Stability bounds for higher order linear dynamical systems

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Abstract

This paper derives analytic expressions for the real stability radius of polynomial matrices with respect to an arbitrary region in the complex plane. We are also discussing numerical issues for computing these radii for different perturbation structures, with application to robust stability of Hurwitz and Schur polynomial matrices.

1 Introduction

A fundamental problem in robustness analysis is to determine the ability of a system matrix to maintain its "stability" under a certain class of perturbations. Since the entries of such matrices frequently depend on some physical parameters, it seems natural to consider *real perturbations*. In many applications it is more convenient to deal with the characteristic polynomial of the (closed-loop) system matrix, as, for instance, in the SIMO (or MISO) cases (see, for instance, [1]). The problem of the stability robustness of polynomials with affine coefficient perturbations has been considered by [5] and solved for arbitrary norms on $\mathbb{R}^n$ spaces. It is worthwhile to be mentioned that robustness to parametric perturbations has been an important topic not only in control theory [12], but also in linear algebra and numerical analysis [9], for over two decades.
In the present paper we consider the stability robustness problem of time–invariant linear systems described by *higher order* dynamical equations, the continuous time case

$$P_k \frac{\mathrm{d^k x(t)}}{\mathrm{dt^k}} + \cdots + P_1 \frac{\mathrm{dx(t)}}{\mathrm{dt}} + P_0 x(t) = 0, \ \ t \in \mathbb{R}_+ \quad (1)$$

as well as the discrete time case,

$$P_k x(t+k) + \cdots + P_1 x(t+1) + P_0 x(t) = 0 \ \ t \in \mathbb{N} \quad (2)$$

Associate with (1) or (2) the matrix polynomial

$$P(\lambda) = P_k \lambda^k + \cdots + P_1 \lambda + P_0 \quad (3)$$

The spectrum of P is defined as

$$\Lambda(P) = \{\lambda \in \mathbb{C} : \det P(\lambda) = 0\}$$

and the elements of $\Lambda(P)$ are also known as the *eigenvalues* or *zeros* of P. Let $\mathbb{C}_g \subset \mathbb{C}$ be an open and connected subset of $\mathbb{C}$. The matrix polynomial $P(\lambda)$ is called $\mathbb{C}_g$– stable (or, simply, stable) iff $\Lambda(P) \subset \mathbb{C}_g$. The typical regions considered for $\mathbb{C}_g$ are the open left half–plane $\mathbb{C}^- = \{s \in \mathbb{C}; \mathrm{Re\, s} < 0\}$ and the open unit disk $\mathbb{C}^1 = \{s \in \mathbb{C}; |s| < 1\}$.
The differential system in (1) is asymptotically stable if and only if $\Lambda(P)$ is a subset of $\mathbb{C}^-$ in which case P is called a *Hurwitz matrix polynomial*. Similarly the discrete time system (2) is asymptotically stable if and only if $\Lambda(P)$ lies inside $\mathbb{C}^1$ and then P is called a *Schur matrix polynomial*.

As we shall see, a natural stability robustness measure for our systems (1) and (2) is given by the *norm* of the smallest perturbation

$$\delta P(\lambda) = \delta P_0 + \delta P_1 \lambda + \cdots + \delta P_k \lambda^k, \ \ \delta P_i \in \mathbb{R}^{n \times n}, \ i = \overline{0,k} \quad (4)$$

needed to "destabilize" $P(\lambda)$, and hence causing at least one zero of $P(\lambda) + \delta P(\lambda)$ to leave the stability region $\mathbb{C}_g$. Denote by $\mathbb{C}_b$ the complement of $\mathbb{C}_g$. The perturbations will be measured via the norm of a constant matrix Δ depending on the matrix coefficients of $\delta P(\lambda)$. Throughout this paper we consider mainly Euclidean norms, that is, $\|\Delta\| = \|\Delta\|_2 = \sigma_1(\Delta)$, where $\sigma_i(\cdot)$ denotes the i–th singular value. A detailed problem formulation will be given in Section 2. As we shall see, the structure of Δ strongly influences the computation of the different stability robustness measures.

The paper is organized as follows. The next section is devoted to the problem formulation and some known results concerning the *real perturbation values* of complex matrices. We also introduce the *real stability radius* for matrix polynomials, as a robustness measure. The concept of stability radius for polynomial matrices has been first investigated in the pioneering work of Pappas and Hinrichsen [8], but in the complex case only and for monic polynomial matrices in the non–monic case. In Section 3 we derive analytic expressions for real stability radii of polynomial matrices. Several open problems are also stressed. More elaborate formulas are derived in Section 4 which prove to be useful when discussing several computational aspects. Consequently, we point out in Section 5 that real stability radii can be efficiently computed for Hurwitz and Schur polynomial matrices. Some additional comments on computational complexity conclude this section. Future research directions along with some final remarks complete the paper.

2 Problem formulation

Assume that the dynamical systems described by (1) and (2) are asymptotically stable (or $\mathbb{C}_g$–stable, for a given $\mathbb{C}_g$). A fundamental question in robust analysis is to which extent these systems can tolerate perturbations of the matrix coefficients without losing their stability, *i.e.* keeping the spectrum $\Lambda(P)$ in the stability region $\mathbb{C}_g$. The answer to this question is given by the *distance to instability* of the associated (stable) matrix polynomial P.

Let us assume that $P(\lambda)$ given by (3) is $\mathbb{C}_g$–stable, regular (*i.e.* $\det P(\lambda) \not\equiv 0$) and that P_k is *nonsingular*. The *real stability radius* of such polynomial matrices is the norm of the smallest perturbation

$$\delta P(\lambda) = \delta P_0 + \delta P_1 \lambda + \cdots + \delta P_k \lambda^k, \quad \delta P_i \in \mathbb{R}^{n\times n}, \; i = \overline{0,k} \tag{5}$$

needed to "destabilize" $P(\lambda)$, and hence causing at least one zero of $P(\lambda) + \delta P(\lambda)$ to leave the stability region $\mathbb{C}_g$.

