CONVEX OPTIMIZATION OVER POSITIVE POLYNOMIALS AND FILTER DESIGN

Y. Genin, Y. Hachez, Yu. Nesterov, P. Van Dooren

CESAME, Université catholique de Louvain
Bâtiment Euler, Avenue G. Lemaître 4-6
B-1348 Louvain-la-Neuve, Belgium
{genin,hachez,nesterov,vdooren}@csam.ucl.ac.be

ABSTRACT

Positive polynomial matrices play a fundamental role in systems and control theory: they represent e.g. spectral density functions of stochastic processes and show up in spectral factorizations, robust control and filter design problems. Positive polynomials obviously form a convex set and were recently studied in the area of convex optimization [1, 5]. It was shown in [2, 5] that positive polynomial matrices can be parametrized using block Hankel and Toeplitz matrices. In this paper, we use this parametrization to derive efficient computational algorithms for optimization problems over positive polynomials. Moreover, we show that filter design problems can be solved using these results.

Keywords: convex optimization, positive polynomials, trigonometric polynomials, filter design.

1 OPTIMIZATION PROBLEMS OVER POSITIVE POLYNOMIALS

Let us first consider scalar positive polynomials over the real line. Note that other relevant classes of positive polynomials (over the imaginary axis and over the unit circle) can be obtained via an appropriate transformation of variables. In the literature, positive polynomials are also often referred to as "nonnegative" polynomials. We have chosen the former denomination.

Let $\mathcal{K}$ be the cone of coefficients $\mathbf{p} \in \mathbb{R}^{2n+1}$ of polynomials positive on the whole real line. Then, many important optimization problems can be written in the following standard form:

$$\min_{\mathbf{p}} \{\langle \mathbf{c}, \mathbf{p} \rangle : \ A\mathbf{p} = \mathbf{b}, \ \mathbf{p} \in \mathcal{K}\}, \tag{1}$$

where A is a $k \times (2n+1)$ matrix with rows $\mathbf{a}_i$. The system of linear equations in (1) may for example express interpolation conditions on our polynomials (or their derivatives) at certain points x_i of the real line. Note that an interpolation condition at $x_0 \in \mathbb{R}$ is a linear constraint of the following form:

$$\langle \mathbf{p}, \pi_{2n}(x_0) \rangle = b, \quad \pi_\ell(x) = (1, x, \ldots, x^\ell)^T \in \mathbb{R}^{\ell+1}. \tag{2}$$

The number of rows k of the matrix A must be smaller or equal to $2n+1$ in order to have a solution for the problem, but quite often k is a much smaller value. The main complexity of such problems then comes from the dimension of the polynomial $\mathbf{p}$.

Using the scalar product $\langle X, Y \rangle \doteq \mathrm{Trace}(XY^T)$, the inclusion $\mathbf{p} \in \mathcal{K}$ can be represented in the following form [5] :

$$p_i = \langle Y, H_i \rangle, \ i = 1, \ldots, 2n+1, \tag{3}$$

$$Y \succeq 0 \in \mathbb{R}^{n(n+1)/2}, \tag{4}$$

where the matrices H_i form the natural basis of the $(n+1) \times (n+1)$ symmetric Hankel matrices. If $\mathbf{p}$ satisfies the above linear equations, we have

$$\langle \mathbf{a}, \mathbf{p} \rangle = \langle H(\mathbf{a}), Y \rangle, \tag{5}$$

where $H(\mathbf{a})$ is the Hankel matrix defined by the vector $\mathbf{a}$, i.e. $H(\mathbf{a}) = \sum_{i=1}^{2n+1} a_i H_i$. Thus, the initial optimization problem (1) can be written in the equivalent matrix form:

$$\begin{array}{ll} \min\limits_{Y} & \langle H(\mathbf{c}), Y \rangle \\ \mathrm{s.\,t.} & \langle H(\mathbf{a}_i), Y \rangle = b_i, \quad i = 1, \ldots, k \\ & Y \succeq 0. \end{array} \tag{6}$$

However, in this case, the dimension of the space of variables is significantly increased. Moreover, in order to apply interior point schemes to this formulation, we need to compute the values and the derivatives of the barrier $F(Y) = -\ln \det Y$ with arbitrary positive definite matrix Y [6]. Therefore the complexity of one iteration of such a scheme will be at least $\mathcal{O}(kn^3)$ arithmetic operations.

