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Maximal stability region of a perturbed nonnegative matrix

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SUMMARY

For a class of positive matrices $A+K$ with a stable positive nominal part A and a structured positive perturbation part K, we address the problem of finding the largest set of admissible perturbations such that the global matrix remains stable. Theoretical bounds are derived and an algorithm for constructing this set is presented. As an example, this algorithm is applied to the regulation of water flow in open channels. Copyright © 2007 John Wiley & Sons, Ltd.

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1. Introduction

A linear time-invariant discrete-time system

$$x(k+1) = Ax(k) \tag{1}$$

is known to be stable if and only if $\rho(A) < 1$.

Models of real world dynamical phenomena often involve positive quantities. A dynamical system (1) is called *positive* if any trajectory of the system starting in the positive orthant $\mathbb{R}^n_+$ remains in $\mathbb{R}^n_+$. This is the case if and only if the matrix A has only real nonnegative entries. In many cases, it may be useful to consider systems with a known "nominal" part A and a unknown part K which may represent uncertainty :

$$x(k+1) = (A+K)x(k). \tag{2}$$

The robustness of (2) will then depend on the size of the set S such that

$$\rho(A+K) < 1 \qquad \forall K \in S.$$

One particular approach consists of considering structured matrices $K = E_1 \Delta E_2^T$ where Δ is the unknown disturbance and E_1 and E_2 are fixed matrices. The problem is then to find the *stability radius of A with respect to nonnegative perturbations of structure (E_1, E_2^T)* which is defined by

$$r_{\mathbb{R}_+}(A; E_1, E_2^T) := \inf\{\|\Delta\|; \Delta \geq 0, \rho(A + E_1 \Delta E_2^T) \geq 1\}.$$

All perturbations in the following set

$$S := \left\{ E_1 \Delta E_2^T \,|\, \|\Delta\| < r_{\mathbb{R}_+}(A; E_1, E_2^T) \right\}$$

are then shown to yield a stable system $A+K$. This problem is solved in [5] and a computable formula is provided.

In this paper we extend these results into a particular direction. We will only consider perturbations matrices Δ in the set $\mathscr{D}$ of nonnegative *diagonal* matrices $\mathscr{D} = \{\mathrm{diag}\{k_1,\dots,k_m\} \mid k_i \geq 0\}$. The parameters k_i are the so-called free parameters occurring in the matrix K, E_1 and E_2 are two matrices placing the elements in appropriate positions in K. The two matrices E_1 and E_2^T have the following properties : there is a non-zero element in row i and column j of E_1 if k_j is present in row i of K and of E_2^T if k_i is present in column j of K. We clarify this by an example : if

$$K = \begin{pmatrix} 2k_1 & 0 & 0 \\ 0 & 0 & k_2 \\ k_1 & 0 & 0 \end{pmatrix},$$

then

$$\Delta = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E_2^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We will restrict ourselves to matrices K for which both E_1 and E_2 are nonnegative as well : $E_1 \geq 0,\ E_2 \geq 0$. Notice that if one of the parameters appears in several rows and columns, it will be repeated several times in the diagonal matrix Δ.

The problem is to find the biggest set $S_{\mathscr{D}} \subseteq \left\{ E_1 \Delta E_2^T \;\middle|\; \Delta \in \mathscr{D} \right\}$ containing the origin and all the perturbations such that the system remains stable :

$$S_{\mathscr{D}} := \left\{ K = E_1 \Delta E_2^T \mid \Delta \in \mathscr{D},\ \rho(A+K) < 1 \right\}$$

where A, E_1, E_2, Δ are nonnegative matrices. Let us point out that this is in fact a starlike set.

Theorem 1.1. *The set $S_{\mathscr{D}}$ is a starlike set.*

Proof From [1], we know that if $A, B \geq 0$ then $\rho(A+B) \geq \rho(A)$. It implies that if $K \in S_{\mathscr{D}}$

$$\rho(A+K) < 1 \Rightarrow \rho(A+\alpha K) < 1 \qquad \forall \alpha \quad 0 \leq \alpha \leq 1$$

and $\alpha K \in S_{\mathscr{D}}$. If $K \notin S_{\mathscr{D}}$ then

$$\rho(A+K) \geq 1 \Rightarrow \rho(A+\beta K) \geq 1 \qquad \forall \beta \quad 1 \leq \beta$$

and $\beta K \notin S_{\mathscr{D}}$. ∎

Since the spectral radius is a continuous function of the parameter K, the boundary of the set $S_{\mathscr{D}}$ is described by

$$\partial S_{\mathscr{D}} = \{K \mid \exists i,\ k_i = 0 \ \&\ \rho(A+K) \leq 1 \qquad \text{or} \qquad \rho(A+K) = 1 \ \&\ K \geq 0\}$$

Examples show later that this set is in general not convex.

