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Elliptic and hyperbolic quadratic eigenvalue problems and associated distance problems

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Abstract

Two important classes of quadratic eigenvalue problems are composed of elliptic and hyperbolic problems. In [Linear Algebra Appl., 351–352 (2002) 455], the distance to the nearest non-hyperbolic or non-elliptic quadratic eigenvalue problem is obtained using a global minimization problem. This paper proposes explicit formulas to compute these distances and the optimal perturbations. The problem of computing the nearest elliptic or hyperbolic quadratic eigenvalue problem is also solved. Numerical results are given to illustrate the theory.

Keywords: Self-adjoint quadratic eigenvalue problem; Elliptic system; Hyperbolic system; Distance problems

1. Self-adjoint quadratic eigenvalue problems

Given $A, B, C \in \mathbb{C}^{n\times n}$, the *quadratic eigenvalue problem* (QEP)

$$Q(\lambda)x = (\lambda^2 A + \lambda B + C)x = 0 \tag{1}$$

has a wide range of applications, from vibration analysis to fluid dynamics. The recent survey [1] contains a list of its many applications, its mathematical properties and several numerical methods for this problem class. For self-adjoint QEP, the

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matrices A, B and C are Hermitian. The leading coefficient A is positive definite and is generally associated to the kinetic energy in mechanical engineering.

In this paper, distance problems related to two important classes of self-adjoint QEP are studied. A QEP is said to be *elliptic* if $(x^*Bx)^2 < 4(x^*Ax)(x^*Cx)$ for all non-zero $x \in \mathbb{C}^n$ and is said to be *hyperbolic* if $(x^*Bx)^2 > 4(x^*Ax)(x^*Cx)$ for all non-zero $x \in \mathbb{C}^n$. Given an elliptic or hyperbolic QEP, our aim is to compute the smallest perturbation such that the corresponding property is lost by the perturbed problem. This is clearly a distance problem (how much the problem can be altered without losing its defining property?). The converse problem is also of interest (what is the nearest QEP having the desired property?).

In [2], the first distance problem is tackled by making use of the Hermitian matrix

$$W(x, A, B, C) = \begin{bmatrix} 2x^*Ax & x^*Bx \\ x^*Bx & 2x^*Cx \end{bmatrix}. \tag{2}$$

Indeed, $\det W(x, A, B, C)$ is strictly positive for all non-zero $x \in \mathbb{C}^n$ if the QEP is elliptic and $\det W(x, A, B, C)$ is strictly negative for all non-zero $x \in \mathbb{C}^n$ if the QEP is hyperbolic. The minimal distance is computed by solving a non-convex global optimization problem, from which the optimal perturbations can then be recovered. As there is no guarantee to obtain a global optimum, this optimization problem can be considered to be difficult to solve efficiently. Moreover, the perturbations have no easy interpretation in the original polynomial setting.

In the sequel, we propose a simpler approach based on the trigonometric matrix polynomial

$$P(\omega) = \sin^2(\omega)A + \cos(\omega)\sin(\omega)B + \cos^2(\omega)C \tag{3a}$$

$$= \begin{bmatrix} \sin(\omega)I & \cos(\omega)I \end{bmatrix} \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} \sin(\omega)I \\ \cos(\omega)I \end{bmatrix} \tag{3b}$$

from which optimal perturbations can be efficiently obtained. Note that $P(\omega) \equiv Q(\lambda)/(\lambda^2 + 1)$ with $\lambda = \tan(\omega)$.

Our solutions are based on the minimal and maximal eigenvalues of $P(\omega)$, regarded as a matrix function of ω. Once the critical frequency $\hat{\omega}$ is identified, an appropriate eigenvector $\hat{x}$ of $P(\hat{\omega})$ allows us to construct the optimal perturbation $\Delta Q(\lambda)$. Sections 2 and 3 deal with elliptic-related and hyperbolic-related distance problems, respectively.

