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Stable Partial Realizations via an Implicitly Restarted Lanczos Method *

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Abstract

The nonsymmetric Lanczos method has recently received attention as a model reduction technique for large-scale systems. Unfortunately, the Lanczos method may generate an unstable partial realization for a given stable system. To remedy this situation, inexpensive implicit restarts are developed to stabilize a Lanczos generated model.

1. Introduction

This paper employs a modified Lanczos method to acquire a stable reduced order model for a stable, SISO system described by the state space equations

$$\dot{x} = Ax + bu \quad (1)$$
$$y = cx + du. \quad (2)$$

It is important to note that the $A \in \Re^{n\times n}$ matrix will always be large and sparse in the following. System matrices of this type arise, for example, out of finite-difference discretizations of various plants including chemical processes and physical structures.

Standard model reduction techniques are ill-suited for large, sparse problems due to the sheer size of A. For example, many of the "optimal" reduction strategies (balanced realization [10], etc.) require knowledge of the solutions to the Lyapunov equations

$$AG_c + G_cA^T + bb^T = 0 \quad (3)$$
$$A^TG_o + G_oA + c^Tc = 0. \quad (4)$$

Conventional computational techniques for solving (3) or (4) entail $O(n^3)$ operations [1], an unacceptable cost when n is large.

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As an alternative, this paper will employ the oblique projector $\pi_k = V_kW_k^T$ to yield a k^{th} order model

$$\dot{\hat{x}} = (W_k^TAV_k)\hat{x} + (W_k^Tb)u = \hat{A}\hat{x} + \hat{b}u$$
$$\hat{y} = (cV_k)\hat{x} + du = \hat{c}\hat{x} + du$$

for the original system in (1) and (2). The matrices $V_k \in \Re^{n\times k}$ and $W_k \in \Re^{n\times k}$ are biorthogonal, i.e. $W_k^TV_k = I$. Moreover, V_k and W_k are related to Krylov spaces, $\mathcal{K}_k$, in that

$$\text{COLSP}(V_k) = \mathcal{K}_k(A, v_1) \quad (5)$$
$$= \text{span}\{v_1, Av_1, \ldots, A^{k-1}v_1\}$$
$$\text{COLSP}(W_k) = \mathcal{K}_k(A^T, w_1). \quad (6)$$

The utility of the Krylov projector comes from the fact that both V_k and W_k can be generated with only inner-products and matrix-vector multiplications. By taking advantage of the fact that the A is sparse, one can compute the projector relatively cheaply. But regardless of how quickly π_k can be computed, one is certainly also interested in the correspondence between the original and reduced order systems. A major insight into this relationship comes from [7,15].

Theorem 1 *Let the reduced order system $(\hat{A}, \hat{b}, \hat{c})$ be a restriction of the system (A, b, c) by the projector π_k where V_k and W_k are defined as in (5) and (6). If the starting vectors, v_1 and w_1, are parallel to b and c^T respectively, then the first $2k$ Markov parameters of the original and reduced-order systems are identical.*

Restating Theorem 1, the reduced order model is a Padé approximation (partial realization) which matches the first $2k$ moments $(cA^{i-1}b, 1 \le i \le 2k)$ of the original system.

Model reduction via Padé approximation (moment matching) has a long history in the literature [11,17]. Thus the observations of [15] are certainly of interest. But the concept of using oblique projectors for

Padé approximation can be taken one step further by forming V_k and W_k via a nonsymmetric Lanczos method [9]. The Lanczos algorithm (fully reorthogonalized) simultaneously computes the projector, π_k, and a tridiagonal $\hat{A}$ with only $O(k^2n)$ operations. Lanczos model reduction is discussed in a multitude of recent papers including [2,8,13,14].

