On Σ-Lossless Transfer Functions and Related Questions*

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ABSTRACT

This paper is concerned with a systematic approach to the properties of Σ-lossless rational transfer functions in the discrete as well as in the continuous time case. As a result, a unifying framework is revealed where several known results fit naturally. Special attention is given to the embedding problem of the Lyapunov equation in view of its direct application to generalized Levinson algorithms.

1. INTRODUCTION

An $r \times r$ rational matrix $R(z)$ is said to be Σ-lossless in the *discrete time case* provided it satisfies the following set of conditions:

$$\begin{aligned} R(z)\Sigma\tilde{R}(z) &\leqslant \Sigma \qquad \text{in} \quad |z|>1, \\ R(z)\Sigma\tilde{R}(z) &= \Sigma \qquad \text{at} \quad |z|=1, \\ R(z)\Sigma\tilde{R}(z) &\geqslant \Sigma \qquad \text{in} \quad |z|<1 \end{aligned} \tag{1}$$

with Σ a signature matrix of the form $I_p \oplus -I_q$ and $r = p+q$. As a matter of fact, the third constraint is actually redundant in the above definition (See section 2), and the identity $R(1/z) \equiv \Sigma\tilde{R}^{-1}(\bar{z})\Sigma$ readily follows by analytic continuation from the middle equality (1).

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Σ-lossless rational matrices $T(p)$ are similarly defined in the *continuous time case* by replacing the constraints (1) by

$$\begin{aligned} T(p)\Sigma\tilde{T}(p) &\leqslant \Sigma \qquad &&\text{in} \quad \operatorname{Re} p > 0 \\ T(p)\Sigma\tilde{T}(p) &= \Sigma \qquad &&\text{at} \quad \operatorname{Re} p = 0 \\ T(p)\Sigma\tilde{T}(p) &\geqslant \Sigma \qquad &&\text{in} \quad \operatorname{Re} p < 0 \end{aligned} \tag{2}$$

In the same way as in the discrete case, these constraints are redundant and the identity $T(-p) \equiv \Sigma\tilde{T}^{-1}(\bar{p})\Sigma$ is a direct consequence of the definition.

It turns out that Σ-lossless matrices enjoy remarkable properties. The three following results discussed in details in this paper are intimately connected.

RESULT I (State space realization).

(a) *Discrete Time Case. Let $\{F, G, H, J\}$ be a minimal realization of a proper transfer function $R(z)$. Then $R(z)$ is Σ-lossless if and only if there exists a positive definite matrix P satisfying*

$$\begin{bmatrix} F & G \\ H & J \end{bmatrix} \begin{bmatrix} P & \\ & \Sigma \end{bmatrix} \begin{bmatrix} \tilde{F} & \tilde{H} \\ \tilde{G} & \tilde{J} \end{bmatrix} = \begin{bmatrix} P & \\ & \Sigma \end{bmatrix}. \tag{3}$$

Moreover, this matrix P is unique.

(b) *Continuous time case. If $\{A, B, C, D\}$ is a minimal realization of a proper transfer function $T(p)$, the above result remains the same except that the conditions* (3) *are replaced by*

$$\begin{bmatrix} AP + P\tilde{A} & P\tilde{C} \\ CP & 0 \end{bmatrix} = \begin{bmatrix} B \\ D \end{bmatrix} \Sigma \begin{bmatrix} \tilde{B} & \tilde{D} \end{bmatrix} = \begin{bmatrix} 0 & \\ & \Sigma \end{bmatrix}. \tag{4}$$

RESULT II (Factorization). *Any Σ-lossless rational matrix of degree n can be factorized as a product of n Σ-lossless rational matrices of degree one.*

RESULT III (Embedding of the Lyapunov equation).

(a) *Discrete time case. Let F and P be two $n \times n$ arbitrary matrices with P positive definite. Clearly, the difference P-$FP\tilde{F}$ can be written as*

$$P - FP\tilde{F} = G\Sigma\tilde{G} \tag{5}$$

in an infinite number of different ways. Note that neither G, nor the signature matrix Σ, nor even their dimensions are uniquely defined by (5).

However, for any admissible pair (G, Σ), *there always exists an embedding* $\{F, G, H, J\}$ *satisfying* (3).

(b) *Continuous time case. Replacing F by A and* (5) *by*

$$AP + P\tilde{A} = -B\Sigma\tilde{B}, \tag{6}$$

the continuous time equivalent of the above result is that one can always find an arbitrary $\{A, B, C, D\}$ *satisfying* (4) *for any admissible pair* (B, Σ).

Let us first make a few comments about the very formulation of the results presented. The restriction in the realization problem that $R(z)$ and $T(p)$ have to be proper is in fact inessential; as is well known, one can always reduce the problem to this situation by an appropriate variable transformation. Also, since the continuous time and the discrete time formulations are related through simple transformation as described in Appendix 1, we will concentrate on the discrete time version and supply the additional material to accommodate the continuous time case only when specifically needed. Finally, since P is positive definite in (3)–(6) and as a state space realization of a transfer function is only defined up to an arbitrary state space transformation, P can be reduced to the identity matrix by using any transformation N such that $P = N\tilde{N}$. In the sequel, such a state space transformation will be emphasized by using script letters $\{\mathcal{F}, \mathcal{G}, \dots\}$ instead of $\{F, G, \dots\}$. In connection with this, it is interesting to point out that a realization $\{\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{J}\}$ is balanced, as described by Moore [1].

The interest in Σ-lossless transfer functions is by no means new in the technical literature. Let us first mention classical network theory, where the case $\Sigma = I_r$ ($p = r$, $q = 0$) is standard. Indeed, $T(p)$ can be viewed as the scattering matrix of a lossless r-port, whose factorization properties are well known [2]. Any realization of such a scattering matrix satisfies (4), which is nothing but the lossless form of the bounded real lemma [3]. Such realizations were extensively used by Youla and Tissi in a particular synthesis procedure [4]. Still in the same context, the case $\Sigma \neq I$ has also been considered in connection with the factorization problem of the so-called transfer matrix [5] and is at least implicitly touched on in any cascade synthesis procedure. The situation is quite similar in linear estimation theory. For example, ladder filters for stationary stochastic processes are known to be a direct implementation of the factorization of a related Σ-lossless transfer function with $\Sigma = I_p \oplus -I_p$ and thus $r = 2p$ [6]. Renewed interest has been recently aroused by the same subject from the viewpoint of α-stationary stochastic processes; as a matter of fact, Equation (5), particularized to F being the shift operator and P a covariance matrix, leads to the concept of displacement rank [7], yielding a generalized Levinson algorithm [8] and hopefully ladder structures for nonstationary stochastic processes. Needless to say, there are numerous publications devoted to the Lyapunov equation, mostly in connection with stability analysis; however, the embedding of this equation in a matrix satisfying (3) or (4) does not seem to have received much attention to date. Let us finally remark that the factorization problem for Σ-lossless functions can be considered as a particular case of the minimal factorization of arbitrary rational transfer matrices; the reader is especially referred to [9] and the references therein for more details about that question.

Special attention has to be directed to the more abstract mathematical literature on the subject, especially in the context of operator theory. Let us mention in this direction five particularly relevant references. In the discrete time case, the factorization problem for arbitrary (not necessarily rational) Σ-lossless functions was discussed and completely solved by Potapov [10]. In the continuous time case, the factorization as well as the embedding problems were considered by Livsic and Brodskii [11, 13] in the abstract setting of colligations; in particular, these authors introduced so-called triangular operator models, yielding straightforward factorization solutions. These results were translated to the discrete time case for the particular situation $\Sigma \equiv I$ in [12]. The embedding of the Lyapunov equation can be viewed as a problem of completing a set of independent vectors in an indefinite metric, which is, e.g., extensively treated by Bognar [16].

