THE ANALYTIC CENTER OF LMI'S AND RICCATI EQUATIONS

Y. Genin, Y. Nesterov, P. Van Dooren

CESAME, Université Catholique de Louvain
Bâtiment Euler, Avenue G. Lemaître 4-6
B-1348 Louvain-La-Neuve, Belgium
Fax : +32-10-472180 and e-mail : vdooren@csam.ucl.ac.be

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Abstract

In this paper we derive formulas for constructing the analytic center of the linear matrix inequality defining a positive (para-hermitian) transfer function. The Riccati equations that are usually associated with such positive transfer functions, are related to boundary points of the convex set. In this paper we show that the analytic center is also described by a closely related equation, and we analyze its spectral properties.

1 Introduction

Positive transfer functions play a fundamental role in systems and control theory: they represent e.g. spectral density functions of stochastic processes, show up in spectral factorizations, and are also related to the algebraic Riccati equation. Positive transfer functions also form a convex set, and this property has lead in systems and control theory to the extensive use of convex optimization techniques in this area (especially for so-called linear matrix inequalities [1]). In order to optimize a certain function $F(X)$ over such a convex set, one defines a barrier $B(X)$ that becomes infinite near the boundary of the set, and then finds the minimum of $c \cdot F(X) + B(X)$, $c \geq 0$, as $c \to +\infty$. These minima (which are function of the parameter c) are called the points of the *central path*. The starting point of this path ($c = 0$) is called the *analytic center* of the set. In this paper, we give an explicit equation for the central point of the domain of the linear matrix inequality defining a positive transfer function. We also show how it relates to the solution of the algebraic Riccati equation that typically arises in the spectral factorization of this positive transfer function. We treat the case of positive transfer functions defined on the unit circle (i.e. the discrete-time case) as well as on the imaginary axis (i.e. the continuous-time case).

2 Continuous-time spectral factorizations

Much of the material of this section follows [3] [4]. Let $\Phi\,(s)$ be an $m \times m$ rational para-hermitian transfer function, i.e.

$$\Phi^*(-\bar{s}) = \Phi\,(s),$$

which admits a minimal realization of the form

$$\Phi\,(s) = \left[\begin{array}{cc} B^*(-s\,I_n - A^*)^{-1}, & I_m \end{array}\right] H \left[\begin{array}{c} (s\,I_n - A)^{-1}B \\ I_m \end{array}\right], \tag{1}$$

where

$$H = \left[\begin{array}{cc} Q & S \\ S^* & R \end{array}\right]. \tag{2}$$

Such a realization implies that $\Phi(s)$ is proper (i.e. $\Phi(\infty) = R$ is bounded). It is well known that $\Phi(s)$ does not define H uniquely and that H can be replaced by

$$H(X) = H + \left[\begin{array}{cc} X\,A + A^*X & X\,B \\ B^*X & 0 \end{array}\right] \tag{3}$$

where X is any $n \times n$ hermitian matrix, without affecting $\Phi(s)$. If moreover, one imposes the condition that $\Phi(s)$ must be nonnegative definite on the imaginary axis ($\Re s = 0$) :

$$\Phi(\jmath\,\omega) = \Phi^*(\jmath\,\omega) \geq 0 \qquad \text{for } -\infty \leq \omega \leq \infty, \tag{4}$$

then one shows [4] that there exists a hermitian matrix X such that $H(X) \geq 0$. Then, clearly $H(X)$ can be factorized into

$$H(X) = \left[\begin{array}{c} L^* \\ W^* \end{array}\right] \left[\begin{array}{cc} L, & W \end{array}\right] \tag{5}$$

for appropriate $r \times n$ and $r \times m$ matrices L and W, respectively, and with r the rank of $H(X)$. Therefore, if $G(s)$ is defined as

