A GENERALIZED STATE-SPACE APPROACH FOR THE ADDITIVE DECOMPOSITION OF A TRANSFER MATRIX

BO KÅGSTRÖM
Institute of Information Processing
University of Umeå, S-901 87 Umeå, Sweden

PAUL VAN DOOREN
Coordinated Science Laboratory
University of Illinois, Urbana IL 61801, USA

We dedicate this paper to Gene Golub, a true source of inspiration for our work, but also a genuine friend, at the occasion of his sixtieth birthday.

ABSTRACT

Robust and reliable algorithms are presented for computing the stable projection with respect to a specified region Γ in the complex plane of a transfer matrix given by its generalized state space realization. The algorithms are based on a block-diagonalization of $\lambda E - A$ (in generalized Schur form) with optimally conditioned transformation matrices. A first direct elimination approach reduces to solving a generalized Sylvester equation. In a second approach an equivalence transformation is constructed from unitary bases of pairs of deflating subspaces obtained from two reorderings of the eigenvalues. The sensitivity of the problem and the stability of the proposed algorithms are discussed and compared with a more classical approach. Results from numerical experiments that evaluate the algorithms and confirm the theory are also reported.

Keywords: Transfer matrix, additive decomposition, generalized state space realization, generalized Sylvester equation, block-diagonalization.

1 Introduction

Constructing the stable projection $H_\gamma(\lambda)$ of a given transfer matrix $H(\lambda)$ is a problem which appears in several analysis and design problems of systems and control theory ([11], [18], [6]). It could be briefly described as follows. Let Γ be a region of the complex plane $\mathcal{C}$ and let $H_{mp}(\lambda)$ be a $m \times p$ rational transfer matrix with poles inside and outside the region Γ, ($\mathcal{C} = \Gamma \cup \bar{\Gamma}, \Gamma \cap \bar{\Gamma} = \emptyset$). Then $H(\lambda)$ always has an additive decomposition

$$H(\lambda) = H_\gamma(\lambda) + H_{\bar{\gamma}}(\lambda) \tag{1.1}$$

where $H_\gamma(\lambda)$ has only poles inside Γ, and $H_{\bar{\gamma}}(\lambda)$ has only poles outside Γ (i.e. inside $\bar{\Gamma}$). The term $H_\gamma(\lambda)$ is then also called the *stable projection* of $H(\lambda)$ on Γ. Notice that this decomposition is not unique. Indeed, let $H_\gamma(\lambda)$ and $H_{\bar{\gamma}}(\lambda)$ be a solution to this problem then $H_\gamma(\lambda) + M$ and $H_{\bar{\gamma}}(\lambda) - M$ with M an arbitrary $m \times p$ constant matrix, is also a solution to (1.1), since adding or subtracting a constant matrix does

not affect the poles of a transfer matrix. The classical construction of a solution to the above problem is the *partial fraction expansion* of $H(\lambda)$ (see e.g. [6], p.56):

$$H(\lambda) = \sum_{i=1}^{k} H_{\alpha_i}(\lambda) + H_0 + H_\infty(\lambda) \tag{1.2}$$

where

$$H_\infty(\lambda) \equiv H_1 \cdot \lambda + H_2 \cdot \lambda^2 + \cdots + H_d \cdot \lambda^d \tag{1.3}$$

and

$$H_{\alpha_i}(\lambda) \equiv \frac{H_{\alpha_i,1}}{\lambda - \alpha_i} + \cdots + \frac{H_{\alpha_i,d_i}}{(\lambda - \alpha_i)^{d_i}}. \tag{1.4}$$

The poles of $H(\lambda)$ are clearly the $\alpha_i, i = 1, \ldots, k$ and $\alpha_{k+1} = \infty$ (depending on the existence of the H_∞-part). It is clear that $H(\lambda)$ and $H_\gamma(\lambda)$ must have coinciding polar parts for $\alpha_i \in \Gamma$ and similarly $H(\lambda)$ and $H_{\bar{\gamma}}(\lambda)$ must have coinciding polar parts for $\alpha_i \in \bar{\Gamma}$. One now obtains a construction for $H_\gamma(\lambda)$ and $H_{\bar{\gamma}}(\lambda)$ by merely grouping in $H_\gamma(\lambda)$ all polar parts corresponding to a pole inside Γ and grouping in $H_{\bar{\gamma}}(\lambda)$ all polar parts corresponding to a pole outside Γ. This immediately reflects the degree of freedom in the constant term H_0 which can arbitrarily be distributed over both $H_\gamma(\lambda)$ and $H_{\bar{\gamma}}(\lambda)$. The problem becomes uniquely defined though, when one adds the constraint

$$H_\gamma(\lambda_0) = \bar{H_0}, \lambda_0 \in \bar{\Gamma} \tag{1.5}$$

for an arbitrary point λ_0 outside Γ, since that fixes the constant part of $H_\gamma(\lambda)$.

The rest of the paper is outlined as follows. In section 2 a generalized state-space approach of the additive decomposition of a transfer matrix is described leading to a block decomposition of the system matrix. Deflating subspaces and optimal block-diagonalization are introduced in section 3. In section 4 two algorithms for computing a block-decomposition with optimally conditioned transformation matrices are presented. Section 5 discusses the sensitivity of the problem and the stability of the proposed algorithms. Section 6 reports some numerical experiments and, finally, some conclusions are summarized in section 7. This paper is based and extends on an earlier conference paper [8].

The following notation is used in the paper. $\lambda(E, A)$ denotes the spectrum of a regular matrix pencil $\lambda E - A$. $\|A\|$ denotes the spectral norm (2-norm) of a matrix A induced by the Euclidean vector norm. $\|A\|_E$ denotes the Frobenius (or Euclidean) matrix norm. $\kappa(A) = \|A\| \cdot \|A^{-1}\|$ denotes the spectral condition number of a matrix A. $\sigma_{max}(A)$ and $\sigma_{min}(A)$ denote the largest and smallest singular values of a matrix A, respectively. For a square matrix A we have that $\|A\| = \sigma_{max}(A)$ and $\|A^{-1}\| = {\sigma_{min}}^{-1}(A)$. $A \otimes B$ denotes the Kronecker product of two matrices A and B whose (i, j)-th block element is $a_{ij}B$. $rank(A)$ denotes the rank of a matrix A. A^H denotes the conjugate transpose of A.

