Linear Algebra and its Applications xxx (2013) xxx–xxx

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Stratification of full rank polynomial matrices ☆

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ARTICLE INFO

Article history:
Received 15 April 2012
Accepted 24 December 2012
Available online xxxx

Submitted by Peter Semrl

Dedicated to the memory of Vera Nikolaevna Kublanovskaya—a great personality and source of inspiration [31].

AMS classification:
15A18
15A21
15A22
65F15
65F35
93B18

Keywords:
Polynomial matrices
Matrix pencils
Linearization
Perturbations
Stratification
Closure hierarchy
Cover relations
StratiGraph

ABSTRACT

We show that perturbations of polynomial matrices of full normal-rank can be analyzed via the study of perturbations of companion form linearizations of such polynomial matrices. It is proved that a full normal-rank polynomial matrix has the same structural elements as its right (or left) linearization. Furthermore, the linearized pencil has a special structure that can be taken into account when studying its stratification. This yields constraints on the set of achievable eigenstructures. We explicitly show which these constraints are. These results allow us to derive necessary and sufficient conditions for cover relations between two orbits or bundles of the linearization of full normal-rank polynomial matrices. The stratification rules are applied to and illustrated on two artificial polynomial matrices and a half-car passive suspension system with four degrees of freedom.

☆ A preprint appears as *Report UMINF 11.11*. The scientific responsibility rests with the authors.

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http://dx.doi.org/10.1016/j.laa.2012.12.013

1. Introduction

Polynomial matrices play an important role in the study of dynamical systems described by sets of differential-algebraic equations (DAEs) with constant coefficient matrices

$$P_d x^{(d)}(t) + \cdots + P_1 x^{(1)}(t) + P_0 x(t) = f(t), \quad t \geqslant 0, \tag{1}$$

where $x(t) \in \mathbb{C}^n$, $f(t) \in \mathbb{C}^m$, $P_i \in \mathbb{C}^{m\times n}$, and $x^{(i)}(t)$ is the i-th derivative of the vector $x(t)$. Taking the Laplace transform of a DAE system (1) and imposing zero initial conditions, yields the algebraic equation

$$P(s)\hat{x}(s) = \hat{f}(s) \quad \text{with} \quad P(s) := P_d s^d + \cdots + P_1 s + P_0, \quad s \in \mathbb{C},$$

where d is the degree of $P(s)$, and $\hat{x}(s)$ and $\hat{f}(s)$ are the Laplace transforms of $x(t)$ and $f(t)$, respectively. Throughout the paper, we assume that the leading coefficient matrix P_d is nonzero so that the highest degree is indeed d (we say it has *exact degree d*). The importance of using polynomial models is widely recognized and can be found in basic references such as [18,30,38,40]. For example, polynomial matrices appear when studying linearizations of mechanical systems [41], multibody dynamics [14], and vibration analysis of buildings, machines, and vehicles [33].

When the polynomial matrix $P(s)$ is square and *regular* (this is when $\det(P(s))$ is not identically zero) then the solutions of the set of differential equations (1) with zero initial conditions mainly depend on the zeros of $P(s)$ and their multiplicities. The fine structure of this so-called zero structure is described in more detail by the elementary divisors of $P(s)$. But if $P(s)$ is *singular* (this is when $\det(P(s))$ is identically zero for any s or when $P(s)$ is non-square) then the solution set of (1) becomes more complex and depends on the *left and right nullspaces* of $P(s)$. These null spaces describe, respectively, constraints one needs to impose on $f(t)$ for (1) to have compatible solutions, and degrees of freedom in the solution set of (1). It is therefore crucial to understand well the complete eigenstructure of $P(s)$ since this will determine the properties of the solution set of (1).

In general, the eigenstructure of $P(s)$ is quite sensitive to perturbations in the matrix coefficients P_i and one wants therefore to accurately describe how that structure can change when small variations are applied to the matrix coefficients. Such a study can be performed by so called versal deformations of the eigenstructure of the Jordan and Kronecker canonical forms as introduced in [1] for square matrices. One tool that can be used to analyze the qualitative information of nearby systems is the theory of stratification [12,11,15,28]. A stratification reveals the closure hierarchy of orbits and bundles of nearby canonical structures and gives important qualitative information about the underlying dynamical system. It shows which canonical structures can be reached by a small perturbation and the relation among these structures. A stratification can be represented as a graph where each node represents an orbit or bundle of a canonical structure and an edge corresponds to a covering relation. When two orbits (or bundles) of canonical structures are nearest neighbors in the closure hierarchy they fulfill a cover relation. Such cover relations can be expressed as combinatorial rules acting on integer sequences representing a subset of the structural elements.

Closure and cover relations have been studied, e.g., in [1,3,5,11,12,23,36,37] for matrices and first order polynomial matrices (matrix pencils), in [15,17,20,24] for system pencils associated with state-space systems, and in [10,24,25] for system pencils associated with descriptor and singular systems. In this paper, we extend these results to the case of polynomial matrices by making use of companion form linearizations. These linearizations are matrix pencils, but with the constraint that some elements of the coefficient matrices are fixed to 0 or 1, which reduces the set of possible eigenstructures that can be achieved. For numerical algorithms see [31] (and references therein) and [43].

Recently, problems related to stratification of polynomial matrices have been addressed in [35,7]. In [35], it is shown that the map between the orbit space of a controllable matrix pair (A, B) and a polynomial matrix $P(s)$ is a homeomorphism under stated assumptions. The orbits considered are the orbits of matrix pairs under system similarity and the orbits of polynomial matrices under right equivalence. Moreover, necessary and sufficient conditions for a polynomial matrix to be in the closure of another are derived.

The rest of the paper is organized as follows. In Section 2, we start by reviewing the different eigenstructure elements that a polynomial matrix can have and make the link with the eigenstructure elements of matrix pencils. In Section 3, we describe standard companion form linearizations of polynomial matrices that preserve these eigenstructure elements. Sections 4 and 5 analyze in more detail the constraints on the eigenstructures for the so-called scalar and matrix cases. In Section 6, we discuss the relations between the polynomial and the system pencil representations. We continue in Section 7 to introduce integer partitions and minimal coin moves that are used to represent the structure integer partitions in Section 8 and which appear in the covering rules. Section 9 introduces the generalized Sylvester space and we define concepts like orbits and bundles for polynomial matrix linearizations and their codimensions expressed in terms of the structure integer partitions. We also show that perturbations of a polynomial matrix can be analyzed via the study of perturbations in its companion form linearization. In Section 10, the cover relations for orbits and bundles of full normal-rank polynomial matrix linearizations are derived. Finally, in Section 11 we illustrate and apply the stratification theory on a few examples, including a passive suspension system model.

2. Structural elements of $P(s)$

The eigenstructure elements of a polynomial matrix require the definition of the Smith normal form and of unimodular matrices (e.g., see [16]).

Definition 2.1. A square polynomial matrix $M(s)$ is said to be *unimodular* if its determinant is constant and nonzero.

Definition 2.2. Two polynomial matrices $P(s)$ and $\widetilde{P}(s)$ of the same size are called *equivalent* if

$$P(s) = M(s)\widetilde{P}(s)N(s),$$

for some unimodular matrices $M(s)$ and $N(s)$ of conforming sizes.

Notice that unimodular matrices have a polynomial inverse that is also unimodular and that products of unimodular matrices are also unimodular, from which it follows that they form a transformation group. Under this transformation group a unique canonical form of an arbitrary polynomial matrix can be obtained.

Definition 2.3 [16]**.** The *Smith normal form* of an arbitrary $m \times n$ polynomial matrix $P(s)$ is the quasi diagonal matrix obtained under unimodular transformations $M_l(s)$ and $M_r(s)$ applied to the rows and columns of $P(s)$:

$$M_l(s)P(s)M_r(s) = \left[\begin{array}{cccc|c} e_1(s) & 0 & \dots & 0 & \\ 0 & e_2(s) & \ddots & \vdots & O_{r,n-r} \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \dots & 0 & e_r(s) & \\ \hline & & & & \\ & O_{m-r,r} & & & O_{m-r,n-r} \\ & & & & \end{array}\right] \tag{2}$$

where each $e_j(s)$ is monic and divides $e_{j+1}(s)$ for $j = 1, \dots, r-1$. The polynomials $e_j(s)$ are unique and are called the *invariant polynomials* of $P(s)$. We will call an invariant polynomial *trivial* if $e_j(s) = 1$, otherwise *non-trivial* (i.e., $e_j(s)$ is a polynomial of degree $\geqslant 1$).

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A *zero* $\alpha \in \mathbb{C}$ of $P(s)$ is a zero of any $e_j(s)$ and its *finite elementary divisors* are the factors $(s-\alpha)^{h_j}$ of each $e_j(s)$; their powers are non-decreasing:

$$h_1 \leqslant h_2 \leqslant \cdots \leqslant h_r \leqslant 0. \tag{3}$$

The index r is called the *normal-rank* of $P(s)$ and it is equal to the rank of $P(s)$ at any value of $s \in \mathbb{C}$ which is not a zero of $P(s)$. Elementary divisors associated with a non-trivial invariant polynomial are also called non-trivial.

We say that an $m \times n$ polynomial matrix has *full normal-rank* if its normal-rank $r = \min(m, n)$. Consequently, if $r = m$ then $n \geqslant m$ and if $r = n$ then $n \leqslant m$.

Remark 2.1. In the rest of the paper, we use a permuted version of the Smith form where the elementary divisors (3) are defined in reverse order:

$$h_1 \geqslant h_2 \geqslant \cdots \geqslant h_r \geqslant 0. \tag{4}$$

This is important for compatibility with the theorems in earlier papers [12,15].

For the zero $s = \infty$, there are several different characterizations. We will use here the definition based on the so-called *reversed* polynomial matrix.

Definition 2.4. For a polynomial matrix $P(s)$ of degree d, the *reversed* polynomial matrix $\mathrm{rev}P(\mu)$ is

$$\mathrm{rev}P(\mu) := \mu^d P\left(\frac{1}{\mu}\right) = P_d + P_{d-1}\mu + \cdots + P_0\mu^d, \tag{5}$$

which is obtained from the substitution $s = \frac{1}{\mu}$ in the polynomial matrix $P(s)$.

Definition 2.5. The finite elementary divisors μ^{h_j} of the zero $\mu = 0$ of $\mathrm{rev}P(\mu)$ are the *infinite elementary divisors* $1/s^{h_j}$ of the polynomial matrix $P(s)$.

Notice there exist other definitions of the infinite zero structure [30,43] but one can easily find relations between them [44].

A polynomial matrix $P(s)$ that has normal-rank r smaller than m and/or n, has also left and right null spaces that can be represented by polynomial bases as one can see from (2). In order to define the null space structure, we need to define *minimal polynomial bases*.

Definition 2.6. The $n \times r$ polynomial matrix $N(s)$ with the highest column degrees $\{d_1, \ldots, d_r\}$ is *column reduced*, if the *highest degree coefficient matrix* N_h, whose j-th column is the coefficient of s^{d_j} in the j-th column of $N(s)$, also has full column rank. Its normal-rank is therefore also equal to r.

We recall here a lemma about column reduced matrices, that will be useful in the rest of the paper. Proof can be found in, e.g., [30].

Lemma 2.7. *Every $n \times r$ polynomial matrix $N(s)$ of normal-rank r can be transformed by a unimodular column transformation $V(s)$ to a column reduced matrix $N(s)V(s)$ with non-increasing column degrees $d_j, j = 1, \ldots, r$. An additional constant and invertible row transformation R will transform the highest degree coefficient matrix of $RN(s)V(s)$ to* $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$.

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Remark 2.2. The dual result obviously holds as well. Every $r \times m$ polynomial matrix $N(s)$ of normal-rank r can be transformed by a unimodular row transformation $U(s)$ to a row reduced matrix $U(s)N(s)$ with non-increasing row degrees $d_j, j = 1, \ldots, r$. An additional *constant and invertible* column transformation C will transform the highest degree coefficient matrix of $U(s)N(s)C$ to $\begin{bmatrix} I_r & 0 \end{bmatrix}$.

Definition 2.8. The $n \times r$ polynomial matrix $N(s)$ is called a *minimal basis* for the space spanned by its columns if $N(s)$ has full column rank for all finite $s \in \mathbb{C}$ and if it is column reduced. The column degrees $\{d_1, \ldots, d_r\}$ of any minimal basis for a particular space, are unique and are called the *minimal indices* of that space.

We are now ready to define the remaining eigenstructure elements of $P(s)$.

Definition 2.9. Let $P(s)$ be an $m \times n$ polynomial matrix of normal-rank r and let

$$N_\ell^T(s)P(s) = 0, \quad P(s)N_r(s) = 0, \tag{6}$$

where the $m \times (m-r)$ polynomial matrix $N_\ell(s)$ and the $n \times (n-r)$ polynomial matrix $N_r(s)$ are column reduced. The left and right null space structures of $P(s)$ are then the column degrees $\{\eta_1, \ldots, \eta_{m-r}\}$ and $\{\epsilon_1, \ldots, \epsilon_{n-r}\}$ of $N_\ell(s)$ and $N_r(s)$, respectively.

The column degrees $\{\eta_1, \ldots, \eta_{l_0}\}$ and $\{\epsilon_1, \ldots, \epsilon_{r_0}\}$ are called the *left (row)* and *right (column) minimal indices*, respectively, where $l_0 = m - r$ and $r_0 = n - r$.