If we measure the perturbations via the norm of a constant matrix depending on the matrix coefficients of $\delta P(\lambda)$

$$\|\Delta\| = G(\delta P_0, \dots, \delta P_k) \tag{6}$$

then the *real stability radius* of $P(\lambda)$ with respect to $\mathbb{C}_g$ can be expressed

$$r_{\mathbb{R}}(P;\mathbb{C}_g) = \inf_{\Delta}\{\|\Delta\| : \Lambda(P) \cap \mathbb{C}_b \neq \emptyset\} \tag{7}$$

Let

$$\det P(\lambda) = a_{nk}\lambda^{nk} + \cdots + a_1\lambda + a_0, \tag{8}$$

where $a_{nk} = \det P_k$. If $\det P_k \neq 0$ then the zeros of $P(\lambda)$ move continuously with the perturbation of the coefficients δP_i. Then the above formula (6) becomes

$$r_{\mathbb{R}}(P;\mathbb{C}_g) = \inf_{\Delta}\{\|\Delta\| : \exists \lambda \in \partial\mathbb{C}_g \;\; s.t. \;\; \det(P(\lambda) + \delta P(\lambda)) = 0\} \tag{9}$$

For any polynomial matrix P and for every $\lambda_0 \in \mathbb{C}$ introduce

$$\nu_{\mathbb{R}}(P, \lambda_0) := \inf_{\Delta}\{\|\Delta\| : \det(P(\lambda_0) + \delta P(\lambda_0)) = 0\}, \tag{10}$$

where $\nu_{\mathbb{R}}(P, \lambda_0)$ is the norm of the smallest perturbation needed to cause at least one eigenvalue of P to be precisely λ_0. Hence

$$r_{\mathbb{R}}(P;\mathbb{C}_g) = \inf_{\lambda \in \partial\mathbb{C}_g} \nu_{\mathbb{R}}(P, \lambda) \tag{11}$$

The above equality (11) states that the computation of $\nu_{\mathbb{R}}(P, \lambda)$ becomes a key issue in evaluating the real stability radius of P.

If P_k is singular, then we distinguish two cases:

a) $\det P(\lambda) \not\equiv 0$, that is, P is regular. In this case the polynomial in (8) has at least one zero at ∞ since $a_{nk} = \det P_k = 0$. Arbitrarily small perturbations can place this root anywhere on a (sufficiently) large circle in $\mathbb{C}$ and can have positive or negative real part.

Consequently, the "singular" case when $\det P_k = 0$ becomes irrelevant to stability radii problems, since the zeros at infinity can be destabilized by arbitrarily small perturbations. This means that in such situations the distance to *instability* is zero. The above argument also shows that the distance to *instability* is always lower or equal to the distance to *singularity*.

b) $\det P(\lambda) \equiv 0$ which implies that there exist arbitrarily small perturbations that place zeros anywhere in the complex plane (see [11]) and hence make it unstable.

As a conclusion we may say that polynomial matrices that have a leading coefficient that is singular and that is allowed to be perturbed has distance zero from instability, since arbitrarily small perturbations will make it unstable. In other words, if P_k is singular then $r_{\mathbb{R}}(P;\mathbb{C}_g) = 0$. For the same reason, when $\det P_k \neq 0$ one should not worry about these perturbations δP_k that will make $P_n + \delta P_n$ singular: for essentially the same perturbation, the polynomial matrix will be also unstable.

The above argument shows that the only relevant case in robust stability analysis is when P_k is nonsingular and $P(\lambda)$ is regular. Moreover, in such cases, the zeros of $P(\lambda)$ move continuously with the perturbation.

However, let us finally notice that by restricting to perturbations that *do not change* infinite zeros, then a different analysis has to be performed. This kind of problem will be addressed separately and does not make the object of the present paper.

We shall consider basically the following perturbation structures:

$$\Delta_1 = \begin{bmatrix} \delta P_0 & \cdots & \delta P_k \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} \delta P_0 \\ \vdots \\ \delta P_k \end{bmatrix}$$

$$\Delta_3 = \begin{bmatrix} \delta P_0 & & \\ & \ddots & \\ & & \delta P_k \end{bmatrix} \tag{12}$$

The polynomial matrix perturbation $\delta P(\lambda)$ can be expressed as:

$$\delta P(\lambda) = \Delta_1 \begin{bmatrix} I \\ \lambda I \\ \vdots \\ \lambda^k I \end{bmatrix} = \begin{bmatrix} I & \lambda I & \cdots & \lambda^k I \end{bmatrix} \Delta_2$$

$$= \begin{bmatrix} I & \xi_1^{-1} I & \cdots & \xi_k^{-1} I \end{bmatrix} \Delta_3 \begin{bmatrix} I \\ \xi_1 \lambda I \\ \vdots \\ \xi_k \lambda^k I \end{bmatrix} \tag{13}$$

where $\xi_i \in \mathbb{C}$ are arbitrary, $\xi_i \neq 0$. Let $\xi = [\xi_1 \cdots \xi_k]^T$.