Notation. The spectral norm of a matrix X is denoted by $\|X\|_2$ and its Frobenius norm by $\|X\|_{\mathrm{F}}$. The minimal and maximal eigenvalues of a Hermitian matrix X are denoted by $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$, respectively. Given perturbations ΔA, ΔB, ΔC of the coefficients, $\Delta Q(\lambda) \doteq \lambda^2 \Delta A + \lambda \Delta B + \Delta C$ and $\Delta P(\omega) \doteq \sin^2(\omega) \Delta A + \cos(\omega) \sin(\omega) \Delta B + \cos^2(\omega) \Delta C$. The identity matrix is denoted by I.

2. Distance problems related to elliptic QEP

The original definition of ellipticity is not convenient when dealing with distance problems. However, an important characterization of elliptic systems is obtained by considering the matrix polynomial $Q(\lambda)$.

Theorem 1. *The self-adjoint QEP* (1) *with A positive definite is elliptic if and only if $Q(\lambda)$ is positive definite for all $\lambda \in \mathbb{R}$*

Proof. This standard proof is based on the quadratic polynomial

$$x^*Q(\lambda)x = (x^*Ax)\lambda^2 + (x^*Bx)\lambda + (x^*Cx), \tag{4}$$

where x is any non-zero vector. Since A is positive definite, this polynomial is positive for all non-zero vector x if and only if $(x^*Bx)^2 - 4(x^*Ax)(x^*Cx)$ is strictly negative for all non-zero vector x. Thus, the matrix polynomial $Q(\lambda)$ is positive definite on the real line if and only if the QEP (1) is elliptic. □

Corollary 2. *The self-adjoint QEP* (1) *with A positive definite is elliptic if and only if $P(\omega)$ is positive definite for all frequencies $\omega \in [-(\pi/2), \pi/2]$.*

A Hermitian quadratic polynomial $Q(\lambda)$ (or the associated trigonometric matrix polynomial $P(\omega)$) is therefore said to be elliptic if the corresponding QEP (1) is elliptic.

Two distance problems related to elliptic QEP are:

- If $Q(\lambda)$ is elliptic, find
$$\Delta Q(\lambda) = \lambda^2\,\Delta A + \lambda\,\Delta B + \Delta C \tag{5}$$
of smallest norm
$$\left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\| \tag{6}$$
such that $Q(\lambda) + \Delta Q(\lambda)$ is not elliptic.
- If $Q(\lambda)$ is not elliptic, find
$$\Delta Q(\lambda) = \lambda^2\,\Delta A + \lambda\,\Delta B + \Delta C \tag{7}$$
of smallest norm
$$\left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\| \tag{8}$$
such that $Q(\lambda) + \Delta Q(\lambda)$ is elliptic.

The first distance problem is solved in both spectral and Frobenius norms by the following theorem.

Theorem 3. *Let $Q(\lambda)$ be elliptic. Any perturbation $\Delta Q(\lambda)$ such that $Q(\lambda) + \Delta Q(\lambda)$ is not elliptic satisfies the inequality*

$$r_{\mathrm{E}} \leqslant \left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_2 \leqslant \left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_{\mathrm{F}}, \tag{9}$$

where $r_{\mathrm{E}} = \min_\omega \lambda_{\min} P(\omega) > 0$. Moreover, equality holds for the rank-one perturbations

$$\begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} = -r_{\mathrm{E}} \left(\begin{bmatrix} \sin(\hat{\omega}) \\ \cos(\hat{\omega}) \end{bmatrix} \begin{bmatrix} \sin(\hat{\omega}) & \cos(\hat{\omega}) \end{bmatrix} \right) \otimes (\hat{x}\hat{x}^*) \tag{10}$$

with $\hat{\omega} = \arg\min_\omega \lambda_{\min} P(\omega)$ and $P(\hat{\omega})\hat{x} = r_{\mathrm{E}}\hat{x}$ ($\|\hat{x}\|_2 = 1$).

