A,B}(w_s,\delta_s)Y_s\right]. \tag{3.50}$$

The proof will again be done block by block. Let us prove it for the block of $\mathcal{C}_{\hat{A},\hat{B}}(I_r)Y$ corresponding to w_γ. Two cases must be considered.

Assuming that w_γ is finite yields

$$\begin{aligned} & A\mathcal{C}_{C,A}(w_\gamma,\delta_\gamma)Y_\gamma \\ (3.51)\qquad & = (A - w_\gamma I + w_\gamma I)\mathcal{C}_{C,A}(w_\gamma,\delta_\gamma)Y_\gamma \\ (3.52)\qquad & = -\left[B \ldots (w_\gamma I_N - A)^{-\delta_\gamma+1}B\right]Y_\gamma + w_\gamma\mathcal{C}_{C,A}(w_\gamma,\delta_\gamma)Y_\gamma \\ (3.53)\qquad & = -B\left[y^{[0]} \ldots y^{[\delta_\gamma-1]}\right] + \mathcal{C}_{C,A}(w_\gamma,\delta_\gamma)Y_\gamma J_{w_\gamma,\delta_\gamma}. \end{aligned}$$

This allows us to write that

$$\begin{aligned} & X\mathcal{O}_{\hat{C},\hat{A}}\hat{A}\mathcal{C}_{\hat{A},\hat{B}}(w_\gamma,\delta_\gamma)Y_\gamma \\ (3.54)\qquad & = X\mathcal{O}_{\hat{C},\hat{A}}(-\hat{B}[y^{[0]} \ \ldots \ y^{[\delta_\gamma-1]}] + \mathcal{C}_{\hat{A},\hat{B}}(w_\gamma,\delta_\gamma)Y_\gamma J_{w_\gamma,\delta_\gamma}) \\ (3.55)\qquad & = X\mathcal{O}_{C,A}(-B[y^{[\delta_\gamma-1]} \ \ldots \ y^{[0]}] + \mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)Y_\gamma J_{w_\gamma,\delta_\gamma}) \\ (3.56)\qquad & = X\mathcal{O}_{C,A}A\mathcal{C}_{A,B}(w_\gamma,\delta_\gamma)Y_\gamma, \end{aligned}$$

where the first part of (3.55) is a consequence of Lemma 3.5 and the second part of (3.55) is a consequence of Lemma 3.7.

Second, assume that $w_\gamma = \infty$. Two cases must be considered. If z_α is finite, then the proof is done by transposing the preceding results. If $\xi_{\alpha,\gamma} = \infty$, then this follows from Lemma 3.7. ∎

Putting together the preceding results, we obtain the following theorem that gives the main result of the section.

THEOREM 3.10. *Let (C_1, A_1, B_1) be a minimal state space realization of the strictly proper transfer function $T_1(s)$ and (C_2, A_2, B_2) be a minimal state space realization of the strictly proper transfer function $T_2(s)$. Let the interpolation set $I = \{I_l, I_r\}$ be $T_1(s)$- and $T_2(s)$-admissible (i.e., the interpolation points are neither poles of $T_1(s)$ nor $T_2(s)$). Then, $T_1(s)$ interpolates $T_2(s)$ at I if and only if the following equations are satisfied:*

$$C_1\mathcal{C}_{A_1,B_1}(I_r)Y(I_r) = C_2\mathcal{C}_{A_2,B_2}(I_r)Y(I_r), \tag{3.57}$$

$$X(I_l)\mathcal{O}_{C_1,A_1}(I_l)B_1 = X(I_l)\mathcal{O}_{C_2,A_2}(I_l)B_2, \tag{3.58}$$

$$X(I_l)\mathcal{O}_{C_1,A_1}(I_l)\mathcal{C}_{A_1,B_1}(I_r)Y(I_r) = X(I_l)\mathcal{O}_{C_2,A_2}(I_l)\mathcal{C}_{A_2,B_2}(I_r)Y(I_r), \tag{3.59}$$

$$X(I_l)\mathcal{O}_{C_1,A_1}(I_l)A_1\mathcal{C}_{A_1,B_1}(I_r)Y(I_r) = X(I_l)\mathcal{O}_{C_2,A_2}(I_l)A_2\mathcal{C}_{A_2,B_2}(I_r)Y(I_r). \tag{3.60}$$