Model reduction via a Krylov projector is certainly cheaper, $O(k^2n)$, than the "optimal" reduction techniques, $O(n^3)$, as $n \gg k$. However, three significant difficulties are associated with utilizing the projector, π_k: singularities in the Padé table (serious Lanczos breakdowns), large steady-state response error due to matching only Markov parameters, and the potential of unstable models for stable systems [4]. At least in the SISO case, the first problem can be avoided by incorporating look-ahead into the Lanczos method [6]. And to overcome the second difficulty, many of the above references propose moment matching about multiple frequencies. This paper will not dwell on these first two difficulties (although the second issue especially is in need of further work); rather it will concentrate on the stability of the reduced-order model. Note that we are not the first to do so. In [13,15], the stability of the reduced-order model is insured by incorporating an inverted grammian, G_c^{-1}, into the projector. However, solving (3) and inverting G_c are $O(n^3)$ operations. The unacceptable cost of this fix overshadows the efficiency of Lanczos-based model reduction.

As an alternative, we propose handling the stability issue by modifying the choice for the projector. If the results with the projector, π_k, are unstable, a related projector, $\bar{\pi}_k = \bar{V}_k \bar{W}_k^T$, is selected which corresponds to the new starting vectors,

$$\bar{v}_1 = \zeta_{\bar{v}}(A - \mu_p I)\dots(A - \mu_1 I)v_1 \qquad (7)$$

$$\bar{w}_1 = \zeta_{\bar{w}}(A^T - \mu_p I)\dots(A^T - \mu_1 I)w_1. \qquad (8)$$

The parameters, μ_i, provide the freedom for modifying the projector. In §3, a new and inexpensive technique, implicitly restarting the Lanczos algorithm, is developed for directly generating this modified projector, $\bar{\pi}_k$, from π_k. An example of this technique is provided in §4 which demonstrates the potential of altering the projector. The paper concludes with some final remarks in §5 and §6.

2. The Standard Lanczos Method

Before exploring restarts, a brief review of the standard nonsymmetric Lanczos algorithm will be provided. For a more detailed discussion of the algorithm, the reader is referred to [5].

Given the starting vectors v_1 and w_1, the Lanczos algorithm produces the matrices $V_k = [v_1, \dots, v_k]$ and $W_k = [w_1, \dots, w_k]$ satisfying the recursive identities

$$AV_k = V_k T_k + \beta_{k+1} v_{k+1} e_k^T \qquad (9)$$

$$A^T W_k = W_k T_k^T + \gamma_{k+1} w_{k+1} e_k^T. \qquad (10)$$

The vector e_k is the k^{th} standard basis vector and T_k is a truncated reduction of A that is tridiagonal. Generally, the elements β_i and γ_i are chosen so that $V_{k+1}^T W_{k+1} = I$. When this biorthogonality condition holds, multiplying (9) on the left by W_k^T yields the relationship $W_k^T A V_k = T_k$. It will also be convenient in the following to denote the residuals $\beta_{k+1} v_{k+1}$ and $\gamma_{k+1} w_{k+1}$ as the vectors r_k and q_k, respectively. The relationships $r_k \in \mathcal{K}_{k+1}(A, v_1)$ and $q_k \in \mathcal{K}_{k+1}(A^T, w_1)$ come from (9) and (10).

With the Lanczos method defined, it is a simple matter to connect it to model reduction via a Krylov projector. The initial vectors should be chosen as $v_1 = b/\beta_1$ and $w_1 = c^T/\gamma_1$ so that π_k corresponds to the Krylov spaces $\mathcal{K}(A, b)$ and $\mathcal{K}(A^T, c^T)$ respectively. Then $\hat{A} = W^T A V = T_k$, $\hat{b} = W^T b = e_1 \beta_1$, $\hat{c} = cV_k = e_1^T \gamma_1$ is the desired partial realization.

3. Implicitly Restarted Lanczos

The degree of success achieved in applying a Lanczos-type method is dependent upon the choice of starting vectors, v_1 and w_1. In some cases, such as the model reduction problem, one can make an educated initial guess for these starting vectors ($v_1 = b/\beta_1$ and $w_1 = c^T/\gamma_1$). But the stable plant, unstable reduced model issue demonstrates that what may appear as a good choice for the starting vectors can yield disastrous results. To overcome the results of a poor starting vector, one could repeatedly and explicitly recompute Krylov spaces with a modified pair of initial vectors. For lack of better data, one should use information from past results to refine these new starting vectors. Yet such an approach becomes computationally expensive when several such restarts are required. Each restart costs $O(k^2n)$ flops.

