STABILITY RADIUS AND OPTIMAL SCALING OF DISCRETE-TIME PERIODIC SYSTEMS

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Abstract: Robust stability properties of periodic discrete time systems are investigated. Analytic expressions are derived for the stability radius in the scalar case.

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1. INTRODUCTION

Let us consider a discrete periodic system of the form

$$E_k x_{k+1} = A_k x_k \quad k = 0, 1, 2, \dots \tag{1}$$

where x_0 is the given initial state and where the matrices $E_k,\ A_k \in \mathbb{C}^{n\times n}$ vary periodically over a period of length K, i.e.

$$E_{k+K} = E_k, \quad A_{k+K} = A_k, \text{for all } \ k \geq 0.$$

When x_0 is given, one can solve this initial value problem provided the E_k matrices are invertible, which we will assume throughout this paper. We define the monodromy matrix

$$\Phi := E_{K-1}^{-1} A_{K-1} \dots E_1^{-1} A_1 E_0^{-1} A_0 \tag{2}$$

and point out that the behaviour over K steps is easily found from (1) to be time invariant :

$$x_{(i+1)K} = \Phi x_{iK} \quad i = 0, 1, \dots \tag{3}$$

The system (1) is said to be stable, if all the eigenvalues of Φ are in the open unit disc, *i.e* $\Lambda(\Phi) \subset \mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$. The eigenvalues of Φ can also be obtained from the bicyclic eigenvalue problem

$$\lambda\mathcal{E} - \mathcal{A}\mathcal{Z} := \begin{bmatrix} \lambda E_0 & & & -A_0 \\ -A_1 & \ddots & & \\ & \ddots & \ddots & \\ & & -A_{K-1} & \lambda E_{K-1} \end{bmatrix} \tag{4}$$

where

$$\mathcal{E} = \mathrm{diag}\{E_0, \cdots, E_{K-1}\},$$
$$\mathcal{A} = \mathrm{diag}\{A_0, \cdots, A_{K-1}\},$$
$$\mathcal{Z} = \begin{bmatrix} 0_n & & & I_n \\ I_n & \ddots & & \\ & \ddots & \ddots & \\ & & I_n & 0_n \end{bmatrix}.$$

Indeed, one easily finds that for $N := nK$

$$\begin{aligned} \det(\lambda\mathcal{E} - \mathcal{A}\mathcal{Z}) &= \det\mathcal{E}\ \det(\lambda I_N - \mathcal{E}^{-1}\mathcal{A}\mathcal{Z}) \\ &= \det\mathcal{E}\ \det(\lambda^K I_n - \Phi) \end{aligned}$$

and hence that the generalized eigenvalues of $(\lambda\mathcal{E} - \mathcal{A}\mathcal{Z})$ are the K-th roots of the eigenvalues of Φ. But the system (1) is not a unique representation of the difference equation. Scaling the equations with a scalar $\alpha_k \neq 0$ will not alter the solution x_k, and substituting $\hat{x}_{k+1} = \beta_k x_{k+1}$ with scalars $\beta_k \neq 0$ always allows to retrieve x_{k+1} from $\hat{x}_{k+1}$.

If we choose a K–periodic scaling $\alpha_k = \alpha_{k+K}$ and $\beta_k = \beta_{k+K}$, for all $k \geq 0$, then we obtain a new periodic system characterized by

$$\lambda\hat{\mathcal{E}} - \hat{\mathcal{A}}\mathcal{Z} = \begin{bmatrix} \alpha_0 I_n & & \\ & \ddots & \\ & & \alpha_{k-1} I_n \end{bmatrix} (\lambda\mathcal{E} - \mathcal{A}\mathcal{Z}) \begin{bmatrix} \beta_0 I_n & & \\ & \ddots & \\ & & \beta_{k-1} I_n \end{bmatrix}^{-1} = D_\alpha(\lambda\mathcal{E} - \mathcal{A}\mathcal{Z})D_\beta^{-1} \tag{5}$$

corresponding to the scaled difference equations

$$\underbrace{(\alpha_k\ E_k\ \beta_k^{-1})}_{\hat{E}_k}\ \hat{x}_{k+1} = \underbrace{(\alpha_k\ A_k\ \beta_{k-1}^{-1})}_{\hat{A}_k}\ \hat{x}_k. \tag{6}$$