On the one side, the problem solved in [5] is more general because it does not assume that the perturbation Δ is diagonal. But, on the other side, when Δ is diagonal, their problem is more restrictive than the one addressed in this paper. All the operator norms induced by an arbitrary monotonic norm on $\mathbb{R}^n$ of a diagonal matrix are equal to the maximum of the elements of the matrix (see [6]). It means that the set S considered in [5] is a box with $k_i \leq k_i^m$ and $k_1^m = \ldots = k_n^m$ and hence a convex subset of $S_{\mathscr{D}}$. We will show that there exists a maximum starlike set $S_{\mathscr{D}} = \{E_1 \Delta E_2^T \mid \Delta \in \mathscr{D}\}$ for which all matrices K in $S_{\mathscr{D}}$ lead to stable $A+K$ and we will describe the boundary of this set in terms of polynomial equations.

2. Maximal perturbation of nonnegative matrices

First we develop some new theoretical results and we then present an algorithm for computing $\partial S_{\mathscr{D}}$.

2.1. Theoretical results

In this section we show that the problem may be decoupled in smaller subproblems involving only a subset of the parameters k_i. To each of these subproblems there corresponds a starlike set $S_{\mathscr{D}_i}$ for which we obtain an analytical expression. The set $S_{\mathscr{D}}$ is the intersection of the sets $S_{\mathscr{D}_i}$.

Since $K = E_1 \Delta E_2^T$ is nonnegative and since the eigenvalues are continuous functions of the matrix elements, we have that the critical switch between $\rho < 1$ and $\rho \geq 1$ will occur when

$$\rho(A + E_1 \Delta E_2^T) = 1.$$

Working only with positive matrices, we have that the spectral radius is also an eigenvalue and hence the above condition is equivalent to

$$\det(A + E_1 \Delta E_2^T - I) = 0$$

and

$$\det(E_1 \Delta E_2^T - (I - A)) = 0.$$

Since $\det(I - A) \neq 0$ ($\rho(A) < 1$) we can multiply the previous equation by $\det(I - A)^{-1}$ to obtain

$$\det((I - A)^{-1} E_1 \Delta E_2^T - I) = 0.$$

Using the fact that

$$\det(MN - I) = 0 \Leftrightarrow \det(NM - I) = 0$$

this is also equivalent to

$$\det(E_2^T (I - A)^{-1} E_1 \Delta - I) = 0$$

where $M := E_2^T (I - A)^{-1} E_1$ is nonnegative since $(I - A)^{-1} = \sum_{i=1}^{\infty} A^i$ and E_1, E_2 are nonnegative. We can use Lemma 2.1 (see [4]) to transform M to a normal form $\hat{M}$.

Lemma 2.1. *Every nonnegative matrix A has a normal form which can obtained under congruent permutations :*

$$\hat{A} = PAP^T = \begin{pmatrix} \hat{A}_{11} & & 0 \\ & \ddots & \\ * & & \hat{A}_{mm} \end{pmatrix} \tag{3}$$

where each diagonal block $\hat{A}_{ii}$ *is square, irreducible or just a* 1×1 *zero bock.*

Applying the same permutation to Δ, we define $\hat{\Delta} = P\Delta P^T$ and have

$$\det(\hat{M}\hat{\Delta} - I) = 0.$$

This clearly decomposes in a number of decoupled problems

$$\det(\hat{M}_{ii}\hat{\Delta}_{ii} - I) = 0.$$

This polynomial equation describes a part of $\partial S_{\mathscr{D}_i}$. The intersection of all these sets leads to the admissible set $S_{\mathscr{D}}$. Let us solve the subproblems :