$$G(s) = L(s\,I_n - A)^{-1}B + W, \tag{6}$$

one has

$$\Phi(s) = G^*(-\bar{s})G(s). \tag{7}$$

Clearly, neither the matrix X nor the factorizations (5) (7) are uniquely defined. An important subset of hermitian matrices X satisfying $H(X) \geq 0$ are those where the rank r is minimal (i.e. equal to the rank of $\Phi(s)$). If $\Phi(\infty) = R$ happens to be nonsingular this subset is easy to characterize. Since R is a submatrix of $H(X) \geq 0$ and it is nonsingular, it follows that $R > 0$. The minimum rank solutions to $H(X) \geq 0$ are then those for which rank $H(X) =$ rank $R = m$ and this is obtained if and only if the Schur complement of R in $H(X)$ equals zero. It turns out that this Schur complement is the celebrated algebraic Riccati equation (ARE)

$$\mathrm{Ricc}(X) \doteq Q + XA + A^*X - (S + XB)R^{-1}(S^* + B^*X) = 0 \tag{8}$$

which yields a spectral factorization of $\Phi(s)$, where $G(s)$ is now square and invertible.

In [4] the solutions of this ARE are described in great detail and it is shown that each hermitian solution X corresponds to a block-triangular decomposition of the so-called Hamiltonian matrix :

$$\mathcal{H} \doteq \begin{bmatrix} A - BR^{-1}S^* & -BR^{-1}B^* \\ -Q + SR^{-1}S^* & -A^* + SR^{-1}B^* \end{bmatrix}.$$

There always exists a similarity transformation of the type

$$\begin{bmatrix} I_n & 0 \\ -X & I_n \end{bmatrix} \mathcal{H} \begin{bmatrix} I_n & 0 \\ X & I_n \end{bmatrix} = \begin{bmatrix} A - BF & -BR^{-1}B^* \\ 0 & -A^* + F^*B^* \end{bmatrix}, \tag{9}$$

where

$$F \doteq R^{-1}(S^* + B^*X).$$

This also implies that $\mathcal{X} \doteq \mathrm{Im} \begin{bmatrix} I_n \\ X \end{bmatrix}$ is an invariant subspace of the Hamiltonian matrix whose associated spectrum are the eigenvalues of the *closed loop* matrix

$$A_F \doteq A - BF.$$

Each solution X of the ARE therefore corresponds to a selection of n eigenvalues of the $2n \times 2n$ Hamiltonian matrix. It is well known that these eigenvalue are *symmetric* with respect to the imaginary axis, and that none of them lies *on* the imaginary axis under the assumption $R > 0$. Let us define by X_- and X_+ the ARE solutions such that the spectrum of A_F lies in the left half plane and right half plane, respectively. Then it is proven in [4] that all solutions X to the linear matrix inequality satisfy

$$X_- \geq X \geq X_+.$$

Notice that this implies boundedness of the domain of $H(X) \geq 0$. The proof of this result relies on the ARE and does not go through anymore when R is singular. For a discussion on generalized Riccati equations defined for singular R, we refer to [6].

3 Discrete-time spectral factorizations

The disctrete-time analogue to the above section is very similar and hence only briefly developed here (see also [3] [6]). Let $\Phi(z)$ be an $m \times m$ rational para-hermitian transfer function, i.e.

$$\Phi^*(\bar{z}^{-1}) = \Phi(z),$$

which admits a minimal realization of the form

$$\Phi(z) = \begin{bmatrix} B^*(z^{-1}I_n - A^*)^{-1}, & I_m \end{bmatrix} H \begin{bmatrix} (zI_n - A)^{-1}B \\ I_m \end{bmatrix}, \tag{10}$$

where

$$H = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}. \tag{11}$$

Again, H is not defined uniquely from $\Phi(z)$ and can be replaced by

$$H(X) = H + \begin{bmatrix} A^*XA - X & A^*XB \\ B^*XA & B^*XB \end{bmatrix} \tag{12}$$

where X is any $n \times n$ hermitian matrix, without affecting $\Phi(z)$. If moreover, one imposes the condition that $\Phi(z)$ must be nonnegative definite on the unit circle ($|z| = 1$) :