We point out here that if we apply the above definitions to a first order (or linear) polynomial matrix $P(s)$ we retrieve the definitions of the structural elements obtained from the *Kronecker canonical form* (KCF) of a *matrix pencil* $sH + G$. Any general $m_{\mathrm{p}} \times n_{\mathrm{p}}$ matrix pencil $sH + G$ can be transformed into KCF in terms of an equivalence transformation with two nonsingular matrices U and V [16]:

$$U(sH + G)V^{-1} = \mathrm{diag}(L_{\epsilon_1}, \ldots, L_{\epsilon_{r_0}}, J(\lambda_1), \ldots, J(\lambda_q), N_{h_1}, \ldots, N_{h_{g_\infty}}, L_{\eta_1}^T, \ldots, L_{\eta_{l_0}}^T), \tag{7}$$

where $J(\lambda_i) = \mathrm{diag}(J_{h_1}(\lambda_i), \ldots, J_{h_{g_i}}(\lambda_i)), i = 1, \ldots, q$, and g_i is the geometric multiplicity of the finite eigenvalue λ_i and g_∞ the geometric multiplicity of the infinite eigenvalue. The matrix pencil $sH + G$ has q distinct finite eigenvalues and each eigenvalue λ_i coincides with a zero of $P(s)$ in Definition 2.3. The four types of *canonical blocks* are:

$$J_{h_k}(\lambda_i) := \begin{bmatrix} s-\lambda_i & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & s-\lambda_i \end{bmatrix}, \quad N_{h_k} := \begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix},$$

$$L_{\epsilon_k} := \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & & s & -1 \end{bmatrix}, \quad \text{and} \quad L_{\eta_k}^T := \begin{bmatrix} s & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix},$$

where

- $J_{h_k}(\lambda_i)$ is a $h_k \times h_k$ *Jordan block* of a finite eigenvalue λ_i, corresponding to a finite elementary divisor of degree h_k, namely $(s-\lambda_i)^{h_k}$,
- N_{h_k} is a $h_k \times h_k$ Jordan block of the infinite eigenvalue, corresponding to an infinite elementary divisor of degree h_k, namely $1/s^{h_k}$,
- L_{ϵ_k} is an $\epsilon_k \times (\epsilon_k + 1)$ *right singular block*, corresponding to a right (column) null vector of minimal degree ϵ_k, namely $[1, s, \ldots, s^{\epsilon_k}]^T$, and
- $L^T_{\eta_k}$ is an $(\eta_k + 1) \times \eta_k$ *left singular block*, corresponding to a left (row) null vector of minimal degree η_k, namely $[1, s, \ldots, s^{\eta_k}]$.

L_0 and L_0^T blocks are of size 0×1 and 1×0, respectively, and each of them contributes with a column (L_0) or row (L_0^T) of zeros in the KCF.

In Section 3, we present a linearization of the polynomial matrix in the form of a matrix pencil and we show that most of the structural elements of $P(s)$ are preserved as the structural elements of the linear pencil.

3. Linearizations

The classical approach to analyze and determine the structural elements of (1) is to study linearizations of polynomial matrices $P(s)$, which result in a large linear matrix pencil $sH + G$ [2,18]. A linearization is not unique, instead there exist several different, e.g., see [4,6,8,34]. Here we only consider the so called right and left linearizations (also called second and first companion linearizations, respectively). We remark that the generalized eigenvalues of the companion linearizations are potentially more ill-conditioned compared to the eigenvalues of $P(s)$. However, when the 2-norms of the coefficient matrices of $P(s)$ are all around one, they are almost equally conditioned [22].

The *right linearization* of an $m \times n$ polynomial matrix $P(s)$, which is equivalent to the so called *second companion form*, has the matrix pencil representation

$$sH_r + G_r := s \begin{bmatrix} I_m & & & \\ & \ddots & & \\ & & I_m & \\ & & & P_d \end{bmatrix} + \begin{bmatrix} 0 & & & P_0 \\ -I_m & \ddots & & P_1 \\ & \ddots & 0 & \vdots \\ & & -I_m & P_{d-1} \end{bmatrix}. \tag{8}$$

In this section, we derive the relations between the eigenstructure elements of $P(s)$ of normal-rank m and those of the matrix pencil $sH_r + G_r$ of size $dm \times (m(d-1)+n)$. To do this we make use of the following lemma.

Lemma 3.1 [16]. *Two polynomial matrices $P(s)$ and $Q(s)$ are equivalent if and only if they have the same invariant polynomials.*

When left multiplying $sH_r + G_r$ (8) with an appropriate unimodular matrix we obtain

$$\begin{bmatrix} I_m & sI_m & \ldots & s^{d-1}I_m \\ & \ddots & \ddots & \vdots \\ & & \ddots & sI_m \\ & & & I_m \end{bmatrix} (sH_r + G_r) = \begin{bmatrix} 0 & & & P(s) \\ -I_m & \ddots & & X_2(s) \\ & \ddots & 0 & \vdots \\ & & -I_m & X_d(s) \end{bmatrix}, \tag{9}$$

where the $X_i(s)$, for $i = 2, \ldots, d$, are polynomial matrices as well. An additional unimodular right transformation then gets rid of the matrices $X_i(s)$:

$$\begin{bmatrix} 0 & & & P(s) \\ -I_m & \ddots & & X_2(s) \\ & \ddots & 0 & \vdots \\ & & -I_m & X_d(s) \end{bmatrix} \begin{bmatrix} I_m & & X_2(s) \\ & \ddots & \vdots \\ & & I_m \; X_d(s) \\ & & I_m \end{bmatrix} = \begin{bmatrix} & P(s) \\ -I_{m(d-1)} & \end{bmatrix}.$$

Together with Lemma 3.1 we have now shown that $sH_r + G_r$ and $P(s)$ have the same finite elementary divisors.

For the infinite elementary divisors, we need to compare the elementary divisors of the eigenvalue $\mu = 0$ of the reversed pencil $H_r + \mu G_r$ with those of the reversed polynomial $\mathrm{rev}P(\mu)$ defined in (5). We now multiply $H_r + \mu G_r$ on the left with an appropriate unimodular matrix in the variable μ:

$$\begin{bmatrix} I_m & & \\ \mu I_m & \ddots & \\ \vdots & \ddots & \ddots \\ \mu^{d-1} I_m & \ldots \; \mu I_m & I_m \end{bmatrix} (H_r + \mu G_r) = \begin{bmatrix} I_m & & Y_d(\mu) \\ & \ddots & \vdots \\ & I_m & Y_2(\mu) \\ & & \mathrm{rev}P(\mu) \end{bmatrix},$$

where now the matrices $Y_i(\mu)$ are polynomial matrices in μ and can be eliminated by an additional unimodular transformation applied to the right:

$$\begin{bmatrix} I_m & & Y_d(\mu) \\ & \ddots & \vdots \\ & I_m & Y_2(\mu) \\ & & \mathrm{rev}P(\mu) \end{bmatrix} \begin{bmatrix} I_m & & -Y_d(\mu) \\ & \ddots & \vdots \\ & I_m & -Y_2(\mu) \\ & & I_m \end{bmatrix} = \begin{bmatrix} I_{m(d-1)} & \\ & \mathrm{rev}P(\mu) \end{bmatrix}$$

showing that $H_r + \mu G_r$ and $\mathrm{rev}P(\mu)$ have the same elementary divisors. We have thus derived the following theorem (see also [18]).

Theorem 3.2. *The polynomial matrix $P(s)$ and the linearized pencil $sH_r + G_r$ defined in (8), have the same finite and infinite elementary divisors.*

In order to address the null space structure, we recall a lemma, proved in [44].

Lemma 3.3. *Let $\begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = 0$ and let X_1 and $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ have full column rank, then Y_2 must also have full column rank.*

We now use this lemma to prove the following theorem for the right null space structures of $P(s)$ and $sH_r + G_r$.

Theorem 3.4. *Let $N(s)$ be a minimal basis for the right null space of the pencil $sH_r + G_r$ and partition it as follows (where $N_2(s)$ has n rows):*

$$\left[\begin{array}{cccc|c} sI_m & & & & P_0 \\ -I_m & \ddots & & & \vdots \\ & \ddots & sI_m & & P_{d-2} \\ & & -I_m & & sP_d + P_{d-1} \end{array}\right] \left[\begin{array}{c} N_1(s) \\ \hline N_2(s) \end{array}\right] = 0. \tag{10}$$

Then $N_2(s)$ is a right minimal basis of $P(s)$ with the same minimal indices as $N(s)$.

Proof. We first apply the same left transformation as in (9) to (10), yielding

$$\left[\begin{array}{ccc|c} 0 & & & P(s) \\ -I_m & \ddots & & X_2(s) \\ & \ddots & 0 & \vdots \\ & & -I_m & X_d(s) \end{array}\right] \left[\begin{array}{c} N_1(s) \\ \hline N_2(s) \end{array}\right] = 0. \tag{11}$$

Clearly, this implies that $P(s)N_2(s) = 0$ and applying Lemma 3.3 to this for any finite value s, implies that $N_2(s)$ has full column rank for any finite value of s.

Let us now partition the highest degree coefficient matrix N_h in a similar fashion. Then, equating the highest degree coefficients of the top $m(d-1)$ equations of (10) yields

$$\left[\begin{array}{ccc|c} I_m & & & 0 \\ & \ddots & & \vdots \\ & & I_m & 0 \end{array}\right] \left[\begin{array}{c} N_{h1} \\ \hline N_{h2} \end{array}\right] = 0.$$

This implies that $N_{h1} = 0$ and N_{h2} has full column rank. Therefore, $N_2(s)$ is a minimal basis with the same minimal indices as $N(s)$. $\square$

Remark 3.1. The following example shows that one cannot say the same for the left minimal indices of $P(s)$ and $sH_r + G_r$:

$$P(s) := \begin{bmatrix} s & s^2 \\ 1 & s \end{bmatrix} = \begin{bmatrix} s \\ 1 \end{bmatrix} \begin{bmatrix} 1 & s \end{bmatrix}, \quad sH_r + G_r = \left[\begin{array}{cc|cc} s & 0 & 0 & 0 \\ 0 & s & 1 & 0 \\ \hline -1 & 0 & 1 & s \\ 0 & -1 & 0 & 1 \end{array}\right].$$

Indeed, the minimal left null spaces of $P(s)$ and $sH_r + G_r$ are respectively $\begin{bmatrix} 1 & -s \end{bmatrix}$ and $\begin{bmatrix} 1 & -s & s & -s^2 \end{bmatrix}$ and their minimal index is different.

Thus we have proved that an $m \times n$ polynomial matrix $P(s)$ of normal-rank m has the same structural elements as the so-called *right* linearization $sH_r + G_r$ (8).

For the left minimal indices we consider the *left* linearization

$$sH_\ell + G_\ell := s \begin{bmatrix} I_n & & & \\ & \ddots & & \\ & & I_n & \\ & & & P_d \end{bmatrix} + \begin{bmatrix} 0 & -I_n & & \\ & \ddots & \ddots & \\ & & 0 & -I_n \\ P_0 & P_1 & \dots & P_{d-1} \end{bmatrix}, \tag{12}$$

for which the dual result holds. Notably, the matrix pencil $sH_\ell + G_\ell$ is equivalent to the so called *first companion form*.

We synthesize the results of this section in the following theorem.

Theorem 3.5. *Let $P(s)$ be an $m \times n$ polynomial matrix of normal-rank r, then*

1. *if $r = m$, $P(s)$ has the same structural elements as $sH_r + G_r$ defined in (8),*
2. *if $r = n$, $P(s)$ has the same structural elements as $sH_\ell + G_\ell$ defined in (12),*
3. *for any r, $P(s)$ has the same elementary divisors as $sH_r + G_r$ and as $sH_\ell + G_\ell$, and*
4. *for any r, $P(s)$ has the same right minimal indices as $sH_r + G_r$ and the same left minimal indices as $sH_\ell + G_\ell$.*

4. Scalar case

In this section, we consider the case where $m = 1$ and we assume that the polynomial matrix has exact degree d (nonzero leading coefficient P_d). This of course implies that the polynomial matrix has normal-rank 1 as well since it is nonzero. The Smith form of such a polynomial matrix is quite special since it contains exactly one polynomial $e(s)$, which is the greatest common divisor of the scalar polynomials in $P(s)$:

$$P(s) := \begin{bmatrix} p_1(s) & \dots & p_n(s) \end{bmatrix}, \quad e(s) := \gcd\{p_1(s), \dots, p_n(s)\}.$$

If k is the degree of $e(s)$ then there are $n-2$ right minimal indices equal to 0 and one equal to $d-k$. The other structure elements are all the possible structures one can find in a scalar polynomial of degree k. We synthesize the conclusions in the following theorem.

Theorem 4.1. *A $1 \times n$ polynomial matrix $P(s)$ of exact degree d has only one elementary divisor $(s-\lambda_i)^{h_i}$ for each zero λ_i, $n-2$ right minimal indices equal to zero, and one right minimal index equal to ϵ_1 satisfying*

$$\sum_i h_i + \epsilon_1 = d.$$

All structures satisfying these constraints are possible for such a polynomial matrix.

Corollary 4.2. *A 1×1 scalar polynomial $p(s)$ of exact degree d has only one elementary divisor $(s-\lambda_i)^{h_i}$ for each zero λ_i satisfying*

$$\sum_i h_i = d.$$

All structures satisfying these constraints are possible for such a polynomial.