2. STABILITY RADIUS OF DISCRETE-TIME PERIODIC SYSTEMS

The robustness issue is a crucial problem for the application of control theory; for example, one of the basic goals of feedback control is to enhance system robustness. Robust stability is also an important topic in linear algebra as well as in numerical analysis. A fundamental problem in robustness analysis is to determine the ability of a system matrix to maintain its stability under a certain class of perturbations. A natural robustness measure is the *distance* of a stable system to the set of unstable systems, defined by Hinrichsen and Pritchard (Hinrichsen and Pritchard, 1986) as the *stability radius* of the system.

Assuming the system (1) to be stable implies that $(\lambda\mathcal{E} - \mathcal{A}\mathcal{Z})$ has only generalized eigenvalues inside the unit circle. Therefore $\mathcal{E}$ is invertible, Φ is well defined and Φ has its eigenvalues inside the unit circle. One is interested in determining the smallest perturbations of the coefficients E_k, A_k that will make the system unstable. Equivalently, one analyzes the sensitivity of the generalized eigenvalues of $\lambda\mathcal{E} - \mathcal{A}\mathcal{Z}$ to structured perturbations in this pencil. To be more specific, one has to find the smallest perturbations

$$\Delta\,\mathcal{E} = \mathrm{diag}\{\delta E_0, \cdots, \delta E_{K-1}\},$$
$$\Delta\,\mathcal{A} = \mathrm{diag}\{\delta A_0, \cdots, \delta A_{K-1}\}$$

such that the system matrix $[\lambda\,(\mathcal{E} + \Delta\mathcal{E}) - (\mathcal{A} + \Delta\mathcal{A})\,\mathcal{Z}]$ has at least one generalized eigenvalue in the *unstable* part of the complex plane, *i.e.* outside the open unit disc. Measured in term of the 2-norm, the minimality of the perturbations in question is characterized by the **stability radius** defined as follows:

$$r_{\mathcal{E},\mathcal{A}} = \inf_{\Delta\mathcal{E},\,\Delta\mathcal{A}} \{\, \max\left(\|\Delta\mathcal{E}\|_2, \|\Delta\mathcal{A}\|_2\right)\ : \exists\lambda \notin \mathbb{D} \ \text{s.t.}\ \det[\lambda(\mathcal{E} + \Delta\mathcal{E}) - (\mathcal{A} + \Delta\mathcal{A})\,\mathcal{Z}] = 0\}. \tag{7}$$

Two subproblems of special interest can derived from this general setting by imposing either the constraint $\Delta\,\mathcal{E} = 0$ or the constraint $\Delta\,\mathcal{A} = 0$. The corresponding stability radii then take respectively the form

$$r_{\mathcal{E}} = \inf_{\Delta\mathcal{E}} \{\, \|\Delta\mathcal{E}\|_2\ : \exists\lambda \notin \mathbb{D} \ \text{s.t.}\ \det[\lambda(\mathcal{E} + \Delta\mathcal{E}) - \mathcal{A}\,\mathcal{Z}] = 0\} \tag{8}$$

or

$$r_{\mathcal{A}} = \inf_{\Delta\mathcal{A}} \{\, \|\Delta\mathcal{A}\|_2)\ : \exists\lambda \notin \mathbb{D} \ \text{s.t.}\ \det[\lambda\,\mathcal{E} - (\mathcal{A} + \Delta\mathcal{A})\,\mathcal{Z}] = 0\}. \tag{9}$$

Because eigenvalues move continuously with $\Delta\mathcal{E}$, $\Delta\mathcal{A}$, equalities (7), (8) and (9) can be rewritten into the form

$$r_{\mathcal{E},\mathcal{A}} = \inf_{|\lambda|=1} \inf_{\Delta\mathcal{E},\,\Delta\mathcal{A}} \{\, \max\left(\|\Delta\mathcal{E}\|_2, \|\Delta\mathcal{A}\|_2\right)\ : \det[\lambda(\mathcal{E} + \Delta\mathcal{E}) - (\mathcal{A} + \Delta\mathcal{A})\,\mathcal{Z}] = 0\}. \tag{10}$$

Note that one has the relations

$$\|\Delta\,\mathcal{E}\|_2 = \max_k \|\delta E_k\|_2, \quad \|\Delta\mathcal{A}\|_2 = \max_k \|\delta A_k\|_2$$

by definition, since the matrices $\Delta\,\mathcal{E}$ and $\Delta\,\mathcal{A}$ are diagonal.