$$\Phi(e^{\jmath\omega}) = \Phi^*(e^{\jmath\omega}) \geq 0 \qquad \text{for } 0 \leq \omega \leq 2\pi, \tag{13}$$

then one shows [4] that there exists a hermitian matrix X such that $H(X) \geq 0$. Then $H(X)$ can be factorized as

$$H(X) = \begin{bmatrix} L^* \\ W^* \end{bmatrix} \begin{bmatrix} L, & W \end{bmatrix} \tag{14}$$

for appropriate $r \times n$ and $r \times m$ matrices L and W, respectively, and with r the rank of $H(X)$. Therefore, if $G(z)$ is defined as

$$G(z) = L(zI_n - A)^{-1}B + W, \tag{15}$$

one has

$$\Phi(z) = G^*(\bar{z}^{-1})G(z). \tag{16}$$

Again the Riccati equation $\mathrm{Ricc}(X)$, defined as :

$$Q + A^*XA - X - (S + A^*XB)(R + B^*XB)^{-1}(S^* + B^*XA) = 0 \tag{17}$$

plays a cucial role since it yields a spectral factorization of $\Phi(z)$, with $G(z)$ square and invertible. The solutions can be obtained this time from a block-triangular decomposition of the so-called Symplectic matrix :

$$\mathcal{S} \doteq \begin{bmatrix} I & BR^{-1}B^* \\ 0 & A^* - SR^{-1}B^* \end{bmatrix}^{-1} \begin{bmatrix} A - BR^{-1}S^* & 0 \\ -Q + SR^{-1}S^* & I \end{bmatrix}.$$

There always exists a similarity transformation of the type

$$\begin{bmatrix} I_n & 0 \\ -X & I_n \end{bmatrix} \mathcal{S} \begin{bmatrix} I_n & 0 \\ X & I_n \end{bmatrix}$$

$$= \begin{bmatrix} I & B(R+B^*XB)^{-1}B^* \\ 0 & A^*-F^*B^* \end{bmatrix}^{-1} \begin{bmatrix} A-BF & 0 \\ 0 & I \end{bmatrix}, \tag{18}$$

where

$$F \doteq (R+B^*XB)^{-1}(S^*+B^*XA).$$

This also implies that $\mathcal{X} \doteq \mathrm{Im} \begin{bmatrix} I_n \\ X \end{bmatrix}$ is an invariant subspace of the Symplectic matrix whose associated spectrum are the eigenvalues of the *closed loop* matrix

$$A_F \doteq A - BF.$$

Notice that this could have been retrieved from the bilinear transform $s = (z-1)/(z+1)$ which reduces every continuous-time system $\{A, B, Q, R, S\}$ to a discrete-time system $\{\hat{A}, \hat{B}, \hat{Q}, \hat{R}, \hat{S}\}$ where

$$\begin{aligned} \hat{A} &\doteq (I-A)^{-1}(I+A) \qquad (19) \\ \hat{B} &\doteq \sqrt{2}(I-A)^{-1}B \\ \hat{H} &\doteq T^*HT, \quad T \doteq \begin{bmatrix} \sqrt{2}(I-A)^{-1} & (I-A)^{-1}B \\ 0 & I \end{bmatrix} \end{aligned}$$

This kind of transformation preserves stability, but requires A to have no eigenvalues at 1. Notice that using this transformation one can also relate the domains of the continuous-time and discrete-time matrix inequalities. In order to do this, we first rewrite them as follows :

$$H_c \doteq \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} + \begin{bmatrix} A^* & I \\ B^* & 0 \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$$

and

$$H_d \doteq \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} + \begin{bmatrix} A^* & I \\ B^* & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} = \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} \frac{X}{2} & 0 \\ 0 & -\frac{X}{2} \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$

we can also rewrite H_c as follows

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} + \begin{bmatrix} A+I & B \\ A-I & B \end{bmatrix}^* \begin{bmatrix} \frac{X}{2} & 0 \\ 0 & -\frac{X}{2} \end{bmatrix} \begin{bmatrix} A+I & B \\ A-I & B \end{bmatrix}.$$