Clearly this is not reflected in the general form $sH + G$ of the pencil $sH_r + G_r$, but it is a result of the fact that $sH_r + G_r$ has *fixed elements* equal to 0 and 1. This problem is also related to the controllability of a generalized state-space system with $n-1$ inputs. For this, we relabel the polynomials as follows:

$$a(s) := p_1(s), \quad B(s) := \left[p_2(s) \ \ldots \ p_n(s) \right],$$

where we assume for simplicity that the highest degree coefficient of $a(s)$ is nonzero and that of $B(s)$ is equal to zero. This can be achieved by a constant column transformation of $P(s)$ (which does not affect the conclusions), where the highest degree coefficient of $a(s)$ is used as pivot to eliminate those of $B(s)$. One can then consider the following partitioning of the linearization of $P(s)$:

$$\left[sE + A \middle| B \right] := \left[\begin{array}{cccc|c} s & & & a_0 & B_0 \\ -1 & \ddots & & a_1 & B_1 \\ & \ddots & s & \vdots & \vdots \\ & & -1 & sa_d + a_{d-1} & B_{d-1} \end{array} \right],$$

where $B_0, \ldots, B_{d-1}$ are $1 \times (n-1)$ matrices. The controllability of this generalized state-space pair is equivalent to the existence of a gcd $e(s)$ of the polynomials of $P(s) = [\ a(s)\ B(s)\ ]$. The dimension of the uncontrollable space is also equal to the degree of $e(s)$. Rather than analyzing this using the perturbations of $\left[sE - A \middle| B \right]$ one can look at perturbations for this row vector of polynomials to have a common divisor.

5. Matrix case

In this section, we derive similar conditions for full normal-rank polynomial matrices as for the scalar case presented in Theorem 4.1 and Corollary 4.2. We will prove the following results.

Theorem 5.1. *An $m \times m$ polynomial matrix $P(s)$ of exact degree d and normal-rank m has m finite elementary divisors $(s-\lambda_i)^{h_j^{(i)}}$, $j = 1, \ldots, m$, for each zero λ_i, $i = 1, \ldots, q$, and m infinite elementary divisors $1/s^{h_j^{(\infty)}}$ (some of these indices can be trivially zero) satisfying*

$$\sum_{i=1}^{q} \sum_{j=1}^{m} h_j^{(i)} + \sum_{j=1}^{m} h_j^{(\infty)} = dm. \tag{13}$$

All structures satisfying the constraints (13) are possible for such a polynomial matrix.

Theorem 5.2. *An $m \times n$ polynomial matrix $P(s)$ of exact degree d and normal-rank m has m finite elementary divisors $(s-\lambda_i)^{h_j^{(i)}}$, $j = 1, \ldots, m$, for each zero λ_i, $i = 1, \ldots, q$, m infinite elementary divisors $1/s^{h_j^{(\infty)}}$, and $n-m$ right minimal indices ϵ_j, $j = 1, \ldots, n-m$ (some of these indices can be trivially zero) satisfying*

$$\sum_{i=1}^{q} \sum_{j=1}^{m} h_j^{(i)} + \sum_{j=1}^{m} h_j^{(\infty)} + \sum_{j=1}^{n-m} \epsilon_j = dm. \tag{14}$$

All structures satisfying the constraints (14) are possible for such a polynomial matrix.

Remark 5.1. The dual result for when $P(s)$ has normal-rank n is obtained by interchanging m and n, and replacing the right minimal indices ϵ with the left minimal indices η in the theorem above.

We thus need to show that all these structures may occur in an $m \times n$ polynomial matrix $P(s)$ of exact degree d. The fact that these constraints are necessary, is evident since an $m \times n$ polynomial matrix of rank $r = m$ can have only r non-trivial elementary divisors for each zero. The fact that these constraints are sufficient, on the other hand, requires a proof. Our proof is based on unimodular transformations, which leave the finite elementary divisors unchanged, but may change the infinite elementary divisors. We therefore make a change of variables, such that the polynomial matrix has no elementary divisors at infinity. For this we need the following lemma.

Lemma 5.3. *Let $P(s)$ be an $m \times n$ polynomial matrix of exact degree d and full normal-rank m which has no zero at $s = \omega$. Then putting $s = \frac{1}{\mu} + \omega$, the transformed polynomial matrix*

$$P_\omega(\mu) := \mu^d P\left(\frac{1}{\mu} + \omega\right) = P_d(1 + \mu\omega)^d + P_{d-1}\mu(1 + \mu\omega)^{(d-1)} + \cdots + P_0\mu^d$$

has the same right nullspace structure as $P(s)$, no zero at infinity, and its finite elementary divisors are given by

$$\left(\mu - \frac{1}{(\lambda_i - \omega)}\right)^{h_j^{(i)}}, \quad j = 1, \ldots, m, \quad \text{and} \quad \mu^{h_j^{(\infty)}}, \quad j = 1, \ldots, m. \tag{15}$$

Proof. The lemma follows directly from the correspondence with the Kronecker structure of the linearized pencil $sH + G$. The linearization of the transformed polynomial matrix $P_\omega(\mu)$ is given by $(1 + \mu\omega)H + \mu G = \mu(\omega H + G) + H$, which has the same right null space structure as $sH + G$ and the same elementary divisors except for the transformations given in (15). $\square$

Note that a full normal-rank polynomial matrix $P(s)$ without zeros at infinity must have a highest degree coefficient matrix P_h which has full rank as well.

Proof of Theorem 5.1. As discussed above, following Lemma 5.3 we can transform a polynomial matrix with full normal rank into one without elementary divisors at infinity and still keep the right nullspace intact. Consequently, we can show that the result of Theorem 5.1 holds by considering a polynomial matrix $P(s)$ without infinite elementary divisors.

Let $P(s)$ be in Smith canonical form and $h_1 \geqslant \cdots \geqslant h_m \geqslant 0$ be the degrees of its elementary divisors, then the highest degree coefficient matrix is the identity matrix I_m. Moreover, since $P(s)$ has no elementary divisors at infinity, the conditions of Theorem 5.1 implies that $\sum_{i=1}^m h_i = dm$. We now show that $P(s)$ can be transformed using unimodular transformations to one of degree d with highest degree coefficient matrix $P_h = I_m$ with degrees all equal to d (hence $P_h = P_d$). This proves that there always is a polynomial matrix of degree d that satisfies Theorem 5.1.

We construct such a polynomial matrix by recursively reducing the difference between h_1 and h_m while $\sum_{i=1}^m h_i$ remains the same. At the end of this process all h_i will be equal to d. Assume for this that $h_1 > d$ then we must have $h_m < d$, otherwise the ordered sequence of h_i could not sum up to dm. Moreover, throughout this process $P(s)$ is assumed to be in column reduced form. Consequently, in the transformation $U(s)P(s)V(s) = \widehat{P}(s)$ below we only show the elements of highest column degrees in $P(s)$ and $\widehat{P}(s)$.

$$\begin{bmatrix} & & 1 \\ & I & \\ 1 & & as \end{bmatrix} \begin{bmatrix} s^{h_1} & & \\ & \ddots & \\ & & s^{h_m} \end{bmatrix} \begin{bmatrix} -a & & \\ & I & \\ s^\delta & & 1 \end{bmatrix} = \begin{bmatrix} bs^{h_1-1} & & \\ & \ddots & \\ xs^{h_1-1} & & as^{h_m+1} \end{bmatrix}, \tag{16}$$

where $\delta = h_1 - h_m - 1$, $a, b \neq 0$, and x is arbitrary. Clearly, this transformation yields a new column reduced matrix but with the smallest column degree h_m increased by one and the largest column degree h_1 decreased by one. The reduction can be continued since we can use Lemma 2.7 to put again the new matrix in normalized column reduced form. Eventually we obtain an $m \times m$ polynomial matrix of degree d with prescribed elementary divisors and $P_d = I_m$. $\square$

Proof of Theorem 5.2. In order to prove Theorem 5.2, we have to construct an $m \times n$ polynomial matrix $P(s)$ with given Smith form, but also with given right minimal indices. Let $N_r(s)$ be a $n \times (n-m)$ column reduced polynomial matrix with prescribed column indices $\epsilon_1, \ldots, \epsilon_{n-m}$. Following Lemma 2.7 we can assume without loss of generality that the indices ϵ_i are non-increasing and that its highest column degree matrix is $\begin{bmatrix} I_{n-m} & 0 \end{bmatrix}^T$. In [30], it is then shown how to construct a row reduced polynomial matrix $P_r(s)$ with a right minimal basis spanned by $N_r(s)$, and such that $P_r(s)$ has the highest row degree matrix $\begin{bmatrix} 0 & I_m \end{bmatrix}$ with non-increasing row degrees $r_1, \ldots, r_m$ satisfying $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n-m} \epsilon_j$. The construction is based on the concepts of left and right coprime factorizations in column and row reduced forms (see [30, pp. 381–385]). Since $P_r(s)$ is row reduced it has no non-trivial elementary divisors, but if we premultiply it with a diagonal matrix $P_f(s)$ of (monic) finite elementary divisors with prescribed degrees $h_1, \ldots, h_m$, then $P(s) := P_f(s)P_r(s)$ satisfies $P(s)N_r(s) = 0$ and it has the prescribed elementary divisors as well. Moreover the row degrees are now $d_i = h_i + \epsilon_i$, $i = 1, \ldots, m$ and the highest row degree matrix is still $\begin{bmatrix} 0 & I_m \end{bmatrix}$.

We now assume that we start with a matrix product $P(s)N_r(s) = 0$ with the above conditions on the highest degree coefficient matrices. The conditions of Theorem 5.2 imply that $\sum_{i=1}^{m} d_i = \sum_{i=1}^{m}(r_i + h_i) = \sum_{j=1}^{n-m} \epsilon_j + \sum_{i=1}^{m} h_i = dm$. If all the coefficients d_i are not equal to d we update the matrices in the product recursively until they become equal.

We need to update simultaneously the matrices $P(s)$ and $N_r(s)$ while making sure that: (i) $N_r(s)$ remains a right minimal basis of $P(s)$ with the same minimal indices; and (ii) $P(s)$ has the same invariant polynomials. In the transformation $U(s)P(s)V(s) = \widehat{P}(s)$ displayed below, we assume that $P(s)$ is in row reduced form and therefore only show the elements of highest row degrees in $P(s)$ and $\widehat{P}(s)$:

$$\begin{bmatrix} -a & & s^{\delta} \\ & I & \\ & & 1 \end{bmatrix} \left[\begin{array}{c|ccc} & s^{d_1} & & \\ 0 & & \ddots & \\ & & & s^{d_m} \end{array}\right] \left[\begin{array}{c|ccc} I_{n-m} & & 0 & \\ \hline & & & 1 \\ 0 & & I & \\ & 1 & & as \end{array}\right] = \left[\begin{array}{c|ccc} & bs^{d_1-1} & & xs^{d_1-1} \\ 0 & & \ddots & \\ & & & as^{d_m+1} \end{array}\right], \tag{17}$$

where $\delta = d_1 - d_m - 1$, $a, b \neq 0$, and x is arbitrary. Transformations are unimodular and hence the elementary divisors of $P(s)$ and $\widehat{P}(s)$ are the same but the smallest row index d_m increased by 1 and the largest row index d_1 decreased by 1. Meanwhile, the right minimal indices did not change, because the corresponding right nullspace underwent the transformation $V^{-1}(s)N_r(s) = \widehat{N}_r(s)$. Below we only show the highest column degree elements in $N_r(s)$ and $\widehat{N}_r(s)$:

$$\left[\begin{array}{c|ccc} I_{n-m} & & 0 & \\ \hline & -as & & 1 \\ 0 & & I & \\ & 1 & & \end{array}\right] \left[\begin{array}{ccc} s^{\epsilon_1} & & \\ & \ddots & \\ & & s^{\epsilon_{n-m}} \\ \hline & 0 & \end{array}\right] = \left[\begin{array}{ccc} s^{\epsilon_1} & & \\ & \ddots & \\ & & s^{\epsilon_{n-m}} \\ \hline xs^{\epsilon_1} & \cdots & xs^{\epsilon_{n-m}} \\ & 0 & \end{array}\right].$$

Clearly, only row $(n-m)+1$ of $\widehat{N}_r(s)$ may contribute to the highest degree matrix, but it will not affect the minimal indices. Again, we can continue the recursive updating transformations until all powers $d_i = d$, which completes the proof. $\square$

For completeness, we include the following two corollaries for polynomial matrices with a full rank highest degree coefficient matrix P_d. The proofs are omitted since, as shown in Lemma 5.3, a full normal-rank polynomial matrix can always be transformed to a polynomial matrix with a full-rank coefficient P_d via a change of variables.

Corollary 5.4. *Let $P(s) = P_d s^d + \cdots + P_1 s + P_0$ be an $m \times m$ polynomial matrix of exact degree d, normal-rank m, and $\det(P_d) \neq 0$. Possible structural elements of $P(s)$ are those of Theorem 5.1 excluding the infinite elementary divisors.*

Corollary 5.5. *Let $P(s) = P_d s^d + \cdots + P_1 s + P_0$ be an $m \times n$ polynomial matrix of exact degree d, normal-rank m, and with P_d of full row rank. Possible structural elements of $P(s)$ are those of Theorem 5.2 excluding the infinite elementary divisors.*

6. Polynomial versus system pencil representation

The matrix pencils $sH_r + G_r$ and $sH_\ell + G_\ell$ corresponding to the right and left linearizations of a full normal-rank $m \times n$ polynomial matrix $P(s) = P_d s^d + \cdots + P_1 s + P_0$, can be expressed as the system pencils

$$\mathbf{S}_{\mathrm{R}}(s) = sH_r + G_r = s\begin{bmatrix} E & 0 \end{bmatrix} + \begin{bmatrix} A & B \end{bmatrix} \quad \text{and} \quad \mathbf{S}_{\mathrm{L}}(s) = sH_\ell + G_\ell = s\begin{bmatrix} E \\ 0 \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix}, \tag{18}$$

respectively. If the highest degree coefficient matrix P_d has full row or column rank, the system pencils in (18) can be transformed into

$$\mathbf{S}_{\mathrm{C}}(s) = s\begin{bmatrix} I_{dm} & 0 \end{bmatrix} + \begin{bmatrix} \widetilde{A} & \widetilde{B} \end{bmatrix} \quad \text{or} \quad \mathbf{S}_{\mathrm{O}}(s) = s\begin{bmatrix} I_{dn} \\ 0 \end{bmatrix} + \begin{bmatrix} \widetilde{A} \\ \widetilde{C} \end{bmatrix}, \tag{19}$$

respectively, where $\mathbf{S}_{\mathrm{C}}(s)$ has full row rank and $\mathbf{S}_{\mathrm{O}}(s)$ has full column rank. The structural elements of $\mathbf{S}_{\mathrm{C}}(s)$ only depend on the matrix pair $(\widetilde{A}, \widetilde{B})$ and those of $\mathbf{S}_{\mathrm{O}}(s)$ on $(\widetilde{A}, \widetilde{C})$. The stratification rules for $\mathbf{S}_{\mathrm{R}}(s)$ and $\mathbf{S}_{\mathrm{L}}(s)$ can be derived from the stratification rules for general matrix pencils $sH + G$, and the rules for $\mathbf{S}_{\mathrm{C}}(s)$ and $\mathbf{S}_{\mathrm{O}}(s)$ from matrix pairs (A, B) and (A, C), respectively. See Section 10.