In view of the abundant literature available on the subject, it should be clear from the outset that the main interest of this paper does not lie in the originality of the material presented. However, there seems to be a definite need first for a unifying approach to Σ-lossless transfer functions in the rational case, secondly to fill in some missing gaps in the literature, and last but not least to provide a comprehensive account of the theory, based upon arguments as simple as possible. An important by-product of this development is the direct application to the problem of generalized Levinson algorithms, which in this set up allow for an easier description and better insights [17–20].

2. THE REALIZATION PROBLEM

Let $\{F, G, H, J\}$ be a minimal realization of an $r \times r$ Σ-lossless transfer function of degree n, i.e.

$$R(z) = J + H(zI_n - F)^{-1}G. \tag{12}$$

Partitioning $R(z)$ conformably with Σ,

$$R(z) = \begin{bmatrix} R_{11}(z) & R_{12}(z) \\ R_{21}(z) & R_{22}(z) \end{bmatrix}, \tag{13}$$

one can introduce a new $r \times r$ transfer function $S(z)$ defined by

$$S(z) = \begin{bmatrix} R_{11}(z) - R_{12}(z)R_{22}^{-1}(z)R_{21}(z) & R_{12}(z)R_{22}^{-1}(z) \\ -R_{22}^{-1}(z)R_{21}(z) & R_{22}^{-1}(z) \end{bmatrix}. \tag{14}$$

In view of the constraints (1) and by using the results of Appendix 2 with $X = R(z)$, $Y = S(z)$, $Z_1 = I_p$, and $Z_2 = -I_q$, the derived transfer function $S(z)$ is readily seen to be lossless, i.e. I-lossless.

It turns out that a state space realization of $S(z)$,

$$S(z) = J_s + H_s(zI_n - F_s)^{-1}G_s, \tag{15}$$

can easily be deduced from (12). Indeed, the first constraint (1) imposes in particular

$$R_{22}(z)\tilde{R}_{22}(z) \geqslant I_q + R_{21}(z)\tilde{R}_{21}(z) \tag{16}$$

in $|z| > 1$, so that $R_{22}(\infty) = J_{22}$ is nonsingular. With the appropriate partitioning of G, H, J, one thus obtains after elementary algebraic manipulations

$$\begin{aligned} F_s &= F - G_2J_{22}^{-1}H_2, \\ G_s &= \begin{bmatrix} G_1 - G_2J_{22}^{-1}J_{21} & G_2J_{22}^{-1} \end{bmatrix}, \\ \tilde{H}_s &= \begin{bmatrix} \tilde{H}_1 - \tilde{H}_2\tilde{J}_{22}^{-1}\tilde{J}_{12} & -\tilde{H}_2\tilde{J}_{22}^{-1} \end{bmatrix}, \\ J_s &= \begin{bmatrix} J_{11} - J_{12}J_{22}^{-1}J_{21} & J_{12}J_{22}^{-1} \\ -J_{22}^{-1}J_{21} & J_{22}^{-1} \end{bmatrix}. \end{aligned} \tag{17}$$

LEMMA 1. *If a realization of a Σ-lossless transfer function $R(z)$ is minimal, so is the realization of its associated lossless transfer function $S(z)$* [2].

Proof. From the identity

$$\begin{bmatrix} zI_n - F & G \end{bmatrix} \begin{bmatrix} I_n & & \\ 0 & I_p & \\ J_{22}^{-1}H_2 & -J_{22}^{-1}J_{21} & J_{22}^{-1} \end{bmatrix} = \begin{bmatrix} zI_n - F_s & G_s \end{bmatrix} \tag{18}$$

one first deduces that the realization (15) is controllable together with the realization (12). The dual argument can clearly be applied for the observability condition. ■

LEMMA 2. *For any minimal realization of a lossless transfer function* $S(z)=J_s+H_s(zI_n-F_s)^{-1}G_s$, *there exists a unique positive definite matrix P satisfying*

$$\begin{bmatrix} F_s & G_s \\ H_s & J_s \end{bmatrix}\begin{bmatrix} P & \\ & I_r \end{bmatrix}\begin{bmatrix} \tilde{F}_s & \tilde{H}_s \\ \tilde{G}_s & \tilde{J}_s \end{bmatrix}=\begin{bmatrix} P & \\ & I_r \end{bmatrix}. \tag{19}$$

Proof. The proof of the existence of such a positive definite matrix P is trivial, since (19) is nothing but the lossless form of the bounded real lemma [3] in the discrete case (see Appendix 1). The uniqueness of P is proved *ab absurdo* by considering the top left block entry of (19) for two matrices P_1 and P_2, which implies $P_1-P_2=F_s^k(P_1-P_2)\tilde{F}_s^k$ for all positive integers k; since all the poles of a lossless transfer function lie necessarily in the open unit disk, one is left with the identity $P_1\equiv P_2$ by letting $k\to\infty$. ■

THEOREM 3. *A transfer function* $R(z)$ *is* Σ*-lossless if and only if to any of its minimal realizations* $R(z)=J+H(zI_n-F)^{-1}G$, *there corresponds a positive definite matrix P satisfying*

$$\begin{bmatrix} F & G \\ H & J \end{bmatrix}\begin{bmatrix} P & \\ & \Sigma \end{bmatrix}\begin{bmatrix} \tilde{F} & \tilde{H} \\ \tilde{G} & \tilde{J} \end{bmatrix}=\begin{bmatrix} P & \\ & \Sigma \end{bmatrix}. \tag{20}$$

Moreover, the matrix P is unique and identical to the matrix P of Lemma 2 *provided the realization of* $S(z)$ *is deduced from* $R(z)$ *via* (15) *and* (17).

Proof. The proof is immediate in one direction by using Lemma 2 and Theorem 16 of Appendix 2 with $Z_1=P\oplus I_p$, $Z_2=-I_q$,

$$X_{12}=\begin{bmatrix} G_2 \\ J_{12} \end{bmatrix}, \qquad X_{21}=\begin{bmatrix} H_2 & J_{21} \end{bmatrix}, \qquad X_{22}=J_{22},$$

and

$$X_{11}=\begin{bmatrix} F & G_1 \\ H_1 & J_{11} \end{bmatrix}. \tag{21}$$

The proof of the converse statement is achieved by noting that the difference

$\Sigma - R(z)\Sigma\tilde{R}(z)$ can be written as

$$\Sigma - R(z)\Sigma\tilde{R}(z) = (|z|^2 - 1)H(zI - F)^{-1}P(\bar{z}I - \tilde{F})^{-1}\tilde{H} \tag{22}$$

with the help of (20). ■

COROLLARY 3. *There exist minimal balanced realizations* $\{\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{J}\}$ *of any* Σ*-lossless transfer function* $R(z)$ $(P \equiv I_n)$.