Alternatively, we can try to solve the problem dual to (1) :

$$\max_{\mathbf{s},\mathbf{y}}\{\langle \mathbf{b},\mathbf{y}\rangle :\ \mathbf{s}+A^T\mathbf{y}=\mathbf{c},\ \mathbf{s}\in\mathcal{K}^*\}, \qquad (7)$$

where $\mathcal{K}^*$ is the cone dual to $\mathcal{K}$: $\mathcal{K}^*=\{\mathbf{s}:\langle\mathbf{s},\mathbf{p}\rangle\geq 0,\forall\mathbf{p}\in\mathcal{K}\}$. It is well-known that, under some natural hypothesis ("strict feasibility of the primal-dual problem"), both problems have the same optimal value.

In our case [5] :

$$\mathcal{K}^*=\{\mathbf{s}\in\mathbb{R}^{2n+1}\ :\ H(\mathbf{s})\succeq 0\}. \qquad (8)$$

Eliminating $\mathbf{s}$ in (7), we obtain the following dual problem:

$$\max_{\mathbf{y}\in\mathbb{R}^k}\{\langle\mathbf{b},\mathbf{y}\rangle :\ H(\mathbf{c}-A^T\mathbf{y})\succeq 0\}. \qquad (9)$$

Note that the dimension of this problem is only k. Moreover, in order to treat the constraints we can use the following natural barrier function [6] :

$$f(\mathbf{y})=-\ln\det H(\mathbf{c}-A^T\mathbf{y}). \qquad (10)$$

The interior point scheme applied to (9) requires the evaluation of the first and second derivatives of the function $f(\mathbf{y})$. It can be easily seen that the derivatives can be expressed as follows ($i,j=1,\ldots,k$) :

$$[f'(y)]_i=\langle H^{-1}(\mathbf{s}),H(\mathbf{a}_i)\rangle, \qquad (11)$$

$$[f''(y)]_{i,j}=\langle H^{-1}(\mathbf{s})H(\mathbf{a}_i)H^{-1}(\mathbf{s}),H(\mathbf{a}_j)\rangle, \qquad (12)$$

where $\mathbf{s}=\mathbf{c}-A^T\mathbf{y}$. In interior point schemes we need to invert the matrix $f''(\mathbf{y})$, which takes $\mathcal{O}(k^3)$ arithmetic operations. However, since k is typically small, the main complexity comes from the computation of the elements of the objects $f'(\mathbf{y})$ and $f''(\mathbf{y})$.

Using the displacement structure of Hankel matrices [3] and convolution, all the scalar products ($i,j=1,\ldots,k$) :

$$\langle H^{-1}(\mathbf{s}),H_j\rangle, \qquad (13)$$

$$\langle H^{-1}(\mathbf{s})H(\mathbf{a}_i)H^{-1}(\mathbf{s}),H_j\rangle \qquad (14)$$

can be computed in $\mathcal{O}(kn\ln^2 n)$ arithmetic operations. Constructing the Hessian from this requires an additional $\mathcal{O}(k^2n)$ operations which brings the total to $\mathcal{O}(kn\ln^2 n+k^2n)$ arithmetic operations for computing the whole Hessian $f''(\mathbf{y})$.

Since interior point schemes that compute the solution of problem (7) with relative accuracy $\epsilon\in(0,1]$ require a number of steps that can be bounded by $\mathcal{O}(\sqrt{n}\ln\frac{1}{\epsilon})$ [6], we obtain a total complexity of $\mathcal{O}(kn^{1.5}(\ln^2 n+k)\ln\frac{1}{\epsilon})$ arithmetic operations. That is a remarkable result for solving an optimization problem in an n-dimensional vector space.

We point out that all these developments carry over with minor changes to the case of positive polynomials over the imaginary axis $j\mathbb{R}$, while for positive polynomials over the unit circle Δ, one can use fast algorithms for Toeplitz matrices. The complexity of these other cases is essentially the same. The authors have also extended the results contained in this section to positive pseudo-polynomial matrices on these particular curves of the complex plane [2].