Let us now consider systems that can be represented by a polynomial fraction $D(s)x(t) = N(s)u(t)$, where $D(s)$ and $N(s)$ are polynomial matrices with degree $D(s) >$ degree $N(s)$. We illustrate how such a polynomial fraction can be expressed in the form of a system pencil using three examples. The first example is of general form and the remaining two are taken from applications.

Example 6.1. Consider the differential equation system

$$\begin{aligned} &D_d x^{(d)}(t) + D_{d-1} x^{(d-1)}(t) + \cdots + D_1 x^{(1)}(t) + D_0 x(t) \\ &\quad = -N_{d-1} u^{(d-1)}(t) - \cdots - N_1 u^{(1)}(t) - N_0 u(t), \end{aligned} \tag{20}$$

where $D_k \in \mathbb{C}^{m \times m}$ and $N_k \in \mathbb{C}^{m \times p}$. The system (20) can equivalently be expressed as

$$\begin{bmatrix} D_d & 0 \end{bmatrix} \begin{bmatrix} x^{(d)}(t) \\ u^{(d)}(t) \end{bmatrix} + \begin{bmatrix} D_{d-1} & N_{d-1} \end{bmatrix} \begin{bmatrix} x^{(d-1)}(t) \\ u^{(d-1)}(t) \end{bmatrix} + \cdots + \begin{bmatrix} D_0 & N_0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = 0, \tag{21}$$

with the associated $m \times (m+p)$ polynomial matrix $P(s) = \begin{bmatrix} D_d & 0 \end{bmatrix} s^d + \cdots + \begin{bmatrix} D_0 & N_0 \end{bmatrix}$ of degree d. With $P(s)$ of full normal-rank m the corresponding right linearization is

$$s \left[\begin{array}{cccc|c} I_m & & & & 0 \\ & \ddots & & & \vdots \\ & & I_m & & \\ & & & D_d & 0 \end{array}\right] + \left[\begin{array}{cccc|c} 0 & & & D_0 & N_0 \\ -I_m & \ddots & & D_1 & N_1 \\ & \ddots & 0 & \vdots & \vdots \\ & & -I_m & D_{d-1} & N_{d-1} \end{array}\right], \tag{22}$$

which is a system pencil of the form $\mathbf{S}_{\mathrm{R}}(s)$. Furthermore, if D_d is nonsingular and well-conditioned then the linearization can be transformed to a system pencil of the form $\mathbf{S}_{\mathrm{C}}(s)$ by multiplying all coefficient matrices of $P(s)$ by D_d^{-1} from left. Similarly, the left linearization of a polynomial matrix with full normal-rank n and $\det(D_d) \neq 0$ corresponds to the system pencil $\mathbf{S}_{\mathrm{O}}(s)$. However, since the long-time goal is to apply and implement the stratification theory in numerical algorithms for robust computation of structure information, we in general recommend to keep to the matrix pencil representation rather than transforming the system pencil (18) to the standard matrix pair form (19).

Example 6.2. Consider a controlled dynamical system which can be expressed by its equation of motion in the form

$$M\ddot{x} + C\dot{x} + Kx = Fu,$$

where M, C, and K are the mass, damping, and stiffness matrices, respectively, F is the input (control) matrix, x is a vector of positive variables, and u is a vector of control variables. Assuming that the mass matrix M is positive definite, the linearization of the associated polynomial matrix can be expressed by the companion form

$$\left[\begin{array}{c|c} sI + A & B \end{array}\right] = \left[\begin{array}{cc|c} sI & M^{-1}K & M^{-1}F \\ -I & sI + M^{-1}C & 0 \end{array}\right],$$

where I is the identity matrix of conforming size. The $(2, 3)$-block is a zero matrix since $\dot{u}$ does not appear in the equation of motion.

Example 6.3. Consider an LTI system represented by the state-space model

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{23}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $x(t)$ is the state vector, and $u(t)$ is the input vector.

The controllability of an LTI system only depends on the matrices A and B, hence the matrix pair (A, B) is usually referred to as the *controllability pair* [15,28]. The system (23) has the corresponding controllability pencil $\mathbf{S}_{\mathrm{C}}(s) = \begin{bmatrix} sI_n + A & B \end{bmatrix}$. For the definition of controllability see any standard textbook on control theory, e.g., [30,40].

7. Coin moves and integer partitions

In the next section we use integer partitions to represent the structural elements of a matrix or matrix pencil and coin moves to define the stratification rules. Here, we recall the definitions by quoting excerpts from [12,15].

An *integer partition* $\kappa = (\kappa_1, \kappa_2, \ldots)$ of an integer K is a monotonically non-increasing sequence of integers ($\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant 0$) where $\kappa_1 + \kappa_2 + \cdots = K$. The *union* $\tau = (\tau_1, \tau_2, \ldots)$ of two integer

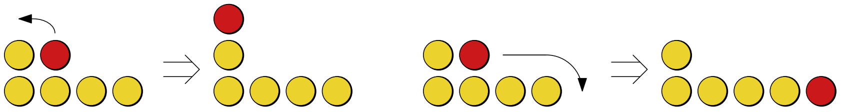

Fig. 1. Minimum leftward and rightward coin moves illustrate that $\kappa = (2, 2, 1, 1)$ is covered by $\tau = (3, 1, 1, 1)$ and $\kappa = (2, 2, 1, 1)$ covers $\nu = (2, 1, 1, 1, 1)$.

partitions κ and ν is defined as $\tau = \kappa \cup \nu$ where $\tau_1 \geqslant \tau_2 \geqslant \cdots$. Furthermore, the *conjugate partition* of κ is defined as $\nu = \text{conj}(\kappa)$, where ν_i is equal to the number of integers in κ that are equal to or greater than i, for $i = 1, 2, \ldots$.

If ν is an integer partition, not necessarily of the same integer K as κ, and $\kappa_1+\cdots+\kappa_i \geqslant \nu_1+\cdots+\nu_i$ for $i = 1, 2, \ldots$, then $\kappa \geqslant \nu$. When $\kappa \geqslant \nu$ and $\kappa \neq \nu$ then $\kappa > \nu$. If κ, ν and τ are integer partitions of the same integer K and there does not exist any τ such that $\kappa > \tau > \nu$ where $\kappa > \nu$, then κ *covers* ν.

An integer partition $\kappa = (\kappa_1, \ldots, \kappa_n)$ can also be represented by n piles of coins, where the first pile has κ_1 coins, the second κ_2 coins and so on. An integer partition κ *covers* ν if ν can be obtained from κ by moving one coin *one* column rightward or *one* row downward, and keep κ monotonically non-increasing. Or equivalently, an integer partition κ *is covered by* τ if τ can be obtained from κ by moving one coin *one* column leftward or *one* row upward, and keep κ monotonically non-increasing. These two types of coin moves are defined in [12] and called *minimum rightward* and *minimum leftward coin moves*, respectively (see Fig. 1).

8. Structure integer partitions

We can now represent the structural elements of matrix pencils defined in Section 2 as integer partitions (notation from [15]):

(i) The *column minimal indices* as $\epsilon = (\epsilon_1, \ldots, \epsilon_{r_0})$, where $\epsilon_1 \geqslant \epsilon_2 \geqslant \cdots \geqslant \epsilon_{r_1} > \epsilon_{r_1+1} = \cdots = \epsilon_{r_0} = 0$. From the conjugate partition $(r_1, \ldots, r_{\epsilon_1}, 0, \ldots)$ of ϵ we define the integer partition $\mathcal{R}(sH + G) = (r_0) \cup (r_1, \ldots, r_{\epsilon_1})$.

(ii) The *row minimal indices* as $\eta = (\eta_1, \ldots, \eta_{l_0})$, where $\eta_1 \geqslant \eta_2 \geqslant \cdots \geqslant \eta_{l_1} > \eta_{l_1+1} = \cdots = \eta_{l_0} = 0$. From the conjugate partition $(l_1, \ldots, l_{\eta_1}, 0, \ldots)$ of η we define the integer partition $\mathcal{L}(sH + G) = (l_0) \cup (l_1, \ldots, l_{\eta_1})$.

(iii) For each distinct finite eigenvalue λ_i, $i = 1, \ldots, q$, with the *finite elementary divisors* on the form $(s - \lambda_i)^{h_1^{(i)}}, \ldots, (s - \lambda_i)^{h_{g_i}^{(i)}}$, where $h_1^{(i)} \geqslant \cdots \geqslant h_{g_i}^{(i)} \geqslant 1$, we introduce the integer partition $h_{\lambda_i} = (h_1^{(i)}, \ldots, h_{g_i}^{(i)})$ which is known as the *Segre characteristic*. The conjugate partition $\mathcal{J}_{\lambda_i}(sH + G) = (j_1, j_2, \ldots)$ of h_{λ_i} is the *Weyr characteristic* of λ_i.

(iv) For the infinite eigenvalue with the *infinite elementary divisors* on the form $\mu^{h_1}, \mu^{h_2}, \ldots, \mu^{h_{g_\infty}}$, with $h_1 \geqslant \cdots \geqslant h_{g_\infty} \geqslant 1$, we introduce the integer partition $h_\infty = (h_1, \ldots, h_{g_\infty})$ which is known as the *Segre characteristic* for the infinite eigenvalue. The conjugate partition $\mathcal{N}(sH + G) = (n_1, n_2, \ldots)$ of h_∞ is the *Weyr characteristic* of the infinite eigenvalue.

The integer partitions above are referred to as *structure integer partitions* [15]. In addition, the condensed notation $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, and $\mathcal{N}$ is used for the integer partitions corresponding to the right and left singular structures, and the Jordan structures of the finite and infinite eigenvalues, respectively, when it is obvious from the context.

9. A geometric view of the polynomial matrix linearizations

Consider the $(2d^2m^2 + 2dm(n - m))$-dimensional space of $dm \times (m(d - 1) + n)$ complex matrix pencils $sH + G$ with Frobenius inner product $\langle sH + G, s\widetilde{H} + \widetilde{G}\rangle \equiv \text{tr}(G\widetilde{G}^* + H\widetilde{H}^*)$. Let us for now only consider matrix pencils with $d = 1$ (see also [11,12]). The *orbit* of a general $m \times n$ matrix pencil:

$$\mathcal{O}_P(sH + G) \equiv \{U(sH + G)V^{-1} \ : \ \det(U) \cdot \det(V) \neq 0\}, \tag{24}$$

is the manifold of all equivalent matrix pencils, i.e., a manifold in the $2mn$-dimensional space. All matrix pencils in the same orbit have the same canonical form, with the eigenstructure fixed. A *bundle* defines the union of all orbits with the same canonical form but with the eigenvalues unspecified, $\mathcal{B}_P(sH + G) := \bigcup_{\lambda_i} \mathcal{O}_P(sH + G)$ [1].

The dimension of $\mathcal{O}_P(sH + G)$ is equal to the dimension of the *tangent space* to $\mathcal{O}_P(sH + G)$ at $sH + G$:

$$\tan(sH + G) = \{sT_H + T_G = s(XH - HY) + (XG - GY)\},$$

where $X \in \mathbb{C}^{m\times m}$ and $Y \in \mathbb{C}^{n\times n}$. The orthogonal complement of the tangent space is the *normal space*, $\operatorname{nor}(sH + G) = \{sZ_H + Z_G\}$ where $Z_H H^* + Z_G G^* = 0$ and $H^* Z_H + G^* Z_G = 0$. The dimension of the normal space is called the *codimension* of $\mathcal{O}_P(sH + G)$, denoted by $\operatorname{cod}(\mathcal{O}_P(sH + G))$. The codimension of the corresponding bundle is one less for each unspecified distinct eigenvalue. For example, a matrix pencil with k unspecified eigenvalues and the rest with known specified eigenvalues has $\operatorname{cod}(\mathcal{B}_P(sH + G)) = \operatorname{cod}(\mathcal{O}_P(sH + G)) - k$.