2 A Generalized State-Space Approach

Assume that $H_{mp}(\lambda)$ is given via its generalized state-space realization:

$$H(\lambda) = C(\lambda E - A)^{-1}B + D \tag{2.1}$$

where $\{\lambda E - A, B, C, D\}$ is a *minimal realization*, i.e. $rank(E)$ equals the McMillan degree of $H(\lambda)$ [11]. Algorithms for obtaining such realizations are e.g. given in [15]. If $H(\lambda)$ happens to be bounded at $\lambda = \infty$, then a realization can always be chosen with $E = I$, which is then a (standard) state space realization. Notice that a minimal realization such as (2.1) is not unique. Several such realizations can in fact be derived from (2.1) via the equivalence relation:

$$\{\lambda E - A, B, C, D\} \stackrel{U,V}{\longrightarrow} \{U^{-1}(\lambda E - A)V, U^{-1}B, CV, D\} \tag{2.2}$$

where U and V are nonsingular matrices. If we define

$$S(\lambda) = \begin{bmatrix} \lambda E - A & B \\ -C & D \end{bmatrix} \tag{2.3}$$

to be the *system matrix* of the realization (2.1) then the equivalent realization (2.2) has a system matrix:

$$S_{U,V}(\lambda) = \begin{bmatrix} U^{-1} & 0 \\ 0 & I_m \end{bmatrix} \cdot \begin{bmatrix} \lambda E - A & B \\ -C & D \end{bmatrix} \cdot \begin{bmatrix} V & 0 \\ 0 & I_p \end{bmatrix} \tag{2.4}$$

which is a compact notation always used in the sequel. That both $S(\lambda)$ and $S_{U,V}(\lambda)$ correspond to the same transfer function $H(\lambda)$ is easily verified from (2.1). By choosing U and V to be the transformation matrices reducing the regular pencil $\lambda E - A$ to its *Weierstrass canonical form* (WCF):

$$U^{-1}(\lambda E - A)V = diag\{\lambda I - J, \lambda N - I\} \tag{2.5}$$

where $\lambda I - J$ is in Jordan canonical form (finite eigenvalues α_i) and N is nilpotent and in Jordan canonical form (infinite eigenvalue), the corresponding realization $S_{U,V}(\lambda)$ yields information which is implicitly equivalent to the partial fraction expansion of $H(\lambda)$. Denoting $B_U = U^{-1}B$ and $C_V = CV$ one then has

$$H(\lambda) = C_V \cdot diag\{(\lambda I - J)^{-1}, (\lambda N - I)^{-1}\} \cdot B_U + D. \tag{2.6}$$

Let $(\lambda - \alpha_i)I - N_i$ be the diagonal block of $\lambda I - J$ corresponding to a given eigenvalue $\lambda = \alpha_i$ (N_i is thus nilpotent) and let B_{U,α_i} be the corresponding block rows of B_U and C_{V,α_i} the corresponding block columns of C_V. Then, clearly

$$H_{\alpha_i}(\lambda) = C_{V,\alpha_i} \cdot [(\lambda - \alpha_i)I - N_i]^{-1} \cdot B_{U,\alpha_i} \tag{2.7}$$

is the polar section of $H_{\alpha_i}(\lambda)$ at $\lambda = \alpha_i$ and the coefficients $H_{\alpha_i,j}$ of (1.4) can easily be derived from appropriate columns and rows of C_{V,α_i} and B_{U,α_i}, respectively. Notice that $rank(H_{\alpha_i,d_i})$ is equal to the number of Jordan blocks of size

d_i corresponding to α_i. The discussion above shows the connection between the Weierstrass form of a generalized state-space realization and the partial fraction expansion of its transfer matrix. At the same time we also indicate the weakness of the partial fraction expansion, since it is related to the WCF: its computation may be *extremely* sensitive when $H(\lambda)$ has multiple poles or poles that are close to each other.

3 Deflating Subspaces and Optimal Block-diagonalization

The comparison in section 2 suggests that the solution to the additive decomposition is in fact solved by a block decomposition of the type:

$$S_{U,V}(\lambda) = \begin{bmatrix} \lambda E_{11} - A_{11} & 0 & B_1 \\ 0 & \lambda E_{22} - A_{22} & B_2 \\ -C_1 & -C_2 & D \end{bmatrix}. \tag{3.1}$$

It is readily verified that the transfer function of $S_{U,V}(\lambda)$ is:

$$H(\lambda) = C_1 \cdot (\lambda E_{11} - A_{11})^{-1} \cdot B_1 + C_2 \cdot (\lambda E_{22} - A_{22})^{-1} \cdot B_2 + D \tag{3.2}$$

and it thus suffices to find a decomposition of the type (3.1) where $\lambda E_{11} - A_{11}$ has only eigenvalues inside Γ and $\lambda E_{22} - A_{22}$ only outside Γ. Such block decompositions boil down to the computation of two complementary deflating subspaces of the original pencil $\lambda E - A$ [5]. For explaining this we first recall the definition of a deflating subspace of a regular pencil $\lambda E - A$ [14]. Any subspace $\mathcal{X}$ satisfying

$$\mathcal{Y} = E\mathcal{X} + A\mathcal{X}; dim\mathcal{Y} = dim\mathcal{X} \tag{3.3}$$

is called a *deflating subspace* of the pencil $\lambda E - A$. The name stems from the following decomposition

$$Q^H(\lambda E - A)Z = \begin{bmatrix} \lambda E_{11} - A_{11} & \lambda E_{12} - A_{12} \\ 0 & \lambda E_{22} - A_{22} \end{bmatrix} \tag{3.4}$$

which is directly obtained from (3.3) using any *unitary* transformations Q and Z such that the first n_1 columns of Z and Q span the subspaces $\mathcal{X}$ and $\mathcal{Y}$, respectively [14]. The block decomposition (3.4) shows that the generalized eigenvalue problem $\lambda E - A$ is now "deflated" into two smaller dimensional problems $\lambda E_{ii} - A_{ii}, i = 1, 2$. The spectrum of $\lambda E_{11} - A_{11}$ associated with the subspace $\mathcal{X}$ uniquely determines the space $\mathcal{X}$ if and only if it is disjoint from the spectrum of $\lambda E_{22} - A_{22}$ [14]. Indeed $\mathcal{X}$ is then the subspace spanned by the eigenvectors and principal vectors of higher grade in the WCF associated with those eigenvalues. But since the spectra of $\lambda E_{11} - A_{11}$ and $\lambda E_{22} - A_{22}$ are disjoint there is also a unique deflating subspace $\mathcal{X}_c$ with the spectrum of $\lambda E_{22} - A_{22}$. In [5] it is shown that these spaces are then *complementary* (i.e. they are linear independent and add up to the whole space) as well as their ranges $\mathcal{Y} = E\mathcal{X} + A\mathcal{X}$ and $\mathcal{Y}_c = E\mathcal{X}_c + A\mathcal{X}_c$. Further, there exists a