Proof. The proof follows from the preceding results. ∎

4. The multipoint Padé reduced order transfer function. In this section, we give a practical way of constructing a minimal state space realization of the transfer function of minimal McMillan degree that interpolates $T(s)$ at the interpolation set I when the corresponding Loewner matrix $\mathcal{L}_{T(s)}(I)$ is invertible. The interpolating transfer function of minimal McMillan degree will be called the multipoint Padé reduced order transfer function $\hat{T}_{MP}(s)$. A minimal state space realization $(\hat{C}_{MP}, \hat{A}_{MP}, \hat{B}_{MP})$ of $\hat{T}_{MP}(s)$ will be obtained by a projection technique. More precisely, the state space realization $(\hat{C}_{MP}, \hat{A}_{MP}, \hat{B}_{MP})$ will be constructed by projecting a minimal state space realization (C, A, B) of $T(s)$ with two projecting matrices $Z, V \in \mathbb{C}^{N\times n}$ as follows:

$$\hat{C}_{MP} = CV, \quad \hat{A}_{MP} = Z^T AV, \quad \hat{B}_{MP} = Z^T B, \quad Z^T V = I_n.$$

It will be shown that the projecting matrices Z, V can be obtained by solving Sylvester equations.

In order to prove these facts, we first introduce two new pairs of matrices. Let us consider the left tangential interpolation set I_l defined in (1.13). For any integer α such that $1 \le \alpha \le k_{left}$, define the matrices $(L_\alpha^{(l)}, L_\alpha^{(r)})$ as follows:

1. If the interpolation point z_α is finite, then take

$$L_\alpha^{(l)} \doteq I_{\beta_\alpha}, \quad L_\alpha^{(r)} \doteq J^T_{z_\alpha,\beta_\alpha}. \tag{4.1}$$

2. If the interpolation point z_α is infinite, then define

$$L_\alpha^{(l)} \doteq -J^T_{0,\beta_\alpha}, \quad L_\alpha^{(r)} \doteq I_{\beta_\alpha}. \tag{4.2}$$

Moreover, define the matrix $\mathcal{X}_\alpha$ as follows:

$$\mathcal{X}_\alpha = \begin{bmatrix} x_\alpha^{[0]} \\ \vdots \\ x_\alpha^{[\beta_\alpha - 1]} \end{bmatrix}. \tag{4.3}$$

Finally, define the matrices $L^{(l)}(I_l)$, $L^{(r)}(I_l)$, and $\mathcal{X}(I_l)$ as follows:

$$L^{(l)}(I_l) \doteq \operatorname{diag}\{L_\alpha^{(l)}\}_{\alpha=1}^{k_{left}}, \quad L^{(r)}(I_l) \doteq \operatorname{diag}\{L_\alpha^{(r)}\}_{\alpha=1}^{k_{left}}, \tag{4.4}$$

$$\mathcal{X}(I_l) \doteq \begin{bmatrix} \mathcal{X}_1 \\ \vdots \\ \mathcal{X}_{k_{left}} \end{bmatrix}. \tag{4.5}$$

Let us consider the right tangential interpolation set I_r defined in (1.16). For any integer α such that $1 \le \alpha \le k_{right}$, define the matrices $(R_\alpha^{(l)}, R_\alpha^{(r)})$ as follows:

1. If the interpolation point w_α is finite, then take

$$R_\alpha^{(l)} \doteq I_{\delta_\alpha}, \quad R_\alpha^{(r)} \doteq J_{w_\alpha,\delta_\alpha}. \tag{4.6}$$

2. If the interpolation point w_α is infinite, then define

$$R_\alpha^{(l)} \doteq -J_{0,\delta_\alpha}, \quad R_\alpha^{(r)} \doteq I_{\delta_\alpha}. \tag{4.7}$$

Moreover, define

$$\mathcal{Y}_\alpha \doteq \left[y_\alpha^{[0]} \dots y_\alpha^{[\delta_\alpha - 1]}\right]. \tag{4.8}$$