While a general matrix pencil of size $dm \times (m(d-1) + n)$ belongs to the complete pencil space, a linearization of a polynomial matrix of degree $d > 1$ only resides in a affine subspace of the pencil space. An intuitive way to realize this is to consider, e.g., the right linearization $sH_r + G_r$ in (8) of a polynomial matrix (dual arguments also hold for the left linearization). The right linearization is a matrix pencil with several fixed elements, where each fixed element decreases the degree of freedom by one. Following [13,32], the set of $dm \times (m(d-1)+n)$ $sH_r + G_r$ forms a $(d+1)mn$-dimensional affine subspace in the pencil space, which we call the *generalized Sylvester space* and denote by $\operatorname{gsyl}(sH_r + G_r)$, see Fig. 2. The manifold of pencils equivalent to $sH_r + G_r$ (see Lemma 9.2) and belonging to the generalized Sylvester space is the orbit defined as

$$\begin{aligned}\mathcal{O}_R(sH_r + G_r) \equiv \{&s\widetilde{H}_r + \widetilde{G}_r = U(sH_r + G_r)V^{-1} \ : \ s\widetilde{H}_r + \widetilde{G}_r \in \operatorname{gsyl}(sH_r + G_r),\\ &\det(U) \cdot \det(V) \neq 0\}.\end{aligned} \tag{25}$$

Equivalently, the orbit of $sH_\ell + G_\ell$ is defined as

$$\begin{aligned}\mathcal{O}_L(sH_\ell + G_\ell) \equiv \{&s\widetilde{H}_\ell + \widetilde{G}_\ell = U(sH_\ell + G_\ell)V^{-1} \ : \ s\widetilde{H}_\ell + \widetilde{G}_\ell \in \operatorname{gsyl}(sH_\ell + G_\ell),\\ &\det(U) \cdot \det(V) \neq 0\}.\end{aligned} \tag{26}$$

So, does a perturbation in the linearization correspond to a perturbation in the polynomial matrix? This was proven for square polynomial matrices in [43, Section 4]; we treat the general case below. The theorem and the following lemma are stated for the right linearization, but obviously they hold for the left linearization as well.

Theorem 9.1. *Let $sH_r + G_r$ be the right linearization (8) of the matrix polynomial $P(s)$, which is perturbed by an arbitrary pencil $s\Delta H + \Delta G$ with $\| \begin{bmatrix} \Delta H & \Delta G \end{bmatrix} \| = O(\epsilon)$. Assuming that $\| \begin{bmatrix} P_0 & \cdots & P_d \end{bmatrix} \| = O(1)$, then there exist matrices X ($dm \times dm$) and Y ($m(d-1)+n \times m(d-1)+n$) such that $\|X\|, \|Y\| = O(\epsilon)$, satisfying*

$$\begin{aligned}&(I + X)(H_r + \Delta H) = (H_r + \widehat{\Delta} H)(I + Y) \text{ and}\\ &(I + X)(G_r + \Delta G) = (G_r + \widehat{\Delta} G)(I + Y),\end{aligned} \tag{27}$$

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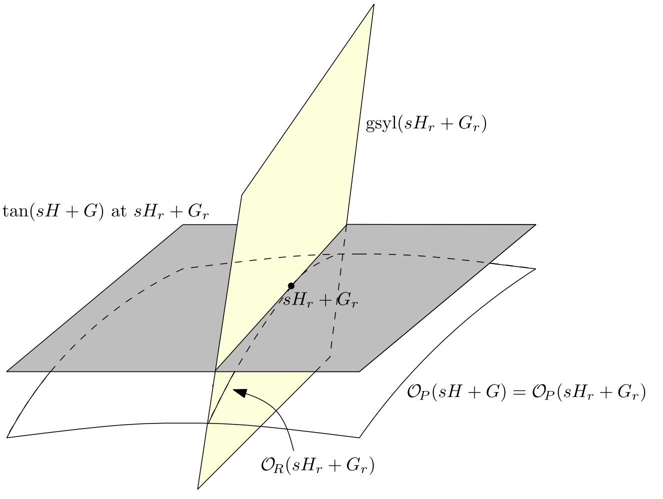

Fig. 2. Illustration of the matrix pencil space, in which we represent a particular right linearization $sH_r + G_r$ as a point. The orbit $\mathcal{O}_R(sH_r + G_r)$ is the intersection between gsyl$(sH_r + G_r)$ and the orbit $\mathcal{O}_P(sH_r + G_r)$.

where

$$H_r + \widehat{\Delta}H = \begin{bmatrix} I_m & & & \\ & \ddots & & \\ & & I_m & \\ & & & P_d + \Delta P_d \end{bmatrix}, \quad G_r + \widehat{\Delta}G = \begin{bmatrix} 0 & & & P_0 + \Delta P_0 \\ -I_m & \ddots & & \vdots \\ & \ddots & 0 & \\ & & -I_m & P_{d-1} + \Delta P_{d-1} \end{bmatrix},$$

and $\| \begin{bmatrix} \Delta P_0 & \cdots & \Delta P_d \end{bmatrix} \| = O(\epsilon)$.

Proof. We prove this by showing how to construct solution matrices X and Y of (27). We solve (27) by imposing that X and Y satisfy the *nonlinear* system of equations in the unknown variables X, Y, $\widehat{\Delta}H$, and $\widehat{\Delta}G$:

$$\begin{aligned} (H_r + \widehat{\Delta}H)Y - X(H_r + \Delta H) &= \Delta H - \widehat{\Delta}H, \\ (G_r + \widehat{\Delta}G)Y - X(G_r + \Delta G) &= \Delta G - \widehat{\Delta}G. \end{aligned} \tag{28}$$

Let (X, Y) be a solution of this system with $O(\epsilon)$ norm (by continuity, such a solution must exist since X, Y, $\widehat{\Delta}G$ and $\widehat{\Delta}H$ go to zero when ΔG and ΔH go to zero). Then it follows from (28) that $\widehat{\Delta}H$ and $\widehat{\Delta}G$ must also have $O(\epsilon)$ norm. If we now neglect all $O(\epsilon^2)$ terms in (28), we obtain a *linear* system of equations

$$\begin{aligned} H_r\widehat{Y} - \widehat{X}H_r &= \Delta H - \widehat{\Delta}H, \\ G_r\widehat{Y} - \widehat{X}G_r &= \Delta G - \widehat{\Delta}G, \end{aligned} \tag{29}$$

which we can restrict to the blocks where the matrices $\widehat{\Delta}G$ and $\widehat{\Delta}H$ are zero:

$$\begin{aligned} H_r\widehat{Y} - \widehat{X}H_r &= \Delta H, \\ G_r\widehat{Y} - \widehat{X}G_r &= \Delta G. \end{aligned} \tag{30}$$

In order to make the solution of this last system unique for $\widehat{X}$ and $\widehat{Y}$, we choose $\widehat{X}_{11} = 0$ and $\widehat{Y}_{dj} = 0$, for all j. For simplicity, we construct them for a polynomial matrix of degree $d = 3$. The general case can be treated similarly. The linear system (30) then yields the following system of equations:

$$\begin{bmatrix} \widehat{Y}_{11} & \widehat{Y}_{12} & \widehat{Y}_{13} \\ \widehat{Y}_{21} & \widehat{Y}_{22} & \widehat{Y}_{23} \\ 0 & 0 & ? \end{bmatrix} - \begin{bmatrix} 0 & \widehat{X}_{12} & \widehat{X}_{13}P_3 \\ \widehat{X}_{21} & \widehat{X}_{22} & \widehat{X}_{23}P_3 \\ \widehat{X}_{31} & \widehat{X}_{32} & ? \end{bmatrix} = \begin{bmatrix} \Delta H_{11} & \Delta H_{12} & \Delta H_{13} \\ \Delta H_{21} & \Delta H_{22} & \Delta H_{23} \\ \Delta H_{31} & \Delta H_{32} & ? \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & ? \\ \widehat{Y}_{11} & \widehat{Y}_{12} & ? \\ \widehat{Y}_{21} & \widehat{Y}_{22} & ? \end{bmatrix} - \begin{bmatrix} \widehat{X}_{12} & \widehat{X}_{13} & ? \\ \widehat{X}_{22} & \widehat{X}_{23} & ? \\ \widehat{X}_{32} & \widehat{X}_{33} & ? \end{bmatrix} = \begin{bmatrix} \Delta G_{11} & \Delta G_{12} & ? \\ \Delta G_{21} & \Delta G_{22} & ? \\ \Delta G_{31} & \Delta G_{32} & ? \end{bmatrix},$$

where "?" indicates a block that does not need to be considered in order to compute the solutions $\widehat{X}$ and $\widehat{Y}$. This is now a square system of equations in the block variables $\widehat{X}_{ij}$ and $\widehat{Y}_{ij}$ of conforming sizes, which can easily be solved in the following order:

$$\begin{aligned} \begin{bmatrix} \widehat{X}_{12} & \widehat{X}_{13} \end{bmatrix} &= \begin{bmatrix} \Delta G_{11} & \Delta G_{12} \end{bmatrix}, \\ \begin{bmatrix} \widehat{Y}_{11} & \widehat{Y}_{12} & \widehat{Y}_{13} \end{bmatrix} &= \begin{bmatrix} \Delta H_{11} & \Delta H_{12} & \Delta H_{13} \end{bmatrix} + \begin{bmatrix} 0 & \widehat{X}_{12} & \widehat{X}_{13}P_3 \end{bmatrix}, \\ \begin{bmatrix} \widehat{X}_{22} & \widehat{X}_{23} \end{bmatrix} &= \begin{bmatrix} \Delta G_{21} & \Delta G_{22} \end{bmatrix} + \begin{bmatrix} \widehat{Y}_{11} & \widehat{Y}_{12} \end{bmatrix}, \\ \begin{bmatrix} \widehat{Y}_{22} & \widehat{Y}_{23} \end{bmatrix} &= \begin{bmatrix} \Delta H_{22} & \Delta H_{23} \end{bmatrix} + \begin{bmatrix} \widehat{X}_{22} & \widehat{X}_{23}P_3 \end{bmatrix}, \\ \begin{bmatrix} \widehat{X}_{33} \end{bmatrix} &= \begin{bmatrix} \Delta G_{32} \end{bmatrix} + \begin{bmatrix} \widehat{Y}_{22} \end{bmatrix}, \\ \begin{bmatrix} \widehat{X}_{31} & \widehat{X}_{32} \end{bmatrix} &= -\begin{bmatrix} \Delta H_{31} & \Delta H_{32} \end{bmatrix}, \\ \begin{bmatrix} \widehat{Y}_{21} \end{bmatrix} &= -\begin{bmatrix} \Delta G_{31} \end{bmatrix} + \begin{bmatrix} \widehat{X}_{32} \end{bmatrix}, \\ \begin{bmatrix} \widehat{X}_{21} \end{bmatrix} &= -\begin{bmatrix} \Delta H_{21} \end{bmatrix} + \begin{bmatrix} \widehat{Y}_{21} \end{bmatrix}. \end{aligned} \tag{31}$$

This linear system of equations is clearly well conditioned (all singular values of the corresponding linear map associated to (31) are $O(1)$); with the block ordering of the above equations the system is upper triangular with diagonal blocks that are $\pm I$ of conforming sizes. Since $\| \begin{bmatrix} \Delta H & \Delta G \end{bmatrix} \| = O(\epsilon)$ and $\| \begin{bmatrix} P_0 & \cdots & P_d \end{bmatrix} \| = O(1)$ the solution matrices of this approximate linear system (31) clearly satisfy $\|\widehat{X}\|, \|\widehat{Y}\| = O(\epsilon)$. Subtracting (29) from (28), yields the nonlinear system

$$\begin{aligned} H_r(Y - \widehat{Y}) - (X - \widehat{X})H_r &= X \cdot \Delta H - \widehat{\Delta} H \cdot Y, \\ G_r(Y - \widehat{Y}) - (X - \widehat{X})G_r &= X \cdot \Delta G - \widehat{\Delta} G \cdot Y. \end{aligned}$$

Since the (nonlinear) right hand side is $O(\epsilon^2)$ and the left hand side is a well conditioned linear system, this implies that the norm of the solution $(X - \widehat{X}, Y - \widehat{Y})$ is $O(\epsilon^2)$. The solution (X, Y) of the nonlinear equations (28) therefore is $O(\epsilon^2)$ close to the solution $(\widehat{X}, \widehat{Y})$ of the linear system (29). It finally follows that $\| \begin{bmatrix} \Delta P_0 & \cdots & \Delta P_d \end{bmatrix} \| = O(\epsilon)$, since it is made of submatrices of $\widehat{\Delta} H$ and $\widehat{\Delta} G$. This completes the proof. $\square$

We remark that any other deformation can be obtained by an appropriate smooth change of parameters and an equivalence transformation smoothly dependent of parameters [1,11]. We can now show that the orbit of the linearization is a manifold, as we have indicated already.