A continuous time version of Theorem 3, trivially obtained via Appendix 1, turns out to be

THEOREM 4. *A transfer function* $T(p)$ *is* Σ*-lossless if and only if to any of its minimal realizations* $T(p) = D + C(pI_n - A)^{-1}B$, *there corresponds a positive definite matrix* P *satisfying*

$$\begin{bmatrix} AP + P\tilde{A} & P\tilde{C} \\ CP & 0 \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix}\Sigma[\tilde{B} \quad \tilde{D}] = \begin{bmatrix} 0 & \\ & \Sigma \end{bmatrix} \tag{23}$$

and this matrix P *is unique. Moreover, a minimal realization of* $T(p)$ *can always be balanced.*

To end up this section, let us indicate how the matrix P can be determined from a given minimal realization $\{F, G, H, J\}$. By considering the realization $\{F_s, G_s, H_s, J_s\}$ deduced from $\{F, G, H, J\}$ via (17), P can be obtained from the top left block entry of (20) as the exponentially convergent series

$$P = \sum_{k=0}^{\infty} F_s^k G_s \tilde{G}_s \tilde{F}_s^k. \tag{24}$$

The matrix P can also be computed directly from the realization $\{F, G, H, J\}$ by the following argument. Let us introduce V_c and V_o, the controllability and observability matrices respectively. With $I_n \otimes \Sigma = \Sigma \oplus \Sigma \oplus \Sigma \oplus \cdots$ the Kronecker product of Σ and I_n, one easily verifies from (20) that the matrix R^* of dimension $(r+1)n$ defined by

$$R^* = \begin{bmatrix} F^n & V_c E \\ V_o & K \end{bmatrix} \tag{25}$$

is $[P \oplus (I_n \otimes \Sigma)]$-unitary, where E is the permutation matrix consisting of identity matrices I_n on the second block diagonal, while K stands for the $r \times n$ matrix

$$K = \begin{bmatrix} J & & & \\ HG & J & & \\ HFG & HG & & \\ \vdots & \vdots & \ddots & \\ HF^{n-2}G & HF^{n-3}G & \cdots & J \end{bmatrix}. \tag{26}$$

From the $[P \oplus (I_n \otimes \Sigma)]$-unitary relations, one then obtains $V_o P \tilde{V}_o = I_n \otimes \Sigma - K(I_n \otimes \Sigma)\tilde{K}$ as well as $\tilde{V}_c P^{-1} V_c = E[I_n \otimes \Sigma - \tilde{K}(I_n \otimes \Sigma)K]\, E$, so that P can be expressed as

$$P = \left(\tilde{V}_o V_o\right)^{-1} \tilde{V}_o \left[I_n \otimes \Sigma - K(I_n \otimes \Sigma)\tilde{K}\right] V_o \left(\tilde{V}_o V_o\right)^{-1}, \tag{27}$$

and its inverse P^{-1} as

$$P^{-1} = \left(V_c \tilde{V}_c\right)^{-1} V_c E\left[I_n \otimes \Sigma - \tilde{K}(I_n \otimes \Sigma)K\right] E \tilde{V}_c \left(V_c \tilde{V}_c\right)^{-1}. \tag{28}$$

An explicit form of the matrix P in the continuous time case can be achieved essentially in the same way; the identity $V_o P = L\tilde{V}_c$ deduced from (23) with

$$L = \begin{bmatrix} -D\Sigma & & & \\ -CB\Sigma & D\Sigma & & \\ -CAB\Sigma & CB\Sigma & & \\ \vdots & \vdots & \ddots & \\ -CA^{n-2}B\Sigma & CA^{n-3}B\Sigma & \cdots & (-1)^{n-2}D\Sigma \end{bmatrix} \tag{29}$$

yields directly

$$P = \left(\tilde{V}_o V_o\right)^{-1} \tilde{V}_o L \tilde{V}_c. \tag{30}$$

3. THE FACTORIZATION PROBLEM

The factorization properties of Σ-lossless transfer functions can be described in a simple manner with the help of the so-called triangular models introduced by Livsic and Bordskii [11–13]. To explain this, let us first introduce the notation

$$\mathcal{R} = \begin{bmatrix} \mathcal{F} & \mathcal{G} \\ \mathcal{H} & \mathcal{J} \end{bmatrix} \tag{31}$$

corresponding to a balanced realization of a Σ-lossless transfer function $R(z) = \mathcal{J} + \mathcal{H}(zI_n - \mathcal{F})^{-1}\mathcal{G}$ so as to have $\mathcal{R}(I_n \oplus \Sigma)\tilde{\mathcal{R}} = (I_n \oplus \Sigma)$. Next assume $\mathcal{F}$ to be lower-block triangular:

$$\mathcal{F} = \begin{bmatrix} \mathcal{F}_{\alpha\alpha} & 0 \\ \mathcal{F}_{\beta\alpha} & \mathcal{F}_{\beta\beta} \end{bmatrix}, \tag{32}$$

where α and β are the dimensions of the blocks $\mathcal{F}_{\alpha\alpha}$ and $\mathcal{F}_{\beta\beta}$ respectively $(\alpha + \beta = n)$; define the conformal partitions $\tilde{\mathcal{G}} = [\tilde{\mathcal{G}}_\alpha, \tilde{\mathcal{G}}_\beta]$ and $\mathcal{H} = [\mathcal{H}_\alpha, \mathcal{H}_\beta]$. From the algebraic property of Z-unitary matrices proved in Appendix 3, an $(I_n \oplus \Sigma)$-unitary matrix $\mathcal{R}_1$ of the form

$$\mathcal{R}_1 = \begin{bmatrix} \mathcal{F}_1 & 0 & G_1 \\ 0 & I_\beta & 0 \\ \mathcal{H}_1 & 0 & \mathcal{J}_1 \end{bmatrix} \tag{33}$$

with $\mathcal{F}_1 = \mathcal{F}_{\alpha\alpha}$, $\mathcal{G}_1 = \mathcal{G}_\alpha$ is known to exist; as a result, the transfer function $R_1(z) = \mathcal{J}_1 + \mathcal{H}_1(zI_\alpha - \mathcal{F}_1)^{-1}\,\mathcal{G}_1$ is Σ-lossless.

LEMMA 5. *The matrix $\mathcal{R}_2 = \mathcal{R}\mathcal{R}_1^{-1}$ is $(I_n \oplus \Sigma)$-unitary and has the form*

$$\mathcal{R}_2 = \begin{bmatrix} I_\alpha & 0 & 0 \\ 0 & \mathcal{F}_2 & \mathcal{G}_2 \\ 0 & \mathcal{H}_2 & \mathcal{J}_2 \end{bmatrix}, \tag{34}$$

and the Σ-lossless transfer function $R(z)$ can be factorized as the product of the two Σ-lossless transfer functions

$$R(z) = R_2(z)R_1(z)$$

with $R_2(z) = \mathcal{J}_2 + \mathcal{H}_2(zI_\beta - \mathcal{F}_2)^{-1}\mathcal{G}_2$.