Finally, define the matrices $R^{(l)}(I_r)$, $R^{(r)}(I_r)$, and $\mathcal{Y}(I_r)$ as follows:

$$R^{(l)}(I_r) \doteq \mathrm{diag}\{R_\alpha^{(l)}\}_{\alpha=1}^{k_{right}} \;\;,\;\; R^{(r)}(I_r) \doteq \mathrm{diag}\{R_\alpha^{(r)}\}_{\alpha=1}^{k_{right}} \tag{4.9}$$

$$\mathcal{Y}(I_r) \doteq \left[\mathcal{Y}_0 \dots \mathcal{Y}_{k_{right}}\right]. \tag{4.10}$$

As a consequence of these definitions we have

$$L^{(l)}L^{(r)} = L^{(r)}L^{(l)}, \quad R^{(l)}R^{(r)} = R^{(r)}R^{(l)} \tag{4.11}$$

and we can now derive the following lemma that introduces the related Sylvester equations.

LEMMA 4.1. *Let (A, B, C) be a state-space realization of the transfer function $T(s)$. Let us consider a $T(s)$-admissible interpolation set $I = \{I_l, I_r\}$. Then,*

$$N = \mathcal{C}_{A,B}(I_r)Y(I_r) \Longleftrightarrow ANR^{(l)}(I_r) - NR^{(r)}(I_r) + B\mathcal{Y}(I_r) = 0, \tag{4.12}$$

$$M = X(I_l)\mathcal{O}_{C,A}(I_l) \Longleftrightarrow L^{(l)}(I_l)MA - L^{(r)}M + \mathcal{X}C = 0. \tag{4.13}$$

Proof. Let us prove (4.12) for only one interpolation condition $I_r = \{(w, y(s))\}$ at a finite point w.

$$\begin{aligned} &ANR^{(l)}(I_r) - NR^{(r)}(I_r) + B\mathcal{Y}(I_r) = 0 \\ &\Longleftrightarrow A\left[n_1 \dots n_k\right] - \left[n_1 \dots n_k\right] J_{w,k} \\ &\qquad +B\left[y^{[0]} \dots y^{[k-1]}\right] = 0. \end{aligned} \tag{4.14}$$

Let us solve this linear equation for N column by column from n_1 up to n_k. We find recursively that

$$(wI - A)n_1 = By^{[0]} \tag{4.15}$$

$$(wI - A)n_{i+1} = By^{[i]} + n_i. \tag{4.16}$$

Moreover, the matrix $wI - A$ is invertible because we always assume here that the interpolation set I is $T(s)$-admissible. This proves that $N = \mathcal{C}_{A,B}(I_r)Y(I_r)$ for one finite interpolation condition $I_r = \{(w, y(s))\}$.

Let us prove (4.12) for only one interpolation condition $I_r = \{(w, y(s))\}$ at an infinite point $w = \infty$.

$$\begin{aligned} &ANR^{(l)}(I_r) - NR^{(r)}(I_r) + B\mathcal{Y}(I_r) = 0 \\ &\Longleftrightarrow A\left[n_1 \dots n_k\right] \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} \\ &\qquad - \left[n_1 \dots n_k\right] + B\left[y^{[0]} \dots y^{[k-1]}\right] = 0. \end{aligned} \tag{4.17}$$

Again, by solving this equation column by column we find that $N = \mathcal{C}_{A,B}(I_r)Y(I_r)$ for one interpolation condition $I_r = \{(\infty, y(s))\}$. If the interpolation set I_r contains more than one pair, say k_r pairs, because of the block diagonal structure of $R^{(l)}, R^{(r)}$ and $Y(I_r)$, and the block structure of $\mathcal{Y}(I_r)$, we can split the columns of N into k_r blocks and prove the result for each pair $(w_\gamma, y_\gamma(s)) \in I_r$ in order to prove that

$$\begin{aligned} N &= \left[N_1 \dots N_{k_r}\right] \\ &= \left[\mathcal{C}_{A,B}\left(w_1, y_1(s)\right) Y\left(w_1, y_1(s)\right) \dots \mathcal{C}_{A,B}\left(w_{k_r}, y_{k_r}(s)\right) Y\left(w_{k_r}, y_{k_r}(s)\right)\right] \\ &= \mathcal{C}_{A,B}(I_r)Y(I_r). \qquad \square \end{aligned} \tag{4.18}$$

The main result of this paper can now be formalized.