Proof. In order to get compact mathematical expressions, let us define the matrix function

$$f(\omega) = \begin{bmatrix} \sin(\omega)I \\ \cos(\omega)I \end{bmatrix}. \tag{11}$$

Any perturbation of $Q(\lambda)$ which makes it non-elliptic must also perturb the appropriate eigenvalues of $P(\omega)$ so that it is not a strictly positive polynomial anymore. For a given frequency ω, standard perturbation theory can be applied to $P(\omega)$. Because of the inequality

$$\lambda_{\min}(P(\omega)) - \lambda_{\max}(\Delta P(\omega)) \leqslant \lambda_{\min}(P(\omega) + \Delta P(\omega)), \tag{12}$$

any perturbation such that $P(\omega)$ loses its definiteness satisfies the following inequality

$$\lambda_{\min}\left(f(\omega)^* \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} f(\omega) \right) \leqslant \left\| f(\omega)^* \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} f(\omega) \right\|_2. \tag{13}$$

From norm consistency,

$$\left\| f(\omega)^* \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} f(\omega) \right\|_2 \leqslant \left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_2 \|f(\omega)\|_2^2. \tag{14}$$

As $f(\omega)^* f(\omega) = I$, we have that $\|f(\omega)\|_2 = 1$. Therefore, a minimization with respect to ω yields

$$r_{\mathrm{E}} \leqslant \left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_2. \tag{15}$$

The inequality $\|\cdot\|_2 \leqslant \|\cdot\|_{\mathrm{F}}$ is well known.

Let $\hat{\omega} = \arg\min_\omega \lambda_{\min} P(\omega)$ and $P(\hat{\omega})\hat{x} = r_{\mathrm{E}}\hat{x}$ ($\|\hat{x}\|_2 = 1$). The perturbations defined by

$$\begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} = -r_{\mathrm{E}} \left(\begin{bmatrix} \sin(\hat{\omega}) \\ \cos(\hat{\omega}) \end{bmatrix} \begin{bmatrix} \sin(\hat{\omega}) & \cos(\hat{\omega}) \end{bmatrix} \right) \otimes (\hat{x}\hat{x}^*) \tag{16}$$

satisfy

$$\left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_{2,\mathrm{F}} = r_{\mathrm{E}}$$

and produce a non-elliptic polynomial. Indeed, by construction, we have

$$\hat{x}^*(P(\hat{\omega}) + \Delta P(\hat{\omega}))\hat{x} = 0 \tag{17}$$

from which we conclude that the polynomial $P(\omega) + \Delta P(\omega)$ is not strictly positive definite. As the quadratic polynomial $Q(\lambda) + \Delta Q(\lambda)$ is non-elliptic, our perturbation is optimal. □

Remark 4. The above theorem also handles the case where only the matrix A is perturbed. In that case, the non-elliptic QEP is obtain by modifying the matrix A so that it loses positive definiteness.

Example 5. The QEP defined by

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 7/4 & 0 & 0 \\ 0 & 15/2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 7/2 & 1 & 0 \\ 1 & 8 & 1 \\ 0 & 1 & 4 \end{bmatrix} \tag{18}$$

is elliptic, see Fig. 1. Its minimal distance to a non-elliptic QEP is $r_{\mathrm{E}} = 0.8460$ and the critical frequency is $\hat{\omega} = -1.0011$. The optimal perturbations are obtained via the eigenvector $\hat{x} = \begin{bmatrix} 0.3281 & -0.8972 & 0.2956 \end{bmatrix}^{\mathrm{T}}$.

A straightforward consequence of Theorem 1 is that the set of elliptic QEP is an *open* convex set. Computing the distance between a non-elliptic QEP and this set, which is our second distance problem, is therefore a badly defined problem.

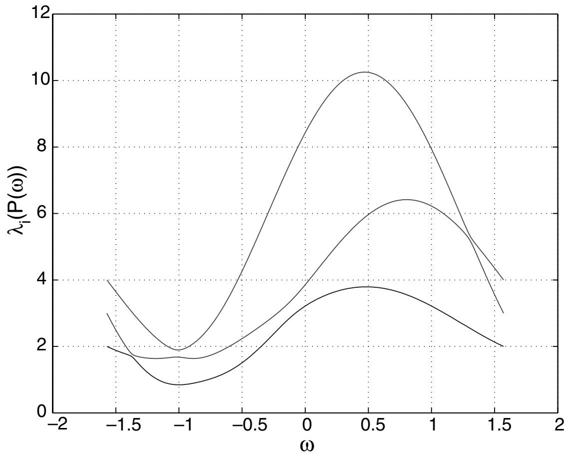

Fig. 1. Eigenvalues of $P(\omega)$ for Example 5.