In this section, an implicit approach (an analogue to implicitly restarted Arnoldi [12]) is developed for generating the modified projector corresponding to the starting vectors in (7,8). It will be shown that given V_k and W_k, one can generate $\bar{V}_k$ and $\bar{W}_k$ more efficiently with implicit restarts. Also, experiments indicate a higher precision in $\bar{\pi}_k$ for the implicit method.

As a simple step between the standard Lanczos method and the new factorization corresponding to (7,8), we will first derive a technique for implicitly obtaining a $\tilde{V}_k$ and $\tilde{W}_k$ which correspond to the starting vectors $\tilde{v}_1 = \rho_v(A - \mu I)v_1$ and $\tilde{w}_1 = \rho_w(A^T - \mu I)w_1$. For the time being, the parameter, μ, is assumed to be real.

The first step in performing an implicit restart is obtaining the two LR-decompositions $L_v R_v = (T_k - \mu I)$ and $L_w R_w = (L_v^{-1} T_k L_v - \mu I)^T$ where in each case L is unit lower-triangular and R is upper-triangular. Due to the band preserving properties of the LR-decomposition [16], L and R are bidiagonal. With these LR-decompositions defined, (9) and (10) can be updated to

$$\begin{aligned} AV_k L_v L_w^{-T} &= V_k L_v L_w^{-T} (L_w^T L_v^{-1} T_k L_v L_w^{-T}) \\ &\quad + r_k e_k^T L_v L_w^{-T} \qquad (11) \\ A^T W_k L_v^{-T} L_w &= W_k L_v^{-T} L_w (L_w^{-1} L_v^T T_k^T L_v^{-T} L_w) \\ &\quad + q_k e_k^T L_v^{-T} L_w. \qquad (12) \end{aligned}$$

If one defines $\tilde{V}_k = V_k L_v L_w^{-T}$, $\tilde{W}_k = W_k L_v^{-T} L_w$ and $\tilde{T}_k = L_w^T L_v^{-1} T_k L_v L_w^{-T}$, then (11) and (12) become

$$\begin{aligned} A\tilde{V}_k &= \tilde{V}_k \tilde{T}_k + r_k e_k^T L_v L_w^{-T} \qquad (13) \\ A^T \tilde{W}_k &= \tilde{W}_k \tilde{T}_k^T + q_k e_k^T L_v^{-T} L_w. \qquad (14) \end{aligned}$$

To see the relationship between the new and old starting vectors (i.e., v_1 and w_1 versus $\tilde{v}_1$ and $\tilde{w}_1$), rewrite (9) as

$$\begin{aligned} (A - \mu I)V_k &= V_k(T_k - \mu I) + r_k e_k^T \\ &= V_k L_v R_v + r_k e_k^T. \qquad (15) \end{aligned}$$

Multiplying (15) on the right by $e_1 = L_w^{-T} e_1$ gives the relation

$$(A - \mu I)V_k e_1 = V_k L_v e_1 \rho_v^{-1} = V_k L_v L_w^{-T} e_1 \rho_v^{-1}$$

where $\rho_v^{-1} = e_1^T R_v e_1$. A similar derivation may be applied to (10) to yield that the two new starting vectors are in fact $\tilde{v}_1 = \rho_v(A - \mu I)v_1$ and $\tilde{w}_1 = \rho_w(A^T - \mu I)w_1$ where $\rho_w^{-1} = e_1^T R_w e_1$.