Lemma 9.2. *The orbit $\mathcal{O}_R(sH_r + G_r)$ in (25) forms a manifold in the matrix pencil space.*

Proof. Consider two smooth manifolds of a given finite dimensional embedding (Euclidean) space. Following [21], these two manifolds intersect transversally if at every point of intersection their separate tangent spaces at the selected point together generate the ambient space at that point. In our case we apply this fact to the generalized Sylvester space (which is its own tangent space since it is an affine space) and to $\mathcal{O}_P(sH_r + G_r)$ whose tangent space is described by the pair $(H_rY - XH_r, G_rY - XG_r)$. In the proof of Theorem 9.1, we precisely showed that these two affine spaces are transversal since the linearized system considered there has always a solution. Indeed, the linear equations (29) in $(\widehat{X}, \widehat{Y})$ imply that any pencil in the ambient space can be written as a combination of a pencil in $\operatorname{gsyl}(sH_r + G_r)$ and a pencil in the tangent space of $\mathcal{O}_P(sH_r + G_r)$:

$$(H_r + \Delta H, G_r + \Delta G) = \overbrace{(H_r\widehat{Y} - \widehat{X}H_r, G_r\widehat{Y} - \widehat{X}G_r)}^{\in \tan(sH+G) \text{ at } sH_r+G_r} + \overbrace{(H_r + \widehat{\Delta}H, G_r + \widehat{\Delta}G)}^{\in \operatorname{gsyl}(sH_r+G_r)} . \tag{32}$$

Consequently, $\mathcal{O}_P(sH_r + G_r)$ is also transversal to $\operatorname{gsyl}(sH_r + G_r)$ and the intersection $\mathcal{O}_R(sH_r + G_r)$ between $\mathcal{O}_P(sH_r + G_r)$ and $\operatorname{gsyl}(sH_r + G_r)$ is a (sub-)manifold. See Fig. 2 for an illustration. □

The codimension of the $\mathcal{O}_R(sH_r + G_r)$ is obtained directly from the transversality theory. In general, the codimension of an intersection between two transversal manifolds is equal to the sum of the codimensions of these two manifolds, e.g., see [21]. Hence, by first considering $\mathcal{O}_R(sH_r + G_r)$ in the general matrix pencil space we have (following the arguments in the proof of Lemma 9.2):

$$\operatorname{cod}(\mathcal{O}_R(sH_r + G_r)) = \operatorname{cod}(\mathcal{O}_P(sH_r + G_r)) + \operatorname{cod}(\operatorname{gsyl}(sH_r + G_r)).$$

Since the manifold $\mathcal{O}_R(sH_r + G_r)$ is indeed restricted to the generalized Sylvester space it follows that $\operatorname{cod}(\mathcal{O}_R(sH_r + G_r)) = \operatorname{cod}(\mathcal{O}_P(sH_r + G_r))$. Moreover, the linearization $sH_r + G_r$ is a matrix pencil without any left minimal indices. Consequently, knowing the eigenstructure elements of $P(s)$ the codimension of $\mathcal{O}_R(sH_r + G_r)$ can be computed using parts of the explicit expression derived in [9] for general matrix pencils, assuming that no left minimal indices exist:

$$\operatorname{cod}(\mathcal{O}_R(sH_r + G_r)) = c_{\text{Right}} + c_{\text{Jor}} + c_{\text{Jor,Right}}, \tag{33}$$

where

$$c_{\text{Right}} = \sum_{\epsilon_k > \epsilon_l} (\epsilon_k - \epsilon_l - 1), \; c_{\text{Jor}} = \sum_{i=1}^{q}\sum_{j=1}^{g_i}(2j-1)h_j^{(i)} + \sum_{j=1}^{g_\infty}(2j-1)h_j^{(\infty)}, \text{ and}$$
$$c_{\text{Jor,Right}} = r_0 \left(\sum_{i=1}^{q}\sum_{j=1}^{g_i} h_j^{(i)} + \sum_{j=1}^{g_\infty} h_j^{(\infty)} \right).$$

Similarly, the codimension of $\mathcal{O}_L(sH_\ell + G_\ell)$ can be computed using parts of the expression for general matrix pencils, assuming that no right minimal indices exist:

$$\operatorname{cod}(\mathcal{O}_L(sH_\ell + G_\ell)) = c_{\text{Left}} + c_{\text{Jor}} + c_{\text{Jor,Left}}, \tag{34}$$

where

$$c_{\text{Left}} = \sum_{\eta_k > \eta_l} (\eta_k - \eta_l - 1), \; c_{\text{Jor,Left}} = l_0 \left(\sum_{i=1}^{q}\sum_{j=1}^{g_i} h_j^{(i)} + \sum_{j=1}^{g_\infty} h_j^{(\infty)} \right),$$

and c_{Jor} is computed as in (33). Note that some of the terms in (33) and (34) can be empty.

10. Stratifications

In this section, we begin by introducing the stratification theory of orbits and bundles, and reviewing some known results. These results are then extended with stratification theory for linearizations of full normal-rank polynomial matrices $P(s)$.

The closure hierarchy of orbits (or bundles) is a stratification that we represent by a connected graph [12,15]. The nodes of the graph correspond to orbits (or bundles) of specified canonical structures and the edges to their covering relations. The organization of the graph is from bottom to top (or top to bottom) with nodes in decreasing (or increasing) order of codimension; see Figs. 3 and 5.

Besides the orbit (or bundle) itself, the *closure* includes all orbits (or bundles) represented by the nodes which can be reached by a *downward path* in the graph. With a downward path we mean a path that starts out from a (specified) node and follows downward edges until it terminates in another (specified) node further down in the graph. Similarly, a path in the opposite direction is called an *upward path*.

We remark that by adding a small perturbation to a matrix pencil (e.g., corresponding to a linearization of $P(s)$), it is always possible to make it more generic corresponding to a node along an upward path from the orbit (or bundle). In general, it is not possible to insist on a downward move by just adding a small perturbation of a given matrix pencil. However, the cases when a structure below in the hierarchy actually is nearby is often of particular interest, as it shows that a more degenerate structure can be found by a small perturbation. In a practical application this could mean that a controllable system is close to an uncontrollable one, which eventually could lead to a disaster.

By picking random matrix pencils of the same size, they will almost all have the same canonical structure, corresponding to the *most generic* case with the lowest codimension in the closure hierarchy. On the other side, the *most degenerate* case, or equivalently, the least generic case has the highest codimension. These extreme cases are represented by the topmost node (most generic) and the bottom node (least generic) in the closure hierarchy graph. For example, general rectangular matrix pencils may have several generic cases, but only one least generic case corresponding to a matrix pencil with only zero entries.

Our approach to deal with the stratification of linearizations of polynomial matrices extends on earlier work for general matrix pencils [12] and for controllability (and observability) pairs [15]. The results in [12,15] are stated in a set of rules expressed as minimal coin moves within and between the structure integer partitions $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, and $\mathcal{N}$ (as defined in Section 8).

For example, necessary and sufficient conditions for an orbit of two matrix pencils $sH + G$ and $s\widetilde{H} + \widetilde{G}$ to be closest neighbors in a closure hierarchy are established in [12]. In other words, conditions when $\mathcal{O}_P(sH + G)$ covers $\mathcal{O}_P(s\widetilde{H} + \widetilde{G})$, where the orbit is the manifold of strictly equivalent matrix pencils (24). The corresponding set of rules for bundles of matrix pencils are also presented in [12].

In [15], the results for general matrix pencils are extended to necessary and sufficient conditions for orbits and bundles of two controllability pairs (A, B) and $(\widetilde{A}, \widetilde{B})$ or two observability pairs (A, C) and $(\widetilde{A}, \widetilde{C})$ to be closest neighbors in a closure hierarchy, where the orbits considered are under feedback equivalence.

The structure elements of the most and least generic orbits are, e.g., considered in [42,9] for matrix pencils and [19,15] for matrix pairs.

10.1. Stratification of polynomial matrix linearizations

Theorem 9.1 and Lemma 9.2 enable us to formulate the covering relations between orbits and bundles of linearizations of $m \times n$ full normal-rank polynomial matrices $P(s)$ in terms of coin rules on the structure integer partitions of the linearizations. We show that these relations can be derived from the covering relations for general matrix pencils. By transforming the zero at infinity to a finite zero (as shown in Lemma 5.3), the results in this section could as well be derived from the stratification theory for matrix pairs, e.g., see [15]. However, since such a transformation can be arbitrarily ill-conditioned it is in general recommended to keep the general form of the linearizations. See also Remark 10.1.

For consistency with earlier results in [12,15] our new findings in Theorem 10.1 below and corollaries are stated using the same notation and similar formulations, see also Sections 7 and 8. For example,

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throughout the following sections we use structure integer partitions ($\mathcal{R}$, $\mathcal{L}$, and $\mathcal{J}$) and the KCF notation introduced in (7) to represent the eigenstructure elements (minimal indices and elementary divisors) of a polynomial matrix and its linearization. Notably, in Theorem 10.1 the eigenvalue λ_i corresponding to the structure integer partition $\mathcal{J}_{\lambda_i}$ belongs to $\overline{\mathbb{C}}$, i.e., $\lambda_i \in \mathbb{C} \cup \{\infty\}$, and the restriction on rules (1) and (2) fixate the number of right and left singular blocks.

Theorem 10.1. *Let $sH_r + G_r$ be the right linearization (8) of an $m \times n$ polynomial matrix $P(s)$ of exact degree d and normal-rank m, where $m < n$. Given the structure integer partitions $\mathcal{R}$ and $\mathcal{J}_{\lambda_i}$ of $sH_r + G_r$, where $\lambda_i \in \overline{\mathbb{C}}$, one of the following if-and-only-if rules finds $s\widetilde{H}_r + \widetilde{G}_r$ fulfilling orbit or bundle covering relations with $sH_r + G_r$.*

A. *$\mathcal{O}_R(sH_r + G_r)$ covers $\mathcal{O}_R(s\widetilde{H}_r + \widetilde{G}_r)$:*

(1) *Minimum* rightward *coin move in $\mathcal{R}$.*
(2) *If the rightmost column in $\mathcal{R}$ consists of one coin only, move that coin to a new rightmost column of some $\mathcal{J}_{\lambda_i}$ (which may be empty initially).*
(3) *Minimum* leftward *coin move in any $\mathcal{J}_{\lambda_i}$ as long as $j_1^{(i)}$ does not exceed m.*

Rules 1 and 2 are not allowed to do coin moves that affect r_0 (first column in $\mathcal{R}$).

B. *$\mathcal{B}_R(sH_r + G_r)$ covers $\mathcal{B}_R(s\widetilde{H}_r + \widetilde{G}_r)$:*

(1) *Same as rule 1 on the left.*
(2) *Same as rule 2 on the left, except it is only allowed to start a new set $\mathcal{J}_{\lambda_i}$ corresponding to a new eigenvalue (i.e., no appending to non-empty sets).*
(3) *Same as rule 3 on the left.*
(4) *Let any pair of eigenvalues coalesce, i.e., take the union of their sets of coins.*

C. *$\mathcal{O}_R(sH_r + G_r)$ is covered by $\mathcal{O}_R(s\widetilde{H}_r + \widetilde{G}_r)$:*

(1) *Minimum* leftward *coin move in $\mathcal{R}$, without affecting r_0.*
(2) *If the rightmost column in some $\mathcal{J}_{\lambda_i}$ consists of one coin only, move that coin to a new rightmost column in $\mathcal{R}$.*
(3) *Minimum* rightward *coin move in any $\mathcal{J}_{\lambda_i}$.*

D. *$\mathcal{B}_R(sH_r + G_r)$ is covered by $\mathcal{B}_R(sH_r + G_r)$:*

(1) *Same as rule 1 on the left.*
(2) *Same as rule 2 on the left, except that $\mathcal{J}_{\lambda_i}$ must consist of one coin only.*
(3) *Same as rule 3 on the left.*
(4) *For any $\mathcal{J}_{\lambda_i}$, divide the set of coins into two new partitions so that their union is $\mathcal{J}_{\lambda_i}$.*

The corresponding rules for $\mathcal{O}_L(sH_\ell + G_\ell)$ and $\mathcal{B}_L(sH_\ell + G_\ell)$ of the dual left linearization $sH_\ell + G_\ell$ (12), associated with an $m \times n$ polynomial matrix of normal-rank n ($n < m$), are obtained by exchanging $\mathcal{R}$ with $\mathcal{L}$ and m with n in the above rules.

Proof. The new restrictions in the cover rules, with respect to the rules for general matrix pencils $sH + G$ [12], follow directly from Theorem 5.2. The restrictions are: (i) no L^T blocks can exist; (ii) since there can at most be m finite and m infinite elementary divisors, $j_1^{(i)}$ in $\mathcal{J}_{\lambda_i}$ for each $\lambda_i \in \overline{\mathbb{C}}$ can at most be m; (iii) r_0 in $\mathcal{R}$ must be $n - m$, which implies that the number of L blocks remains fixed and is $n - m$.

The covered-by rules follows directly by reversing the cover rules. □

Remark 10.1. The rules in Theorem 10.1 and the covering rules for matrix pairs [15] are similar. However, the orbits in the stratifications are different, and as we will see in Theorem 10.4 their least generic (most degenerate) canonical structures are not the same. We have chosen to include all four cases (A to D) in Theorem 10.1 to make it self-contained.

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We also remark that the set of A-rules in Theorem 10.1 is in line with the necessary conditions for covering relations of matrix pencils with no left minimal indices [24].

Now, the cover relations for square full normal-rank polynomial matrices follow from Theorem 10.1 together with the restrictions of Theorem 5.1. Notably, these rules coincide with the cover rules for regular matrix pencils [23,12] with the exception that the number of Jordan blocks is restricted by the normal-rank.

Corollary 10.2. *Let $sH_r + G_r$ be the right linearization of a square $m \times m$ polynomial matrix $P(s)$ of exact degree d and normal-rank m. Then the covering relations are given by rule (3) of* A *and rules (3) and (4) of* B *in Theorem 10.1.*

Remark 10.2. Obviously, when considering square polynomial matrices of full normal-rank, the right and left linearizations are equivalent with the same structural elements (same elementary divisors and no minimal indices, see Theorem 3.5).

In addition, the cover relations for $m \times n$ full normal-rank polynomial matrices with a full rank highest degree coefficient matrix follow straightforwardly. The only restriction, with respect to the cover rules in Theorem 10.1, is that there cannot exist infinite eigenvalues since H_r has full row rank (or H_ℓ has full column rank).