For any λ for which $P(\lambda)$ is is invertible introduce

$$M_1(\lambda) := \begin{bmatrix} I \\ \lambda I \\ \vdots \\ \lambda^k I \end{bmatrix} P^{-1}(\lambda), \quad M_3(\lambda, \xi) :=$$

$$= \begin{bmatrix} I \\ \xi_1 \lambda I \\ \vdots \\ \xi_k \lambda^k I \end{bmatrix} P^{-1}(\lambda) \begin{bmatrix} I & \xi_1^{-1} I & \cdots & \xi_k^{-1} I \end{bmatrix} \text{ and}$$

$$M_2(\lambda) := P^{-1}(\lambda) \begin{bmatrix} I & \lambda I & \cdots & \lambda^k I \end{bmatrix}. \tag{14}$$

By using the well–known equality

$$\det(I + AB) = \det(I + BA),$$

one can deduce from (13) and (14) that for $i = 1, 2, 3$

$$\det(P(\lambda) + \delta P(\lambda)) = 0 \Leftrightarrow$$

$$\det(I+\delta P(\lambda)P^{-1}(\lambda)) = 0 \Leftrightarrow \det(I+P^{-1}(\lambda)\delta P(\lambda)) = 0$$

$$\Leftrightarrow \det(I + \Delta_i M_i(\lambda)) = 0. \tag{15}$$

Let us check for instance (15) for $i = 2$. One has

$$\det(P(\lambda) + \delta P(\lambda)) = 0 \Leftrightarrow \det(I + P^{-1}(\lambda)\delta P(\lambda)) = 0$$

$$\Leftrightarrow \det(I + P^{-1}(\lambda) \begin{bmatrix} I & \lambda I & \cdots & \lambda^k I \end{bmatrix} \Delta_2) = 0$$

$$\Leftrightarrow \det(I + M_2(\lambda)\Delta_2) = 0,$$

which is the same as (15) for $i = 2$. Hence an important issue in the computation of the real stability radius is to solve the following linear algebra problem: Given a *complex* matrix $M \in \mathbb{C}^{l\times m}$, determine

$$\inf_{\Delta\in\mathbb{R}^{m\times l}} \{\|\Delta\|_2 \ : \ \det(I - \Delta M) = 0\}.$$

This problem is solved by the next theorem.
Define the largest *real perturbation value* of M by

$$\mu_{\mathbb{R}}(M) := \Big[\inf_{\Delta\in\mathbb{R}^{m\times l}} \{\|\Delta\|_2 \ : \ \det(I - \Delta M) = 0\}\Big]^{-1} \tag{16}$$

Notice that $\mu_{\mathbb{R}}(M) = 0$ if and only if there is no Δ such that $\det(I - \Delta M) = 0$.
For any complex matrix (vector, scalar) $M \in \mathbb{C}^{l\times m}$, let $M_x \in \mathbb{R}^{l\times m}$, $M_y \in \mathbb{R}^{l\times m}$ denote its real and imaginary part, respectively, that is $M = M_x + jM_y$. Associate to M the $2l \times 2m$ *real* matrix depending on the real parameter $\gamma \in (0, 1]$

$$N_M(\gamma) := \begin{bmatrix} M_x & -\gamma M_y \\ \gamma^{-1} M_y & M_x \end{bmatrix}. \tag{17}$$

Then the following result holds.

Theorem 1 *[6] Let $M \in \mathbb{C}^{l\times m}$. Then*

$$\mu_{\mathbb{R}}(M) = \inf_{\gamma\in(0,1]} \sigma_2\Big(N_M(\gamma)\Big) \tag{18}$$

The function to be minimized on the right hand-side of (18) is a unimodal function on $(0, 1]$.
Furthermore, assume that the optimum in (18) is attained for some $\gamma_{opt} \in (0, 1]$. Then, the "optimal" perturbation, i.e. the minimum norm real matrix Δ such that $\det(I-\Delta M) = 0$ is given by

$$\Delta = \sigma_{2,opt}^{-1} \begin{bmatrix} v_x & v_y \end{bmatrix} \begin{bmatrix} u_x & u_y \end{bmatrix}^{\dagger} \tag{19}$$

where $u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$ and $v = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ are a pair of left and right singular vectors of the matrix $N_M(\gamma_{opt})$ corresponding to $\sigma_{2,opt}$. Moreover, $u_x^T u_x = v_x^T v_x$, $u_x^T u_y = v_x^T v_y$ and $u_y^T u_y = v_y^T v_y$.

■

We have used $A^{\dagger}$ to denote the Moore-Penrose (generalized) inverse of the matrix A.

Remark 2

1. *The minimization in (18) is quite easy due to the fact that $\sigma_2\Big(N_M(\gamma)\Big)$ is unimodal on $(0, 1]$: any local minimum is a global one.*

2. *The map $M \mapsto \mu_{\mathbb{R}}(M)$ is upper semi-continuous. The only discontinuity points are at real M ($M_y = 0$).*

■

Now the following preliminary result holds.

Lemma 3 *The real stability radius (9) of the matrix polynomial $P(\lambda)$ with respect to the perturbation matrix Δ_i, $i = 1, 2$, is given by*

$$r_{\mathbb{R}}(P, \mathbb{C}_g; \Delta_i) = \Big[\sup_{\lambda\in\partial\mathbb{C}_g} \mu_{\mathbb{R}}(M_i(\lambda))\Big]^{-1}, \ \ i = 1, 2 \tag{20}$$

Proof. Since $P(\lambda)$ is $\mathbb{C}_g$-stable, $P(\lambda)$ is invertible for any $\lambda \in \partial\mathbb{C}_g$, so $M_i(\lambda)$, $i = 1, 2$ is well defined. For the perturbation structures Δ_i, $i = 1, 2$ relation (10) reads

$$\nu_{\mathbb{R}}(P, \lambda; \Delta_i) = \inf_{\Delta_i}\{\|\Delta_i\| : \det(P(\lambda)+\delta P(\lambda)) = 0\} \tag{21}$$

Looking at the equivalences in (15), the above equality (21) becomes

$$\nu_{\mathbb{R}}(P, \lambda; \Delta_i) = \inf_{\Delta_i}\{\|\Delta_i\| : \det(I + \Delta_i M_i(\lambda)) = 0\}$$

$$= [\mu_{\mathbb{R}}(M_i(\lambda))]^{-1}, \ \ i = 1, 2, \tag{22}$$

taking also into account that $\mu_{\mathbb{R}}(M) = \mu_{\mathbb{R}}(-M)$, as definition (16) shows. The conclusion follows now immediately from (22) and (11). ■

One can express $\mu_{\mathbb{R}}(M_1(\lambda))$ and $\mu_{\mathbb{R}}(M_2(\lambda))$ via Theorem 1. The third problem $\mu_{\mathbb{R}}(M_3(\lambda, \xi))$ is a constrained problem which is much more difficult to solve due to the block diagonal structure of Δ_3. We shall give in this case some lower and upper bounds, but there is no closed formula. It is only conjectured that the degree of freedom offered by ξ_i, $i = \overline{1, k}$, might lead to such a formula, when considering Euclidean norms.