- Let $\hat{M}_{ii} = 0$ then $\det(\hat{M}_{ii}\hat{\Delta}_{ii} - I) \neq 0$ for all *bounded* $\hat{\Delta}_{ii}$;
- Let $\hat{M}_{ii} \neq 0$ and irreducible. If the size n_i of $\hat{M}_{ii} = [m_{r,c}]_{r,c=1}^{n_i}$ is small enough, the problem can be exactly solved.
  - If $n_i = 1$, the solution is trivial $\det(m_{11}k_1 - 1) = 0 \quad$ for $k_1 = m_{11}^{-1}$.
  - If $n_i = 2$, we have $\det\left(\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} - I\right) = 0$ or equivalently :
  $$(m_{11}m_{22} - m_{12}m_{21})k_1k_2 - m_{11}k_1 - m_{22}k_2 + 1 = 0. \quad (4)$$
  The stable region for the k_1, k_2 is thus a starlike set with boundary defined by (4) and $k_{1,2} = 0$ (see Fig. 1).
  - If $n_i = 3$, we have
  $$\det(\hat{M}_{ii}\hat{\Delta}_{ii} - I) = 0$$
  or equivalently :
  $$\det(\hat{M}_{ii})k_1k_2k_3 + (m_{21}m_{12} - m_{11}m_{22})k_1k_2 + (m_{13}m_{31} - m_{11}m_{33})k_1k_3 + (m_{23}m_{32} - m_{22}m_{33})k_2k_3 + m_{11}k_1 + m_{22}k_2 + m_{33}k_3 - 1 = 0. \quad (5)$$

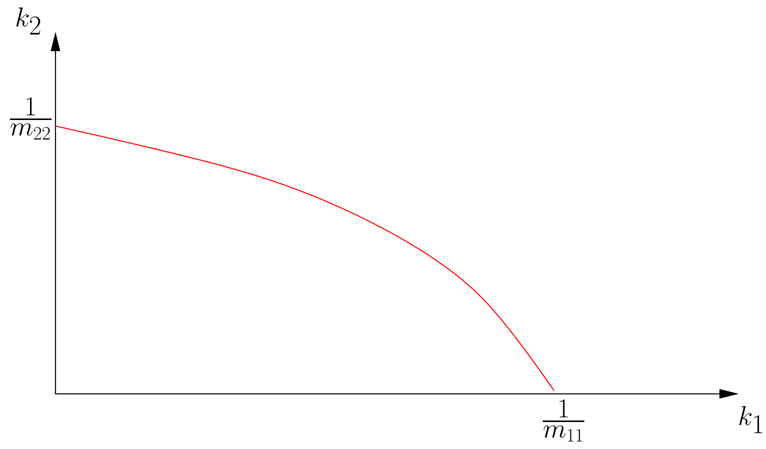

Figure 1. The largest set of the parameter k_1 and k_2 containing the origin such that $A+E_1\Delta E_2^T$ is stable.

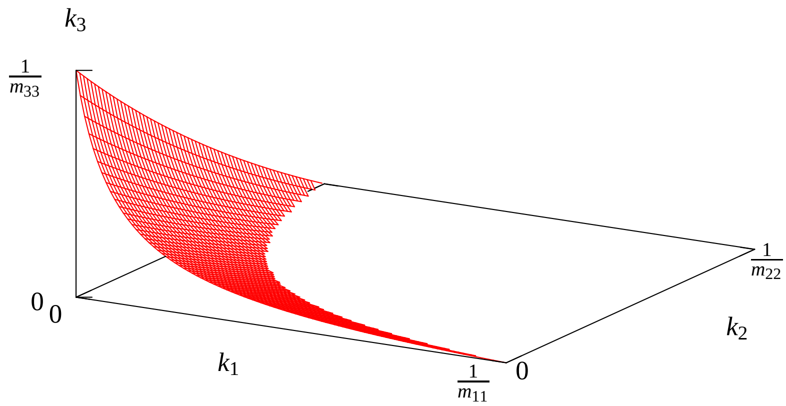

Figure 2. The boundary of the largest set (k_1,k_1,k_3) containing the origin such that $A+E_1\Delta E_2^T$ is stable

The stable region for the k_i is thus also a starlike set which boundaries are defined by (5) and $k_{1,2,3} = 0$ (see Fig. 2).