Apply now the congruence transformation T defined in (19) then we have $T^*H_cT = \hat{H}_d$ with

$$\hat{H}_d \doteq \begin{bmatrix} \hat{Q} & \hat{S} \\ \hat{S}^* & \hat{R} \end{bmatrix} + \begin{bmatrix} \hat{A}^* & I \\ \hat{B}^* & 0 \end{bmatrix} \begin{bmatrix} \hat{X} & 0 \\ 0 & -\hat{X} \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \end{bmatrix},$$

with $\hat{A}, \hat{B}, \hat{Q}, \hat{S}$ and $\hat{R}$ defined as in (19), and $\hat{X} = X$. So the bilinear transformation also preserves the solution of the Riccati equation as well as domain of the linear matrix inequality.

The development of this section requires $R + B^*XB$ to be nonsingular, but we refer to [6] for a generalized Riccati approach when this matrix is singular.

4 Boundedness of the domain of an LMI

As mentioned in the preceding section, it is of importance for numerical reasons to analyze the domain (or solution set) of the linear matrix inequality $H(X) \geq 0$ and to identify conditions which guarantee its boundedness. It turns out that this is precisely the case if the pair (A, B) is controllable. In order to prove this, we introduce the notation $H_0(X) = H(X) - H$ for the homogeneous part of $H(X)$.

Lemma
The continuous-time system $\dot{x}(t) = Ax(t) + Bu(t)$ is controllable if and only if

$$H_0(X) = \begin{bmatrix} A^*X + XA & XB \\ B^*X & 0 \end{bmatrix} \geq 0, X = X^* \tag{20}$$

implies $X = 0$.

Proof :
To prove the *only if* part, define $z^*(t) = [x^*(t), u^*(t)]$ so that $z^*(t)H_0(X)z(t) \geq 0$ is found to be equivalent to

$$\frac{d}{dt}(x(t)^*X\,x(t)) \geq 0,$$

a relation which can be integrated over an interval $[0, T]$ so as to yield the inequality

$$x(T)^*X\,x(T) - x(0)^*X\,x(0) \geq 0.$$

Since $x(0)$ ad $x(T)$ can be chosen arbitrarily if (A, B) is controllable, $H_0(X)$ cannot have any other solution than $X = X^* = 0$.

The *if* part is proved by contradiction. If $X = 0$ is the only solution of $H_0(X) \geq 0$ and if the pair (A, B) was not controllable, then an isometry U, i.e. $U^*U = I_k$, could be found whose columns would span the uncontrollable system subspace. Therefore, one would have $U^*B = 0$ and $U^*A = A_uU^*$ with $A_u \doteq U^*A\,U$ Setting $X = U\,X_uU^*$ where X_u is some $k \times k$ matrix, one would then obtain

$$H_0(X) = \begin{bmatrix} U(A_u^*X_u + X_u\,A_u)U^* & 0 \\ 0 & 0 \end{bmatrix}.$$

As a nonzero solution X_u of the inequality $(A_u^*X_u + X_u\,A_u) \geq 0$ can always be easily constructed, there would exist a non zero solution $X = U\,X_uU^*$, contradicting our assumption.

■

The discrete time version of this lemma is very similar :

Lemma
The discrete-time system $x(k+1) = Ax(k) + Bu(k)$ is controllable if and only if

$$H_0(X) = \begin{bmatrix} A^*X\,A - X & A^*X\,B \\ B^*X\,A & B^*X\,B \end{bmatrix} \geq 0, X = X^* \tag{21}$$

implies $X = 0$.

Proof :

To prove the *only if* part, define $z^*(k) = [x^*(k),\, u^*(k)]$ so that $z^*(k)H_0(X)z(k) \geq 0$ is found to be equivalent to

$$x(k+1)^* X\, x(k+1) \geq x(k)^* X\, x(k),$$

a relation which can be repeated over an interval $[0, K-1]$ so as to yield the inequality

$$x(K)^* X\, x(K) - x(0)^* X\, x(0) \geq 0.$$

Since $x(0)$ ad $x(K)$ can be chosen arbitrarily if (A, B) is controllable, $H_0(X)$ cannot have any other solution than $X = X^* = 0$.