THEOREM 4.2. *Consider a transfer function $T(s)$ and a $T(s)$-admissible tangential interpolation set I and assume that the corresponding Loewner matrix $\mathcal{L}_{T(s)}(I) \in \mathbb{C}^{n\times n}$ is invertible. Define then two invertible matrices $M, N \in \mathbb{C}^{n\times n}$ such that*

$$\mathcal{L}_{T(s)} \doteq X\mathcal{O}_{C,A}\mathcal{C}_{A,B}Y = MN, \tag{4.19}$$

and define the "multipoint Padé" reduced order transfer function $\hat{T}_{MP}(s)$ via its state space realization $\{\hat{A}_{MP}, \hat{B}_{MP}, \hat{C}_{MP}\}$ given by the equations

$$\hat{C}_{MP}N = C\mathcal{C}_{A,B}Y, \tag{4.20}$$

$$M\hat{B}_{MP} = X\mathcal{O}_{C,A}B, \tag{4.21}$$

$$M\hat{A}_{MP}N = X\mathcal{O}_{C,A}A\mathcal{C}_{A,B}Y. \tag{4.22}$$

If the interpolation points are not poles of $\hat{T}_{MP}(s)$, i.e., if the interpolation set I is $\hat{T}_{MP}(s)$-admissible, then $\hat{T}_{MP}(s)$ interpolates $T(s)$ at I. Moreover, $\hat{T}_{MP}(s)$ is the unique transfer function of McMillan degree $s(I_l) = s(I_r)$ that interpolates $T(s)$ at I and there exists no such transfer function of lower McMillan degree.

Proof. First, note that it is always possible to find a couple of invertible matrices M, N that satisfy (4.19) because of the invertibility of $\mathcal{L}_{T(s)}(I)$. Second, it can be verified that $\hat{T}_{MP}(s)$ is uniquely defined and does not depend on the particular choice of matrices M, N satisfying (4.19).

The proof consists of showing that $M = X(I_l)\mathcal{O}_{\hat{C}_{MP},\hat{A}_{MP}}(I_l)$ and that $N = \mathcal{C}_{\hat{A}_{MP},\hat{B}_{MP}}(I_r)Y(I_r)$. From the preceding results, it is equivalent to show that M and N are solutions of the Sylvester equations of Lemma 4.1. First, from (4.19)–(4.22) and Lemma 4.1, we have

$$\begin{aligned} &\hat{A}_{MP}NR^{(l)} - NR^{(r)} + \hat{B}_{MP}\mathcal{Y} \\ &= M^{-1}X\mathcal{O}_{C,A}(A\mathcal{C}_{A,B}YR^{(l)} - \mathcal{C}_{A,B}YR^{(r)} + B\mathcal{Y}) = 0. \end{aligned} \tag{4.23}$$

This implies also from Lemma 4.1 that $N = \mathcal{C}_{\hat{A}_{MP},\hat{B}_{MP}}(I_r)Y(I_r)$. Analogously, $M = X(I_l)\mathcal{O}_{\hat{C}_{MP},\hat{A}_{MP}}(I_l)$. The proof follows now from Proposition 3.10. Indeed, (4.20) is equivalent to saying that the right tangential interpolation conditions are satisfied, (4.21) corresponds to the left tangential equations and (4.19) and (4.22) are equivalent to the two-sided interpolation conditions. Hence, $\hat{T}_{MP}(s)$ interpolates $T(s)$ at I.