decomposition

$$U^{-1}(\lambda E - A)V = \begin{bmatrix} \lambda E_{11} - A_{11} & 0 \\ 0 & \lambda E_{22} - A_{22} \end{bmatrix} \tag{3.5}$$

where V and U are no longer unitary, but are such that their first n_1 columns span $\mathcal{X}$ and $\mathcal{Y}$, respectively, and their last n_2 columns span $\mathcal{X}_c$ and $\mathcal{Y}_c$. The complementarity of these pairs of subspaces also ensures that U and V are invertible [5], and U and V are given by:

$$U = [Y|Y_c]\,; V = [X|X_c] \tag{3.6}$$

where X, X_c, Y and Y_c are submatrices whose columns span the spaces $\mathcal{X}, \mathcal{X}_c, \mathcal{Y}$ and $\mathcal{Y}_c$, respectively. From (3.5-3.6) it follows that our problem is solved by the computation of the subspaces

$$\mathcal{X} = \mathcal{X}_\gamma; \mathcal{X}_c = \mathcal{X}_{\bar{\gamma}} \tag{3.7}$$

corresponding to the complementary regions Γ and $\bar{\Gamma}$ we are interested in. These spaces are unique, as follows from our discussion and the matrices U and V derived from (3.6) are then invertible and yield the derived decomposition (3.5).

By minimizing the condition numbers of the transformation matrices we will compute an additive decomposition which is less sensitive to rounding errors. So, in this context we are interested to have U and V in (3.5) as well-conditioned as possible subject to the constraint that the block columns of U and V span $\mathcal{X}, \mathcal{X}_c, \mathcal{Y}$ and $\mathcal{Y}_c$, respectively. UD_1 and VD_2, where $D_i = diag\{D_{11}^{(i)}, D_{22}^{(i)}\}$ has a block partitioning compatible with (3.5), also satisfy (3.5) and we define (3.5) to be an *optimal block-diagonalization* if U and V fulfill

$$\kappa(P) = \inf_D \kappa(PD) \;\text{ for } P = U, V \tag{3.8}$$

where infimum is taken over all nonsingular block diagonal matrices D. In our case we only have two blocks and then it is well-known that any transformation matrices U and V with unitary bases for $\mathcal{X}, \mathcal{X}_c, \mathcal{Y}$ and $\mathcal{Y}_c$, respectively, satisfy (3.8) (see e.g. [1],[4]).

4 Block Decomposition Algorithms

In order to construct a decomposition of the type (3.1) we first transform $\lambda E - A$ to generalized Schur form by applying the QZ-algorithm [12]. Then the eigenvalues of $\lambda E - A$ are reordered so that the n_1 eigenvalues inside Γ are placed in the $(1,1)$-block of the transformed pencil and consequently the remaining eigenvalues (outside Γ) are placed in the $(2,2)$-block of the transformed pencil. This reordering can also be accomplished with unitary matrices [16], [17]. After this generalized state-space transformation (accomplished with unitary transformation matrices Q and Z) we have a system matrix of the form:

$$S_{Q,Z}(\lambda) = \begin{bmatrix} \lambda E_{11} - A_{11} & \lambda E_{12} - A_{12} & B_1 \\ 0 & \lambda E_{22} - A_{22} & B_2 \\ -C_1 & -C_2 & D \end{bmatrix} \tag{4.1}$$

Now it remains to decouple the diagonal blocks by annihilating the $(1,2)$-block of (4.1). We present two approaches for this decoupling. The first is a direct elimination approach which is performed by solving a generalized Sylvester equation. In the second approach the required equivalence transformation is constructed from unitary bases of pairs of deflating subspaces obtained from two reorderings of the eigenvalues of the generalized Schur form of $\lambda E - A$.

4.1 Block-diagonalization by Solving the Generalized Sylvester Equation

Here, $\lambda E_{12} - A_{12}$ is zeroed out with an equivalence transformation of the form:

$$\begin{bmatrix} I_{n_1} & -L & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_m \end{bmatrix} \cdot \begin{bmatrix} \lambda E_{11} - A_{11} & \lambda E_{12} - A_{12} & B_1 \\ 0 & \lambda E_{22} - A_{22} & B_2 \\ -C_1 & -C_2 & D \end{bmatrix} \cdot \begin{bmatrix} I_{n_1} & R & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_p \end{bmatrix}. \quad (4.2)$$

This leads to solving for R and L in the *generalized Sylvester equation*

$$\begin{aligned} A_{11}R - LA_{22} &= -A_{12} \\ E_{11}R - LE_{22} &= -E_{12} \end{aligned} \quad (4.3)$$

which is a linear system with $2n_1 \cdot n_2$ unknowns, the entries of L and R. In [10], [9] efficient and reliable algorithms and software for solving this matrix equation are presented. One of the algorithms uses the generalized Schur form of both $\lambda E_{ii} - A_{ii}, i = 1, 2$ which happen to be provided by the triangular form described in the previous step (4.1). Notice that B_1 and C_2 are also affected by the equivalence transformation (4.2). In the following we summarize the main steps of the proposed algorithm.

Algorithm ADTF1
(*Additive Decomposition of a Transfer Function no. 1*)

1. Apply the QZ-algorithm to $\lambda E - A$.
2. Reorder the eigenvalues so that the n_1 eigenvalues inside Γ are in $\lambda E_{11} - A_{11}$. After steps 1-2 we have the following decomposition of $\lambda E - A$:
$$Q^H(\lambda E - A)Z = \begin{bmatrix} \lambda E_{11} - A_{11} & \lambda E_{12} - A_{12} \\ 0 & \lambda E_{22} - A_{22} \end{bmatrix} \quad (4.4)$$
3. Zero out $\lambda E_{12} - A_{12}$ by solving the generalized Sylvester equation (4.3).
4. Complete the equivalence transformation from steps 1-2 giving a system matrix of the form (3.1):
$$B \equiv \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} := Q^H B; C \equiv \begin{bmatrix} C_1 & C_2 \end{bmatrix} := CZ \quad (4.5)$$
5. Complete the equivalence transformation (4.2):
$$B_1 := B_1 - LB_2; C_2 := C_1 R + C_2 \quad (4.6)$$

6. Construct U and V as in (3.6) from steps 1, 2 and 3:

$$\begin{array}{l} U \equiv [\ U_1 \quad U_2\ ] := [\ Q_1 \quad Q_1 L + Q_2\ ] \\ V \equiv [\ V_1 \quad V_2\ ] := [\ Z_1 \quad Z_1 R + Z_2\ ] \end{array} \tag{4.7}$$

where $Q \equiv [\ Q_1 \quad Q_2\ ], Z \equiv [\ Z_1 \quad Z_2\ ]$, and Q_1, Z_1, U_1 and V_1 have n_1 columns, respectively, and Q_2, Z_2, U_2 and V_2 have n_2 columns, respectively.