Proof. The first statement is obvious, since $(I_n \oplus \Sigma)$-unitary matrices clearly form a group for the multiplication. The last statement is readily proved by direct verification, so that one is left with establishing the particular form (34). Denoting by $X_{i,j}$ $(1 \leqslant i, j \leqslant 3)$ the block entries of $\mathcal{R}_2$, one deduces from the equality $\mathcal{R}_2 \mathcal{R}_1 = \mathcal{R}$ the values $X_{12} = 0$, $X_{22} = \mathcal{F}_{\beta\beta}$, $X_{32} = \mathcal{H}_\beta$ on the one hand and the equations

$$(X_{11} - I_\alpha)\mathcal{F}_1 + X_{13}\mathcal{H}_1 = 0, \qquad (X_{11} - I_\alpha)\mathcal{G}_1 + X_{13}\mathcal{J}_1 = 0$$

on the other hand; since $\mathcal{R}_1$ is nonsingular, this forces $X_{11} = I_\alpha$, $X_{13} = 0$. Finally, the relations $X_{21} = X_{31} = 0$ are found to be direct consequences of the $(I_n \oplus \Sigma)$-unitarity of $\mathcal{R}_2$. ■

By using the identity $\mathcal{R}_1^{-1} = (I_n \oplus \Sigma)\tilde{\mathcal{R}}_1 \ (I_n \oplus \Sigma)$, the nontrivial block entries of $\mathcal{R}_2$ are expressed in terms of $\mathcal{R}$ and $\mathcal{R}_1$ as

$$\begin{aligned} \mathcal{F}_2 &= \mathcal{F}_{\beta\beta}, \qquad & \mathcal{G}_2 &= \mathcal{F}_{\beta\alpha}\tilde{\mathcal{H}}_1\Sigma + \mathcal{G}_\beta \Sigma \tilde{\mathcal{J}}_1 \Sigma, \\ \mathcal{H}_2 &= \mathcal{G}_\beta, \qquad & \mathcal{J}_2 &= \mathcal{H}_\alpha \tilde{\mathcal{H}}_1 \Sigma + \mathcal{J}\Sigma\tilde{\mathcal{J}}_1\Sigma. \end{aligned} \tag{35}$$

On the basis of the above lemma, we are now in a position to prove the following important factorization theorem.

THEOREM 6. *Any Σ-lossless transfer function $R(z)$ of degree n can be factorized as a product of n Σ-lossless transfer functions of degree one.*

Proof. Let $R(z) = \mathcal{J} + \mathcal{H}(zI_n - \mathcal{F})^{-1}\mathcal{G}$ be a balanced minimal realization of the Σ-lossless transfer function. From the Schur lemma, any matrix is known to be reducible to a lower triangular form by unitary transformation. Let us then perform the appropriate unitary state transformation so as to reduce $\mathcal{F}$ to this form; note that such a transformation does not destroy the balanced property of the realization. Let f_{11} and the vector $\tilde{g}_1$ stand for the top left entry of $\mathcal{F}$ and the first row of $\mathcal{G}$ respectively. A direct application of Lemma 5 with $\alpha = 1$, $\beta = n-1$, $\mathcal{F}_1 = f_{11}$, and $\mathcal{G}_1 = \tilde{g}_1$ yields the proof of the theorem, since the lower triangular form of $\mathcal{F}_2 = \mathcal{F}_{\beta\beta}$ is unaltered, so that the above process can be iterated to exhaust the degree. ■

It is interesting to write down the explicit form of the first degree factors. With $\alpha = 1$, $\mathcal{F}_1 = f_{11}$, $\mathcal{G}_1 = \tilde{g}_1$, an admissible form for the $(I_n \oplus \Sigma)$-unitary matrix $\mathcal{R}_1$ is easily found to be

$$\mathcal{R}_1 = \begin{bmatrix} f_{11} & 0 & \tilde{g}_1 \\ 0 & I_{n-1} & 0 \\ -e_1 \Sigma g_1 & 0 & I_r - \gamma_1 \Sigma g_1 \tilde{g}_1 \end{bmatrix} \tag{36}$$

with the notation $e_1 = f_{11}/|f_{11}|$ if $f_{11} \neq 0$, $e_1 = 1$ if $f_{11} = 0$, and $\gamma_1 = (1 + |f_{11}|)^{-1}$. As a consequence, we have:

COROLLARY 7. *Any Σ-lossless transfer function of degree n can be factorized as the product* $R(z) = W_0 \prod_{i=1}^{n} R_{n-i+1}(z)$, *where* W_0 *is a Σ-unitary constant matrix and the* $R_i(z)$ *are Σ-lossless transfer functions of degree one and of the form*

$$R_i(z) = I_r - \Sigma g_i \left[\gamma_i + e_i (z - f_{ii})^{-1} \right] \tilde{g}_i, \tag{37}$$

with $e_i = f_{ii}/|f_{ii}|$ ($= 1$ *if* $f_{ii} = 0$) *and* $\gamma_i = (1 + |f_{ii}|)^{-1}$.

Note in (37) that the f_{ii} are the diagonal entries of the lower triangular matrix F, in other words the poles of $R(z)$, and that the r-vectors g_i, iteratively computed by (35), satisfy the equalities

$$|f_{ii}|^2 + \tilde{g}_i \Sigma g_i = 1. \tag{38}$$

Note from (37) and the minimality of $R(z)$ that $g_i \neq 0$.

There exist, as shown by Potapov [10], remarkable forms for the first degree factors $R_i(z)$. To put these canonical forms into evidence via the triangular model approach, we need first the following preparatory lemma concerning Σ-isotropic vectors, i.e. nonzero r-vectors g satisfying $\tilde{g} \Sigma g = 0$.

LEMMA 8. *Let* g *be a nonzero vector such that* $\tilde{g} \Sigma g = 0$. *Then there exist decompositions of* g *as* $g = u + v$ *with the following properties*:

$$\tilde{u} \Sigma u = 1, \qquad \tilde{v} \Sigma v = -1, \qquad \tilde{v} \Sigma u = 0. \tag{39}$$

Proof. Let us partition g as $[g_p^T \; g_q^T]^T$ conformably with Σ. Clearly, there exist unitary matrices U_p and U_q of dimensions p and q respectively such as to have $g_p^1 = U_p g_p = K_1 [1, 0, \dots, 0]^T$ and $g_q^1 = U_q g_q = K_2 [1, 0, \dots, 0]^T$ with K_1 and K_2 positive real constants. Note that these constants cannot be zero, for otherwise g would be; moreover, $K_1 = K_2 = K$, since $\tilde{g}\Sigma g = \tilde{g}_p^1 g_p^1 - \tilde{g}_q^1 g_q^1 = K_1^2 - K_2^2 = 0$. It is then easily verified that vectors u, v, satisfying (39) are defined by

$$[u,v] = \frac{1}{2}\left[\begin{array}{c|c} \tilde{U}_p & \\ \hline & \tilde{U}_q \end{array}\right]\left[\begin{array}{cc} K+K^{-1} & K-K^{-1} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \hline K-K^{-1} & K+K^{-1} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{array}\right]. \tag{40}$$

THEOREM 9. *Any Σ-lossless transfer function $R(z)$ of degree n can be obtained as*

$$R(z) = W_0 \prod_{i=n}^{1} \Lambda(z, z_i) W_i, \tag{41}$$

where the z_i are the poles of $R(z)$, the W_i are Σ-unitary constant matrices, and the $\Lambda(z, z_i)$ are first degree Σ-lossless factors of the form

$$\Lambda(z, z_i) = \begin{cases} \dfrac{1-\bar{z}_i z}{z - z_i} \oplus I_{p-1} \oplus I_q & \text{if} \quad |z_i| < 1, \\ I_{p-1} \oplus I_2 - \dfrac{z+z_i}{z-z_i}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \oplus I_{q-1} & \text{if} \quad |z_i| = 1, \\ I_p \oplus \dfrac{1-\bar{z}_i z}{z-z_i} \oplus I_{q-1} & \text{if} \quad |z_i| > 1. \end{cases} \tag{42}$$

Proof. To achieve the proof of this theorem, it is clearly sufficient to show that there exist appropriate left and right Σ-unitary matrices reducing $R_i(z)$ as given by (37) to $\Lambda(z, z_i)$. Let us first assume $z_i \equiv f_{ii}$ to lie in the unit disk ($|z_i| < 1$), which implies $\tilde{g}_i \Sigma g_i = K^2 > 0$ by (38). From the results of Appendix 3, a Σ-unitary matrix V is known to exist with the vector g_i / K as its first column, whence $\tilde{V} \Sigma g_i = K[1,0,\ldots,0]^T$. From (37) and since