Next, we focus our attention on the *scalar* case, when $n = 1$ and E_k, A_k are real numbers, which can be simply denoted by e_k, a_k. Accordingly, for the perturbations matrices δE_k, δA_k we write δe_k, δa_k.

To deal with the problem, let us introduce from the data the two polynomials of the x variable:

$$P_e(x) = \prod_{k=0}^{K-1} (1 - x/|e_k|), \ P_a(x) = \prod_{k=0}^{K-1} (1 + x/|a_k|). \tag{11}$$

Define also the constant $a \in \mathbb{C}$ by

$$a^K = \frac{a_0\, a_1, \ldots, a_{K-1}}{e_0\, e_1, \ldots, e_{K-1}} \tag{12}$$

which is just another way to write (2) in the scalar case. Note that, since the unperturbed system (1) is assumed to be stable, one has the property $|a| < 1$.

A closed formula for the stability radius (10) is given by the next theorem and it is expressed in terms of a polynomial equation involving both P_e and P_a introduced above.

Theorem 1. Let ζ_0 be the *smallest positive real zero* of the polynomial equation

$$P_e(x) - |a|^K\, P_a(x) = 0. \tag{13}$$

Then

$$r_{\mathcal{E},\mathcal{A}} = \zeta_0 \tag{14}$$

i.e the stability radius (10) is the smallest positive real root of the polynomial equation (13).
Moreover, a minimal perturbation is obtained by setting for $k = 0, 1, \ldots, K-1$:

$$\delta e_k = -\, e_k\, \zeta_0/|e_k|,\ \delta a_k = +\, a_k\, \zeta_0/|a_k|,\ \hat{\lambda} = a/|a|. \tag{15}$$

As an immediate consequence, one has

Corollary 2. The spectral radii $r_{\mathcal{E}}$ and $r_{\mathcal{A}}$ will be determined as the *smallest positive real zero* of the polynomial equations $P_e(x) - |a|^K = 0$ and $1 - |a|^K P_a(x) = 0$, respectively, while minimal coefficient perturbations can be still defined by the first or second relation (15).

The case $a = 0$ is treated separately.

3. OPTIMAL SCALING OF BICYCLIC MATRICES

The second part of the paper is dedicated to an alternative derivation of Theorem 1 and Corollary 2. Recall the definitions of $r_{\mathcal{E},\mathcal{A}}$, $r_{\mathcal{E}}$ and $r_{\mathcal{A}}$, respectively. Introduce the following matrices, parametrized by a variable λ:

$$M_\lambda := \lambda\mathcal{E} - \mathcal{A}\mathcal{Z}, \tag{16}$$

$$N_\lambda := (\lambda\mathcal{E} - \mathcal{A}\mathcal{Z})\mathcal{Z}^{-1}, \tag{17}$$

$$L_\lambda := \begin{bmatrix} I \\ \mathcal{Z} \end{bmatrix} M_\lambda^{-1} \begin{bmatrix} \lambda I & -I \end{bmatrix}. \tag{18}$$

Using structured perturbation results (Van Dooren and Vermaut, 1997), one can show that appropriate *lower bounds* for the stability radii (7), (8), (9) are given by the following optimization problems

$$r_{\mathcal{E}} \geq \left\{ \sup_{|\lambda|=1} \inf_{D}\ \sigma_{\max}\left(DM_\lambda^{-1}D^{-1}\right)\right\}^{-1} \tag{19}$$

$$r_{\mathcal{A}} \geq \left\{ \sup_{|\lambda|=1} \inf_{\tilde{D}}\ \sigma_{\max}\ \left(\tilde{D}N_\lambda^{-1}\tilde{D}^{-1}\right)\right\}^{-1} \tag{20}$$

$$r_{\mathcal{E},\mathcal{A}} \geq \left\{ \sup_{|\lambda|=1} \inf_{\hat{D}}\ \sigma_{\max}\ \left(\hat{D}L_\lambda\hat{D}^{-1}\right)\right\}^{-1} \tag{21}$$

where $\hat{D} = \mathrm{diag}\{D_1, D_2\}$. For more details, see (Van Dooren and Vermaut, 1997).