Notice that only U_2 and V_2 are affected by the transformation matrices of the equivalence transformation (4.2). We have only introduced U and V in step 6 for clarity. They should be stored in Q and Z, respectively. The algorithm is of course also possible to apply to the standard state space realization $H(\lambda) = C(\lambda I - A)^{-1}B + D$. The use of the Sylvester equation in this particular case was also suggested in [13].

In Table 1 we summarize the amount of work in flops required by algorithm ADTF1 (one flop is the amount of work associated with one floating point add and one floating point multiply). The pencil $\lambda E - A$ is of dimension $n_1 + n_2$ by $n_1 + n_2$. For the operation counts of step 1 see [7], [12]; for step 2 see [16], [17]; for step 3 see [10]. Step 4 to 6 are just matrix multiplications. By adding the flops for all steps in Table 1 we get an estimate of the overall workcount for the ADTF1-algorithm (cubic in n_1+n_2, since normally $n_1+n_2 > m$ and p). The solution of the generalized

Table 1: *The amount of work in flops for algorithm ADTF1*

Step	No. Flops	Step	No. Flops
1	$15(n_1+n_2)^3$	4	$(n_1+n_2)^2(m+p)$
2	$5(n_1+n_2)^3$ on average	5	$n_1 n_2(m+p)$
3	${n_1}^2 n_2 + n_1 {n_2}^2 + O(n_1 n_2)$	6	$2(n_1+n_2)n_1 n_2$

Sylvester equation in step 3 give us L and R in the block-decomposition (4.2). It is now possible to compute an optimal diagonalizing equivalence transformation of $\lambda E - A$ in generalized Schur form (4.4) [4],[5]. Without loss of generality the right and left transformation matrices are of the form SD with

$$S = \begin{bmatrix} I_{n_1} & M \\ 0 & I_{n_2} \end{bmatrix}, D = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}. \tag{4.8}$$

Then the choice [5]

$$D_{11} = I_{n_1}, D_{22} = (1 + \|M\|^2)^{-\frac{1}{2}} \cdot I_{n_2} \tag{4.9}$$

makes $\kappa(SD)$ satisfy the optimality condition (3.8) and

$$\kappa(SD) = \|M\| + (1 + \|M\|^2)^{\frac{1}{2}}. \tag{4.10}$$

Let $l = (1 + \|L\|^2)^{\frac{1}{2}}$ and $r = (1 + \|R\|^2)^{\frac{1}{2}}$. Then the optimal block-diagonalization imposes the following modifications (scalings) in algorithm ADTF1: $\lambda E_{22} - A_{22}$ in

(4.4) will be replaced by $\frac{l}{r}(\lambda E_{22} - A_{22})$; B_2 in (4.5) by lB_2; and C_2 in (4.6) by $(C_1 R + C_2)/r$. The extra work concerns mainly the computation of l and r which both require $O(4n_1^2 n_2)$ flops. The work for the scalings are only second order terms.

4.2 Block-diagonalization with Unitary Bases from Two Reorderings of the Eigenvalues

It is in fact possible to construct an optimal block diagonalizing equivalence transformation of $\lambda E - A$ directly from two reorderings of the eigenvalues of the generalized Schur form. The following theorem gives an algorithm for computing the transformation matrices that block diagonalizes $\lambda E - A$ with the desired splitting of the spectrum $\lambda(E, A)$.

Theorem 4.1 *Let (Q, Z) be unitary matrices that transform $\lambda E - A$ to the generalized Schur form:*

$$Q^H(\lambda E - A)Z = \begin{bmatrix} Q_1^H \\ Q_2^H \end{bmatrix} (\lambda E - A) \begin{bmatrix} Z_1 & Z_2 \end{bmatrix} = \begin{bmatrix} \lambda E_{11}^{(1)} - A_{11}^{(1)} & \lambda E_{12}^{(1)} - A_{12}^{(1)} \\ 0 & \lambda E_{22}^{(1)} - A_{22}^{(1)} \end{bmatrix} \tag{4.11}$$

with $\lambda(E_{11}^{(1)}, A_{11}^{(1)}) \subseteq \Gamma$ and $\lambda(E_{11}^{(1)}, A_{11}^{(1)}) \bigcap \lambda(E_{22}^{(1)}, A_{22}^{(1)}) = \emptyset$. Similarly, let (U, V) be unitary matrices that transform $\lambda E - A$ to the generalized Schur form:

$$U^H(\lambda E - A)V = \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} (\lambda E - A) \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} \lambda E_{11}^{(2)} - A_{11}^{(2)} & \lambda E_{12}^{(2)} - A_{12}^{(2)} \\ 0 & \lambda E_{22}^{(2)} - A_{22}^{(2)} \end{bmatrix} \tag{4.12}$$

with $\lambda(E_{22}^{(2)}, A_{22}^{(2)}) \subseteq \Gamma$ and $\lambda(E_{11}^{(2)}, A_{11}^{(2)}) \bigcap \lambda(E_{22}^{(2)}, A_{22}^{(2)}) = \emptyset$. Then

$$X^{-1} = \begin{bmatrix} U_2^H \\ Q_2^H \end{bmatrix}, Y = \begin{bmatrix} Z_1 & V_1 \end{bmatrix} \tag{4.13}$$

are transformation matrices with optimal condition numbers (3.8) that block diagonalize $\lambda E - A$ such that the $(1,1)$-block of the transformed pencil has all (n_1) eigenvalues inside Γ and the $(2,2)$-block has all eigenvalues outside Γ (i.e. inside $\bar{\Gamma}$).