Conjecture 4 *Let $\xi = [\xi_1 \dots \xi_k]^T$, where $\xi_i \neq 0$, $i = \overline{1,k}$. Then for any λ for which $P(\lambda)$ is invertible*

$$\inf_{\Delta_3}\{\|\Delta_3\|_2 \ : \ \det(I + \Delta_3 M_3(\lambda, \xi)) = 0\}$$

$$= \min_{\xi} \mu_{\mathrm{IR}}^{-1}\big(M_3(\lambda, \xi)\big) = \mu_{\mathrm{IR}}^{-1}\big(M_3(\lambda, \xi_{opt})\big) \tag{23}$$

where ξ_{opt} is the optimal scaling *attaining the minimum. Notice that the optimal scaling depends on λ as well.*
Furthermore, the real stability radius (9) of the matrix polynomial $P(\lambda)$ with respect to the perturbation matrix Δ_3 is given by

$$r_{\mathrm{IR}}(P, \mathbb{C}_g; \Delta_3) = \Big[\sup_{\lambda \in \partial\mathbb{C}_g} \mu_{\mathrm{IR}}\big(M_3(\lambda, \xi_\lambda)\big)\Big]^{-1} \tag{24}$$

where ξ_λ is the optimal scaling *obtained for given $\lambda \in \partial\mathbb{C}_g$.*

■

Further we derive some upper and lower bounds for the real stability radius of $P(\lambda)$ in the Δ_3 case. These bounds are expressed in terms of the real stability radii determined in Lemma 3, by using the available structure and by choosing an appropriate vector ξ.
We start by stating without proof two additional results, which hold for arbitrarily p–Holder norms, $\|\Delta\| = \sup_{x\neq 0} \frac{\|\Delta x\|_p}{\|x\|_p}$.

Proposition 5 *Let $\Delta = \mathrm{diag}\, \Delta_i$, $i = \overline{1,k}$, $\Delta_i \in \mathbb{C}^{l_i \times m_i}$. Then*

$$\|\Delta\|_p = \max_{i=\overline{1,k}} \|\Delta_i\|_p \tag{25}$$

■

The next Proposition is a known fact in linear algebra (see [3]).

Proposition 6

1. *Let $\Delta = \begin{bmatrix} \Delta_1 & \Delta_2 & \dots & \Delta_k \end{bmatrix} \in \mathbb{C}^{l\times m}$, $\Delta_i \in \mathbb{C}^{l\times m_i}$, $m = \sum_{i=1}^k m_i$. Then*

$$\max_{i=\overline{1,k}} \|\Delta_i\|_p \leq \|\Delta\|_p \leq k^{\frac{p-1}{p}} \max_{i=\overline{1,k}} \|\Delta_i\|_p \tag{26}$$

2. *Let $\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_k \end{bmatrix} \in \mathbb{C}^{l\times m}$, $\Gamma_i \in \mathbb{C}^{l_i\times m}$, $l = \sum_{i=1}^k l_i$.*

   Then

$$\max_{i=\overline{1,k}} \|\Gamma_i\|_p \leq \|\Gamma\|_p \leq k^{\frac{1}{p}} \max_{i=\overline{1,k}} \|\Gamma_i\|_p \tag{27}$$

The second inequality in (26) and (27) becomes an equality if $\Delta_i = \Delta_j$ and $\Gamma_i = \Gamma_j$, respectively, $\forall i, j \in \overline{1,k}$.

■

Lemma 7 *Let $\nu_{\mathrm{IR}}(P, \lambda; \Delta_i)$, $i = 1, 2, 3$ be the function $\nu_{\mathrm{IR}}(P, \lambda)$ read with respect to the perturbation structures Δ_i. Then, for all λ for which $P(\lambda)$ is invertible, the following inequalities hold:*

$$(k+1)^{\frac{1-p}{p}} \nu_{\mathrm{IR}}(P, \lambda; \Delta_1) \leq \nu_{\mathrm{IR}}(P, \lambda; \Delta_3) \leq \nu_{\mathrm{IR}}(P, \lambda; \Delta_1) \tag{28}$$

and

$$(k+1)^{-\frac{1}{p}} \nu_{\mathrm{IR}}(P, \lambda; \Delta_2) \leq \nu_{\mathrm{IR}}(P, \lambda; \Delta_3) \leq \nu_{\mathrm{IR}}(P, \lambda; \Delta_2) \tag{29}$$

Proof. Let

$$\Delta_1 = \begin{bmatrix} \Delta_{1,0} & \Delta_{1,1} & \dots & \Delta_{1,k} \end{bmatrix},$$

$$\Delta_2 = \begin{bmatrix} \Delta_{2,0} \\ \Delta_{2,1} \\ \vdots \\ \Delta_{2,k} \end{bmatrix}, \quad \Delta_3 = \begin{bmatrix} \Delta_{3,0} & & & \\ & \Delta_{3,1} & & \\ & & \ddots & \\ & & & \Delta_{3,k} \end{bmatrix} \tag{30}$$

be the minimum norm real matrices such that

$$\det(I + \Delta_1 M_1(\lambda)) = \det(I + \Delta_2 M_2(\lambda))$$

$$= \det(I + \Delta_3 M_3(\lambda, \xi)) = 0$$

for some complex scaling ξ and with M_1, M_2, M_3 introduced by (14). In other words,

$$\nu_{\mathrm{IR}}(P, \lambda; \Delta_i) = \|\Delta_i\|_p, \ \ i = 1, 2, 3 \tag{31}$$