We have still to prove that $\hat{T}_{MP}(s)$ is the unique transfer function of McMillan degree n that satisfies the interpolation conditions with respect to $T(s)$, and that there exist no transfer function of McMillan degree smaller than n that satisfies the interpolation conditions. To do this, first assume that there exists $\hat{T}(s)$ of McMillan degree $k < n$ that satisfies the interpolation conditions. Let $(\hat{C}, \hat{A}, \hat{B})$ be a minimal state space realization of $\hat{T}(s)$. Clearly,

$$\operatorname{rank} \mathcal{C}_{\hat{A},\hat{B}}(I_r)Y(I_r) \leq \operatorname{rank} \mathcal{C}_{\hat{A},\hat{B}}(I_r) = \operatorname{rank} Contr(\hat{A}, \hat{B}) = k < n. \tag{4.24}$$

From the interpolation conditions, we must have that $\mathcal{L}_{T(s)}(I) = \mathcal{L}_{\hat{T}(s)}(I)$. This implies that

$$n = \operatorname{rank} \mathcal{L}_{T(s)}(I) = \operatorname{rank} \mathcal{L}_{\hat{T}(s)}(I) \leq k. \tag{4.25}$$

This proves that it is not possible to find an interpolating transfer function of McMillan degree smaller than n.

If we assume that there exists another interpolating transfer function $\hat{T}(s)$ of McMillan degree n, it is not difficult to verify that the procedure given for constructing a minimal state space realization $(\hat{C}, \hat{A}, \hat{B})$ of $\hat{T}(s)$ will produce a state space realization that is similar to $(\hat{C}_{MP}, \hat{A}_{MP}, \hat{B}_{MP})$. This implies that $\hat{T}(s) = \hat{T}_{MP}(s)$ and concludes the proof. $\square$

By inverting the matrices M and N into (4.19)–(4.22), if we define

$$Z^T = M^{-1} X \mathcal{O}_{C,A}, \quad V = \mathcal{C}_{A,B} Y N^{-1}, \tag{4.26}$$

we see that

$$Z^T V = I_n, \quad CV = \hat{C}_{MP}, \quad Z^T B = \hat{B}_{MP}, \quad Z^T A V = \hat{A}_{MP}. \tag{4.27}$$

5. Concluding remarks. An important result that has not been considered in this paper is the following. Assume that a reduced order transfer function $\hat{T}_1(s)$ has been constructed that interpolates the original transfer function $T(s)$ at the interpolation set I_1 with the projecting matrices Z_1 and V_1. If one wants to add new interpolation conditions, say I_2, all that we have to do is to compute the generalized Krylov subspaces corresponding to the new interpolation set I_2 and to construct new projecting matrices Z_2, V_2 that contain, respectively, the column span of Z_1 and V_1 and the new, respectively, left and right generalized Krylov subspaces.

Another important result that can easily be derived is that we only need the projecting matrices Z, V to contain some subspaces, but they can contain other subspaces as well! For instance, Theorem 4.2 can be generalized as follows.

THEOREM 5.1. *Consider a transfer function $T(s) \doteq C(sI - A)^{-1}B$ and a $T(s)$-admissible tangential interpolation set $I \doteq \{I_l, I_r\}$. Let us assume that the projecting matrices Z, V (such that $Z^T V = I_n$) are such that*

$$Colsp(V) \supseteq Colsp\left(\mathcal{C}_{A,B}(I_r)Y(I_r)\right),$$
$$Colsp(Z^T) \supseteq Colsp\left(\mathcal{O}^T_{C,A}(I_l)X^T(I_l)\right).$$

Then, if the interpolation point of I are not poles of $\hat{T}(s) \doteq CV(sI_n - Z^T AV)^{-1} Z^T B$, the transfer function $\hat{T}(s)$ interpolates $T(s)$ at I.

It should also be pointed out that this Krylov technique can easily be extended to generalized state space systems, also called descriptor systems.

Finally, we have shown that the projecting matrices Z, V, constructed in order to compute a state space realization of $\hat{T}_{MP}(s)$, are solutions of Sylvester equations. Actually, it can be shown that, generically, constructing a reduced order transfer function with projecting matrices that are solutions of a Sylvester equation with respect to a state space realization of the original transfer function is equivalent to solving a particular tangential interpolation problem. We refer to [10] for results in this direction.

Appendix.