However, the distance to the closure of this set and the associated boundary point can be easily obtained. This problem can be solved using the recent parametrization of non-negative matrix polynomials by positive semidefinite matrices [3].

Theorem 6. *The Hermitian quadratic polynomial $\lambda^2 A + \lambda B + C$ is non-negative on the real line if and only if there exists a matrix X such that*

$$\begin{bmatrix} A & B/2 - X \\ B/2 + X & C \end{bmatrix} \succeq 0 \tag{19}$$

and $X = -X^*$.

Indeed, the closure of the set of elliptic QEP is exactly the set of Hermitian quadratic matrix polynomials non-negative on the real line, see Theorem 1. Depending on the measure, the following convex problems provide us with the asymptotically optimal perturbations:

- Spectral norm

$$\begin{aligned} \min \quad & \tau, \\ \text{s.t.} \quad & \tau^2 I \succeq \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix}^*, \\ & \begin{bmatrix} A + \Delta A & B/2 + \Delta B/2 - X \\ B/2 + \Delta B/2 + X & C + \Delta C \end{bmatrix} \succeq 0, \\ & \Delta A = \Delta A^*, \quad \Delta B = \Delta B^*, \quad \Delta C = \Delta C^*, \quad X = -X^*. \end{aligned} \tag{20}$$

- Frobenius norm

$$\begin{aligned} \min \quad & \left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_{\mathrm{F}}^2, \\ \text{s.t.} \quad & \begin{bmatrix} A + \Delta A & B/2 + \Delta B/2 - X \\ B/2 + \Delta B/2 + X & C + \Delta C \end{bmatrix} \succeq 0, \\ & \Delta A = \Delta A^*, \quad \Delta B = \Delta B^*, \quad \Delta C = \Delta C^*, \quad X = -X^*. \end{aligned} \tag{21}$$

Both optimization problems can be recast as semidefinite programming problems in a straightforward manner. Therefore, they are efficiently solvable in polynomial time (up to any given accuracy) using modern interior-point methods [4–6].

An arbitrarily close elliptic QEP is then obtained from their solutions using an appropriate shift. For instance, the polynomial $Q(\lambda) + \Delta Q(\lambda) + \epsilon I$ is elliptic, for all $\epsilon > 0$. Note that both optimization problems allow us to handle *structured* perturbations.

If we are only interested in the spectral norm, an optimal unstructured perturbation is easy to obtain.

Theorem 7. *Let $Q(\lambda)$ be non-elliptic. Any perturbation $\Delta Q(\lambda)$ such that $Q(\lambda) + \Delta Q(\lambda)$ is elliptic satisfies the strict inequality*

$$-r_{\mathrm{E}} < \left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_2, \tag{22}$$

where $r_{\mathrm{E}} = \min_\omega \lambda_{\min} P(\omega) \leqslant 0$. *For* $\epsilon > 0$, *an arbitrarily close perturbation* $\Delta Q(\lambda)$ *corresponds to* $\Delta A = \Delta C = (-r_{\mathrm{E}} + \epsilon)I$ *and* $\Delta B = 0$.

Proof. This proof is completely similar to the first part of the proof of Theorem 3 and is therefore omitted. □

Unfortunately, we were not able to obtain an explicit expression of the optimal perturbations for the Frobenius norm.

Example 8. As shown on Fig. 2, the QEP defined by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 7/4 & 0 & 0 \\ 0 & 15/2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 3/2 & 1 & 0 \\ 1 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \tag{23}$$

is neither elliptic nor hyperbolic. Its minimal distance to a boundary point of the set of elliptic QEP's is equal to $-r_{\mathrm{E}} = 0.5163$. The semidefinite formulation for the spectral norm produces the optimal perturbations

$$\begin{aligned} \Delta A &= \begin{bmatrix} 0.1639 & -0.0698 & 0.0455 \\ -0.0698 & 0.3047 & -0.0126 \\ 0.0455 & -0.0126 & 0.2644 \end{bmatrix}, \\ \Delta B &= \begin{bmatrix} -0.2008 & 0.0190 & -0.0463 \\ 0.0190 & -0.3208 & 0.0077 \\ -0.0463 & 0.0077 & -0.4003 \end{bmatrix}, \end{aligned} \tag{24}$$

$$\Delta C = \begin{bmatrix} 0.1062 & -0.1081 & 0.0300 \\ -0.1081 & 0.2476 & -0.0379 \\ 0.0300 & -0.0379 & 0.2278 \end{bmatrix} \tag{25}$$

with

$$\left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_2 = 0.5163 = -r_{\mathrm{E}}.$$