2 CONE OF POSITIVE TRIGONOMETRIC POLYNOMIALS

Remember that the classic definition of trigonometric polynomial is as follows.

Definition 1. A trigonometric polynomial $q(\omega)$ of degree n is defined by

$$q(\omega)=\sum_{k=0}^{n}a_k\cos(k\omega)+b_k\sin(k\omega). \qquad (15)$$

where $a_k,b_k\in\mathbb{R},\ \forall k$.

Note that we can assume that $b_0=0$ without any loss of generality.

This definition does not emphasize the nature of $q(\omega)$, i.e. the fact that $q(\omega)$ is basically a pseudo-polynomial of degree n defined on $\Delta\doteq\{z\in\mathbb{C}:|z|=1\}$. Let us define the scalar product of two complex matrices X and Y by $\langle X,Y\rangle\doteq\mathrm{Re\,Trace}(XY^*)$. In particular, X and Y could be two vectors. The following proposition illustrates this important connection.

Proposition 1. *Any trigonometric polynomial $q(\omega)$ (of degree n) is equivalent to a "complex" trigonometric polynomial $p(z)$ (of degree n) defined by*

$$p(z)=\langle p,\pi_n(z)\rangle,\quad\forall z\in\Delta \qquad (16)$$

where $p\in\mathbb{R}\times\mathbb{C}^n$, $p_k=a_k+jb_k(0\leq k\leq n)$, $\pi_n(z)=(1,z,\ldots,z^n)^T$.

Let $E=\mathbb{R}\times\mathbb{C}^n$ and $\mathcal{H}^{n+1}$ be the set of Hermitian matrices of order $n+1$. Define the cone of trigonometric polynomials of degree n positive on the (complex) unit circle by

$$K_T=\{p\in E:p(z)=\langle p,\pi_n(z)\rangle\geq 0,\forall z\in\Delta\}. \qquad (17)$$

where $\pi_n(z)=(1,z,\ldots,z^n)^T$.

K_T is a cone since $\alpha K_T\subset K_T,\forall\alpha\geq 0$. Moreover it is a convex one since the sum of two positive trigonometric polynomials is always a positive one. The characterization of K_T is based on an old result of Fejér :

Theorem 2 (Fejér). *Let $\pi_n(z) = (1, z, \ldots, z^n)^T$. A trigonometric polynomial $p(z) = \langle p, \pi_n(z)\rangle$ of degree n is positive on the unit circle, i.e. $p(z) \geq 0,\ \forall z \in \Delta$, if and only if there exists a complex polynomial $\gamma(z) = \sum_{i=0}^{n} \gamma_i z^i$ such that $p(z) = |\gamma(z)|^2,\ \forall z \in \Delta$. Moreover, $p \in \mathbb{R}^{n+1}$ if and only if $\gamma_i \in \mathbb{R},\ \forall i$.*

Using Fejér's theorem, Nesterov has parametrized K_T with the set of Hermitian positive semi-definite matrices of order $n+1$. Using the same notation as [5], let us define the linear operator $\Lambda_H : E \to \mathcal{H}^{n+1}$ by

$$\Lambda_H(v) = \frac{1}{2}\sum_{i=1}^{n+1}[T_i v_i + T_i^T \overline{v_i}] \tag{18}$$

where

$$T_1 = I_{n+1}, \tag{19}$$

$$[T_i]_{j,k} = \begin{cases} 2 & \text{if } j-k = i-1 \\ 0 & \text{otherwise} \end{cases}, i = 2, \ldots, n. \tag{20}$$

Note that the operator Λ_H^* dual to the operator $\Lambda_H(v) = \frac{1}{2}\sum_{i=1}^{n+1}[\Lambda_i v_i + \Lambda_i^T \overline{v_i}]$ w.r.t. the inner product $\langle \cdot, \cdot \rangle$ satisfies $[\Lambda_H^*(Y)]_i = \mathrm{Trace}(Y \Lambda_i^*)$.

Theorem 3 (Nesterov [5]).

$$K_T = \{p \in E : p = \Lambda_H^*(Y), Y \succeq 0 \in \mathcal{H}^{n+1}\}$$
$$K_T^* = \{c \in E : \Lambda_H(c) \succeq 0 \in \mathcal{H}^{n+1}\}$$

Moreover, these two cones are proper, i.e. they are pointed and their interior are not empty.