The *if* part is proved by contradiction. If $X = 0$ is the only solution of $H_0(X) \geq 0$ and if the pair (A, B) was not controllable, then an isometry U, i.e. $U^*U = I_k$, could be found whose columns would span the uncontrollable system subspace. Therefore, one would have $U^*B = 0$ and $U^*A = A_u U^*$ with $A_u \doteq U^*A\,U$ Setting $X = U\, X_u U^*$ where X_u is some $k \times k$ matrix, one would then obtain

$$H_0(X) = \begin{bmatrix} U(A_u^* X_u\, A_u - X_u)U^* & 0 \\ 0 & 0 \end{bmatrix}.$$

As a nonzero solution X_u of the inequality $(A_u^* X_u\, A_u - X_u) \geq 0$ can always be easily constructed, there would exist a non zero solution $X = U\, X_u U^*$, contradicting our assumption.

■

On the basis of this lemma, let us show (by contradiction) that $H + H_0(X) \geq 0$ has a bounded solution set for X. Indeed, if the (convex) set of solutions X was supposed to be unbounded, it would contain a "ray" $t \cdot X$ for $t \in [t_0, \infty]$ such that $H + t \cdot H_0(X) \geq 0$, whence $H_0(X) \geq 0$ for $t \to \infty$. Therefore, there would exist a nonzero hermitian solution X to the problem.

5 Analytic center of the convex set

We treat here both the continuous-time and discrete-time cases. Suppose that there exists $\bar{X} = \bar{X}^T$ with $H(\bar{X}) > 0$. Since the domain of $H(X) \geq 0$ is bounded, we can define its central point as follows. We choose a barrier function

$$B(X) \doteq -\ln\det H(X), \tag{22}$$

and define the analytic center of the domain of $H(X) \geq 0$ as the minimizer of this barrier. Such a point is well defined (see [2]). Let us find its characteristic equation.

The *gradient* of the matrix function $B(H)$ equals

$$\partial B(X)/\partial X = -H(X)^{-1}. \tag{23}$$

With the notation $< .,. >$ for the Frobenius scalar product of matrices, it appears that X will be an extremal point of the barrier if an only if

$$< -\partial B(X)/\partial X, \Delta H(X)[Y] >= 0, \quad \forall\, Y = Y^*, \tag{24}$$

where $\Delta H(X)[Y]$ is the incremental step in direction Y.

For continuous-time systems, the increment of $H(X)$ corresponding to a hermitian incremental direction Y of X is found to be

$$\Delta H(X)[Y] = \begin{bmatrix} A^*Y + YA & YB \\ B^*Y & 0 \end{bmatrix}, \quad \forall\, Y = Y^*. \tag{25}$$

The equation for the extremal point then becomes

$$< H(X)^{-1}, \begin{bmatrix} A^*Y + YA & YB \\ B^*Y & 0 \end{bmatrix} >= 0, \quad \forall\, Y = Y^*. \tag{26}$$

Defining

$$F \doteq R^{-1}(B^*X + S^*), \quad P \doteq Q + A^*X + XA - F^*RF,$$

then $H(X)$ factorizes as

$$H(X) = \begin{bmatrix} I & F^* \\ 0 & I \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}, \tag{27}$$

and (26) then becomes equivalent to

$$< \begin{bmatrix} P^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix}, \begin{bmatrix} I & -F^* \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} A^*Y + YA & YB \\ B^*Y & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} >= 0, \quad \forall\, Y = Y^*, \tag{28}$$

or also

$$< P^{-1}, A^*Y + YA - F^*B^*Y - YBF >= 0, \quad \forall\, Y = Y^*. \tag{29}$$

This is equivalent to

$$P^{-1}A_F^* + A_F P^{-1} = 0 \tag{30}$$

where we define

$$A_F = A - BF.$$

We point out now that P is nothing but the Riccati equation $\mathrm{Ricc}(X)$ defined earlier, and that A_F is the corresponding closed loop matrix. For the classical Riccati solutions we have $P = \mathrm{Ricc}(X) = 0$ and the corresponding closed loop matrix is well-known to have its eigenvalues equal to a subset of the corresponding Hamiltonian (which subset depends on the chosen Riccati solution).