If the Frobenius norm is used, we obtain

$$\begin{aligned} \Delta A &= \begin{bmatrix} 0.0247 & -0.1497 & 0.0605 \\ -0.1497 & 0.2314 & -0.1454 \\ 0.0605 & -0.1454 & 0.1092 \end{bmatrix}, \\ \Delta B &= \begin{bmatrix} -0.0194 & 0.1123 & -0.0658 \\ 0.1123 & -0.1680 & 0.1525 \\ -0.0658 & 0.1525 & -0.1298 \end{bmatrix}, \end{aligned} \tag{26}$$

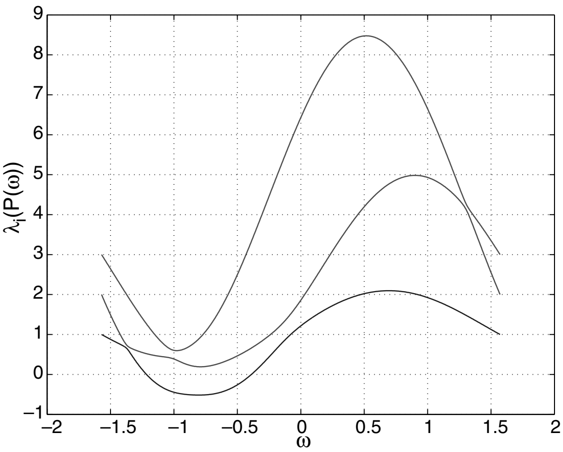

Fig. 2. Eigenvalues of $P(\omega)$ for Example 8.

$$\Delta C = \begin{bmatrix} 0.0163 & -0.0898 & 0.0745 \\ -0.0898 & 0.1283 & -0.1683 \\ 0.0745 & -0.1683 & 0.1548 \end{bmatrix} \tag{27}$$

with

$$\left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_{\mathrm{F}} = 0.5899.$$

In Theorems 3 and 7, our solutions are based on the quantity

$$r_{\mathrm{E}} = \min_{\omega} \lambda_{\min} P(\omega), \tag{28}$$

which can be computed in polynomial time (up to any given accuracy). Indeed, this problem can be recast as a convex optimization problem, for which a global minimum can be easily obtained. Of course, other standard methods in linear algebra can also be adapted. In particular, the bisection and level set methods described in [2,7] can be modified in a straightforward way to obtain a global minimum.

3. Distance problems related to hyperbolic QEP

There also exists a characterization of hyperbolic systems in terms of the matrix polynomial $Q(\lambda)$.

Theorem 9. *The self-adjoint QEP* (1) *with A positive definite is hyperbolic if and only if $Q(\lambda)$ is negative definite for some $\lambda \in \mathbb{R}$.*

Proof. See the proof of Theorem 1 in [8]. □

Corollary 10. *The self-adjoint QEP* (1) *with A positive definite is hyperbolic if and only if $P(\omega)$ is negative definite for some $\omega \in [-(\pi/2), \pi/2]$.*

As before, there are two distance problems related to hyperbolic QEP:

- If $Q(\lambda)$ is hyperbolic, find
$$\Delta Q(\lambda) = \lambda^2\,\Delta A + \lambda\,\Delta B + \Delta C \tag{29}$$
of smallest norm
$$\left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\| \tag{30}$$
such that $Q(\lambda) + \Delta Q(\lambda)$ is not hyperbolic.
- If $Q(\lambda)$ is not hyperbolic, find
$$\Delta Q(\lambda) = \lambda^2\,\Delta A + \lambda\,\Delta B + \Delta C \tag{31}$$
of smallest norm
$$\left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\| \tag{32}$$
such that $Q(\lambda) + \Delta Q(\lambda)$ is hyperbolic.