We shall actually prove only inequalities (28), since relation (29) follows in a similar way. Let $\Delta_1^d = \mathrm{diag}\ (\Delta_{1,i})$ and $\Delta_3^l = \begin{bmatrix} \Delta_{3,0} & \Delta_{3,1} & \dots & \Delta_{3,k} \end{bmatrix}$. Then it follows from (13)–(14) that

$$\det(I + \Delta_3 M_3(\lambda, \xi)) = 0 \Leftrightarrow \det(I + \Delta_1^d\, M_3(\lambda, \xi)) = 0$$

$$\Leftrightarrow \det(I + \Delta_3^l M_1(\lambda)) = 0 \Leftrightarrow \det(I + \Delta_1 M_1(\lambda)) = 0$$

Since Δ_3 is "optimal" with respect to all block diagonal perturbations, one has that $\|\Delta_3\|_p \leq \|\Delta_1^d\|_p$. But $\|\Delta_1^d\|_p = \max_i \|\Delta_{1,i}\|_p$, as Proposition 5 shows. By applying now Proposition 6 to Δ_1 one gets

$$\|\Delta_3\|_p \leq \|\Delta_1^d\|_p \leq \|\Delta_1\|_p \leq (k+1)^{\frac{p-1}{p}} \|\Delta_1^d\|_p \tag{32}$$

On the other hand, Δ_1 is "optimal" with respect to all block line perturbations, hence $\|\Delta_1\|_p \leq \|\Delta_3^l\|_p$. By applying now Proposition 6 to Δ_3^l one gets

$$\|\Delta_1\|_p \leq \|\Delta_3^l\| \leq$$

$$\leq (k+1)^{\frac{p-1}{p}} \max_i \|\Delta_{3,i}\|_p = (k+1)^{\frac{p-1}{p}} \|\Delta_3\|_p \tag{33}$$

as Proposition 5 states. By combining now (32) and (33) one obtains

$$\|\Delta_3\|_p \le \|\Delta_1\|_p \le (k+1)^{\frac{p-1}{p}}\|\Delta_3\|_p,$$

or, equivalently,

$$(k+1)^{\frac{1-p}{p}}\|\Delta_1\|_p \le \|\Delta_3\|_p \le \|\Delta_1\|_p \tag{34}$$

The conclusion (28) follows now immediately by combining the above relation (34) with equality (31). ■

The next result is a direct consequence of formula (11) and of the previous lemma.

Corollary 8 *In the above context*

$$(k+1)^{\frac{1-p}{p}} r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_1) \le r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_3) \le r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_1) \tag{35}$$

and

$$(k+1)^{-\frac{1}{p}} r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_2) \le r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_3) \le r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_2) \tag{36}$$

■

For reasonable values of k all stability radii should thus be quite close to each other.

3 Main results

The main purpose of this section is to derive a closed formula for $r_{\mathbb{R}}(P;\mathbb{C}_g)$ with respect to Δ_1 and Δ_2. The third case will be as usual treated separately.
First, we shall derive appropriate state space realizations for the rational matrix functions M_1 and M_2, respectively. Let us introduce

$$E := \begin{bmatrix} I_n & & & \\ & I_n & & \\ & & \ddots & \\ & & & 0_n \end{bmatrix}$$

$$A_1 := \begin{bmatrix} 0_n & I_n & & \\ & \ddots & \ddots & \\ & & 0_n & I_n \\ -P_0 & \dots & -P_{k-1} & -P_k \end{bmatrix}, \quad B_1 := \begin{bmatrix} 0_n \\ 0_n \\ \vdots \\ I_n \end{bmatrix},$$

$$A_2 := \begin{bmatrix} 0_n & & & -P_0 \\ I_n & \ddots & & \vdots \\ & \ddots & 0_n & -P_{k-1} \\ & & I_n & -P_k \end{bmatrix},$$

$$C_2 := \begin{bmatrix} 0_n & 0_n & \dots & I_n \end{bmatrix} \tag{37}$$

Straightforward computations show that

$$M_1(\lambda) = \begin{bmatrix} I \\ \lambda I \\ \vdots \\ \lambda^k I \end{bmatrix} P^{-1}(\lambda) = (\lambda E - A_1)^{-1} B_1 \quad \text{and} \tag{38}$$

$$M_2(\lambda) = P^{-1}(\lambda)\begin{bmatrix} I & \lambda I & \cdots & \lambda^k I \end{bmatrix} = C_2(\lambda E - A_2)^{-1} \tag{39}$$

Formulas (38)–(39) show that the real stability radii problems of polynomial matrices are equivalent to real structured stability radii problems of "companion" pencils, like (E, A_1) or (E, A_2). Furthermore, realizations (38)–(39) enable us to express the real and imaginary parts of $M_1(\lambda)$ and $M_2(\lambda)$, respectively, in terms of the initial data.
Let

$$\lambda := \lambda_x + j\lambda_y \;\text{ and }\; M_i(\lambda) := M_{i,x} + j\, M_{i,y},\; i = 1,2$$

where $M_{i,x}, \; M_{i,y}$ are real matrices of the same dimension as M_i. Then one has

$$\mathrm{Re}(\lambda E - A_i)^{-1} = \left[(\lambda_x E - A_i) + \lambda_y^2 E(\lambda_x E - A_i)^{-1} E\right]^{-1} \tag{40}$$

and

$$\mathrm{Im}\left[(\lambda E - A_i)^{-1}\right] =$$

$$= -\lambda_y(\lambda_x E - A_i)^{-1}\left[(\lambda_x E - A_i) + \lambda_y^2 E(\lambda_x E - A_i)^{-1} E\right]^{-1} \tag{41}$$

It follows automatically that

$$\begin{array}{rcl} M_{1,x} & = & \mathrm{Re}\left[(\lambda \mathrm{E} - \mathrm{A}_1)^{-1}\right]\mathrm{B}_1 \\ M_{1,y} & = & \mathrm{Im}\left[(\lambda \mathrm{E} - \mathrm{A}_1)^{-1}\right]\mathrm{B}_1 \\ M_{2,x} & = & C_2\mathrm{Re}\left[(\lambda \mathrm{E} - \mathrm{A}_2)^{-1}\right] \\ M_{2,y} & = & C_2\mathrm{Im}\left[(\lambda \mathrm{E} - \mathrm{A}_2)^{-1}\right] \end{array} \tag{42}$$

Explicit formulas for $\mu(M_1(\lambda))$ and $\mu(M_2(\lambda))$ are given below.