We show in this paper that these lower bounds are actually **equalities** in the scalar case ($n = 1$) and that the optimal scaling in (5) can be computed relatively easily. Furthermore, one can also construct the "optimal" perturbations $\Delta\mathcal{E}$, $\Delta\mathcal{A}$ which actually attain the lower bounds in (19)–(21).

The key point in determining the stability radii $r_{\mathcal{E},\mathcal{A}}$, $r_{\mathcal{E}}$ and $r_{\mathcal{A}}$ is to find an optimal scaling of the matrix M_λ introduced by (16), such that the smallest singular value of $D_\alpha\ M_\lambda\ D_\beta^{-1}$ is maximized, and whereby we have special relations between D_α and D_β. We point out that M_λ has bicyclic structure, i.e. it is a linear combination of a block diagonal and a block cyclic matrix. This property will be crucial in proving our extremal properties.

We introduce two classes of diagonal and unitary block scaling matrices:

$$\mathcal{D} = \{D = \mathrm{diag}\{d_1 I_n, \ldots, d_K I_n\}\ \ d_i \in \mathbb{R}_+\} \tag{22}$$

$$\mathcal{U} = \{U = \mathrm{diag}\{U_1, \ldots, U_K\}\ U_i^* U_i = I_n\} \tag{23}$$

which play an important role in this analysis. It is clear that any $V \in \mathcal{U}$ and $D \in \mathcal{D}$ commute with each other. Therefore if $D_\alpha, D_\beta \in \mathcal{D}$ we have

$$\sigma_{\max}\{D_\alpha\ M_\lambda\ D_\beta^{-1}\} = \sigma_{\max}\{D_\alpha\ UM_\lambda\ V^*\ D_\beta^{-1}\}$$

for any $U, V \in \mathcal{U}$. So instead of optimizing the scaling of M_λ one can as well optimize the scaling of $UM_\lambda V^*$. This is exploited in the sequel.
Let us also point out that for any matrix M and scaling $D \in \mathcal{D}$ and $U, V \in \mathcal{U}$, since $\sigma_{\min} \leq |\lambda_{\min}|$ and $\sigma_{\max} \geq |\lambda_{\max}|$, we have the inequalities

$$\begin{aligned} &\sigma_{\min}(DMD^{-1}) \leq |\lambda_{\min}(UMV^*)| \iff \\ &\quad \sigma_{\max}(DM^{-1}D^{-1}) \geq |\lambda_{\max}(VM^{-1}U^*)|. \end{aligned} \tag{24}$$

3.1 *The scalar case* $n = 1$

We consider this case separately since it has a closed form solution. Recall the notation introduced in Section 2 for the special case when $n = 1$.

By using the well-known Floquet transform (see (Sreedhar. and Van Dooren, 1997)) and some elementary algebraic manipulations, one can prove the following important result.

Lemma 3. For the λ-family of matrices M_λ in (16), there exist unitary matrices $U, V \in \mathcal{U}$ such that, for $\hat{\lambda} := a/|a|$, one has

$$V^* M_{\hat{\lambda}}^{-1} U = |M_{\hat{\lambda}}^{-1}|.$$

We are now ready to state the key result of this section. The proof appeals to Perron-Frobenius theory and uses the inequalities (24), as well as Lemma 3.

Lemma 4. Let M_λ be given by (16). Then there exists a unit modulus value $\hat{\lambda}$ such that

$$\sup_{|\lambda|=1} \sup_{U,V\in\mathcal{U}} |\lambda_{\max}(V^*M_\lambda^{-1}U)| = \lambda_{\max}(|M_{\hat{\lambda}}^{-1}|)$$
$$= \sup_{|\lambda|=1} \inf_{D\in\mathcal{D}} \sigma_{\max}(DM_\lambda^{-1}D^{-1}). \quad (25)$$

Moreover, there exist unitary matrices $U, V \in \mathcal{U}$ and a diagonal matrix $D \in \mathcal{D}$ such that actually attain both equalities in (25).