– It may happen that a coefficient k_i appears in different blocks $\hat{\Delta}_{ii}$. For example, if

$$A = \begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & & \\ & & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{pmatrix}, \quad K = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & k_3 & \\ & & & k_1 \end{pmatrix}$$

then

$$\hat{\Delta}_{11} = \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix}, \quad \hat{\Delta}_{22} = \begin{pmatrix} k_3 & \\ & k_1 \end{pmatrix}.$$

In this case, the admissible set for (k_1, k_2, k_3) is simply the intersection of the two sets obtained by analysing the two subproblems. This is illustrated in Figure 3.

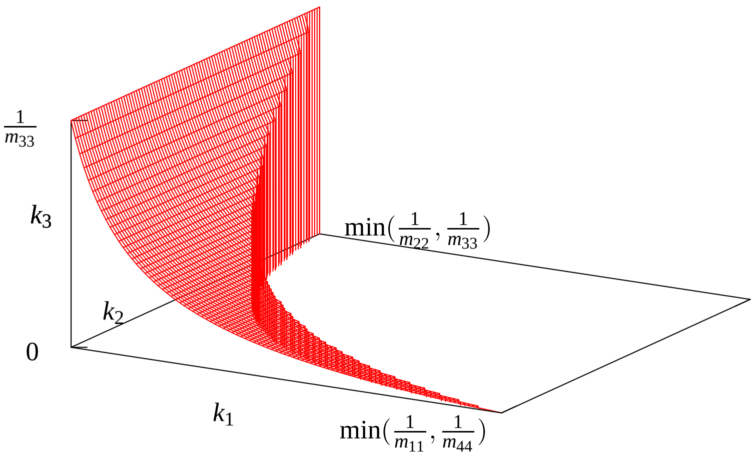

Figure 3. The admissible set is the intersection of the two admissible sets.

2.2. *Modified problem*

In the above analysis the sets $S_{\mathscr{D}_i}$ can become quite complex to describe if n_i becomes large, since one has to solve polynomial equation of degree n_i in several variables. We describe here two approximation of the original problem that are easier to compute. The approximation consists in finding necessary conditions to ensure stability (i.e. a subset of $S_{\mathscr{D}_i}$).

For example, one can freeze one particular k_i and express the conditions on the remaining parameters. This subset will be a slice of $S_{\mathscr{D}_i}$. This subset is still starlike in the leftover parameter.

One other possibility is to express a condition on the maximum of the k_i in the same way as [5]. The following result is a refinement of the global bound obtained in [5]. Let ρ_i be the spectral radius of M_{ii} then

$$\begin{aligned} \det(\hat{M}_{ii}\hat{\Delta}_{ii}-I) &\neq 0 \qquad \text{for } \hat{\Delta}_{ii} < \rho_i^{-1}I \\ \det(\hat{M}_{ii}\hat{\Delta}_{ii}-I) &= 0 \qquad \text{for } \hat{\Delta}_{ii} = \rho_i^{-1}I. \end{aligned}$$

Proof Let x_i be the Perron vector of the irreducible matrix $\hat{M}_{ii}$. It is well known (see [4]) that for an irreducible matrix the so-called Perron vector x_i corresponding to the (positive) Perron root ρ_i is strictly positive. Therefore

$$\hat{M}_{ii}x_i = \rho_i x_i,\ x_i > 0$$

then clearly

$$(\hat{M}_{ii}\rho_i^{-1}I - I)x_i = 0 \qquad \hat{\Delta}_{ii} = \rho_i^{-1}I$$

Also for $\hat{\Delta}_{ii} < \rho_i^{-1}I$

$$\det(\hat{M}_{ii}\hat{\Delta}_{ii} - I) \neq 0$$

since there exists a scaling

$$\left\|D^{-1}\hat{M}_{ii}D\right\|_\infty = \rho_i$$

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and clearly

$$\left\|D^{-1}\hat{M}_{ii}\hat{\Delta}_{ii}D\right\|_{\infty} = \left\|D^{-1}\hat{M}_{ii}D\hat{\Delta}_{ii}\right\|_{\infty} < 1.$$

∎

Therefore we can claim that all matrices Δ in the following set

$$S = \left\{ \Delta \mid \begin{pmatrix} \hat{\Delta}_{11} & & \\ & \ddots & \\ & & \hat{\Delta}_{mm} \end{pmatrix} = P\Delta P^T \qquad \hat{\Delta}_{ii} < \begin{cases} \text{any bounded value} & \text{if } \rho_i = 0 \\ \rho_i^{-1} I & \text{if } \rho_i \neq 0 \end{cases} \right\}$$

are such that

$$\rho(A + E_1 \Delta E_2^T) < 1.$$

The problem may thus be split into several subproblems. If the subproblems are small enough, we may have some analytical necessary and sufficient conditions. If the subproblems are more complex, to ensure that $\rho(A + E_1\Delta E_2^T) < 1$, sufficient conditions may be used such as freezing a k_i or imposing that for each $\hat{M}_{ii} \neq 0$, $\hat{\Delta}_{ii} < \rho_i^{-1} I$.