Lemma 9 *Let $M_{i,x}$ and $M_{i,y}$, $i = 1,2$, be given by (42). Then*

$$\mu(M_i(\lambda)) = \inf_{\gamma\in(0,1]} \sigma_2\left(\begin{bmatrix} M_{i,x} & -\gamma M_{i,y} \\ \gamma^{-1} M_{i,y} & M_{i,x} \end{bmatrix}\right) \tag{43}$$

Proof. The proof of (43) is a direct consequence of Theorem 1 applied to $M_i(\lambda)$, $i = 1,2$ given by (38)–(39). ■

By combining now Lemma 3 and Lemma 9 we obtain the main result of the section.

Theorem 10 *The real stability radius (9) of the polynomial matrix $P(\lambda)$ with respect to the perturbation matrices Δ_i, $i = 1,2$, is given by*

$$r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_i) =$$

$$= \left[\sup_{\lambda\in\partial\mathbb{C}_g} \inf_{\gamma\in(0,1]} \sigma_2\left(\begin{bmatrix} M_{i,x} & -\gamma M_{i,y} \\ \gamma^{-1} M_{i,y} & M_{i,x} \end{bmatrix}\right)\right]^{-1} \tag{44}$$

■

Some additional comments are given below.

Remark 11 *Formulas (42) show that $M_{i,x}$, $M_{i,y}$ depend explicitly on the real and imaginary part of λ, i.e. λ_x and λ_y, respectively. When considering Hurwitz or Schur stability, $M_{i,x}$ and $M_{i,y}$ will depend on a single real parameter, such as ω: $\omega = \lambda_y$ for Hurwitz stability or $e^{j\omega} = \lambda_x + j\lambda_y$ for Schur stability.*
According to Remark 2, since $\mu_{\mathbb{R}}$ is discontinuous when $M_i(\lambda)$ is real, one gets non-continuous function of λ which has also to be minimized on the boundary of the stability region, as (20) shows. Therefore it can be difficult to elaborate appropriate numerical algorithms. More details concerning these aspects can be found in [10].

4 Simplified formulas

Special attention will be payed to the particular structure of both $M_1(\lambda)$ and $M_2(\lambda)$ (see (14)). This structure will be exploited in the light of Theorem 1, in order to reduce the complexity of the minimization over γ when calculating $r_{\mathbb{R}}$ and to obtain simpler expressions for the smallest "destabilizing" perturbations. The third case will be treated separately.

If $\lambda^i P(\lambda)^{-1} := X_i(\lambda) + j\,Y_i(\lambda)$, where $X_i, Y_i \in \mathbb{R}^{n\times n}$, $i = \overline{0,k}$ then

$$M_{1,x} = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_k \end{bmatrix} \quad \text{and} \quad M_{1,y} = \begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_k \end{bmatrix}.$$

Let us also define

$$\Lambda_\gamma := N_\lambda(\gamma) = \begin{bmatrix} \lambda_x & -\gamma\lambda_y \\ \gamma^{-1}\lambda_y & \lambda_x \end{bmatrix},$$

$$N_0(\lambda,\gamma) := N_{P^{-1}(\lambda)}(\gamma) = \begin{bmatrix} X_0 & -\gamma Y_0 \\ \gamma^{-1} Y_0 & X_0 \end{bmatrix}$$

and for $i = 1, 2$

$$N_i(\lambda,\gamma) := N_{M_i(\lambda)}(\gamma) = \begin{bmatrix} M_{i,x} & -\gamma M_{i,y} \\ \gamma^{-1} M_{i,y} & M_{i,x} \end{bmatrix}.$$

From the definition of X_i, Y_i one has that

$$\begin{bmatrix} X_i & -\gamma Y_i \\ \gamma^{-1} Y_i & X_i \end{bmatrix} = (\Lambda_\gamma^i \otimes I_n) \begin{bmatrix} X_0 & -\gamma Y_0 \\ \gamma^{-1} Y_0 & X_0 \end{bmatrix} \quad (45)$$

Then there exists an orthogonal matrix $\Pi \in \mathbb{R}^{(2k+2)n\times(2k+2)n}$ such that

$$\Pi\, N_1 = \begin{bmatrix} N_0 \\ (\Lambda_\gamma \otimes I_n)\, N_0 \\ \vdots \\ (\Lambda_\gamma^k \otimes I_n)\, N_0 \end{bmatrix} = (A_1(\lambda,\gamma) \otimes I_n)\, N_0, \quad (46)$$

where

$$A_1(\lambda,\gamma) := \begin{bmatrix} I_2 \\ \Lambda_\gamma \\ \vdots \\ \Lambda_\gamma^k \end{bmatrix}.$$

Consequently

$$\sigma_2(N_1) = \sigma_2\Big((A_1(\lambda,\gamma) \otimes I_n)\, N_0\Big), \quad \forall\gamma \in (0,1] \quad (47)$$

Since $A_1^T A_1$ is positive definite, one can find a Cholesky factor L_1, that is, $L_1^T L_1 = A_1^T A_1$, with L_1 in upper triangular form. However, L_1 has no rational expression in terms of γ (or λ). With the above considerations in mind, relation (47) becomes

$$\sigma_2(N_1) = \sigma_2\Big((L_1 \otimes I_n)\, N_0\Big) \quad (48)$$

Introduce

$$A_2(\lambda,\gamma) := \begin{bmatrix} I_2 & \Lambda_\gamma & \dots & \Lambda_\gamma^k \end{bmatrix}.$$

If L_2 is a Cholesky factor of $A_2 A_2^T$, since $M_{2,x} = \begin{bmatrix} X_0 & X_1 & \dots & X_k \end{bmatrix}$, and $M_{2,y} = \begin{bmatrix} Y_0 & Y_1 & \dots & Y_k \end{bmatrix}$, one can prove in a similar manner as before that

$$\sigma_2(N_2) = \sigma_2\Big(N_0\, (L_2 \otimes I_n)\Big). \quad (49)$$

The next result is a direct consequence of Lemma 3 and Theorem 1, combined with relations (48) and (49).