Let us focus on the first distance problem. First, note that hyperbolicity of the QEP (1) is lost by adding to A a perturbation of spectral norm equal to $\lambda_{\min}(A)$, which makes this matrix lose its definiteness. Hereafter, these perturbations $\Delta Q(\lambda)$ with $\Delta B \equiv 0$ and , $\Delta C \equiv 0$ are said to be trivial. Of course, there also exist non-trivial perturbations.

Theorem 11. *Let $Q(\lambda)$ be hyperbolic. Any non-trivial perturbation $\Delta Q(\lambda)$ such that $Q(\lambda) + \Delta Q(\lambda)$ is not hyperbolic satisfies the inequality*
$$-r_{\mathrm{H}} \leqslant \left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_2 \leqslant \left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_{\mathrm{F}}, \tag{33}$$
where $r_{\mathrm{H}} = \min_\omega \lambda_{\max} P(\omega) < 0$. Moreover, equality holds for the rank-one perturbations
$$\begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} = -r_{\mathrm{H}} \left(\begin{bmatrix} \sin(\hat\omega) \\ \cos(\hat\omega) \end{bmatrix} \begin{bmatrix} \sin(\hat\omega) & \cos(\hat\omega) \end{bmatrix} \right) \otimes (\hat x \hat x^*) \tag{34}$$
with $\hat\omega = \arg\min_\omega \lambda_{\max} P(\omega)$ and $P(\hat\omega)\hat x = r_{\mathrm{H}} \hat x$ ($\|\hat x\|_2 = 1$).

Proof. As $Q(\lambda)$ is hyperbolic, the matrix $P(\omega)$ is negative definite for at least one frequency ω. In order to get at least one non-negative eigenvalue at all frequencies, the eigenvalues of $P(\omega)$ must be shifted a quantity greater than $-r_{\mathrm{H}} = -\min_\omega \lambda_{\max} P(\omega)$. This is a necessary condition for $P(\omega)$ to have a non-negative eigenvalue at the frequency $\hat{\omega} = \arg\min_{\omega} \lambda_{\max} P(\omega)$. Therefore, the lower bound on the norm of the perturbation is

$$-r_{\mathrm{H}} \leqslant \left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_2 . \tag{35}$$

The inequality $\|\cdot\|_2 \leqslant \|\cdot\|_{\mathrm{F}}$ is well known.

The perturbations defined by

$$\begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} = -r_{\mathrm{H}} \left(\begin{bmatrix} \sin(\hat{\omega}) \\ \cos(\hat{\omega}) \end{bmatrix} \begin{bmatrix} \sin(\hat{\omega}) & \cos(\hat{\omega}) \end{bmatrix} \right) \otimes (\hat{x}\hat{x}^*) \tag{36}$$

satisfy

$$\left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_2 = -r_{\mathrm{H}}$$

and yield a non-hyperbolic polynomial. Indeed, the perturbation $\Delta Q(\lambda)$ is non-trivial so that the leading coefficient $A + \Delta A$ of $Q(\lambda) + \Delta Q(\lambda)$ is still positive definite. Since we have

$$\begin{aligned} &\hat{x}^*(P(\omega) + \Delta P(\omega))\hat{x} \\ &\quad = \begin{bmatrix} \sin(\hat{\omega}) & \cos(\hat{\omega}) \end{bmatrix} \begin{bmatrix} \hat{x}^*(A + \Delta A)\hat{x} & \hat{x}^*(B + \Delta B)\hat{x}/2 \\ \hat{x}^*(B + \Delta B)\hat{x}/2 & \hat{x}^*(C + \Delta C)\hat{x} \end{bmatrix} \begin{bmatrix} \sin(\hat{\omega}) \\ \cos(\hat{\omega}) \end{bmatrix} = 0, \end{aligned} \tag{37}$$