2.3. Algorithm

The results presented in the previous section can be used to construct the set $S_{\mathscr{D}}$, in the following manner :

1. Compute the matrix $M := E_2^T(I - A)^{-1}E_1$ and perform permutations to put it under the normal form (3). This can be done by applying the following algorithm :

   (a) Use Tarjan's algorithm [7] to find the set of strongly connected subgraphs associated to the graph G defined by the Adjacency Matrix M^{ad} ($M^{ad}_{i,j} = 1$ if $M_{i,j} \neq 0$, $M^{ad}_{i,j} = 0$ otherwise).

   (b) Consider a new graph G' whose nodes represent the strongly connected subgraphs : two nodes i and j of G' are connected if there exists one edge between a node of G in the subgraph i and a node of G in the subgraph j. The adjacency matrix of this new graph G' can be computed simply from M^{ad} by first summing up the rows corresponding to the same subgraph and then summing up the columns corresponding to the same subgraph.

(c) Identify a leaf i of the graph G' (which always exists because there is no cycle in G') and permute the columns and the rows of M corresponding to the subgraph i at the beginning of the matrix. Suppress node i from G'. Repeat 1c until M is in canonical form (3).

2. For each of the $\hat{M}_{ii}$ blocks, express the condition $\det(\hat{M}_{ii}\hat{\Delta}_{ii} - I) = 0$ which describes a part of the boundary of $S_{\mathscr{D}}$. If the size of $\hat{M}_{ii}$ is too high, a more restrictive condition such as first freezing a k_i or a condition in term of $\rho(\hat{M}_{ii})$ can be used.

It can be now claimed that, if

$$(k_1, \ldots, k_n) \in S_{\mathscr{D}}$$

then

$$\rho\left(A + E_1 \begin{pmatrix} k_1 & & \\ & \ddots & \\ & & k_n \end{pmatrix} E_2^T\right) < 1.$$

3. Application to the control of hyperbolic systems of conservation laws

As a matter of illustration, we present in this section an application to the control design for hyperbolic systems of conservation laws, with a typical example from waterways networks management (see e.g. [2]). We consider a set of N systems of two linear conservation laws of the general form :

$$\partial_t h_i(t,x) + \partial_x q_i(t,x) = 0 \tag{6}$$

$$\partial_t q_i(t,x) + \alpha\beta\partial_x h_i(t,x) + (\alpha - \beta)\partial_x q_i(t,x) = 0 \tag{7}$$

where :

- $i = 1, \dots, N$
- t and x are two independent variables : a time variable $t \in [0, +\infty)$ and a space variable $x \in [0, L]$ on a finite interval ;
- (h, q) ; $[0, +\infty) \times [0, L] \rightarrow \Omega \subset \mathbb{R}^2$ is the vector of the two dependent variables (i.e. $h(t, x)$ and $q(t, x)$ are the two states of the system) ;
- α and β are two real positive constants.

The first equation (6) is a conservation law with h_i the conserved quantity and q_i the flux. The second equation (7) is a conservation law with q_i the conserved quantity and $\alpha\beta h_i + (\alpha - \beta)q_i$ the flux. A typical example is given by the shallow water equations that are used for the modelling of 1-D water flow in open channels and where (6) is a mass conservation equation and (7) a momentum conservation equation.

As a matter of illustration, we shall consider here the connection of three reaches as represented in Figure 4. In this case, the model (6)–(7) represents the shallow water equations linearised around a steady state with

$$
\begin{aligned}
h_i(t, x) &= H_i(t, x) - \bar{H}_i \\
q_i(t, x) &= Q_i(t, x) - \bar{Q}_i
\end{aligned}
$$

where $H_i(t, x)$ and $Q_i(t, x)$ are the water level and the flow rate in the pools respectively, while $\bar{H}_i$ and $\bar{Q}_i$ are the steady-state set points. The control variables are the flows between the reaches $u_i = q_i$ which can be achieved by choosing the appropriate vertical position of the spillways located between the pools.