Proof: By using (4.11), (4.12) and (4.13) in $X^{-1}(\lambda E - A)Y$ we obtain

$$X^{-1}(\lambda E - A)Y = \begin{bmatrix} U_2^H Q_1(\lambda E_{11}^{(1)} - A_{11}^{(1)}) & U_2^H U_1(\lambda E_{11}^{(2)} - A_{11}^{(2)}) \\ Q_2^H Q_1(\lambda E_{11}^{(1)} - A_{11}^{(1)}) & Q_2^H U_1(\lambda E_{11}^{(2)} - A_{11}^{(2)}) \end{bmatrix} \tag{4.14}$$

Since U and Q are unitary it follows that the $(1,2)$ and $(2,1)$ blocks of (4.14) are zero matrices. The desired spectral properties of the diagonal pencils follow from the fact that the $(1,1)$-block is an equivalence transformation of $\lambda E_{11}^{(1)} - A_{11}^{(1)}$ and similarly the (2,2)-block is equivalent to $\lambda E_{11}^{(2)} - A_{11}^{(2)}$. By construction the two block columns of Y and the two block rows of X^{-1} have orthonormal bases which ensure transformation matrices with optimal condition numbers [1]. □

Theorem 4.1 is of course also applicable to the standard problem $\lambda I - A$ in which case X can be chosen as Y. In the following we summarize the main steps of an algorithm based on Theorem 4.1:

Algorithm ADTF2
(Additive Decomposition of a Transfer Function no. 2)

1. Apply the QZ-algorithm to $\lambda E - A$.

2. Reorder the eigenvalues of the generalized Schur form from step 1 so that the n_1 eigenvalues inside Γ are in the $(1,1)$-block. Save the transformation matrices (Q, Z).
   After steps 1-2 we have transformed $\lambda E - A$ on the form (4.11).

3. Reorder the eigenvalues of the generalized Schur form from step 1 such that the n_1 eigenvalues inside Γ are in the $(2,2)$-block, giving (4.12).

4. Construct X^{-1} and Y as in (4.13) from steps 2 and 3 and compute the diagonal blocks of $X^{-1}(\lambda E - A)Y$.

5. B and C are affected by X^{-1} and Y:

$$B \equiv \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} := X^{-1}B; C \equiv \begin{bmatrix} C_1 & C_2 \end{bmatrix} := CZ \tag{4.15}$$

After steps 1-5 of ADTF2 we have a system matrix $S_{X,Y}(\lambda)$ of the desired form (3.1). Notice that the diagonal pencils of $S_{X,Y}(\lambda)$ may now be full blocks but their spectra will satisfy the properties of Theorem 4.1. By adding the flops for all steps in Table 2 we get an estimate of the overall work count for the ADTF2-algorithm. The operation counts for step 1-3 follow Table 1. The work count for step 4 is based on the expressions of the diagonal blocks in (4.14). Step 5 comprises two matrix multiplications. Besides, the extra storage for (Q, Z) in step 2 the ADTF2-

Table 2: *The amount of work in flops for algorithm ADTF2*

Step	No. Flops	Step	No. Flops
1	$15(n_1+n_2)^3$	4	$(n_1+n_2)^3 + n_1^3 + n_2^3$
2	$5(n_1+n_2)^3$ on average	5	$(n_1+n_2)^2(m+p)$
3	$5(n_1+n_2)^3$ on average		

algorithm requires somewhat more work, mainly due to that the solution of the generalized Sylvester equation in ADTF1 is replaced by a second reordering of the eigenvalues in ADTF2.

Notice that the two reorderings of Theorem 4.1 can be performed independently. So, in a parallel setting it is possible to perform the two reorderings during the time for one.

5 The Sensitivity of the Problem

We only measure here the sensitivity of the block decomposition (3.1), but clearly the matrices B and C also play a role in the sensitivity of the final projection $H_\gamma(\lambda)$ (1.1). Indeed, we can analyze the sensitivity by looking at the expressions for the matrices $\lambda E_{11} - A_{11}, B_1$ and C_1 in terms of the original matrices A, B, C and E and the blocks L_1 and R_1 describing the spaces we compute in the block decomposition:

$$\begin{bmatrix} L_1 & 0 \\ L_2 & 0 \\ 0 & I_m \end{bmatrix} \cdot \begin{bmatrix} \lambda E - A & B \\ -C & D \end{bmatrix} \cdot \begin{bmatrix} R_1 & R_2 & 0 \\ 0 & 0 & I_p \end{bmatrix} = \begin{bmatrix} \lambda E_{11} - A_{11} & 0 & B_1 \\ 0 & \lambda E_{22} - A_{22} & B_2 \\ -C_1 & -C_2 & D \end{bmatrix} \tag{5.1}$$

where

$$B_1 = L_1 B; C_1 = C R_1; \lambda E_{11} - A_{11} = L_1(\lambda E - A) R_1. \tag{5.2}$$

If we perturb A, B, C, E then this will affect the bases L_1 and R_1 and of course also E_{11}, A_{11}, B_1 and C_1. If the perturbations of A, B, C and E are bounded normwise by δ the perturbations of L_1 and R_1 will be of size δ times the sensitivity of the spaces and in turn the latter perturbations will be the most influential in the perturbations of E_{11}, A_{11}, B_1 and C_1 as we can see from (5.2). So we concentrate on the sensitivity of the spaces connected to the block decomposition we are after by introducing a condition number that provides information about the sensitivity of the block decomposition (3.1) to perturbations in data. In section 3 we showed that computing (3.1) is a deflating subspace problem and that the additive decomposition problem is equivalent to computing the subspaces $\mathcal{X}$ and $\mathcal{X}_c$ (3.7). The sensitivity of these subspaces depends on $dif(A_{11}, A_{22}; E_{11}, E_{22})$ [14], which is the smallest singular value of the linear operator that maps (L, R) in (4.3) to $(A_{11}R - LA_{22}, E_{11}R - LE_{22})$. In [5] it is shown that the dif-function also is $\sigma_{min}(Z)$, the smallest singular value of the $2n_1n_2$ by $2n_1n_2$ matrix

$$Z \equiv \begin{bmatrix} I_{n_2} \otimes A_{11} & -A_{22}^T \otimes I_{n_1} \\ I_{n_2} \otimes E_{11} & -E_{22}^T \otimes I_{n_1} \end{bmatrix} \tag{5.3}$$

where Z is one representation of the coefficient matrix of the linear system (4.3). The smallest singular value of Z is nonzero as long as the spectra of $\lambda E_{11} - A_{11}$ and $\lambda E_{22} - A_{22}$ are disjoint (which they are by definition in our application). A consequence of these definitions is that we can bound the solution of the generalized Sylvester equation [5] as

$$\|(L, R)\|_E \leq \frac{\|(A_{12}, E_{12})\|_E}{\sigma_{min}(Z)}. \tag{5.4}$$

So, a small $\sigma_{min}(Z)$ may cause L and R to be large, implying that the deflating subspaces $\mathcal{X}$ and $\mathcal{X}_c$ are not well separated. Clearly, the sensitivity of the block

decomposition (3.1) for perturbations in the data is proportional to $\sigma_{min}^{-1}(Z)$. (For details of the perturbation theory for deflating subspaces see [14], [5] and for details of the perturbation theory for the generalized Sylvester equation see [10]). One interpretation of these perturbation results says that if $\sigma_{min}(Z)$ is small then small changes in the minimal realization (especially perturbations in A and/or E) can induce large changes in $\mathcal{X}$ and $\mathcal{X}_c$ (3.7), i.e. the additive decomposition problem is ill-conditioned. Similarly, a moderate or large value of $\sigma_{min}(Z)$ indicates a well-conditioned additive decomposition problem (1.1) and small changes in the minimal realization will only cause correspondingly small changes in the subspaces $\mathcal{X}$ and $\mathcal{X}_c$ (3.7).