$K^2 = 1 - |z_i|^2$, one then deduces

$$\Sigma\tilde{V}R_i(z)\Sigma V = \left\{1 - K^2\left[\gamma_i + e_i(z - z_i)^{-1}\right]\right\} \oplus I_{n-1}$$

$$= -\frac{z_i}{|z_i|}\frac{1 - \bar{z}_i z}{z - z_i} \oplus I_{n-1},$$

and finally the first form (42), as the matrix $-z_i/|z_i| \oplus I_{n-1}$ is trivially Σ-unitary. The case $|z_i| > 1$ can obviously be treated in a similar way, and its detailed discussion is left to the reader. By using the decomposition of Lemma 8 for isotropic vectors when $|z_i| = 1$ in the form $g_i/\sqrt{2} = u_i + v_i$, we know from Appendix 3 that there exists a Σ-unitary matrix V with u_i and v_i as its pth and $(p+1)$th columns respectively, so that $\tilde{V}\Sigma g_i = \sqrt{2}\,[0,\ldots,0,1,-1,0,\ldots,0]^T$. As a result, one has

$$\Sigma\tilde{V}R_i(z)\Sigma V = I_{p-1} \oplus I_2 - 2\left[\gamma_i + e_i(z - z_i)^{-1}\right]\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \oplus I_{q-1}, \tag{43}$$

which yields directly the middle form (42), since $\gamma_i = \frac{1}{2}$ and $e_i = z_i$ when $|z_i| = 1$. ■

To complete this section, let us briefly indicate how to accommodate the factorization problem in the continuous time case along the same lines. A few modifications with respect to the discrete time case have to be introduced, which actually turn out to be simplifications.

Let $T(p) = \mathcal{D} + \mathcal{C}(pI_n - \mathcal{A})^{-1}\mathcal{B}$ be a minimal balanced realization of a Σ-lossless transfer function ($P \equiv I_n$) with $\mathcal{A}$ in lower block triangular form

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{\alpha\alpha} & \\ \mathcal{A}_{\beta\alpha} & \mathcal{A}_{\beta\beta} \end{bmatrix}. \tag{44}$$

With the conformal partitions $\tilde{\mathcal{B}} = [\tilde{\mathcal{B}}_\alpha, \tilde{\mathcal{B}}_\beta]$ and $\mathcal{C} = [\mathcal{C}_\alpha, \mathcal{C}_\beta]$, define the matrices

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}, \tag{45}$$

$$\mathcal{T}_\beta = \begin{bmatrix} I_\alpha & 0 & 0 \\ 0 & \mathcal{A}_{\beta\beta} & \mathcal{B}_\beta \\ 0 & -\mathcal{D}\Sigma\tilde{\mathcal{B}}_\beta & \mathcal{D} \end{bmatrix}, \qquad \mathcal{T}_\alpha = \begin{bmatrix} \mathcal{A}_{\alpha\alpha} & 0 & \mathcal{B}_\alpha \\ 0 & I_\beta & 0 \\ -\Sigma\tilde{\mathcal{B}}_\alpha & 0 & I_r \end{bmatrix}. \tag{46}$$

Although, contrary to the matrices $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}$ of the discrete time case, none of the above matrices is necessarily invertible, the following lemma, replacing Lemma 5, is established in a straightforward manner.

LEMMA 10. *The matrix* $\mathfrak{T}$ *can be factorized as* $\mathfrak{T} = \mathfrak{T}_\beta \mathfrak{T}_\alpha$; *moreover, the entries of both* $\mathfrak{T}_\alpha$ *and* $\mathfrak{T}_\beta$ *satisfy the relations* (23) *with* $P = I_n$, *so that* $T(p) = T_\beta(p) T_\alpha(p)$ *where* $T_\alpha(p)$ *and* $T_\beta(p)$ *are the* Σ*-lossless transfer functions*

$$T_\alpha(p) = I_r - \Sigma \tilde{\mathfrak{B}}_\alpha (pI_\alpha - \mathfrak{A}_{\alpha\alpha})^{-1} \mathfrak{B}_\alpha,$$

$$T_\beta(p) = \mathfrak{D}\left[I_r - \Sigma \tilde{\mathfrak{B}}_\beta (pI_\beta - \mathfrak{A}_{\beta\beta})^{-1} \mathfrak{B}_\beta \right]. \tag{47}$$

Proof. Clearly, the relations (23) with $P = I_n$ imply for the entries of the matrix T the equalities

$$\mathfrak{A}_{\beta\alpha} = -\mathfrak{B}_\beta \Sigma \tilde{\mathfrak{B}}_\alpha, \qquad \mathfrak{C}_\alpha = -\mathfrak{D}\Sigma\tilde{\mathfrak{B}}_\alpha, \qquad \mathfrak{C}_\beta = -\mathfrak{D}\Sigma\tilde{\mathfrak{B}}_\beta, \tag{48}$$

which imply $\mathfrak{T} = \mathfrak{T}_\beta \mathfrak{T}_\alpha$ as well as the claimed properties of $\mathfrak{T}_\alpha$ and $\mathfrak{T}_\beta$. ∎

The factorization theorem for $T(p)$ is then obvious provided $\mathfrak{A}$ has been put into a lower triangular form by the appropriate unitary transformation.

THEOREM 11. *Let the* a_{ii} *and the* $\tilde{b}_i$ *be respectively the successive diagonal entries of* $\mathfrak{A}$ *and the successive rows of* $\mathfrak{B}$. *Then, any* Σ*-lossless transfer function* $T(p)$ *admits a factorization in first degree* Σ*-lossless factors of the form* $T(p) = \mathfrak{D}\prod_{i=1}^{n} T_{n-i+1}(p)$ *with* $T_i(p) = I_r - \Sigma b_i (p - a_{ii})^{-1} \tilde{b}_i$.

Finally, by using exactly the same technique as in the proof of Theorem 9, the canonical form of the above factorization is found to be:

THEOREM 12. *Any* Σ*-lossless transfer function* $T(p)$ *of degree* n *can be factorized as*

$$T(p) = W_0 \prod_{i=n}^{1} \Omega(p, p_i) W_i, \tag{49}$$

where the p_i *are the poles of* $T(p)$, *the* W_i *are* Σ*-unitary constant matrices,* *and the* $\Omega(p, p_i)$ *are first degree* Σ*-lossless factors of the form*

$$\Omega(p, p_i) = \begin{cases} \dfrac{p+\bar{p}_i}{p-p_i} \oplus I_{p-1} \oplus I_q & \text{if} \quad \operatorname{Re} p_i < 0, \\ I_{p-1} \oplus I_2 - \dfrac{1}{p-p_i}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \oplus I_{q-1} & \text{if} \quad \operatorname{Re} p_i = 0, \\ I_p \oplus \dfrac{p+\bar{p}_i}{p-p_i} \oplus I_{q-1} & \text{if} \quad \operatorname{Re} p_i > 0. \end{cases} \tag{50}$$

4. EMBEDDING OF THE LYAPUNOV EQUATION

Let P and F, respectively a positive definite and an arbitrary $n \times n$ matrix, be given. Clearly, the difference $P - FP\tilde{F}$ is a Hermitian matrix and can be most generally written as

$$P - FP\tilde{F} = G\Sigma\tilde{G}, \tag{51}$$

where G is an $n \times r'$ matrix and Σ a signature matrix $\Sigma = I_{p'} \oplus I_{q'}$ with $p' + q' = r'$. Note that neither p', nor q', nor even the matrix G for a given pair (p', q') is uniquely defined. However, if the matrix $P - FP\tilde{F}$ has p positive and q negative eigenvalues and hence rank $r = p + q$, the following constraints on G and Σ directly result from the Sylvester law of inertia:

$$\operatorname{rank} G \geqslant r, \qquad p' \geqslant p, \qquad q' \geqslant q.$$

THEOREM 13. *For any admissible pair of matrices* (G, Σ), *there exists a* $(P \oplus \Sigma)$*-unitary embedding, in other words a square matrix* R *of dimension* $n + r'$,

$$R = \begin{bmatrix} F & G \\ H & J \end{bmatrix}, \tag{52}$$

satisfying $R(P \oplus \Sigma)\tilde{R} = P \oplus \Sigma$.