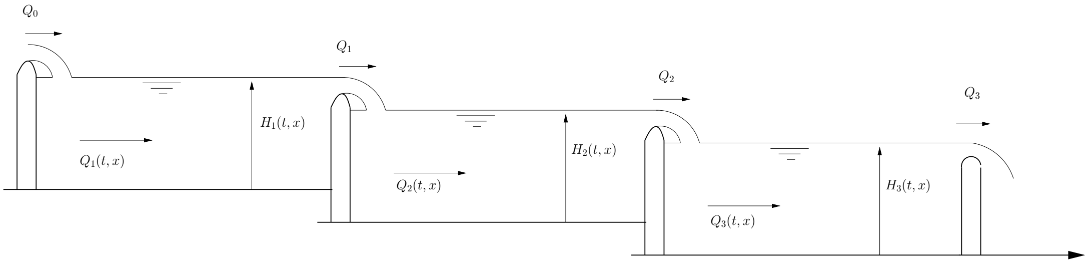

Figure 4. A canal with three reaches and four gates.

The six boundary conditions necessary to have a well-posed system are

$$\begin{aligned} q_1(t,0) &= u_0 \\ q_1(t,L) &= u_1 \\ q_2(t,L) &= u_2 \\ q_3(t,L) &= u_3 \\ q_1(t,L) &= q_2(t,0) \\ q_2(t,L) &= q_3(t,0). \end{aligned}$$

The first four conditions are imposed by the controls. The last two conditions express the flow conservation.

As shown e.g. in [2], it is convenient to work with the *Riemann coordinates* defined by the following change of coordinates :

$$\begin{aligned} a_i(t,x) &= q_i(t,x) + \beta h_i(t,x) \\ b_i(t,x) &= q_i(t,x) - \alpha h_i(t,x). \end{aligned}$$

With these coordinates, the system (6)-(7) is rewritten under the following diagonal form :

$$\partial_t \begin{pmatrix} a_i(t,x) \\ b_i(t,x) \end{pmatrix} + \begin{pmatrix} +\alpha & 0 \\ 0 & -\beta \end{pmatrix} \partial_x \begin{pmatrix} a_i(t,x) \\ b_i(t,x) \end{pmatrix} = 0 \qquad \forall i \in 1,\ldots,3 \tag{8}$$

and the boundary conditions are expressed as :

$$\begin{aligned}
\frac{\alpha a_1(t,0)+\beta b_1(t,0)}{\alpha+\beta} &= u_0 \\
\frac{\alpha a_1(t,L)+\beta b_1(t,L)}{\alpha+\beta} &= u_1 \\
\frac{\alpha a_2(t,L)+\beta b_2(t,L)}{\alpha+\beta} &= u_2 \\
\frac{\alpha a_3(t,L)+\beta b_3(t,L)}{\alpha+\beta} &= u_3 \\
\frac{\alpha a_1(t,L)+\beta b_1(t,L)}{\alpha+\beta} &= \frac{\alpha a_2(t,0)+\beta b_2(t,0)}{\alpha+\beta} \\
\frac{\alpha a_2(t,L)+\beta b_2(t,L)}{\alpha+\beta} &= \frac{\alpha a_3(t,0)+\beta b_3(t,0)}{\alpha+\beta}.
\end{aligned}$$

We consider the situation where each control $u_i(t)$ is a linear function of only one state variable, as follows :

$$\begin{array}{ll}
u_0 \text{ function of } b_1(t,0) & u_0 = k_0' b_1(t,0) \\
u_1 \text{ function of } a_1(t,L) & u_1 = k_1' a_1(t,L) \\
u_2 \text{ function of } a_2(t,L) & u_2 = k_2' a_2(t,L) \\
u_3 \text{ function of } a_3(t,L) & u_3 = k_3' a_3(t,L).
\end{array}$$