In summary, $\sigma_{min}(Z)$ is the relevant condition number for the additive decomposition problem. It measures the separation of $\lambda(E_{11}, A_{11})$ from $\lambda(E_{22}, A_{22})$. It is small if only a small perturbation is needed to make an eigenvalue in $\lambda(E_{11}, A_{11})$ coalesce with one in $\lambda(E_{22}, A_{22})$ [5]. Notice that if $\lambda E_{11} - A_{11}$ and $\lambda E_{22} - A_{22}$ have well-separated spectra and $\sigma_{min}(Z)$ is small then this signals that the original regular pencil $\lambda E - A$ is close to a singular pencil ($det(\lambda E - A) \equiv 0$ for all λ).

It is interesting to know how much $dif(A_{11}, A_{22}; E_{11}, E_{22})$ $(= \sigma_{min}(Z))$ can change due to the scaling for optimally conditioned transformation matrices as in (4.9-4.10). Before we state the result we first prove the following Lemma:

Lemma 5.1 *Let A be an $n \times n$ matrix and $D = diag\{I_{n_1}, \alpha I_{n_2}\}$ where $\alpha > 0$ and $n = n_1 + n_2$. Then $\sigma_{min}(A)$ and $\sigma_{min}(AD)$ will differ at most by a factor α:*

$$min\{1, \alpha\}\sigma_{min}(A) \leq \sigma_{min}(AD) \leq max\{1, \alpha\}\sigma_{min}(A). \tag{5.5}$$

Proof: Since $\|A^{-1}\|^{-1} = \sigma_{min}(A)$, $\|D\| = max\{1, \alpha\}$, and $\|D^{-1}\|^{-1} = 1/min\{1, \alpha\}$ the lemma follows immediately from the following 2-norm inequalities:

$$\frac{\|A^{-1}\|}{\|D\|} \leq \|(AD)^{-1}\| \leq \|A^{-1}\| \cdot \|D^{-1}\| \tag{5.6}$$

□

Theorem 5.1 *The scaling (4.9-4.10) for optimally conditioned transformation matrices U_{opt}, V_{opt} will change the separation between the spectra measured by the dif-function with at most a factor $\frac{l}{r}$:*

$$\begin{array}{c} min\{1, \frac{l}{r}\}dif(A_{11}, A_{22}; E_{11}, E_{22}) \leq dif(A_{11}, \frac{l}{r}A_{22}; E_{11}, \frac{l}{r}E_{22}) \leq \\ max\{1, \frac{l}{r}\}dif(A_{11}, A_{22}; E_{11}, E_{22}) \end{array} \tag{5.7}$$

where $l = (1 + \|L\|^2)^{\frac{1}{2}}$ and $r = (1 + \|R\|^2)^{\frac{1}{2}}$.

Proof: The matrix representation (5.1) of the dif-function will change to

$$Z_{opt} = Z \cdot \begin{bmatrix} I_{n_2} \otimes I_{n_1} & 0 \\ 0 & \frac{l}{r} I_{n_2} \otimes I_{n_1} \end{bmatrix} \tag{5.8}$$

Now (5.7) follows by applying Lemma 5.1 to (5.8). □

The following theorem shows how far from the optimal condition numbers we are before the scaling (4.9-4.10).

Theorem 5.2 *Let U and V be the transformation matrices in (4.7). The improvement of the condition numbers for U and V in relation to U_{opt} and V_{opt}, that fulfill the optimality condition (3.8), can be bounded as follows:*

$$l^2(l - \sqrt{l^2 - 1}) \leq \frac{\kappa(U)}{\kappa(U_{opt})} \leq (l^2 + \sqrt{l^2 - 1})(l - \sqrt{l^2 - 1}) \tag{5.9}$$

$$r^2(r - \sqrt{r^2 - 1}) \leq \frac{\kappa(V)}{\kappa(V_{opt})} \leq (r^2 + \sqrt{r^2 - 1})(r - \sqrt{r^2 - 1}) \tag{5.10}$$

where $l = (1 + \|L\|^2)^{\frac{1}{2}}$ and $r = (1 + \|R\|^2)^{\frac{1}{2}}$.

Proof: Without loss of generality we assume that U and V are on the form:

$$U = \begin{bmatrix} I_{n_1} & L \\ 0 & I_{n_2} \end{bmatrix}, V = \begin{bmatrix} I_{n_1} & R \\ 0 & I_{n_2} \end{bmatrix}. \tag{5.11}$$

In general we have

$$\left\| \begin{bmatrix} I_{n_1} & M \\ 0 & I_{n_2} \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} 1 & \|M\| \\ 0 & 1 \end{bmatrix} \right\|^2 = 1 + 0.5(m^2 + (m^4 + 4m^2)^{\frac{1}{2}}) \tag{5.12}$$

where $m = \|M\|$. By applying (5.12) to (5.11) and using l and r we can bound the condition numbers for U and V as

$$l^2 \leq \kappa(U) \leq l^2 + \sqrt{l^2 - 1}, r^2 \leq \kappa(V) \leq r^2 + \sqrt{r^2 - 1} \tag{5.13}$$

Now, (5.9) and (5.10) follow by using the fact that U_{opt} and V_{opt} satisfy (4.10). □