Proof. This theorem is an immediate consequence of the results of Appendix 3, particularized to $Z_1 = P$, $Z_2 = \Sigma$, $X_1 = [F, G]$. ■

Any admissible pair (G, Σ) does not lead necessarily to a minimal realization of the Σ-lossless transfer function defined on R by $R(z) = J + H(zI_n - F)^{-1}G$; this is readily seen from the example $F = I_n$, $G = H = 0$, and Σ an arbitrary signature matrix. However, there always exist admissible pairs (G, Σ) yielding minimal realizations, as shown by the following argument. Let (G, Σ) be any admissible pair, and assume $[F, G]$ not to be controllable. Extend G to the matrix $G_e = [G, G_a, G_a]$ so that G_e have full rank n, and set $\Sigma_e = \Sigma \oplus I_a \oplus - I_a$ with a the number of columns of G_a. From Theorem 13, there exists a $(P \oplus \Sigma_e)$-unitary matrix R_e corresponding to the admissible pair (G_e, Σ_e), and the realization of the Σ_e-lossless transfer function $R_e(z) = J_e + H_e(zI_n - F)^{-1}G_e$ is controllable by construction. The latter realization is actually minimal on the strength of the following theorem.

THEOREM 14. *Let R be a $(P \oplus \Sigma)$-unitary matrix. If the realization of the corresponding Σ-lossless transfer function $R(z) = J + H(zI_n - F)^{-1} G$ is controllable, it is minimal.*

Proof. From the equations $V_o P \tilde{V}_o = I_n \otimes \Sigma - K(I_n \otimes \Sigma)\tilde{K}$ and $\tilde{V}_c P^{-1} V_c = E[I_n \otimes \Sigma - \tilde{K}(I_n \otimes \Sigma)K]E$ established at the end of Section 2, it is clearly sufficient to show that $I_n \otimes \Sigma - K(I_n \otimes \Sigma)\tilde{K}$ and $I_n \otimes \Sigma - \tilde{K}(I_n \otimes \Sigma)K$ have the same rank. From the identity

$$\begin{bmatrix} I_n \otimes \Sigma & K \\ \tilde{K} & I_n \otimes \Sigma \end{bmatrix} = \begin{bmatrix} & I_{rn} \\ I_{rn} & \end{bmatrix} \begin{bmatrix} I_n \otimes \Sigma & \tilde{K} \\ K & I_n \otimes \Sigma \end{bmatrix} \begin{bmatrix} & I_{rn} \\ I_{rn} & \end{bmatrix}$$

it appears by using Schur complement techniques that the two matrices under consideration have actually the same rank and signature. ■

We next give a *recursive* algorithm to the embedding problem R, directly from the equation $P - FP\tilde{F} = G\Sigma\tilde{G}$, and assuming the pair $[F, G]$ to be controllable. This algorithm is essentially based upon the balanced triangular models introduced in Section 3; as a side result, a complete factorization of the corresponding Σ-lossless transfer function is obtained.

Consider a similarity transformation T reducing F to a lower triangular form $T^{-1}FT$, and let $M\tilde{M}$ be the Cholesky decomposition of $T^{-1}P\tilde{T}^{-1}$ with M lower triangular. Clearly, the matrix $\mathcal{F} = M^{-1}T^{-1}FTM$ is lower triangular and satisfies $I_n - \mathcal{F}\tilde{\mathcal{F}} = \mathcal{G}\Sigma\tilde{\mathcal{G}}$ with $\mathcal{G} = M^{-1}T^{-1}G$. Let f_{11} be the first diagonal entry of F, and $\tilde{g}_1$ be the first row of $\mathcal{G}$:

$$\mathcal{F} = \begin{bmatrix} f_{11} & \\ F_{21} & F_{22} \end{bmatrix}, \qquad \mathcal{G} = \begin{bmatrix} \tilde{g}_1 \\ G_2 \end{bmatrix}. \tag{53}$$

With $\mathcal{R}_1$ as in (36) and $G_2' = G_2 - F_{21}\tilde{g}$, one has

$$[\mathcal{F}, \mathcal{G}] = \begin{bmatrix} 1 & 0 & 0 \\ y & \mathcal{F}_{22} & \mathcal{G}_2' \end{bmatrix} \cdot \mathcal{R}_1, \tag{54}$$

where y is some $(n-1)$-vector, which actually must be zero due to $I - \mathcal{F}\tilde{\mathcal{F}} = \mathcal{G}\Sigma\tilde{\mathcal{G}}$. Since $\mathcal{F}_{22}$ is unmodified in the process and hence still lower triangular, this deflation procedure can be iterated up to

$$[\mathcal{F}, \mathcal{G}] = [I_n, 0]\mathcal{R} \tag{55}$$

with $\mathcal{R} = \prod_{i=1}^{n} \mathcal{R}_{n-i+1}$ and $\mathcal{R}_k$ of the form

$$\mathcal{R}_k = \begin{bmatrix} I_{k-1} & 0 & 0 & 0 \\ 0 & f_{kk} & 0 & \tilde{g}_k \\ 0 & 0 & I_{n-k} & 0 \\ 0 & e_k \Sigma g_k & 0 & I_r - \gamma_k \Sigma g_k \tilde{g}_k \end{bmatrix}. \tag{56}$$

Clearly, the matrix $\mathcal{R}$ is $(I_n \oplus \Sigma)$-unitary and solves the embedding problem associated to the given pair $[F, G]$ at the extra cost of computing $H = \mathcal{H}T^{-1}M^{-1}$. It is essential to point out however that the actual determination of the matrix R is not required to obtain the factorized form of the corresponding Σ-lossless transfer function, which is immediately available from (56) and Corollary 7.

From a computational viewpoint, it is most interesting to keep the dimension of the embedding R as small as possible. From (57), the minimal dimension of the embedding is known to be $n + r$, where r is the rank of $P - FP\tilde{F}$. It has been indicated however that such a choice does not lead to a minimal realization R in all cases. The following strong result is nevertheless available.

THEOREM 15. *If none of the eigenvalues of F lies on the unit circle, all embeddings R yields minimal realizations of the corresponding Σ-lossless transfer functions.*

Proof. Starting from $P = FP\tilde{F} + G\Sigma\tilde{G}$, one easily derives by induction $P = F^k P \tilde{F}^k + \sum_{l=0}^{k-1} F^l G \Sigma \tilde{G} \tilde{F}^l$ for all $k \geqslant 0$. If the pair $[F, G]$ is not controllable, there exists a nonzero vector y with the property $\tilde{y}F^lG=0$ for all l, whence $\tilde{y}Py=\tilde{y}F^kP\tilde{F}^ky$ for all k. Since P is positive definite and $\tilde{y}F^kP\tilde{F}^ky$ tends necessarily to zero or infinity for $k\to\infty$, this leads to a contradiction. The proof is then completed by duality on the observability condition. ∎

Finally, let us stress the following point. There exists an obvious *duality* between R and $\tilde{R}$ in the sense that if R is $(P\oplus\Sigma)$-unitary, then $\tilde{R}$ is $(P^{-1}\oplus\Sigma)$-unitary, and conversely. It is then natural to think of exploiting this duality for inverting a given positive definite matrix P with the help of a suitable operator F. This idea is precisely the cornerstone of the generalized Levinson algorithms for the inversion of matrices close to Toeplitz in the displacement rank sense [7, 8]. In the next section, it will be shown that when P is a Toeplitz matrix and F the shift operator, the related embedding R and its associated Σ-lossless transfer function yield directly the ladder implementation of the classical Levinson algorithm.