Theorem 12 *The real stability radius (9) of the polynomial matrix $P(\lambda)$ with respect to the perturbation structures Δ_1 and Δ_2 is given by*

$$r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_1) = \left[\sup_{\lambda\in\partial\mathbb{C}_g} \inf_{\gamma\in(0,1]} \sigma_2\Big((L_1 \otimes I_n)\, N_{P^{-1}(\lambda)}(\gamma)\Big)\right]^{-1}$$
and
$$r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_2) = \left[\sup_{\lambda\in\partial\mathbb{C}_g} \inf_{\gamma\in(0,1]} \sigma_2\Big(N_{P^{-1}(\lambda)}(\gamma)\, (L_2 \otimes I_n)\Big)\right]^{-1} \quad (50)$$

Let Δ_1 be the minimum norm destabilizing perturbation with respect to $M_1(\lambda)$ and denote by $\Delta_{1,i}$ its i-th $n\times n$ block component, $i = \overline{0,k}$, as in (30). Let also $\begin{bmatrix} u_x \\ u_y \end{bmatrix}$ and $\begin{bmatrix} v_x \\ v_y \end{bmatrix}$ be a pair of left and right singular vectors of the matrix $N_1(\lambda,\gamma_{opt})$ corresponding to the "infimum" $\sigma_{2,opt}(N_1)$, that is

$$N_1 \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \sigma_{2,opt} \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \quad N_1^* \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \sigma_{2,opt} \begin{bmatrix} v_x \\ v_y \end{bmatrix}.$$

Introduce

$$V := \begin{bmatrix} v_x & v_y \end{bmatrix} \quad \text{and} \quad U_0 := \begin{bmatrix} u_x(1:n) & u_y(1:n) \end{bmatrix},$$

respectively.
Similarly, if Δ_2 is the minimum norm destabilizing perturbation with respect to $M_2(\lambda)$, then $\Delta_{2,i}$, $i = \overline{0,k}$, stands for its i-th $n\times n$ block component as in (30). Let, as before, $\begin{bmatrix} u_x' \\ u_y' \end{bmatrix}$ and $\begin{bmatrix} v_x' \\ v_y' \end{bmatrix}$ be a pair of left and right singular

vectors of the matrix $N_2(\lambda,\gamma_{opt})$ corresponding to the "infimum" $\sigma'_{2,opt}(N_2)$, that is,

$$N_2\left[\begin{array}{c} v'_x \\ v'_y \end{array}\right] = \sigma'_{2,opt}\left[\begin{array}{c} u'_x \\ u'_y \end{array}\right], \quad N_2^*\left[\begin{array}{c} u'_x \\ u'_y \end{array}\right] = \sigma'_{2,opt}\left[\begin{array}{c} v'_x \\ v'_y \end{array}\right].$$

With a certain lack of consistency in the notation, let

$$V_0 := \left[\begin{array}{cc} v'_1(1:n) & v'_2(1:n) \end{array}\right] \text{ and } U := \left[\begin{array}{cc} u'_1 & u'_2 \end{array}\right],$$

respectively. Essentially relying on formula (19) and on equality (45), the following result holds.

Theorem 13 *For every $\lambda \in \mathbb{C}$ which is not a root of P and for every $i \in \overline{0,k}$ we have*

$$\begin{array}{ll} \Delta_{1,i} = -\sigma_{2,opt}^{-1}(N_1)\ (V^{\dagger})^T\ \Lambda_{\gamma_{opt}}^i\ U_0^T & \text{and} \\ \Delta_{2,i} = -\sigma_{2,opt}^{-1}(N_2)\ V_0\ \Lambda_{\gamma_{opt}}^i\ U^{\dagger} & \end{array} \tag{51}$$

Furthermore,

$$\begin{array}{l} \delta P(\lambda) = -\sigma_{2,opt}^{-1}(N_1)\ (V^{\dagger})^T \left[\sum_{i=0}^k \lambda^i \Lambda_{\gamma_{opt}}^i\right] U_0^T \text{ and} \\ \delta P(\lambda) = -\sigma_{2,opt}^{-1}(N_2)\ V_0\ \left[\sum_{i=0}^k \lambda^i \Lambda_{\gamma_{opt}}^i\right] U^{\dagger} \end{array} \tag{52}$$

■

As already mentioned, we can derive for the third case lower and upper bounds in terms of $r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_i)$, $i = 1,2$. The result is a direct consequence of Corollary 8 for $p = 2$.

Lemma 14 *The following inequalities hold for $i = \overline{1,2}$:*

$$(k+1)^{-\frac{1}{2}} r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_i) \le r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_3) \le r_{\mathbb{R}}(P;\mathbb{C}_g;\Delta_i) \tag{53}$$

■

Let Δ_3 be the minimal norm perturbation perturbation that attains $\mu_{\mathbb{R}}(M_3(\lambda))$. An appropriate counterpart of Theorem 13 is conjectured below.