LEMMA A.1. *Let $T(s)$ and $\hat{T}(s)$ be two* strictly proper *$p \times m$ transfer functions. $\hat{T}(s)$ tangentially interpolates $T(s)$ at I with respect to Definition* 1.3 *if and only if the three following conditions are satisfied:*
for all finite $z_\alpha, 1 \le \alpha \le r$, for any $1 \le i \le \beta_\alpha$:

$$\frac{d^{i-1}}{ds^{i-1}} \left\{ x_\alpha(s)(T(s) - \hat{T}(s)) \right\} \Big|_{s=z_\alpha} = 0; \tag{A.1}$$

for all $z_\alpha = \infty, 1 \le \alpha \le r$,

$$x_\alpha(s)(T(s) - \hat{T}(s)) = O(s^{-1})^{\beta_\alpha + 1}; \tag{A.2}$$

for all finite w_α, $1 \le \alpha \le s$, for any $1 \le i \le \delta_\alpha$,

$$\frac{d^{i-1}}{ds^{i-1}} \left\{ (T(s) - \hat{T}(s)) y_\alpha(s) \right\} \Big|_{s=w_\alpha} = 0; \tag{A.3}$$

for all $w_\alpha = \infty, 1 \le \alpha \le s$,

$$(T(s) - \hat{T}(s)) y_\alpha(s) = O(s^{-1})^{\delta_\alpha + 1}; \tag{A.4}$$

for all finite $\xi_{\alpha,\gamma}$, for all $f = 1, \ldots, \beta_\alpha$, $g = 1, \ldots, \delta_\gamma$,

$$\frac{d^{f+g-1}}{ds^{f+g-1}} \left\{ x_\alpha^{(f)}(s)(T(s) - \hat{T}(s)) y_\gamma^{(g)}(s) \right\} \Big|_{s=\xi_{\alpha,\gamma}} = 0; \tag{A.5}$$

for all infinite $\xi_{\alpha,\gamma}$, the coefficient $e^{[f+g]}$ of s^{-f-g} of the product

$$x_\alpha^{(f)}(s)(T(s) - \hat{T}(s)) y_\gamma^{(g)}(s) \doteq \sum_{k=1}^{+\infty} e^{[k]} s^{-k} \tag{A.6}$$

is zero, where $f = 1, \ldots, \beta_\alpha$; $g = 1, \ldots, \delta_\gamma$.

Proof of Lemma A.1. It is easy to see that the left tangential interpolation conditions (A.1)–(A.2) and condition 1 of Definition 1.3 are equivalent. For the same reasons, the right tangential interpolation conditions (A.3)–(A.4) and conditions 2 of Definition 1.3 are equivalent. Moreover, it is not difficult to see that the two-sided tangential interpolation condition 3 of Definition 1.3 implies conditions (A.5)–(A.6). The proof will be completed by showing that conditions (A.1)–(A.6) imply conditions 1, 2, and 3 of Definition 1.3.

Let us first consider the case with a finite left and right interpolation point $z \in \mathbb{C}$. As usual, we assume that this point is admissible for $T(s)$ and $\hat{T}(s)$; i.e., it is neither a pole of $T(s)$ nor a pole of $\hat{T}(s)$. So, we assume that we are given two polynomial vectors $x(s)$ and $y(s)$ of respective degree $\beta - 1$ and $\delta - 1$ such that

$$x(s)(T(s) - \hat{T}(s)) = O(s - z)^\beta, \quad x(z) \ne 0, \tag{A.7}$$

$$(T(s) - \hat{T}(s)) y(s) = O(s - z)^\delta, \quad y(z) \ne 0, \tag{A.8}$$

and for all $1 \leq f \leq \beta$, $1 \leq g \leq \delta$,

$$\frac{d^{f+g-1}}{ds^{f+g-1}} \left| \left\{ x^{(f)}(s)(T(s) - \hat{T}(s))y^{(g)}(s) \right\} \right|_{s=z} = 0. \tag{A.9}$$

We want to prove that this implies for all $1 \leq f \leq \beta$, $1 \leq g \leq \delta$,

$$x^{(f)}(s)(T(s) - \hat{T}(s))y^{(g)}(s) = O(s-z)^{f+g}. \tag{A.10}$$

By using Lemma 3.7, (A.10) is equivalent to the equation

$$X\mathcal{O}_{C,A}\mathcal{C}_{A,B}Y = X\mathcal{O}_{\hat{C},\hat{A}}\mathcal{C}_{\hat{A},\hat{B}}Y. \tag{A.11}$$