Corollary 10.3. *Let $P(s)$ be an $m \times n$ polynomial matrix of exact degree d with the highest degree coefficient matrix P_d of full rank. If $m < n$ and normal-rank of $P(s)$ is m, consider the right linearization $sH_r + G_r$ of $P(s)$, else if $n < m$ and normal-rank of $P(s)$ is n, consider the left linearization $sH_\ell + G_\ell$ of $P(s)$. Then it follows that the covering relations are given by the stated rules in Theorem 10.1, where $\lambda_i \in \mathbb{C}$ (no eigenvalues at infinity can exist).*

The canonical structure elements of the most and least generic orbits or bundles in the stratification of a linearization of a full normal-rank polynomial matrix are given by the following theorem.

Theorem 10.4. *Let $sH_r + G_r$ and $sH_\ell + G_\ell$ be the right and left linearizations, respectively, of an $m \times n$ polynomial matrix $P(s)$ of exact degree d and full normal-rank r, where $m \neq n$.*

If $r = m$, the most generic orbit (or bundle) of $sH_r + G_r$ has the structure integer partition $\mathcal{R} = (r_0, \ldots, r_\alpha, r_{\alpha+1})$ where $r_0 = \cdots = r_\alpha = n - m$, $r_{\alpha+1} = (dm) \mod (n - m)$, and $\alpha = \lfloor (dm)/(n - m) \rfloor$.

There exist $p(d)$ least generic orbits of $sH_r + G_r$ (all with the same codimension), where the partition function $p(d)$ is the number of possible partitions of the integer d. All least generic orbits have $\mathcal{R} = (n - m)$, but different $\mathcal{J}_{\lambda_i}$ associated with at most d distinct eigenvalues λ_i, $i = 1, \ldots, d$. For each orbit construct $\mathcal{J}_{\lambda_i}$ as follows. Let $\kappa = (k_1, \ldots, k_d) = (m, \ldots, m)$. Distribute the integer partition κ across $\mathcal{J}_{\lambda_i}$, $i = 1, \ldots, d$, so that the union of the new $\mathcal{J}_{\lambda_i}$ is κ. One or several $\mathcal{J}_{\lambda_i}$ can be empty.

The least generic bundle of $sH_r + G_r$ has $\mathcal{R} = (n - m)$ and $\mathcal{J}_\lambda = (j_1, \ldots, j_d) = (m, \ldots, m)$, i.e., m J_d blocks corresponding to a single eigenvalue of multiplicity m.

If $r = n$, the most and least generic orbits (or bundles) of $sH_\ell + G_\ell$ are obtained by exchanging $\mathcal{R}$ with $\mathcal{L}$ and interchanging m and n in the above expressions.

Proof. Most generic orbit and bundle: Since each term in (33) (or (34)) coincides with the corresponding part of the codimension count for matrix pencils with $m < n$ (or $m > n$) [42,9], it follows that the most generic orbit and bundle have the same eigenstructure as the corresponding matrix pencil. If this would not be the case there would exist another orbit of a polynomial matrix linearization with lower codimension, but in that case there would also exist a matrix pencil with the same codimension (and vice versa).

Least generic orbit: Theorem 5.2 states that all structural elements add up to dm. Moreover, the least generic orbit must have all right minimal indices equal to zero (i.e., only L_0 blocks exist), thus they make no contribution to the sum (14). Consequently, $\sum h_j^{(i)} = dm$, where we for simplicity here assume that any possible infinite elementary divisor is included in the total. Since dm is an integer and we have at most m nonzero elementary divisors for each eigenvalue, there can exist at most d eigenvalues. There are two extreme cases: (i) only one eigenvalue λ_1, which means that there must exist m Jordan blocks of size $d \times d$ with the corresponding eigenvalue λ_1 of multiplicity m ($\mathcal{J}_{\lambda_1} = (m, \ldots, m)$); (ii) d distinct eigenvalues λ_i, $i = 1, \ldots, d$, which means that there must exist m Jordan blocks of size 1×1 for each λ_i ($\mathcal{J}_{\lambda_i} = (m)$). Since the sum of all elementary divisors $\sum h_j^{(i)} = dm$, the number of (distinct) eigenvalues does not change the codimension. Therefore, all possible partitions of $\kappa = (k_1, \ldots, k_d) = (m, \ldots, m)$ across one or several $\mathcal{J}_{\lambda_i}$, $i = 1, \ldots, d$, are possible.

Least generic bundle: Since each distinct (unspecified) eigenvalue decreases the codimension, the least generic bundle has only one multiple eigenvalue corresponding to $\mathcal{J}_{\lambda_1} = \kappa$ (corresponding to case (i) for orbits). $\square$

11. Sample stratification examples

In this section, we apply and illustrate the stratification theory on a few examples, including two artificial polynomial matrices and a half-car passive suspension system with four degrees of freedom.

We have implemented the stratification results presented in Section 10.1 in the software tool StratiGraph [27,26,29], which makes it possible to compute, generate and visualize the closure hierarchy graphs.

Before we move on we introduce a condensed notation for the KCF, used in StratiGraph for representing the eigenstructure elements. A general block diagonal matrix $A = \mathrm{diag}(A_1, A_2, \ldots, A_b)$ with b blocks can be represented as a direct sum

$$A \equiv A_1 \oplus A_2 \oplus \cdots \oplus A_b \equiv \bigoplus_{k=1}^{b} A_k.$$

Now, the KCF (7) can compactly be expressed as $U(sH+G)V^{-1} \equiv \mathbb{L} \oplus \mathbb{L}^T \oplus \mathbb{J}(\lambda_1) \oplus \cdots \oplus \mathbb{J}(\lambda_q) \oplus \mathbb{N}$, where

$$\mathbb{L} = \bigoplus_{k=1}^{r_0} L_{\epsilon_k}, \quad \mathbb{L}^T = \bigoplus_{k=1}^{l_0} L_{\eta_k}^T, \quad \mathbb{J}(\lambda_i) = \bigoplus_{k=1}^{g_i} J_{h_k}(\lambda_i), \quad \text{and} \quad \mathbb{N} = \bigoplus_{k=1}^{g_\infty} N_{h_k}.$$

Notice that blocks of the KCF in the direct sum notation above are, without loss of generality, ordered so that the singular blocks ($\mathbb{L}$ and $\mathbb{L}^T$) appear first.

In the following, we use this direct sum notation of the KCF to represent a linearization of a polynomial matrix with the corresponding eigenstructure elements. Moreover, we denote the orbit of a linearization, e.g., $\mathcal{O}_R(sH_r + G_r)$, having the eigenstructure elements corresponding to the KCF $\mathbb{L} \oplus \mathbb{L}^T \oplus \mathbb{J}(\lambda_1) \oplus \cdots \oplus \mathbb{J}(\lambda_q) \oplus \mathbb{N}$ in the compact form $\mathcal{O}_R(\mathbb{L} \oplus \mathbb{L}^T \oplus \mathbb{J}(\lambda_1) \oplus \cdots \oplus \mathbb{J}(\lambda_q) \oplus \mathbb{N})$.

11.1. Two full normal-rank polynomial matrices

We start by considering a full normal-rank 2×4 polynomial matrix of degree $d = 2$. In Fig. 3 and Table 1, we illustrate some of the results presented in Section 10 applied to this example. From Theorem 10.4 we obtain the most and least generic orbits: the most generic $\mathcal{O}_R(2L_2)$ has codimension 0; the two least generic orbits $\mathcal{O}_R(2L_0 \oplus 2J_1(\mu_1) \oplus 2J_1(\mu_2))$ and $\mathcal{O}_R(2L_0 \oplus 2J_2(\mu_1))$ have both codimension 16. By applying the set of A-rules in Theorem 10.1 we obtain the complete orbit stratification of the right linearization $sH_r + G_r$ shown in Fig. 3. Note that each edge between two nodes in the graph is the result of applying one of these rules.

We choose to illustrate the orbit covering A-rules by starting at $\mathcal{O}_R(2L_2)$ with codimension 0 and following the right-most downward path in Fig. 3. To simplify we use the notation Cod : #, introduced in Table 1, to denote a specific orbit. By applying the rule A.(1) to orbit 0:1 we obtain the orbit 1:1.

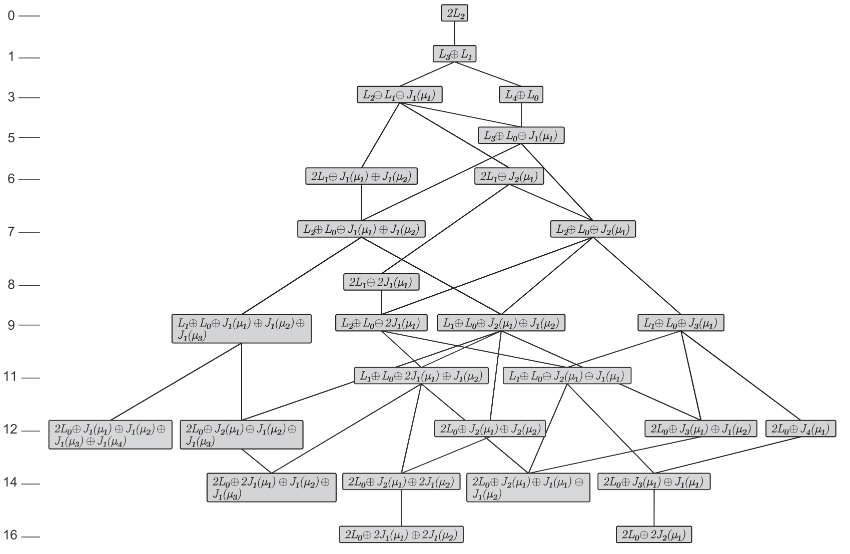

Fig. 3. Orbit stratification of the right linearization $sH_r + G_r$ of a full normal-rank 2×4 polynomial matrix of degree 2 ($m = 2$, $n = 4$, $d = 2$, $r = m$). Each node in the graph represents an orbit of $sH_r + G_r$ with the stated structural elements represented as KCF-blocks. The numbers on the left are the codimensions of the orbits on each level. See also Table 1.

Table 1
The table lists all orbits in the stratification of the right linearization of the full normal-rank 2×4 polynomial matrix of degree 2 shown in Fig. 3. The first column Cod is the codimension of the corresponding orbit and # is a sequence number identifying orbits with the same codimension (numbered from left to right in Fig. 3). The third column Blocks gives the structural elements of the orbit represented in KCF block notation, and the remaining columns show the corresponding structure integer partitions $\mathcal{R}$ and $\mathcal{J}$.

Cod:#	Blocks	$\mathcal{R}$	$\mathcal{J}_{\mu_1}$	$\mathcal{J}_{\mu_2}$	$\mathcal{J}_{\mu_3}$	$\mathcal{J}_{\mu_4}$
0 : 1	$2L_2$	(2, 2, 2)				
1 : 1	$L_3 \oplus L_1$	(2, 2, 1, 1)				
3 : 1	$L_2 \oplus L_1 \oplus J_1(\mu_1)$	(2, 2, 1)	(1)			
3 : 2	$L_4 \oplus L_0$	(2, 1, 1, 1, 1)				
5 : 1	$L_3 \oplus L_0 \oplus J_1(\mu_1)$	(2, 1, 1, 1)	(1)			
6 : 1	$2L_1 \oplus J_1(\mu_1) \oplus J_1(\mu_2)$	(2, 2)	(1)	(1)		
6 : 2	$2L_1 \oplus J_2(\mu_1)$	(2, 2)	(1, 1)			
7 : 1	$L_2 \oplus L_0 \oplus J_1(\mu_1) \oplus J_1(\mu_2)$	(2, 1, 1)	(1)	(1)		
7 : 2	$L_2 \oplus L_0 \oplus J_2(\mu_1)$	(2, 1, 1)	(1, 1)			
8 : 1	$2L_1 \oplus 2J_1(\mu_1)$	(2, 2)	(2)			
9 : 1	$L_1 \oplus L_0 \oplus J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3)$	(2, 1)	(1)	(1)	(1)	
9 : 2	$L_2 \oplus L_0 \oplus 2J_1(\mu_1)$	(2, 1, 1)	(2)			
9 : 3	$L_1 \oplus L_0 \oplus J_2(\mu_1) \oplus J_1(\mu_2)$	(2, 1)	(1, 1)	(1)		
9 : 4	$L_1 \oplus L_0 \oplus J_3(\mu_1)$	(2, 1)	(1, 1, 1)			
11 : 1	$L_1 \oplus L_0 \oplus 2J_1(\mu_1) \oplus J_1(\mu_2)$	(2, 1)	(2)	(1)		
11 : 2	$L_1 \oplus L_0 \oplus J_2(\mu_1) \oplus J_1(\mu_1)$	(2, 1)	(2, 1)			
12 : 1	$2L_0 \oplus J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3) \oplus J_1(\mu_4)$	(2)	(1)	(1)	(1)	(1)
12 : 2	$2L_0 \oplus J_2(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3)$	(2)	(1, 1)	(1)	(1)	
12 : 3	$2L_0 \oplus J_2(\mu_1) \oplus J_2(\mu_2)$	(2)	(1, 1)	(1, 1)		
12 : 4	$2L_0 \oplus J_3(\mu_1) \oplus J_1(\mu_2)$	(2)	(1, 1, 1)	(1)		
12 : 5	$2L_0 \oplus J_4(\mu_1)$	(2)	(1, 1, 1, 1)			
14 : 1	$2L_0 \oplus 2J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3)$	(2)	(2)	(1)	(1)	
14 : 2	$2L_0 \oplus J_2(\mu_1) \oplus 2J_1(\mu_2)$	(2)	(1, 1)	(2)		
14 : 3	$2L_0 \oplus J_2(\mu_1) \oplus J_1(\mu_1) \oplus J_1(\mu_2)$	(2)	(2, 1)	(1)		
14 : 4	$2L_0 \oplus J_3(\mu_1) \oplus J_1(\mu_1)$	(2)	(2, 1, 1)			
16 : 1	$2L_0 \oplus 2J_1(\mu_1) \oplus 2J_1(\mu_2)$	(2)	(2)	(2)		
16 : 2	$2L_0 \oplus 2J_2(\mu_1)$	(2)	(2, 2)			

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Once again rule A.(1) is used on 1:1 to get 3:2. From 3:2 we can recursively apply rule A.(2) four times obtaining 5:1, 7:2, 9:4, and 12:5. We remark that going from 3:2 to 5:1 a new set $\mathcal{J}$ for a new eigenvalue is created, while the remaining edges in the subpath correspond to that the existing $\mathcal{J}$ is extended, creating a Jordan block of larger size (2, 3, and 4, respectively). Finally, the last two edges (from 12:5 to 16:2 via 14:4) are obtained by applying rule A.(3).