the Schur complement of the two-by-two matrix

$$\begin{bmatrix} \hat{x}^*(A + \Delta A)\hat{x} & \hat{x}^*(B + \Delta B)\hat{x}/2 \\ \hat{x}^*(B + \Delta B)\hat{x}/2 & \hat{x}^*(C + \Delta C)\hat{x} \end{bmatrix} \tag{38}$$

with respect to its (1, 1)-entry is necessarily equal to 0. Up to the positive factor $\hat{x}^*(A + \Delta A)\hat{x}$, this is equivalent to

$$(\hat{x}^*(B + \Delta B)\hat{x})^2 - 4(\hat{x}^*(A + \Delta A)\hat{x})(\hat{x}^*(C + \Delta C)\hat{x}) = 0. \tag{39}$$

Since $\hat{x}$ is a non-zero vector, $Q(\lambda) + \Delta Q(\lambda)$ is by definition non-hyperbolic. □

Remark 12. In general, the above theorem does not handle any trivial perturbations. It is therefore of paramount importance to compare the distance $-r_{\mathrm{H}}$ with $\lambda_{\min}(A)$ in order to select the optimal perturbation. The complete procedure is illustrated in Examples 13 and 14.

Example 13. The QEP defined by

$$A = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 5/2 \end{bmatrix}, \quad B = \begin{bmatrix} 7/4 & 0 & 0 \\ 0 & 15/2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \tag{40}$$

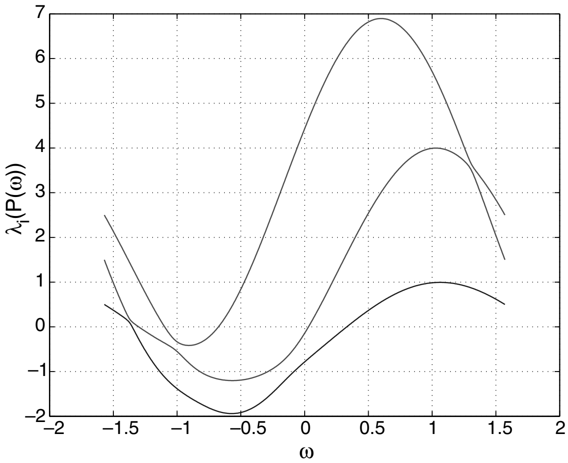

Fig. 3. Eigenvalues of $P(\omega)$ for Example 13.

$$C = \begin{bmatrix} -1/2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is hyperbolic, see Fig. 3. Its minimal distance to a non-hyperbolic QEP is $-r_{\mathrm{H}} = 0.4161$ and the critical frequency is $\hat{\omega} = -0.9080$. The optimal perturbations are obtained via the associated eigenvector $\hat{x} = \begin{bmatrix} 0.6831 & 0.5617 & 0.4667 \end{bmatrix}^{\mathrm{T}}$.

Example 14. The QEP defined by

$$A = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 5/2 \end{bmatrix}, \quad B = \begin{bmatrix} 7/4 & 0 & 0 \\ 0 & 15/2 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$
$$C = \begin{bmatrix} -3/2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \tag{41}$$

is hyperbolic, see Fig. 4. As the distance $-r_{\mathrm{H}} = 0.8263$ is greater than the minimal eigenvalue of A, the optimal perturbation is the trivial one. We get that $\Delta A = -0.5 e_0 e_0^{\mathrm{T}}$, $\Delta B = 0$ and $\Delta C = 0$ where e_0 is the first canonical vector.

Let us now consider the second distance problem. Since the set of hyperbolic QEP's is not closed, we can only expect to compute a boundary point. As before, an arbitrarily close hyperbolic QEP can then be obtained using an appropriate shift.

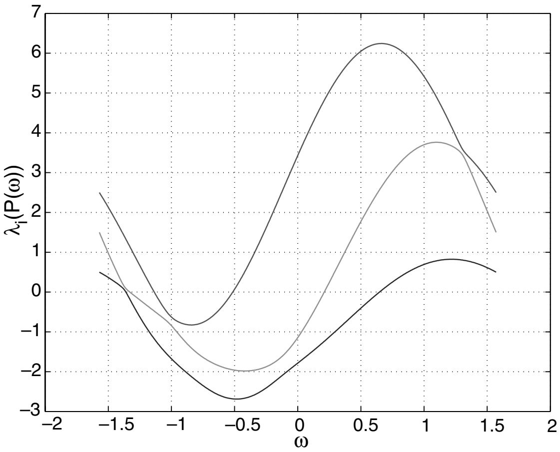

Fig. 4. Eigenvalues of $P(\omega)$ for Example 14.