But for an interior point of the domain of $H(X) > 0$ it is obvious that we also have $P = \mathrm{Ricc}(X) > 0$, and hence P has a Hermitian square root T satisfying $P = T^2$. Multiplying (30) on both sides with the invertible matrix T we obtain

$$T^{-1}A_F^*T + TA_FT^{-1} = 0.$$

Hence TA_FT^{-1} is skew Hermitian and has all its eigenvalues on the imaginary axis, and so does A_F. Therefore, the closed loop matrix A_F of the analytic center has a spectrum that is also "central" in a certain sense.

For discrete-time systems, the increment of $H(X)$ equals

$$\Delta H(X)[Y] = \begin{bmatrix} A^*YA - Y & A^*YB \\ B^*YA & B^*YB \end{bmatrix}, \quad \forall\, Y = Y^*. \tag{31}$$

Defining

$$F \doteq (R + B^*XB)^{-1}(B^*XA + S^*),$$

$$P \doteq Q + A^*XA - X - F^*(R + B^*XB)F,$$

$$A_F \doteq A - BF,$$

then $H(X)$ factorizes as

$$H(X) = \begin{bmatrix} I & F^* \\ 0 & I \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & R + B^*XB \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}, \tag{32}$$

and the equation for the extremal point becomes

$$< \begin{bmatrix} P^{-1} & 0 \\ 0 & (R + B^*XB)^{-1} \end{bmatrix}, \begin{bmatrix} I & -F^* \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} A^*YA - Y & A^*YB \\ B^*YA & B^*YB \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} >= 0, \quad \forall\, Y = Y^*, \tag{33}$$

or also $\forall\, Y = Y^*$,

$$< P^{-1}, A_F^*YA_F - Y > + < (R+B^*XB)^{-1}, B^*YB >= 0. \tag{34}$$

This is equivalent to

$$A_F P^{-1} A_F^* - P^{-1} + B(R + B^*XB)^{-1}B = 0, \tag{35}$$

which is not a homogenous Lyapunov equation anymore. Since (A, B) is controllable (by assumption), so is (A_F, B) and it follows then from (35) that the eigenvalues of A_F are now strictly inside the unit circle. This is clearly different from the continuous-time case where the spectrum of A_F was on the boundary of the stability region, an in some sense "central".

Notice that we could have transformed the solution of the corresponding continuous-time problem via the bilinear transform, which would then yield a feedback that puts all eigenvalues on the unit circle, but the feedback would of course be different.

6 Concluding remarks

In this paper we described the analytic center of the convex domain of

$$H = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} + \begin{bmatrix} A^*X + XA & XB \\ B^*X & 0 \end{bmatrix} \geq 0, X = X^*. \tag{36}$$

and of

$$H = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} + \begin{bmatrix} A^*XA - X & A^*XB \\ B^*XA & B^*XB \end{bmatrix} \geq 0, X = X^*. \tag{37}$$

corresponding to a continuous-time and discrete-time positive transfer function, respectively. We discussed some of its properties as compared to the classical Riccati solutions. This point is of more practical importance when optimizing over a *class* of positive functions, say e.g. the class of functions of the above type where now Q R and S are arbitrary. A typical example are the positive pseudo-polynomials which can be described like this [5]. In such a case, one can use convex optimization techniques and the central path plays a crucial role in those optimization problems.

Another point that might be important is that the analytic center is obviously always an interior point (at least when the domain has a non-empty interior). For this reason, it is also less sensitive to perturbations on A, B, Q, R and S than any boundary point (like those described by the ARE). Such issues still need to be investigated, though.

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