The proof will be completed if we show that for all $1 \leq f \leq \beta$, $1 \leq g \leq \delta$, for all integer k such that $1 \leq k \leq f+g-1$, the derivative

$$\frac{d^{f+g-k-1}}{ds^{f+g-k-1}} \left\{ x^{(f)}(s)(T(s) - \hat{T}(s))y^{(g)}(s) \right\} \Big|_{s=z} = 0. \tag{A.12}$$

Let us first verify (A.12) for $k = 1$. First, straightforward calculation gives

$$\begin{aligned} &\frac{d^{f+g-2}}{ds^{f+g-2}} \left\{ x^{(f)}(s)T(s)y^{(g)}(s) \right\} \Big|_{s=z} \\ &= \sum_{k=0}^{f-1}\sum_{l=0}^{g-1} x^{[k]}C(zI-A)^{k+l-f-g+1}By^{[l]} &\text{(A.13)} \\ &= \sum_{k=0}^{f-1}\sum_{l=0}^{g-1} (x^{[k]}C(zI-A)^{k-f})(zI-A)((zI-A)^{l-g}By^{[l]}) &\text{(A.14)} \\ &= \left(X\mathcal{O}_{C,A}(zI-A)\mathcal{C}_{A,B}Y\right)_{f,g}. &\text{(A.15)} \end{aligned}$$

From Lemmas 3.7 and 3.9,

$$(X\mathcal{O}_{C,A}(zI-A)\mathcal{C}_{A,B}Y) = (X\mathcal{O}_{\hat{C},\hat{A}}(zI-\hat{A})\mathcal{C}_{\hat{A},\hat{B}}Y). \tag{A.16}$$

This concludes the proof for the case $k = 1$. Now, we assume that for all $1 \leq f \leq \beta$ and $1 \leq g \leq \delta$, and for all $0 \leq r \leq \min(k, f+g-1)$,

$$\frac{d^{f+g-r-1}}{ds^{f+g-r-1}} \left\{ x^{(f)}(s)(T(s) - \hat{T}(s))y^{(g)}(s) \right\} \Big|_{s=z} = 0, \tag{A.17}$$

and we want to prove that (A.17) is still true for $r = \min(k+1, f+g-1)$. So, we choose $1 \leq f \leq \beta$ and $1 \leq g \leq \delta$ such that $f+g-1 \geq k+1$. We obtain the following equations:

$$\begin{aligned} &\frac{d^{f+g-k-2}}{ds^{f+g-k-2}} \left\{ x^{(f)}(s)(T(s) - \hat{T}(s))y^{(g)}(s) \right\} \Big|_{s=z} \\ &= \frac{d^{f-1+g-k-1}}{ds^{f-1+g-k-1}} \left\{ x^{(f-1)}(s)(T(s) - \hat{T}(s))y^{(g)}(s) \right\} \Big|_{s=z} &\text{(A.18)} \\ &\quad + \frac{d^{f-1+g-k-1}}{ds^{f-1+g-k-1}} \left\{ (z-s)^{f-1}x^{[f-1]}(T(s) - \hat{T}(s))y^{(g)}(s) \right\} \Big|_{s=z}. &\text{(A.19)} \end{aligned}$$

By the recursive argument,

$$\frac{d^{f-1+g-k-1}}{ds^{f-1+g-k-1}} \left\{ x^{(f-1)}(s)(T(s) - \hat{T}(s))y^{(g)}(s) \right\} \Big|_{s=z} = 0. \tag{A.20}$$

Moreover, we know from (A.3) that

$$(T(s) - \hat{T}(s))y^{(g)}(s) = O(z - s)^g. \tag{A.21}$$

This implies that

$$\frac{d^{f+g-k-2}}{ds^{f+g-k-2}} \left\{ x^{(f)}(s)(T(s) - \hat{T}(s))y^{(g)}(s) \right\} \Big|_{s=z} = 0. \tag{A.22}$$

The case at infinity can be treated in a similar way.

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