Theorem 15. *Let $Q(\lambda)$ be non-hyperbolic with positive definite A. Any perturbation $\Delta Q(\lambda)$ such that $Q(\lambda) + \Delta Q(\lambda)$ is hyperbolic satisfies the strict inequality*

$$r_{\mathrm{H}} < \left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_2, \tag{42}$$

where $r_{\mathrm{H}} = \min_{\omega} \lambda_{\max} P(\omega) \geqslant 0$. For any $\epsilon > 0$, an arbitrarily close perturbation is

$$\begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} = -(r_{\mathrm{H}} + \epsilon) \left(\begin{bmatrix} \sin(\hat{\omega}) \\ \cos(\hat{\omega}) \end{bmatrix} \begin{bmatrix} \sin(\hat{\omega}) & \cos(\hat{\omega}) \end{bmatrix} \right) \otimes I \tag{43}$$

with $\hat{\omega} = \arg\min_{\omega} \lambda_{\max} P(\omega)$.

Proof. As $Q(\lambda)$ is non-hyperbolic, the matrix $P(\omega)$ has at least one non-negative eigenvalue at all frequencies ω. In order to make $P(\omega)$ hyperbolic, we need to shift the eigenvalues of $P(\omega)$ by a quantity strictly greater than $r_{\mathrm{H}} = \min_{\omega} \lambda_{\max} P(\omega)$. At the corresponding frequency $\hat{\omega} = \arg\min_{\omega} \lambda_{\max} P(\omega)$, $P(\omega)$ could then become negative definite. Therefore, the strict lower bound on the norm of the perturbation is

$$r_{\mathrm{H}} < \left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_2. \tag{44}$$

For any $\epsilon > 0$, the perturbations defined by

$$\begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} = -r_{\mathrm{H}}(1 + \epsilon) \left(\begin{bmatrix} \sin(\hat{\omega}) \\ \cos(\hat{\omega}) \end{bmatrix} \begin{bmatrix} \sin(\hat{\omega}) & \cos(\hat{\omega}) \end{bmatrix} \right) \otimes I \tag{45}$$

satisfy

$$\left\| \begin{bmatrix} \Delta A & \Delta B/2 \\ \Delta B/2 & \Delta C \end{bmatrix} \right\|_2 = r_{\mathrm{H}}(1 + \epsilon)$$

and yield an hyperbolic polynomial.

Indeed, at the frequency $\omega = \hat{\omega}$ and for all unit vectors x, $x^*(P(\omega) + \Delta P(\omega))x \leqslant -r_{\mathrm{H}}\epsilon$. At the frequency $\omega = \hat{\omega}$, $P(\omega) + \Delta P(\omega)$ is thus negative definite and the polynomial $Q(\lambda) + \Delta Q(\lambda)$ is hyperbolic. □

Example 16. The QEP defined in Example 8 is neither elliptic nor hyperbolic. Its minimal distance to a boundary point of the set of hyperbolic QEP's is equal to $r_{\mathrm{H}} = 0.5957$. The critical frequency is $\hat{\omega} = -0.9785$ and the associated perturbations are

$$\begin{aligned} &\Delta A = -0.4101(1+\epsilon)I_3, \quad \Delta B = 0.5519(1+\epsilon)I_3, \\ &\Delta C = -0.1857(1+\epsilon)I_3 \end{aligned} \tag{46}$$

with $\epsilon > 0$.

In Theorems 11 and 15, our solutions are based on the quantity

$$r_{\mathrm{H}} = \min_{\omega} \lambda_{\max} P(\omega). \tag{47}$$

Although this problem cannot be recast as a convex optimization problem, other standard methods in linear algebra can be applied to obtained the global minimum. In particular, the bisection and level set methods described in [2,7] can be modified in a straightforward way.

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