Notice that individual eigenvalues of $\lambda E_{ii} - A_{ii}$ and their Jordan canonical forms can be more sensitive for small perturbations in the data than the separation of the spectrum of $\lambda E_{11} - A_{11}$ from the spectrum of $\lambda E_{22} - A_{22}$. In the partial fraction expansion (1.2) the sensitivity of the individual eigenvalues will be reflected in the sensitivity of the computed additive decomposition. This is in contrast to our block decomposition approach, where the inherent conditioning of the additive decomposition problem (measured as $\sigma_{min}(Z)$) will be reflected in the sensitivity of the computed solution which is the best we can ask for. In other words, both algorithms ADTF1 and ADTF2 are numerically stable in the sense that small changes in data will only cause relative errors in the computed quantities of size proportional to the conditioning of the original problem. For steps 1, 2, 4 and 6 of ADTF1 we can make use of Wilkinson's backward error analysis for products of unitary matrices [19]. A rounding error analysis of the generalized Schur algorithm for solving the generalized Sylvester equation is presented in [10]. The conclusion there is that the algorithm is weakly stable [2], meaning that the relative errors in the computed (L, R) are small for all well-conditioned problems. More precisely, the relative errors in the computed (L, R) are proportional to the condition number times a smooth function of the relative machine precision. These errors will finally propagate to the computations where R and L are involved (steps 4-6 in ADTF1). All steps in

ADTF2 include products of unitary matrices and Wilkinson's backward analysis for products of unitary matrices is applicable [19].

In [10], [9] dif^{-1}-estimators (lower bounds on $\sigma_{min}^{-1}(Z)$ in the style of the Linpack condition estimator [3]) that are incorporated into the generalized Schur algorithm for solving the generalized Sylvester equation (4.3) are presented. The heuristic condition estimators need $O(n_1^2 n_2 + n_1 n_2^2)$ flops, which is the same magnitude of work as solving the generalized Sylvester equation (4.3) (see Table 1). This should be compared with $O(10 n_1^3 n_2^3)$ flops for computing the exact value of $\sigma_{min}^{-1}(Z)$. Since solving the generalized Sylvester equation is one step in algorithm ADTF1, this block decomposition approach also provides us with estimates of the conditioning of the original problem and the sensitivity of the computed additive decomposition. For the algorithm ADTF2 the condition estimation must be performed in a post-processing step.

6 Numerical Experiments

Here we report some numerical results from our Fortran 77 implementations of ADTF1 and ADTF2. To assess the reliability and robustness of the algorithms with condition estimators from [10], [9], we ran different sets of test problems, including both well-conditioned and ill-conditioned additive decomposition problems. In all examples we have chosen Γ to be the interior of the unit circle. The first set of test problems are generated by specifying the spectra of $\lambda E - A$, and the upper triangular part of A and E are filled with elements chosen randomly from $[-1.0x, 1.0x]$, where $x > 0$ is an integer that affects the conditioning of the problem. Finally, A and E are obtained by pre- and post-multiplication of random unitary matrices. Examples 1-3 are generated in this way with the following spectra: Ex 1:$\{0.25, 0.5, 1.5, 2.0\}$; Ex 2:$\{0., 0.995i, 1.005i, -0.995i, -1.005i\}$; Ex 3:$\{0., \pm 0.995i, \pm 1.005i, \pm 0.995i, \pm 1.005i\}$. In Example 4 we choose: $A = J_3(1 - \alpha, 1) \oplus J_2(1 + \alpha, 1)$ and $E = I_3 \oplus I_2$, where $J_n(d, s)$ denotes a Jordan block of dimension n with d and s as diagonal and superdiagonal elements, respectively, and α is a real parameter > 0. As before we apply a unitary random equivalence transformation to (A, E). The final set of test problems were set up to evaluate our algorithms on random matrices. In Example 5, A and E are 8 by 8 upper triangular with entries chosen randomly from $[-1.0, 1.0]$ corresponding to an upper triangular generalized Schur form. The other extreme is when all eigenvalues are complex conjugate pairs. So, in Example 6, A is chosen 6 by 6, quasi triangular with 2 by 2 blocks along the diagonal and E is chosen upper triangular. In the random examples we also apply a unitary random equivalence transformation to (A, E). All tests were performed using double precision real arithmetic ($machep \approx 1.E-17$).

In Tables 3-4 results for ADTF1 and ADTF1 with optimal block-diagonalizing equivalence transformation are shown. Quantities displayed are $\sigma_{min}(Z)$, the product $p_F = \sigma_{min}(Z) dif^{-1}$, $\kappa(U), \kappa(V)$ the spectral condition numbers of the transfor-

mation matrices and residuals for the computed block diagonal matrices $\widehat{A}$ and $\widehat{E}$. These residuals reflect the sensitivity of the block decompositions. The reliability of the dif^{-1}-estimator is proved if $0 < p_F \leq 1$ and its accuracy is determined by how close p_F is to 1. We see that the correct dimension n_1 of the stable projection of $H(\lambda)$ on Γ was computed for all examples. Further, the condition estimator was in all cases (except one) within a factor four of the correct value. For some examples we get a drastic improvement of the condition numbers for U and V. In Table 5

Table 3: *Results for ADTF1 with condition estimator*

Ex	x	n_1	$\sigma_{min}(Z)$	p_F	$\kappa(U)$	$\kappa(V)$	$\|\widehat{A} - UAV\|$	$\|\widehat{E} - UEV\|$
1a	1	2	0.409	0.822	1.61	2.66	5.5E-17	0.0E0
1b	10	2	0.045	0.551	19.9	3104.	2.5E-14	4.1E-14
2a	1	3	0.118	0.634	13.5	16.1	8.3E-16	4.5E-16
2b	10	3	0.263	0.736	330.	2737.2	6.8E-14	4.5E-16
3a	1	5	9.7E-3	0.347	3361.	9013.	3.8E-13	4.8E-13
3b	10	5	5.3E-4	0.277	5.0E5	4.5E8	5.3E-10	1.4E-10
4a	-	3	1.7E-9	0.413	1.0	1.0	1.2E-15	7.6E-16
4b	-	3	1.7E-13	0.409	1.0	1.0	2.3E-15	1.2E-15
5	-	3	0.036	0.577	80.4	59.3	1.1E-15	2.2E-15
6	-	2	0.085	0.360	10.5	11.3	2.9E-15	1.9E-15

Table 4: *Results for ADTF1 with optimal block-diagonalizing equivalence transformation and condition estimator*

Ex	x	n_1	$\sigma_{min}(Z)$	p_F	$\kappa(U)$	$\kappa(V)$	$\|\widehat{A} - UAV\|$	$\|\widehat{E} - UEV\|$
1a	1	2	0.389	0.783	1.59	2.45	4.7E-17	3.1E-17
1b	10	2	0.028	0.343	8.59	111.4	2.0E-15	1.3E-15
2a	1	3	0.113	0.605	6.95	7.68	1.6E-16	2.4E-16
2b	10	3	0.209	0.586	36.3	54.3	5.6E-15	4.6E-14
3a	1	5	7.9E-3	0.284	115.9	189.9	6.5E-14	5.0E-14
3b	10	5	6.6E-5	0.035	1.4E3	4.3E4	4.4E-12	3.9E-12
4a	-	3	1.7E-9	0.413	1.0	1.0	1.2E-15	7.6E-16
4b	-	3	1.7E-13	0.409	1.0	1.0	2.2E-15	1.3E-15
5	-	3	0.040	0.656	17.8	15.2	1.8E-15	1.5E-15
6	-	2	0.083	0.352	6.04	6.28	2.3E-15	1.3E-15

we display $\frac{\kappa(U)}{\kappa(U_{opt})}$, $\frac{\kappa(V)}{\kappa(V_{opt})}$ and the lower and upper bounds of Theorem 5.2 of these ratios. In most examples the lower bounds of (5.9-5.10) are almost attained.