All theorems and properties related to the embeddings of the Lyapunov equation in the discrete time case can be translated to the continuous time case without difficulty. Let us just mention that a solution to the embedding problem is particularly simple in the continuous time case. Let us indeed consider the Lyapunov equation

$$AP+P\tilde{A}=-B\Sigma\tilde{B} \tag{57}$$

Assuming the pair $[A,B]$ to be controllable, $C=-\Sigma\tilde{B}P^{-1}$ and $D=I_r$ provide a solution to the embedding problem, as it can be easily verified from (23), and the realization of the corresponding Σ-lossless transfer function $T(p)=I_r-\Sigma\tilde{B}P^{-1}(pI_n-A)^{-1}B$ is minimal. Finally, by using the technique of balanced triangular models, the complete factorization of $T(p)$ can easily be achieved with the help of Theorem 11.

5. EXAMPLE

As an example of application of the proposed theory relative to Σ-lossless transfer functions, we will briefly discuss the standard problem of inverting a Toeplitz matrix as well as the ladder realization of the related Levinson algorithm. These classical questions will be approached via the technique of the displacement ranks [7]; in this context, the power of the methods proposed in this paper will appear in full light and should lead the way to a systematic treatment and a deep understanding of the generalized Levinson algorithms.

Let P be a positive definite Toeplitz matrix

$$P=\begin{bmatrix} c_0 & \bar{c}_1 & \cdots & \bar{c}_n \\ c_1 & c_0 & & \bar{c}_{n-1} \\ \vdots & & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{bmatrix}, \tag{58}$$

where c_0 will be assumed to have been normalized to $c_0 \equiv 1$, and define F as the lower shift operator of dimension $n+1$:

$$F=\begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}. \tag{59}$$

By definition, the displacement rank of P is the rank of the matrix $P - FP\tilde{F}$, and one has clearly

$$P - FP\tilde{F}=\begin{bmatrix} 1 & \bar{c}_1 & \cdots & \bar{c}_n \\ c_1 & & & \\ \vdots & & 0 & \\ c_n & & & \end{bmatrix}, \tag{60}$$

so that the displacement rank of any nontrivial Toeplitz matrix appears to be 2. Note that the eigenvalues of F are all at the origin $z=0$; hence, in view of Theorem 15 one can look for an embedding of the Lyapunov equation of minimum dimension. Since $P - FP\tilde{F}$ has the signature $(1, -1)$, we have to choose Σ as the matrix $1 \oplus -1$, and a matrix G satisfying $P - FP\tilde{F} = G\Sigma\tilde{G}$ is easily found to be

$$\tilde{G}=\begin{bmatrix} 1 & \bar{c}_1 & \cdots & \bar{c}_n \\ 0 & -\bar{c}_1 & \cdots & -\bar{c}_n \end{bmatrix}. \tag{61}$$

The construction of the $(P \oplus \Sigma)$-unitary matrix R (52) embedding the Lyapunov equation considered can easily be achieved with the help of the coefficients $(a_0, a_1, \ldots, a_n)$ satisfying the linear system $P[a_0, a_1, \ldots, a_n]^T =$

$[1,0,\dots,0]^T$. After some elementary calculations, one finds that a suitable choice for the matrices H and J is given by

$$H = \frac{1}{\sqrt{a_0}}\begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 & a_0 \\ \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n & 0 \end{bmatrix}, \tag{62}$$

$$J = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{a_0} \end{bmatrix}. \tag{63}$$

Since R is $(P \oplus \Sigma)$-unitary; then $\tilde{R}$ is $(P^{-1} \oplus \Sigma)$-unitary; one has in particular the Lyapunov equation $P^{-1} - \tilde{F}P^{-1}F = \tilde{H}\Sigma H$, and hence, as $F^l = 0$ for all $l > n$,

$$P^{-1} = \sum_{k=0}^{n} \tilde{F}^k \tilde{H} \Sigma H F^k. \tag{64}$$

By inserting (62) into (64), one derives directly the Gohberg-Semencul formula for the inverse of a Toeplitz matrix [15]:

$$P^{-1} = \frac{1}{a_0}\begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ \vdots & \vdots & \ddots & \\ a_n & a_{n-1} & \cdots & a_0 \end{bmatrix}\begin{bmatrix} \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_n \\ & \bar{a}_0 & & \bar{a}_{n-1} \\ & & \ddots & \vdots \\ & & & \bar{a}_0 \end{bmatrix}$$

$$-\frac{1}{a_0}\begin{bmatrix} 0 & & & \\ \bar{a}_n & 0 & & \\ \vdots & & \ddots & \\ \bar{a}_1 & \cdots & \bar{a}_n & 0 \end{bmatrix}\begin{bmatrix} 0 & \bar{a}_n & \cdots & \tilde{a}_1 \\ & 0 & & \vdots \\ & & \ddots & \bar{a}_n \\ & & & 0 \end{bmatrix}. \tag{65}$$

Let us now consider R as the realization of the Σ-lossless transfer function $R(z) = J + H(zI_{n-1} - F)^{-1}G$. The polynomial $a(z) = \Sigma_{i=0}^{n} a_i z^i$ is clearly identified with the normalized reciprocal of the Szegö orthogonal polynomial of the first kind relative to P; introducing the orthogonal polynomial of the second kind $r(z)$ via the equation $(1 + 2c_1 z + \cdots + 2c_n z^n)a(z) = r(z) + O(z^{n+1})$, one easily verifies that $R(z)$ can be written down as

$$R(z) = \frac{1}{2\sqrt{a_0}}\begin{bmatrix} z^{-(n+1)} & \\ & z^{-n} \end{bmatrix}\begin{bmatrix} a(z)+r(z) & a(z)-r(z) \\ \hat{a}(z)-\hat{r}(z) & \hat{a}(z)+\hat{r}(z) \end{bmatrix} \tag{66}$$

with $\hat{a}(z)=z^n\bar{a}(1/\bar{z})$ and $\hat{r}(z)=z^n\bar{r}(1/\bar{z})$, the reciprocals of $a(z)$ and $r(z)$ respectively.

From the theory of Schur-Szegö parameters associated with P [14], the above matrix $R(z)$ is known to admit the factorization

$$R(z)=\prod_{k=n}^{0}\frac{1}{\left(1-|e_k^2|\right)^{1/2}}\begin{bmatrix} z^{-1} & \\ & 1\end{bmatrix}\begin{bmatrix} 1 & e_k \\ \bar{e}_k & 1\end{bmatrix} \tag{67}$$

with $e_0=0, e_1, e_2,\dots,e_n$ the successive Schur-Szegö parameters relative to P. Note that the above factorization has precisely the canonical form established in Theorem 9.

Besides, it is not difficult to show that this factorized form is directly achieved by the recursive algorithm described in Section 4. Details of this, which imply elementary algebraic manipulations only, are left to the reader.