This characterization of trigonometric polynomials is not sufficient in our context. In fact, many filter design problems involve at least two intervals. The aim of this section is thus to extend this characterization using an earlier result of Markov-Lukacs.

Theorem 4 (Markov-Lukacs (even degree)). *Let $q(x) \in \mathbb{R}[x]$ be a real polynomial of degree $2n$ defined on the whole real line and $[a, b]$ be a segment. $q(x)$ is positive on $[a, b]$ if and only if $q(x) = q_1(x) + (x-a)(b-x)q_2(x)$ where q_1 and q_2 are two real polynomials (of degree $2n$ and $2(n-1)$, respectively) positive on the whole real line.*

Let us fix some a from Δ, such that $0 < \arg(a) < \pi$ and consider the symmetric arc

$$\Delta_a = \{z \in \Delta : -\arg(a) \leq \arg(z) \leq \arg(a)\}. \tag{21}$$

The convex cone of trigonometric polynomials of degree n positive on Δ_a is thus defined by

$$K_T^a = \{p \in E : p(z) = \langle p, \pi_n(z)\rangle \geq 0, \forall z \in \Delta_a\}. \tag{22}$$

Since the well-known bilinear transformation ($z \in \mathbb{C} \leftrightarrow u \in \mathbb{R}$

$$z = \frac{1+ju}{1-ju} \in \Delta \quad \leftrightarrow \quad u = j\frac{1-z}{1+z} \in \mathbb{R}, \tag{23}$$

establishes a one-to-one correspondence between Δ and $\mathbb{R}$, one can easily prove the following result.

Corollary. *A trigonometric polynomial $p(z)$ is positive on Δ_a if and only if this polynomial can be decomposed as follows*

$$p(z) = p_1(z) + \phi_a(z)p_2(z) \tag{24}$$

where p_1 and p_2 are two trigonometric polynomials (of degree n and $n-1$, respectively) positive on Δ and the function ϕ_a is defined by

$$\phi_a(z) = z - (a + a^{-1}) + z^{-1}. \tag{25}$$

In order to characterize K_T^a, we need to define a new linear operator $\Lambda_H(\cdot; a) : E \to \mathcal{H}^n$ by

$$\begin{aligned}\Lambda_H(v; a) = &\frac{-(a+a^{-1})}{2}\left[\sum_{i=1}^{n}(T_i v_i + T_i^T \overline{v_i})\right] \\ &+ [T_1 v_2 + T_1^T \overline{v_2}] \\ &+ \frac{1}{2}\left[\sum_{i=3}^{n+1}(T_{i-1} v_i + T_{i-1}^T \overline{v_i})\right] \\ &+ \frac{1}{2}\left[\sum_{i=1}^{n-1}(T_{i+1} v_i + T_{i+1}^T \overline{v_i})\right].\end{aligned}$$

Using the techniques developed in [5], the following theorem, which is similar to Theorem 3, can be stated

Theorem 5.

$$\begin{aligned}K_T^a = \{&p \in E : p = \Lambda_H^*(Y_1) + \Lambda_H^*(Y_2; a), \\ &Y_1 \succeq 0 \in \mathcal{H}^{n+1}, Y_2 \succeq 0 \in \mathcal{H}^n\}, \\ K_T^{a*} = \{&c \in E : \Lambda_H(c) \succeq 0 \in \mathcal{H}^{n+1}, \\ &\Lambda_H(c; a) \succeq 0 \in \mathcal{H}^n\}.\end{aligned}$$

Moreover, these two cones are proper, i.e. they are pointed and their interior are not empty.

Let us now consider arbitrary arcs of Δ. If b and c are two points belonging to Δ such that $\arg(b) < \arg(c)$, the arc associated to (b, c) is defined by

$$\Delta_{b,c} = \{z \in \Delta : \arg(b) \leq \arg(z) \leq \arg(c)\} \tag{26}$$

The convex cone of trigonometric polynomials of degree n positive on $\Delta_{b,c}$ is thus defined by

$$K_T^{bc} = \{p \in E : p(z) \geq 0, \forall z \in \Delta_{b,c}\}. \tag{27}$$

The following lemma shows the strong connection between positivity on the arbitrary arc $\Delta_{b,c}$ and positivity on a particular symmetric arc Δ_a.