As an immediate consequence, one can find easily an optimal scaling for the λ-matrix of interest showing up in (21), that is, L_λ defined by (18). If we now choose $\hat{V} := \mathrm{diag}\{V, \mathcal{Z}V\mathcal{Z}^T\}$, $\hat{U} := \mathrm{diag}\{\hat{\lambda}^{-1}U, -U\}$, then we have indeed

$$\hat{V}^*L_{\hat{\lambda}}\hat{U} = \begin{bmatrix} I \\ \mathcal{Z} \end{bmatrix} V^*M_{\hat{\lambda}}^{-1}U \begin{bmatrix} I & I \end{bmatrix}$$
$$= \begin{bmatrix} I \\ \mathcal{Z} \end{bmatrix} |M_{\hat{\lambda}}^{-1}| \begin{bmatrix} I & I \end{bmatrix} = |L_{\hat{\lambda}}|$$

which is the required scaling for $L_{\hat{\lambda}}$. From this, the next result trivially follows.

Corollary 5. Let L_λ be given by (18). Then there exists a unit modulus value $\hat{\lambda}$ such that

$$\sup_{|\lambda|=1} \sup_{\hat{U},\hat{V}\in\mathcal{U}} |\lambda_{\max}(\hat{V}^*L_\lambda\hat{U})| = \lambda_{\max}(|L_{\hat{\lambda}}|)$$
$$= \sup_{|\lambda|=1} \inf_{\hat{D}\in\mathcal{D}} \sigma_{\max}(\hat{D}L_\lambda\hat{D}^{-1}). \quad (26)$$

Similarly, one can look at the optimal scaling for $r_{\mathcal{A}}$, which is again a similarity scaling but on the matrix $N_\lambda^{-1} = \mathcal{Z}M_\lambda^{-1}$. Take $W := \mathcal{Z}V\mathcal{Z}^T$ (which is diagonal and unitary) and check that

$$W^*N_{\hat{\lambda}}^{-1}U = \mathcal{Z}V^*M_{\hat{\lambda}}^{-1}U = \mathcal{Z}|M_{\hat{\lambda}}^{-1}| = |N_{\hat{\lambda}}^{-1}|,$$

which is now the optimal scaling for $N_{\hat{\lambda}}^{-1}$.

Let us finally note that the case $a = 0$ is treated separately.

We may now state the main result of the paper.

Theorem 6. Let $D_{\mathcal{E},\mathcal{A}}$, $D_{\mathcal{E}}$ and $D_{\mathcal{A}}$ be the *optimal scalings* for the λ-matrices L_λ, M_λ^{-1} and N_λ^{-1}, respectively; these scalings are *directly* obtained from the Perron vectors of $|L_\lambda|$, $|M_\lambda^{-1}|$ and $|N_\lambda^{-1}|$. Then

$$r_{\mathcal{E},\mathcal{A}} = \left(\lambda_{\max}(|L_{\hat{\lambda}}|)\right)^{-1} \quad (27)$$
$$r_{\mathcal{E}} = \left(\lambda_{\max}(|M_{\hat{\lambda}}^{-1}|)\right)^{-1} \quad (28)$$
$$r_{\mathcal{A}} = \left(\lambda_{\max}(|N_{\hat{\lambda}}^{-1}|)\right)^{-1} \quad (29)$$

Remark 7. One can now check without difficulty that equalities (27) and (14) coincide.

3.2 *The case $n > 1$*

Here we use an algorithm that tries to narrow down the gap between the upper and lower bounds

$$\inf_{D} \sigma_{\max}(DVM_\lambda^{-1}U^*D^{-1})$$
$$\geq \sup_{U,V} |\lambda_{\max}(VDM_\lambda^{-1}D^{-1}U^*)|. \quad (30)$$

and we try to reach a scaled matrix that the gap is zero (and hence satisfies the equal modulus property). Notice that the structure of M is such that every eigenvector has non-zero subvectors (corresponding to the blocks). So the iterative procedure works as follows : compute the dominant left and right eigenvectors of the current matrix M. Scale with D so that subblocks satisfy the equal modulus property. These left and right vectors are not equal but if we "rotate" the subvectors then they become equal. Then iterate on this new M matrix. This appeared to converge quadratically on matrices with real elements.

One should show that at each step the singular value can only decrease and the eigenvalue only increase.

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