Clearly we are nearing the desired result; new starting vectors have been obtained which fit the desired form. Unfortunately, (13) and (14) are not valid Lanczos identities. Define l_v and l_w to be elements of the products $L_v L_w^{-T}$ and $L_v^{-T} L_w$ respectively. Then in (13) and (14), the residuals are multiplied by

$$\begin{aligned} e_k^T L_v L_w^{-T} &= \left(0, 0, \ldots 0, l_v^{(k,k-1)}, l_v^{(k,k)}\right) \text{ or} \\ e_k^T L_v^{-T} L_w &= \left(0, 0, \ldots 0, l_w^{(k,k-1)}, l_w^{(k,k)}\right) \end{aligned}$$

rather than just e_k^T. However, one can obtain a valid Lanczos factorization by simply truncating off a portion of (13) and (14). Rewrite (13) as

$$A\tilde{V}_k = (\tilde{V}_{k-1}, \tilde{v}_k, r_k) \left(\begin{array}{c|c} \tilde{T}_{k-1} & \tilde{\gamma}_k e_{k-1} \\ \tilde{\beta}_k e_{k-1}^T & \tilde{\alpha}_k \\ l_v^{(k,k-1)} e_{k-1}^T & l_v^{(k,k)} \end{array} \right)$$

and (14) as

$$A^T \tilde{W}_k = (\tilde{W}_{k-1}, \tilde{w}_k, q_k) \left(\begin{array}{c|c} \tilde{T}_{k-1}^T & \tilde{\beta}_k e_{k-1} \\ \tilde{\gamma}_k e_{k-1}^T & \tilde{\alpha}_k \\ l_w^{(k,k-1)} e_{k-1}^T & l_w^{(k,k)} \end{array} \right).$$

Equating the first $k-1$ columns of these two expressions yields the new Lanczos identities

$$\begin{aligned} A\tilde{V}_{k-1} &= \tilde{V}_{k-1} \tilde{T}_{k-1} + \tilde{r}_{k-1} e_{k-1}^T \qquad (16) \\ A^T \tilde{W}_{k-1} &= \tilde{W}_{k-1} \tilde{T}_{k-1}^T + \tilde{q}_{k-1} e_{k-1}^T. \qquad (17) \end{aligned}$$

The new starting vectors are still defined as above while the new residual vectors are

$$\begin{aligned} \tilde{r}_{k-1} &= \tilde{\beta}_k \tilde{v}_k + l_v^{(k,k-1)} r_k \\ \tilde{q}_{k-1} &= \tilde{\gamma}_k \tilde{w}_k + l_w^{(k,k-1)} q_k. \end{aligned}$$

One can also show that $\tilde{V}_{k-1}$, $\tilde{W}_{k-1}$, $\tilde{r}_{k-1}$, and $\tilde{q}_{k-1}$ meet the biorthogonality condition. It is further claimed that one can only insure $\tilde{r}_{k-1}^T \tilde{q}_{k-1} \neq 0$ because both LR-decompositions are included in the above development. For example, if *only* L_v was incorporated into the above expressions, μ's would exist (mainly the eigenvalues of T_k) for which the new residual vectors would be orthogonal.

From the above work, an extension to the general case is straightforward. One is now interested in a series of LR-decompositions. Define

$$\begin{aligned} L_{v_i} R_{v_i} &= (\bar{L}_{i-1} T_k \bar{L}_{i-1}^{-1} - \mu_i I) \\ L_{w_i} R_{w_i} &= (L_{v_i}^T \bar{L}_{i-1}^{-T} T_k^T \bar{L}_{i-1}^T L_{v_i}^{-T} - \mu_i I) \end{aligned}$$

where

$$\bar{L}_{i-1} = L_{w_{i-1}}^T L_{v_{i-1}}^{-1} \cdots L_{w_1}^T L_{v_1}^{-1}.$$

Note that in practice, one should determine L_{v_i} and L_{w_i} via an implicit LR approach (see §5). Pairs of complex conjugate shifts would be handled via double LR shifts [16].