As expected the computed residuals in Table 3-4 are of the size $max\{\kappa(U), \kappa(V)\}$ times *machep*. So, the bounds (5.9-5.10) give us apriori information on how much accuracy we can gain by further scaling. In Table 6 $\sigma_{min}(Z_{opt})$ and the upper and lower bounds of Theorem 5.1 are shown. For comparison, $\sigma_{min}(Z)$ and the condition numbers of the equivalence transformation (U, V) computed by ADTF2 are displayed in the last three columns of Table 6. Notice that $\sigma_{min}(Z)$ did not change very much after scaling for optimally conditioned U and V. Knowing an estimate of $\sigma_{min}(Z)$ the bounds (5.7) give us apriori information on how much the difference between the spectra measured by the dif-function can change under scaling for an optimal block diagonalization. Notice Example 3 in Table 6: $\sigma_{min}(Z)$

Table 5: *Lower and upper bounds on the relative changes of $\kappa(U)$ and $\kappa(V)$*

Ex	x	n_1	$lb(5.9)$	$\frac{\kappa(U)}{\kappa(U_{opt})}$	$ub(5.9)$	$lb(5.10)$	$\frac{\kappa(V)}{\kappa(V_{opt})}$	$ub(5.10)$
1a	1	2	0.7740	1.0126	1.2558	0.8339	1.0857	1.8561
1b	10	2	2.2045	2.3155	6.4380	27.8572	27.8654	83.5583
2a	1	3	1.8117	1.9424	5.2179	1.9832	2.1016	5.7528
2b	10	3	9.0769	9.1073	27.1893	13.5696	13.5895	40.6812
3a	1	5	28.987	28.9947	86.9485	47.4663	47.4725	142.3910
3b	10	5	351.822	351.823	1055.5	1.066E4	1.066E4	3.198E4
4a	-	3	1.0	1.0	1.0	1.0	1.0	1.0
4b	-	3	1.0	1.0	1.0	1.0	1.0	1.0
5	-	3	4.4691	4.5267	13.3228	3.8365	3.9000	11.4108
6	-	2	1.5926	1.7450	4.5268	1.6508	1.7962	4.7116

Table 6: *Lower and upper bounds on $\sigma_{min}(Z_{opt})$ with optimal block-diagonalizing equivalence transformation and results for ADTF2*

Ex	x	n_1	$lb(5.7)$	$\sigma_{min}(Z_{opt})$	$ub(5.7)$	$\sigma_{min}(Z)$	$\kappa(U)$	$\kappa(V)$
1a	1	2	0.3108	0.389	0.409	0.364	1.59	2.45
1b	10	2	0.0035	0.028	0.045	0.005	8.59	111.4
2a	1	3	0.1074	0.113	0.118	0.239	6.95	7.68
2b	10	3	0.1762	0.209	0.263	0.009	36.3	54.3
3a	1	5	5.9E-3	7.9E-3	9.7E-3	7.3E-5	115.9	189.9
3b	10	5	1.75E-5	6.6E-5	5.3E-4	1.8E-7	1.4E3	4.3E4
4a	-	3	1.7E-9	1.7E-9	1.7E-9	1.7E-9	1.0	1.0
4b	-	3	1.7E-13	1.7E-13	1.7E-13	1.7E-13	1.0	1.0
5	-	3	0.0360	0.040	0.042	0.014	17.8	15.2
6	-	2	0.0816	0.083	0.085	0.035	6.04	6.28

computed by ADTF2 is a factor 100 smaller than $\sigma_{min}(Z)$ computed by ADTF1. In most cases ADTF2 gives a smaller $\sigma_{min}(Z)$ than ADTF1. One possible explanation is that the diagonal blocks are affected as well by ADTF2.

7 Conclusions

We have presented robust and numerically (backward) stable algorithms ADTF1 and ADTF2 for solving the additive decomposition problem of a transfer matrix given by its generalized state space realization $H(\lambda) = C(\lambda E - A)^{-1}B + D$. The algorithms are based on block-diagonalization of $\lambda E - A$ (in generalized Schur form) with optimally conditioned transformation matrices U_{opt} and V_{opt}. ADTF1 uses a direct elimination approach by solving a generalized Sylvester equation giving U and V. Then U_{opt} and V_{opt} are obtained from a block diagonal scaling of U and V. ADTF2 constructs U_{opt} and V_{opt} from unitary bases of pairs of deflating subspaces obtained from two reorderings of the eigenvalues of $\lambda E - A$ in generalized Schur form. A condition number $\sigma_{min}(Z)$ that measures the sensitivity of the computed block decomposition and the subspaces $\mathcal{X}$ and $\mathcal{X}_c$ (3.7) is introduced. The stability of the proposed algorithms are discussed and compared with the classical partial fraction expansion approach (e.g. see [6]). Algorithm ADTF1 also provides reliable lower

bounds of $\sigma_{min}^{-1}(Z)$ [10], [9] that estimate the sensitivity of the computed additive decomposition (1.1). If dif^{-1} and/or the condition numbers of the transformation matrices are large the computed additive decomposition including the subspaces $\mathcal{X}$ and $\mathcal{X}_c$ (3.7) may be sensitive for small perturbations in A and/or E. For algorithm ADTF1 computable lower and upper bounds that estimate the change of the conditioning of the computed block-decomposition due to the block-diagonal scaling are derived. Results from numerical experiments are reported that evaluate the algorithms and confirm the theory. Based on these results and the time complexity of the two algorithms (measured in flops), we conclude that ADTF1 with condition estimator is the most reliable and robust way of computing a block decomposition of the type (3.1).

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