Lemma 6. *A trigonometric polynomial $p(z)$ is positive on $\Delta_{b,c}$ if and only if the trigonometric polynomial $q(z') = p(z'e^{j\theta})$ is positive on Δ_a where a, θ are defined by*

$$a = e^{j(\arg(c) - \arg(b))/2}, \tag{28}$$

$$\theta = \frac{\arg(c) + \arg(b)}{2}. \tag{29}$$

We can now state a corollary similar to Corollary 2

Corollary. *A trigonometric polynomial $p(z)$ is positive on $\Delta_{b,c}$ if and only if this polynomial can be decomposed as follows*

$$p(z) = p_1(z) + \phi_{a,\theta}(z) p_2(z) \tag{30}$$

where p_1 and p_2 are two trigonometric polynomials (of degree n and $n-1$, respectively) positive on Δ. The parameters a and θ are defined by (28) *and* (29)*, respectively. The function $\phi_{a,\theta}$ is defined by*

$$\phi_a(z) = e^{-j\theta} z - (a + a^{-1}) + e^{j\theta} z^{-1}. \tag{31}$$

Using the same techniques than those presented in [5], a theorem similar to Theorem 3 could be stated. If $c = \overline{b}$ and a, θ are defined by (28) and (29), we get the following result :

Theorem 7.

$$\begin{aligned} K_T^{b\overline{b}} &= \{p \in E : p = \Lambda_H^*(Y_1) + \Lambda_H^*(Y_2; a, \overline{a}), \\ &\qquad Y_1 \succeq 0 \in \mathcal{H}^{n+1}, Y_2 \succeq 0 \in \mathcal{H}^n\}, \\ K_T^{b\overline{b}*} &= \{c \in E : \Lambda_H(c) \succeq 0 \in \mathcal{H}^{n+1}, \\ &\qquad \Lambda_H(c; a, \overline{a}) \succeq 0 \in \mathcal{H}^n\} \end{aligned}$$

where

$$\begin{aligned} \Lambda_H(v; a, \overline{a}) = &\frac{-(a + a^{-1})}{2} \left[\sum_{i=1}^{n} (T_i v_i + T_i^T \overline{v_i}) \right] \\ &- \left[T_1 v_2 + T_1^T \overline{v_2} \right] \\ &- \frac{1}{2} \left[\sum_{i=3}^{n+1} (T_{i-1} v_i + T_{i-1}^T \overline{v_i}) \right] \\ &- \frac{1}{2} \left[\sum_{i=1}^{n-1} (T_{i+1} v_i + T_{i+1}^T \overline{v_i}) \right]. \end{aligned}$$

Moreover, these two cones are proper, i.e. they are pointed and their interior are not empty.

3 FILTER DESIGN

In this section our previous results are applied to a lowpass filter design problem. In fact, this problem is formulated as a feasibility problem w.r.t. an appropriate convex set using the results mentioned above. Finally we discuss the implementation of our method.

Description

Consider a class of systems whose input and output satisfy a linear constant coefficient difference equation of the form

$$\sum_{k=0}^{n} a_k y[t-k] = \sum_{k=0}^{n} b_k u[t-k]. \tag{32}$$

The frequency response, i.e. the system transfer function evaluated on the unit circle, has the form

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{n} a_k e^{-j\omega k}}{\sum_{k=0}^{n} b_k e^{-j\omega k}} \tag{33}$$

In order to design a lowpass filter, the constraints we have to satisfy are best written using the squared magnitude of the filter frequency response [7]

$$R(\omega) = |H(e^{j\omega})|^2. \tag{34}$$

The semi-infinite inequality constraints considered are thus written as

$$\alpha_1 \le R(\omega) \le \alpha_2, \quad \omega \in [0, \omega_a], \tag{35}$$

$$R(\omega) \le \delta, \quad \omega \in [\omega_b, \pi], \tag{36}$$

$$R(\omega) \ge 0, \quad \omega \in [0, \pi]. \tag{37}$$

where $0 < \omega_a < \omega_b < \pi$ and $\alpha_1 > 0$.