Conjecture 15 *Let $N_3(\lambda,\xi,\gamma) := N_{M_3(\lambda,\xi)}(\gamma)$. For every $\lambda \in \mathbb{C}$ which is not a root of P, there exists $\xi^T := \left[\begin{array}{cccc} 1 & \xi_1 & \dots & \xi_k \end{array}\right]$ such that for every $i \in \overline{0,k}$ we have*

$$\Delta_{3,i} = \sigma_{2,opt}^{-1}(N_3)\ V_0\ \Xi_{i,\gamma_{opt}}\ \Lambda_{\gamma_{opt}}^i\ U_0^T, \tag{54}$$

where $\Xi_{i,\gamma} \in \mathbb{R}^{2\times 2}$ depend on ξ_i and γ. Furthermore,

$$\delta P(\lambda) = \sigma_{2,opt}^{-1}(N_3)\ V_0\Big[\sum_{i=0}^k \Xi_{i,\gamma_{opt}}(\lambda\Lambda_{\gamma_{opt}})^i\Big]U_0^T. \tag{55}$$

■

5 Computational aspects

The above real stability radius can be computed efficiently for Hurwitz and Schur polynomial matrices, by exploiting formula (50) and by appealing to some previous work on similar topics (see [10]). First, let us notice that $\partial\mathbb{C}_g$ is, for the Hurwitz and Schur case, the imaginary axis ($\lambda = j\omega$) and the unit circle ($\lambda = e^{j\omega}$), respectively. Therefore, $\lambda \in \mathbb{C}_g$ will be parametrized by a single real variable. The optimization of (50) consists of two subproblems. At a given frequency ω we optimize γ. Despite the higher dimension of M_1 and M_2, this only involves at each step the SVD of a $2n \times 2n$ matrix, as Theorem 12 shows. Moreover,

$$\sigma_2\Big((L_1\otimes I_n)\,N_{P^{-1}(\lambda)}(\gamma)\Big) = \sigma_{n-1}^{-1}\Big(N_{P(\lambda)}(\gamma)\,(L_1^{-1}\otimes I_n)\Big)$$

and

$$\sigma_2\Big(N_{P^{-1}(\lambda)}(\gamma))(L_2\otimes I_n)\Big) = \sigma_{n-1}^{-1}\Big((L_2^{-1}\otimes I_n)N_{P(\lambda)}(\gamma)\Big).$$

which show that the computation of the real and imaginary parts of $P^{-1}(\lambda)$ is replaced by a simple inversion of a 2 by 2 upper or lower triangular matrix L_i.
For a given γ, one then finds a new frequency point according to a scheme proposed in [10]. The crucial point is to realize the transfer functions

$$N_1^*(\lambda,\gamma)N_1(\lambda,\gamma) - \rho^2 I$$

and

$$N_2(\lambda,\gamma)N_2^*(\lambda,\gamma) - \rho^2 I.$$

One can now use the standard realizations (37) from Section 3. Take, for instance, $\lambda = j\omega$ and $i = 1$. Then

$$N_1(j\omega,\gamma) = \widetilde{C}_{1,\gamma}(j\omega\widetilde{E} - \widetilde{A}_1)^{-1}\widetilde{B}_{1,\gamma},$$

where

$$\widetilde{A}_1 = \left[\begin{array}{cc} A_1 & 0 \\ 0 & -A_1 \end{array}\right], \quad \widetilde{B}_{1,\gamma} = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} B_1 & \gamma B_1 \\ -\gamma^{-1}B_1 & B_1 \end{array}\right],$$

$$\widetilde{C}_{1,\gamma} = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} I & \gamma I \\ \gamma^{-1}I & -I \end{array}\right] \text{ and } \widetilde{E}_1 = \left[\begin{array}{cc} E & 0 \\ 0 & E \end{array}\right].$$

It turns out that ρ is a singular value of $N_1(j\omega,\gamma)$ if and only if ω is a (generalized) eigenvalue of the following *Hamiltonian* structure

$$\left[\begin{array}{cc} \widetilde{A}_1 - j\omega\widetilde{E}_1 & \widetilde{B}_{1,\gamma}\widetilde{B}_{1,\gamma}^T/\rho \\ -\widetilde{C}_{1,\gamma}^T\widetilde{C}_{1,\gamma}/\rho & -\widetilde{A}_1^T - j\omega\widetilde{E}_1 \end{array}\right].$$

A similar scheme works for the Schur matrix polynomials, when the above Hamiltonian structure is replaced by a symplectic one. Consider $\lambda = e^{j\omega}$ and $i = 2$. Then

$$N_2(e^{j\omega},\gamma) = \widetilde{C}_{2,\gamma}(e^{j\omega}\widetilde{E}_2 - \widetilde{A}_2)^{-1}\widetilde{B}_{2,\gamma},$$

where

$$\widetilde{A}_2 = \left[\begin{array}{cc} -A_2 & 0 \\ 0 & I \end{array}\right], \quad \widetilde{B}_{2,\gamma} = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} -I & -\gamma I \\ -\gamma^{-1}I & I \end{array}\right],$$

$$\widetilde{C}_{2,\gamma} = \frac{1}{\sqrt{2}} \begin{bmatrix} C_2 & \gamma C e^{j\omega} \\ \gamma^{-1} C & -C e^{j\omega} \end{bmatrix} \text{ and } \widetilde{E}_2 = \begin{bmatrix} -I & 0 \\ 0 & A_2 \end{bmatrix}.$$

Again, ρ is a singular value of $N_2(e^{j\omega}, \gamma)$ if and only if $e^{j\omega}$ is a (generalized) eigenvalue of the following *symplectic* structure

$$\begin{bmatrix} \widetilde{A}_2 - e^{j\omega}\widetilde{E}_2 & e^{j\omega}\widetilde{B}_{2,\gamma}\widetilde{B}_{2,\gamma}^T/\rho \\ \widetilde{C}_{2,\gamma}^T\widetilde{C}_{2,\gamma}/\rho & e^{j\omega}\widetilde{A}_2^T - \widetilde{E}_2^T \end{bmatrix}.$$

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