We could as well have started at the least generic $\mathcal{O}_R(2L_0 \oplus 2J_2(\mu_1))$ (16:2) and applied the covered-by rules in Theorem 10.1 to generate the right-most (now) upward path ending at orbit $\mathcal{O}_R(2L_2)$ (0:1) in Fig. 3.

Next we consider square $m \times m$ polynomial matrices $P(s)$ of exact degree d and with full normal-rank $r = m$, which we only illustrate here with an example where $m = d = r = 2$. There are five most generic orbits where each of them are independent, i.e., they belong to different closure hierarchy graphs. The five cases I–V are listed below together with the corresponding least generic orbit and their codimension.

Case	Cod	Most generic orbit	Cod	Least generic orbit
I	4	$\mathcal{O}_R(J_4(\mu_1))$	8	$\mathcal{O}_R(2J_2(\mu_1))$
II	4	$\mathcal{O}_R(J_3(\mu_1) \oplus J_1(\mu_2))$	6	$\mathcal{O}_R(J_2(\mu_1) \oplus J_1(\mu_1) \oplus J_1(\mu_2))$
III	4	$\mathcal{O}_R(J_2(\mu_1) \oplus J_2(\mu_2))$	8	$\mathcal{O}_R(2J_1(\mu_1) \oplus 2J_1(\mu_2))$
IV	4	$\mathcal{O}_R(J_2(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3))$	6	$\mathcal{O}_R(2J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3))$
V	4	$\mathcal{O}_R(J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3) \oplus J_1(\mu_4))$	–	–

We remark that case V above, indeed, is a stratification with only one node (four distinct eigenvalues corresponding to finite zeros of $P(s)$). The stratification of the five cases can be obtained from the closure hierarchy graph of a regular matrix pencil, with the restriction that there can be at most m elementary divisors associated with each eigenvalue.

11.2. Half-car suspension model

Finally, we apply the stratification theory to a mechanical system in the form of a half-car suspension model with four degrees of freedom as shown in Fig. 4, where k_i are stiffnesses, c_i dampings, l_i lengths, m_i masses, and $J_p \approx m_b l_f l_r$ is the body moment of inertia. The model represents one side of a car (front and rear suspension), where the pitch φ and heave motion z_b of the vehicle body and the vertical translation of the front and rear axles (z_f and z_r, respectively) can be analyzed. Typical values for a passenger sedan can be found in, e.g., [39].

The equations of motion of the half-car suspension model are:

$$m_f\ddot{z}_f = k_{t_f}(z_f - q_f) + k_f(z_b - \varphi l_f - z_f) + c_f(\dot{z}_b - \dot{\varphi} l_f - \dot{z}_f) + m_f g, \tag{35}$$

$$m_r\ddot{z}_r = k_{t_r}(z_r - q_r) + k_r(z_b + \varphi l_r - z_r) + c_r(\dot{z}_b + \dot{\varphi} l_r - \dot{z}_r) + m_r g, \tag{36}$$

$$\begin{aligned} m_b\ddot{z}_b = {} & k_f(z_f - z_b + \varphi l_f) + k_r(z_r - z_b + \varphi l_r) + c_f(\dot{z}_f - \dot{z}_b + \dot{\varphi} l_f) \\ & + c_r(\dot{z}_r - \dot{z}_b - \dot{\varphi} l_r) + m_b g, \end{aligned} \tag{37}$$

$$\begin{aligned} J_p\ddot{\varphi} = {} & -k_f l_f(z_f - z_b + \varphi l_f) - c_f l_f(\dot{z}_f - \dot{z}_b + \dot{\varphi} l_f) + k_r l_r(z_r - z_b + \varphi l_r) \\ & + c_r l_r(\dot{z}_r - \dot{z}_b - \dot{\varphi} l_r). \end{aligned} \tag{38}$$

Let the state vector be $x = \begin{bmatrix} z_f & z_r & z_b \end{bmatrix}^T$, and the input vector be $u = \begin{bmatrix} q_f & q_r & g \end{bmatrix}^T$, where $q_f,\ q_r$ are the road heights. Then the equations (35)–(37) can be represented in matrix form by the second-order differential equation

$$M\ddot{x} + C\dot{x} - Kx - C_p\dot{\varphi} - K_p\varphi = Fu, \tag{39}$$

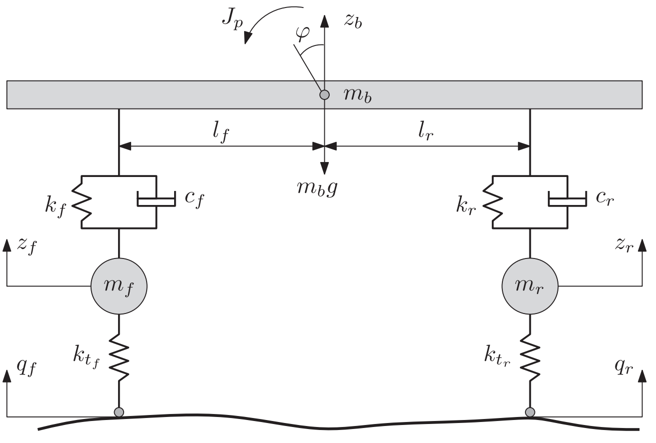

Fig. 4. Half-car passive suspension model.

and (38) as

$$J_p\ddot{\varphi} + k_p\dot{\varphi} + c_p\varphi - C_p^T\dot{x} - K_p^Tx = 0, \tag{40}$$

where

$$M = \operatorname{diag}(m_f, m_r, m_b), \quad F = \begin{bmatrix} 1 & 0 & m_f \\ 0 & 1 & m_r \\ 0 & 0 & m_b \end{bmatrix}, \quad C = \begin{bmatrix} c_f & 0 & -c_f \\ 0 & c_r & -c_r \\ -c_f & -c_r & c_f + c_r \end{bmatrix},$$

$$K = \begin{bmatrix} k_{t_f} - k_f & 0 & k_f \\ 0 & k_{t_r} - k_r & k_r \\ k_f & k_r & -k_f - k_r \end{bmatrix}, \quad C_p = \begin{bmatrix} -c_fl_f & c_rl_r & c_fl_f - c_rl_r \end{bmatrix}^T,$$

$$K_p = \begin{bmatrix} -k_fl_f & k_rl_r & k_fl_f + k_rl_r \end{bmatrix}^T, \quad c_p = c_fl_f^2 + c_rl_r^2, \quad \text{and} \quad k_p = k_fl_f^2 + k_rl_r^2.$$

Using the Laplace variable s, (39) and (40) can be expressed as

$$\begin{aligned} &Ms^2x + Csx - Kx - C_ps\varphi - K_p\varphi = Fu, \quad \text{and} \\ &J_ps^2\varphi + k_ps\varphi + c_p\varphi - C_p^Tsx - K_p^Tx = 0, \end{aligned} \tag{41}$$

respectively. Eliminating φ from (41) leads to the fourth-order differential equation

$$P_4x^{(4)} + P_3x^{(3)} + P_2x^{(2)} + P_1x^{(1)} + P_0x = Q_2u^{(2)} + Q_1u^{(1)} + Q_0u, \tag{42}$$

where

$$\begin{aligned} &P_4 = J_pM, \quad P_3 = k_pM + J_pC, \quad P_2 = k_pC + c_pM - J_pK - C_pC_p^T, \\ &P_1 = c_pC - k_pK - K_pC_p^T - C_pK_p^T, \quad P_0 = c_pK - K_pK_p^T, \quad Q_2 = J_pF, \\ &Q_1 = k_pF, \text{ and } \quad Q_0 = c_pF. \end{aligned}$$

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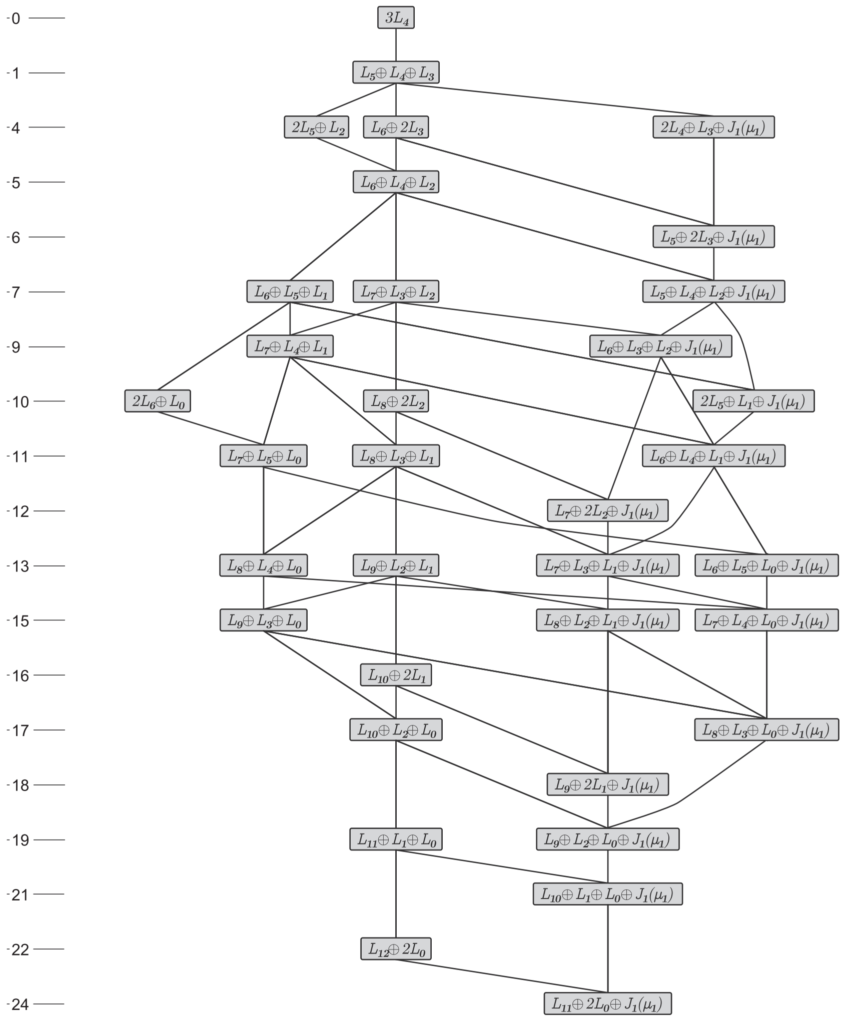

Fig. 5. Subgraph of the orbit stratification of the right linearization (43). The nodes in the left part represent the orbits of controllable systems (only L_k blocks). The nodes in the right part represent the orbits of uncontrollable systems with one uncontrollable mode (only L_k blocks and one $J_1(\mu_1)$ block). The numbers (0–24) to the left show the codimensions of the different orbits in the subgraph.

Using the technique outlined in Example 6.1 we obtain the right linearization of the associated 3×6 polynomial fraction (42) as

$$\left[\begin{array}{cccc|c} sI_3 & & & P_0 & Q_0 \\ -I_3 & sI_3 & & P_1 & Q_1 \\ & -I_3 & sI_3 & P_2 & Q_2 \\ & & -I_3 & sP_4 + P_3 & 0 \end{array}\right], \tag{43}$$

If P_4 (in this case the diagonal matrix J_pM) is well-conditioned one can apply the stratification rules in Corollary 10.3. Otherwise, we keep to the formulation (43) and use the rules in Theorem 10.1. The complete stratification of (43) has 6416 different orbits! Here, we only show a subgraph in Fig. 5

for illustration. The subgraph represents all the controllable orbits (on the left) together with the closest uncontrollable orbits with one uncontrollable mode (on the right) which can be reached by a perturbation of the polynomial matrix coefficients. The most generic orbit with KCF $3L_4$ corresponds to the case when the three transformed inputs $\{\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3\}$ in the linearization control four states each, while the least generic controllable orbit ($\mathcal{O}_R(L_{12} \oplus 2L_0)$ with codimension 22) corresponds to when $\widetilde{u}_1$ controls all twelve states. One example taken from the uncontrollable part of the graph is when the two suspensions do not have any damping ($c_f = c_r = 0$). Such a configuration belongs to $\mathcal{O}_R(L_5 \oplus 2L_3 \oplus J_1(\mu_1))$ with codimension 6. In practice, this means that a suspension system with low damping factor is likely to be close to $\mathcal{O}_R(L_5 \oplus 2L_3 \oplus J_1(\mu_1))$.

Acknowledgments

We would like to acknowledge the fruitful discussions with Pierre-Antoine Absil, Fernando De Téran, Froilan Dopico, Steve Mackey, and Volker Mehrmann. Finally, we are grateful to the referees for their constructive comments.

This work presents research results supported by the Swedish Foundation for Strategic Research under grant A3 02:128, UMIT Research Lab via an EU Mål 2 project, and the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. In addition, support has been provided by eSSENCE, a strategic collaborative e-Science programme funded by the Swedish Research Council.

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