Corresponding to p implicit restarts are the new identities

$$\begin{aligned} A\bar{V}_{k-p} &= \bar{V}_{k-p} \bar{T}_{k-p} + \bar{r}_{k-p} e_{k-p}^T \\ A^T \bar{W}_{k-p} &= \bar{W}_{k-p} \bar{T}_{k-p}^T + \bar{q}_{k-p} e_{k-p}^T \end{aligned}$$

where $\bar{T}_{k-p}$, $\bar{V}_{k-p}$ and $\bar{W}_{k-p}$ are the appropriate submatrices of $\bar{T}_k = \bar{L}_p T_k \bar{L}_p^{-1}$, $\bar{V}_k = V_k \bar{L}_p^{-1}$ and $\bar{W}_k = W_k \bar{L}_p^T$. The new residuals are

$$\begin{aligned} \bar{r}_{k-p} &= \bar{\beta}_{k-p+1} \bar{v}_{k-p+1} + \bar{l}_v^{(k,k-p)} r_k \\ \bar{q}_{k-p} &= \bar{\gamma}_{k-p+1} \bar{w}_{k-p+1} + \bar{l}_w^{(k,k-p)} q_k \qquad (18) \end{aligned}$$

where the $\bar{l}$'s are elements of $\bar{L}_v = L_{v_1} \cdots L_{v_p}$ and $\bar{L}_w = L_{w_1} \cdots L_{w_p}$. Most importantly, the starting vectors do indeed satisfy (7) and (8). Note that only p additional standard Lanczos iterations are required to obtain an order-k Lanczos factorization corresponding to $\bar{v}_1$ and $\bar{w}_1$.

4. Example: The Portable CD Player

The Compact Disc player is a well-known mechanism for reproducing sound from a disc. At the heart of the CD player is an optical unit (consisting of a laser diode, lenses, and photodectors) which is mounted on the end of a radial arm [3]. In particular, we will be interested in the relationship between the voltage applied to the magnetic lens actuator and the resulting lens position. Traditionally, the behavior of the lens position is represented by a third-order set of equations. However, controllers designed from these simple, low-order systems experience difficulties when employed in newer, portable CD players [3].

To obtain a higher-order controller for the CD player, a better model of its behavior is required. Via finite element approximation, various portions of the CD player were modeled and combined to yield a system of equations of order $n = 120$. It is unfortunate that the size of A is relatively small. But, this example is very adequate in demonstrating both the severity of the unstable partial realization problem and the power of implicit restarts in solving this problem.

A very valid concern is the total number of Lanczos realizations (T_k, $W_k^T b$, cV_k, $1 \leq k \leq 120$) which are actually unstable. If there are only a few values of k for which T_k is unstable, then incorporating implicit restarts into the standard Lanczos method is unnecessary work. But Figure 1 demonstrates that T_k stable is the exception, not the rule, for this example. In general, one cannot count on stumbling upon stability at the appropriate recursion step k.

However, employing implicit restarts with appropriate choices for the parameters, μ_i, (see §5) quickly stabilizes the reduced-order model. The number of restarts needed to obtain a stable $\bar{T}$ given various T_k's is indicated in Table 1.

Table 1: Restarts Needed to Stabilize an Order-k Model

	$k = 20$	$k = 30$	$k = 40$	$k = 50$	k=60
Restarts	5	0	2	3	1

It is also important to note that in this example, implicit restarts do not have a detrimental effect on the accuracy of the final, stabilized model (and, in fact, they are extremely beneficial when the original model is unstable). For example, Figure 2 displays the impulse responses for both an initially stable Lanczos model (T_{47}) and a restarted (stabilized) Lanczos model ($\bar{T}_{50}$). Even with a modified projector, $\bar{\pi}_k$, the restarted model's response is closer to that of the actual system.

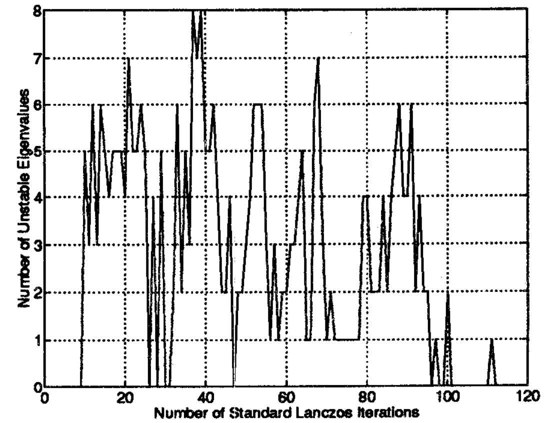

Figure 1: The number of unstable eigenvalues in T_k, where k is the number of Lanczos iterations.