In order to completely formulate the problem, the objective function, which depends on the specific problem we want to solve, should be specified :

- minimize the passband ripple given a stopband attenuation : $\min(\alpha_2 - \alpha_1)$ (δ, n are fixed);
- maximize the stopband attenuation : $\min \delta$ (α_1, α_2, n are fixed);
- minimize the degree of the filter : $\min n$ ($\alpha_1, \alpha_2, \delta$ are fixed).

Since the whole optimization problem (objective and constraints) is not convex but quasi-convex, our approach combines a bisection rule on the parameter ($\alpha_1 - \alpha_2, \delta$ or n) and an optimization scheme solving a feasibility problem.

A straightforward *approximation* of the semi-infinite inequality constraints uses N sampling frequencies

$$0 \le \omega_1 \le \cdots \le \omega_N \le \pi \tag{38}$$

and replaces the semi-infinite inequality constraints with the corresponding ordinary inequalities. A standard rule of thumb is to choose $N \simeq 15n$ linearly spaced sampling frequencies [7]. Note that this is only an approximative procedure while our approach exactly handles such constraints.

Formulation

The key point of our development is the existence of two trigonometric polynomials $p_1(z)$ and $p_2(z)$ such that

$$R(\omega) = |H(e^{j\omega})|^2 = |\frac{\sum_{k=0}^{n} a_k e^{-j\omega k}}{\sum_{k=0}^{n} b_k e^{-j\omega k}}|^2 = \frac{p_1(z)}{p_2(z)}. \tag{39}$$

The coefficients $\{a_k\}$ and $\{b_k\}$ can be recovered by taking the stable spectral factors of $p_1(z)$ and $p_2(z)$, respectively.

Assume that $\alpha_1, \alpha_2, \delta$ and n are fixed and let $l_1^2 = \alpha_1$, $l_2^2 = 0$, $u_1^2 = \alpha_2$, $u_1^2 = \delta$, $\Omega_1 = [0, \omega_a]$, $\Omega_2 = [\omega_b, \pi]$. The feasibility problem corresponding to our rational lowpass filter design problem can then be formulated as follows. Find the coefficients of two trigonometric polynomials $p_1(z)$ and $p_2(z)$ such that the design constraints

$$l_i^2 \leq \frac{p_1(z)}{p_2(z)} \leq u_i^2, \quad z \in \Omega_i,\ i = 1, 2 \tag{40}$$

are satisfied (see Figure 1).

In order to avoid the well-known overshoot phenomenon, we slightly modify our design constraints :

$$l_1^2 \leq \frac{p_1(z)}{p_2(z)}, \quad z = e^{j\omega}, \qquad \omega \in [0, \omega_a], \tag{41}$$

$$\frac{p_1(z)}{p_2(z)} \leq u_1^2, \qquad z = e^{j\omega}, \qquad \omega \in [0, \omega_b], \tag{42}$$

$$0 \leq \frac{p_1(z)}{p_2(z)}, \quad z = e^{j\omega}, \qquad \omega \in [\omega_a, \pi], \tag{43}$$

$$\frac{p_1(z)}{p_2(z)} \leq u_1^2, \qquad z = e^{j\omega}, \quad \omega \in [\omega_b, \pi]. \tag{44}$$

Our previous results allows us to rewrite the design constraints. The feasibility problem becomes : Find two polynomials p_1 and p_2 such that the following inequalities hold :

$$q_1(z) \doteq p_1(z) \geq 0, \qquad z \in \Delta, \tag{45}$$

$$q_2(z) \doteq p_1(z) - l_1^2 p_2(z) \geq 0, \qquad z \in \Delta_a, \tag{46}$$

$$q_3(z) \doteq u_1^2 p_2(z) - p_1(z) \geq 0, \qquad z \in \Delta_b, \tag{47}$$

$$q_4(z) \doteq u_2^2 p_2(z) - p_1(z) \geq 0, \qquad z \in \Delta \setminus \Delta_b. \tag{48}$$

These four constraints are positivity constraints on trigonometric polynomials. Using the convex sets mentioned above, they are therefore equivalent to :

$$q_1 = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in K_T, \tag{49}$$

$$q_2 = \begin{pmatrix} I & -l_1^2 I \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in K_T^a, \tag{50}$$

$$q_3 = \begin{pmatrix} -I & u_1^2 I \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in K_T^b, \tag{51}$$

$$q_4 = \begin{pmatrix} -I & u_2^2 I \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in K_T^{b\bar{b}}. \tag{52}$$