APPENDIX 1

In this appendix, the explicit formulas will be given to derive from a continuous time realization a discrete time equivalent and conversely.

The transformation $p=r(z-1)/(z+1)$ is well known to map the regions $\operatorname{Re} p>0$, $\operatorname{Re} p=0$, $\operatorname{Re} p<0$ onto the regions $|z|>1, |z|=1, |z|<1$ respectively, for any positive value of r. The matrix $\left(r\dfrac{z-1}{z+1}I_n-A\right)^{-1}$ can then be written as

$$\left(r\frac{z-1}{z+1}I_n-A\right)^{-1}=(rI_n-A)^{-1}+X(z) \tag{A.1}$$

provided r is not an eigenvalue of A, which is trivially always possible. The explicit form of the strictly proper matrix $X(z)$ is easily found to be

$$X(z)=2r(rI_n-A)^{-1}\left[zI_n-(rI_n-A)^{-1}(rI_n+A)\right]^{-1}(rI_n-A)^{-1}. \tag{A.2}$$

Let $T(p)=D+C(pI_n-A)^{-1}B$ be a state space realization of a continuous time transfer function; a discrete time copy of $T(p)$, namely $R(z)=J+$

$H(zI_n - F)^{-1}G$, can immediately be deduced from (A.1),(A.2) by setting

$$\begin{aligned} F &= (rI_n + A)(rI_n - A)^{-1} = (rI_n - A)^{-1}(rI_n + A) \\ G &= \sqrt{2}\, r(rI_n - A)^{-1}B, \\ H &= \sqrt{2}\, rC(rI_n - A)^{-1}, \\ J &= D + C(rI_n - A)^{-1}B. \end{aligned} \tag{A.3}$$

It is easily verified that if $T(p)$ is Σ-lossless, so is $R(z)$, and moreover that the matrices P appearing in (3) and (4) are identical for both realizations. Finally, note that a minimal realization of $T(p)$ necessarily yields a minimal realization of $R(z)$.

The same argument can obviously be used to obtain the converse formulas, via the transformation $z = e^{j\theta}(1+p)/(1-p)$ with θ an arbitrary real number. After elementary computations, these formulas are found to be

$$\begin{aligned} A &= \left(F - e^{j\theta}I_n\right)\left(e^{j\theta}I_n + F\right)^{-1} = \left(e^{j\theta}I_n + F\right)^{-1}\left(F - e^{j\theta}I_n\right), \\ B &= \sqrt{2}\, e^{j\theta/2}\left(e^{j\theta}I_n + F\right)^{-1}G, \\ C &= \sqrt{2}\, e^{j\theta/2}H\left(e^{j\theta}I_n + F\right)^{-1}, \\ D &= J - H\left(e^{j\theta}I_n + F\right)^{-1}G. \end{aligned} \tag{A.4}$$

APPENDIX 2

The aim of this appendix is to establish a remarkable property of quadratic forms.

Define Z' and Z'' as the direct sums $Z' = Z_1 \oplus Z_2$ and $Z'' = Z_1 \oplus - Z_2$ where Z_1 and Z_2 are arbitrary nonsingular Hermitian matrices. Let

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{B.1}$$

be an arbitrary square matrix partitioned conformably with Z', Z'', and further assume X_{22} to be invertible, so that the matrix

$$Y = \begin{bmatrix} X_{11} - X_{12}X_{22}^{-1}X_{21} & X_{12}X_{22}^{-1} \\ -X_{22}^{-1}X_{21} & X_{22}^{-1} \end{bmatrix} \tag{B.2}$$

is well defined.

THEOREM 16. *The matrices $Z' - XZ'\tilde{X}$ and $Z'' - YZ''\tilde{Y}$ are congruent.*

Proof. Note that the matrix Y can be written as $Y = W_1^{-1}W_2$ with

$$W_1 = \begin{bmatrix} I & -X_{12} \\ 0 & -X_{22} \end{bmatrix} \quad \text{and} \quad W_2 = \begin{bmatrix} X_{11} & 0 \\ X_{21} & -I \end{bmatrix}. \tag{B.3}$$

Hence, the congruence relation

$$Z' - XZ\tilde{X} = W_1[Z'' - YZ''\tilde{Y}]\tilde{W}_1 \tag{B.4}$$

is straightforward. ■

APPENDIX 3

Let Z_1 and Z_2 be arbitrary nonsingular Hermitian matrices of dimensions n_1 and n_2 respectively. A matrix X is said to be *Z-unitary* with $Z = Z_1 \oplus Z_2$ if $XZ\tilde{X} = Z$. Partitioning X conformably with Z as $X_1 = [X_{11}, X_{12}]$, $X_2 = [X_{21}, X_{22}]$, one can rewrite the Z-unitarity relation as

$$X_1 Z\tilde{X}_1 = Z_1, \tag{C.1}$$

$$X_2 Z\tilde{X}_1 = 0, \qquad X_2 Z\tilde{X}_2 = Z_2. \tag{C.2}$$

The purpose of this appendix is to show that given an arbitrary $n_1 \times (n_1 + n_2)$ matrix X_1 satisfying (C.1), one can always find a Z-unitary embedding X of X_1, by constructing a matrix X_2 satisfying (C.2). Note that in the case where Z is positive definite, this embedding property is trivial via the Gram-Schmidt orthogonalization procedure; such is not the case in general, however, due to the existence of Z-isotropic vectors, i.e. vectors v satisfying $\tilde{v}Zv = 0$ with $v \neq 0$.

To begin with, let us observe that X_1 has full rank n_1 by (C.1); hence there exists a nonsingular matrix T of dimension $n_1 + n_2$ producing

$$X_1 Z T = [Y_1, 0] \tag{C.3}$$

with Y_1 square nonsingular. Since X_2 must have full rank in view of the second constraint (C.2), it must have the form

$$X_2 = [0, Y_2]\tilde{T} \tag{C.4}$$

with Y_2 square nonsingular, so as to satisfy $X_2 Z \tilde{X}_1 = 0$. The second condition (C.2) then becomes

$$Y_2 R \tilde{Y}_2 = Z_2 \tag{C.5}$$

with R the bottom right block entry of $\tilde{T}ZT$. Let us then define two nonsingular congruence transformations M and N so as to have $Z_2 = M\Delta\tilde{M}$ and $R = N\Gamma\tilde{N}$ with $\Delta = I_{s_1} \oplus - I_{s_2}$, $\Gamma = I_{t_1} \oplus - I_{t_2} \oplus 0_{t_3}$, and necessarily $s_1 + s_2 = t_1 + t_2 + t_3 = n_2$. We claim that the matrix $Y_2 = MN^{-1}$ is a solution of the problem at hand. To prove this, we have to establish in view of (C.5) the equality $\Delta = \Gamma$ and hence $t_1 = s_1$, $t_2 = s_2$, and $t_3 = 0$. First, the resulting matrix X is nonsingular: indeed, X_1 and X_2 both have full rank on the one hand, and the equality $\tilde{v}_2 X_2 = \tilde{v}_1 X_1$ would imply $0 = \tilde{v}_2 X_2 Z \tilde{X}_1 = \tilde{v}_1 X_1 Z \tilde{X}_1 = \tilde{v}_1 Z_1$ and hence $v_1 \equiv v_2 \equiv 0$ on the other hand. Secondly, one has by construction $XZ\tilde{X} = Z_1 \oplus M\Gamma\tilde{M}$, which by Sylvester's law of inertia forces $\Delta = \Gamma$.

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