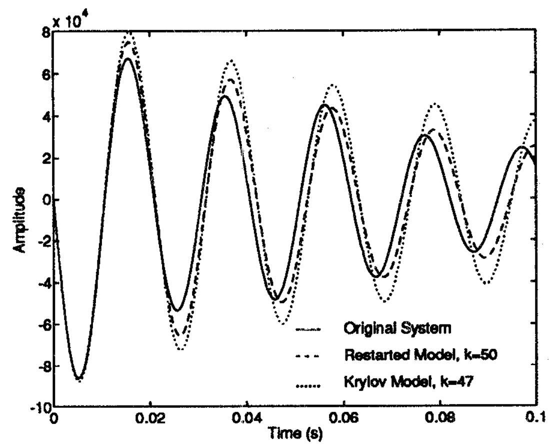

Figure 2: Impulse responses for CD player models.

5. Implementation Remarks

Until now, several important implementation details have been glossed over. This section will quickly address some of these issues.

Paramount in arriving at a stabilizing projector from an initial projector is proper selection of the parameters (shifts), μ_i. Although there is certainly an endless number of possibilities for the shifts, the following theorem (an analogue to one in [12]) indicates a practical policy for choosing the restart parameters.

Theorem 2 *Let* $\{\theta_1, \ldots, \theta_k\} \cup \{\mu_1, \ldots, \mu_p\}$ *be a disjoint partition of the spectrum of* T_{k+p} *and define* $\bar{T}_k$

to be the the tridiagonal matrix resulting from p implicit restarts with shifts μ_1 through μ_p. The eigenvalues of $\bar{T}_k$ are $\{\theta_1, \ldots, \theta_k\}$.

Restarting with exactly p eigenvalues of T_{k+p} as the shifts "tosses out" these p eigenvalues from $\bar{T}_k$. For our application, given that T_k is unstable, one needs to proceed until a T_{k+p} is determined with less than p unstable poles. Then via implicit restarts and Theorem 2, one can remove the unstable poles to yield a stable $\bar{T}_{k+q}$, $0 \leq q \leq p$. Note that the condition "find T_{k+p} with less than p unstable poles" is much less restrictive than finding a stable T_{k+p}.

Although the ultimate goal in choosing the shifts, μ_i, is to obtain a stable realization, there is also a more basic concern, the sensitivity of LR-decompositions. Because the LR-decomposition of T_{k+p} exists only if its $k+p-1$ leading principal minors are nonzero, a shift cannot be an eigenvalue of any of these $k+p-1$ minors of T_{k+p}. But the existence of L is not sufficient. L must be kept well-scaled to maintain the biorthogonality of $\bar{V}$ and $\bar{W}$. In general, selecting the shifts to be the unstable eigenvalues of T_{k+p} seems to generate well-scaled L's. However, one should be aware that a slight perturbation on a shift may be required in some cases.

In practice, these LR-decompositions should be performed implicitly [16]. That is, a series of elementary transformations should be used to chase a bulge down the tridiagonal of T_k. For the single-shift case, generating $\bar{T}_{k-1}$ in this manner costs only $O(k^2)$ flops while $\bar{V}_{k-1}$ and $\bar{W}_{k-1}$ can each be generated with $O(kn)$ flops. An additional $O(kn)$ flops is needed for the single Lanczos iteration (full reorthogonalization) yielding $\bar{T}_k$. Note that an explicit restart (with full reorthogonalization), on the other hand, requires $O(k^2 n)$ operations.

6. Concluding Remarks

In this paper, a novel technique based on implicit Lanczos restarts was developed for stabilizing a partial realization of a large-scale system. Numerical experiments, such as the CD player in §4, indicate that this method is a promising, efficient remedy for the stability problem in Padé approximation.

However, future work is necessary. A more definitive relationship between restarts and the resulting model must be established. The ability of restarts to improve the accuracy of the Lanczos model beyond stability will be explored. Additionally, the issues of singularities in the Padé table and moment matching about multiple frequencies must be readdressed.

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