With the following reparametrization :

$$\begin{aligned}
k_0 &= -\frac{\beta}{\alpha} + \frac{(\alpha+\beta)}{\alpha} k_0' \\
k_i &= -\frac{\alpha}{\beta} + \frac{(\alpha+\beta)}{\beta} k_i' \qquad i = 1,\ldots,3
\end{aligned}$$

the boundary conditions are written as :

$$\begin{pmatrix} b_1(t,L) \\ b_2(t,L) \\ b_3(t,L) \\ a_1(t,0) \\ a_2(t,0) \\ a_3(t,0) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_3 \\ k_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\beta}{\alpha} & 0 & 1+\frac{\beta}{\alpha}(k_1) & 0 & 0 \\ 0 & 0 & -\frac{\beta}{\alpha} & 0 & 1+\frac{\beta}{\alpha}(k_2) & 0 \end{pmatrix}}_{A+K} \cdot \begin{pmatrix} b_1(t,0) \\ b_2(t,0) \\ b_3(t,0) \\ a_1(t,L) \\ a_2(t,L) \\ a_3(t,L) \end{pmatrix} \tag{9}$$

where A is fixed and K has a fixed structure but the values of its non-zero entries are linear combinations of the “free parameters” k_i. In this case, K is the part of the matrix which reflects the choice made by the operator for the control parameters and A reflects the conservation of the flow.

The problem studied here is to find the largest range of values for the control parameters k_i such that the system remains stable. Stability means here that, from any smooth enough initial condition, the Cauchy problem for system (8) with boundary conditions (9) has a unique classical solution that exponentially converges to the origin.

From Theorem 6 in [3], we know that a sufficient stability condition is that :

$$\rho(|A+K|) < 1$$

where $|A+K|$ denotes the matrix whose entries are the absolute values of the entries of $A+K$. We are thus interested in finding a set S such that $\rho(|A+K|) < 1 \quad \forall K \in S$.

From [1], we know that if $A, B \geq 0$ then $\rho(A+B) \geq \rho(A)$. It implies that

$$\rho(|A+K|) \leq \rho(|A|+|K|)$$

and the set S may be found using the algorithm presented in Section 2.3.

If we apply the algorithm proposed in Section 2.3 with the following numerical values :

$$\begin{aligned} \alpha &= 3.6 \\ \beta &= 2.6, \end{aligned}$$

we obtain that there exists three blocks $\hat{M}_{ii}$ (one of dimension 2 and two of dimension one). The different coefficients must be bounded as follow

$$\begin{aligned} |k_0 k_1| &< 1 \\ |k_2| &< 1.38 \\ |k_3| &< 1.38 \end{aligned}$$

in order to guarantee the stability of the system. This decomposition in three blocks is quite natural since only the two first parameters influence the stability of the first reach. If the first reach is stable, only the parameter k_2 has an influence on the stability of the second reach. Eventually k_3 controls the stability of the third reach.

The decoupling of the problem in smaller subproblem allows to increase the possible value of some parameters which may have a positive influence on the global behaviour of the system. In the example of Section 3, if we take the sufficient condition presented in [5] all the parameters must be bounded by 1. The decomposition in subproblems allows us to increase the value of k_3 and k_4 up to 1.38. It also allows to select $k_0 > 1$ provided k_1 is small enough and conversely.

4. Conclusions

In this paper, we have considered the problem of finding the largest set of perturbation such that a positive matrix remains stable. We have extended the results of [5] in the particular case where the perturbation matrix Δ is diagonal. In this case, the problem can be decoupled in smaller subproblems. For each of the subproblems, necessary and sufficient analytical conditions were derived to describe the starlike sets of admissible parameters. Outside of these sets, the perturbation destabilises the system.

These sets, which are not necessarily convex, contain the largest admissible ball described in [5].

REFERENCES

1. A. Berman and R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. siam, 1994.
2. J. de Halleux and G. Bastin. Stabilisation of Saint Venant equations using Riemann invariants : application to waterways with mobile spillways. In *Proceedings 15th IFAC World Congress*, pages 2151–2156, Barcelona, 2002.
3. J. de Halleux, C. Prieur, J.-M. Coron, B. d'Andra Novel, and G. Bastin. Boundary feedback control in networks of open channels. *Automatica*, 39:1365–1376, 2003.
4. F. R. Gantmacher. *The Theory of Matrices*. New York, NY, USA, 1959.
5. D. Hinrichsen and N. K. Son. Stability radii of positive discrete-time systems under affine parameter perturbations. *International J. Robust and Nonlinear Control*, 8:1169–1188, 1998.
6. R.A. Horn and Ch. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.
7. R. Tarjan. Depth first search and linear graph algorithms. *SIAM Journal of Computing*, 1(2):140–160, june 1972.