Defining $K = K_T \times K_T^a \times K_T^b \times K_T^{b\bar{b}}$ and $q = \begin{pmatrix} q_1, & q_2, & q_3, & q_4 \end{pmatrix}^T$, our problem now becomes : Find p_1 and p_2 such that

$$q = \begin{pmatrix} I & 0 \\ I & -l_1^2 I \\ -I & u_1^2 I \\ -I & u_2^2 I \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = A \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in K \tag{53}$$

Algorithm

Let $F_*(w)$ be the usual barrier function of the dual cone $K^* = K_T^* \times K_T^{a*} \times K_T^{b*} \times K_T^{b\bar{b}*}$ [6]. Problem (53) is feasible if and only if the optimization problem

$$\begin{array}{ll} \min & F_*(w) \\ \mathrm{s.t.} & A^* w = A^* \tilde{w} \\ & w \in \operatorname{int} K^* \end{array} \tag{54}$$

is bounded. Our algorithm therefore checks this property. Moreover, using the properties of self-scaled cones and of normal barriers, e.g. the cone of positive semidefinite matrices and its usual barrier, one can show that the gradient of $F_*(w)$ establishes a one-to-one mapping between $\operatorname{int} K$ and $\operatorname{int} K^*$.

For instance, assume that α and n are fixed and that we minimize the stopband attenuation. The corresponding algorithm can be described by the following steps :

Step 0 Choose a starting value of δ.

Step 1 Solve the analytic center problem (54). If its value is bounded, go to Step 2. Otherwise, the problem (53) is not feasible : increase δ and go to Step 1.

Step 2 If δ is small enough (termination criteria), go to Step 3. Otherwise, $\delta := \delta/2$ and go to Step 1.

Step 3 Using the properties of the gradient of F_*, recover p_1 and p_2. Compute the spectral factorization of p_1 and p_2 to get $\{a_k\}$ and $\{b_k\}$.

Note the bisection rule on δ.

Discussion

Even if our formulation is correct, the numerical behavior of this formulation is not always adequate. In fact, this algorithm has been implemented using the LMI Control Toolbox of MATLAB. As soon as $n \geq 7$, it breaks down. As a matter of fact, it turns out that the problem is intrinsically ill-conditioned as put into light by numerical experiments. Therefore an appropriate reformulation of our problem approach should be looked for that keeps the property of our original treatment : it should not be based on approximations of the semi-infinite inequality constraints. Finally, let us stress that optimizing over pseudo-polynomials positive on intervals is a challenging and interesting problem. The authors intend to substantiate this claim in a forthcoming paper.

Acknowledgements

This paper presents research supported by NSF contract CCR-97-96315 and by the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. A research fellowship from the Belgian National Fund for Scientific Research is also gratefully acknowledged by the second author. The scientific responsibility rests with its authors.

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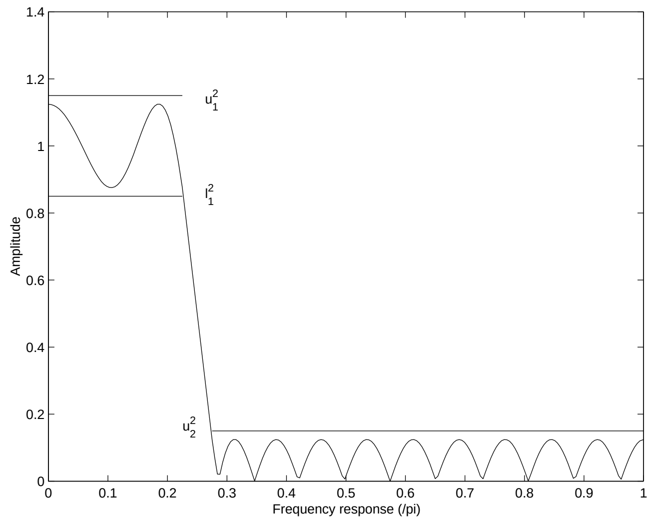

Figure 1: Bandpass filter problem ($\omega_a = 0.225, \omega_b = 0.275, n = 26$)