PERIODIC INVARIANT SUBSPACES IN CONTROL

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Abstract: In this paper we present several different characterizations of invariant subspaces of periodic eigenvalue problems. We analyze their equivalence and discuss their use in control theory.

Keywords: Periodic systems, discrete-time systems, canonical forms, eigenvectors, eigenvalues

1. MAIN RESULTS

Consider the (homogeneous) linear time varying system :

$$E_k x_{k+1} = A_k x_k + B_k u_k, \qquad k \in \mathbf{N} \qquad (1)$$

where $\mathbf{N}$ is the set of natural numbers, x_k is an n-dimensional vector of descriptor variables, u_k is an m-dimensional vector of input variables, the matrices E_k and A_k are $n \times n$, and B_k is $n \times m$. This system is said to be periodic with period K if $E_k = E_{k+K}$, $A_k = A_{k+K}$ and $B_k = B_{k+K}$, for all $k \in \mathbf{N}$, and K is the smallest positive integer for which this holds. If we allow both the E_k and A_k matrices to be singular, then x_k may still uniquely be defined in the context of a two point boundary value problem, as e.g. in optimal control of periodic systems. It is shown in (Sreedhar and Van Dooren, 1999) that a necessary and sufficient condition for this is that the pencil

$$\lambda\mathcal{E} - \mathcal{A} := \begin{bmatrix} \lambda E_1 & & & -A_1 \\ -A_2 & \lambda E_2 & & \\ & \ddots & \ddots & \\ & & -A_K & \lambda E_K \end{bmatrix} \qquad (2)$$

is regular (i.e. $\det(\lambda\mathcal{E} - \mathcal{A}) \not\equiv 0$). We call such periodic systems *regular*.

We can always define a periodic similarity transformation of a regular periodic system (1.1) by multiplying it from the left by the invertible transformation S_k and substituting x_k by $x_k \doteq T_{k-1}\hat{x}_k$, where S_k and T_k are again K-periodic. Let us denote diagonal block transformations $\mathcal{S}$ and $\mathcal{T}$ as follows :

$$\mathcal{S} \doteq \begin{bmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_K \end{bmatrix}, \ \mathcal{T} \doteq \begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_K \end{bmatrix}.$$

Defining the *periodic similarity transformation*

$$\mathcal{S}\begin{bmatrix} \lambda E_1 & & & -A_1 \\ -A_2 & \lambda E_2 & & \\ & \ddots & \ddots & \\ & & -A_K & \lambda E_K \end{bmatrix}\mathcal{T}$$

$$\doteq \begin{bmatrix} \lambda \hat{E}_1 & & & -\hat{A}_1 \\ -\hat{A}_2 & \lambda \hat{E}_2 & & \\ & \ddots & \ddots & \\ & & -\hat{A}_K & \lambda \hat{E}_K \end{bmatrix} \qquad (3)$$

one then obtains a new system

$$\hat{E}_k \hat{x}_{k+1} = \hat{A}_k \hat{x}_k + \hat{B}_k u_k, \qquad k \in \mathbf{N}, \qquad (4)$$

where $\hat{B}_k \doteq S_k B_k$.

One important case of such a periodic similarity transformation is the so-called “Floquet transform” of a periodic system :

$$\begin{bmatrix} \lambda \hat{E}_1 & & & -\hat{A}_1 \\ -\hat{A}_2 & \lambda \hat{E}_2 & & \\ & \ddots & \ddots & \\ & & -\hat{A}_K & \lambda \hat{E}_K \end{bmatrix} = \begin{bmatrix} \lambda I & & & -M \\ -M & \lambda I & & \\ & \ddots & \ddots & \\ & & -M & \lambda I \end{bmatrix} (5)$$

This reduces the homogeneous problem to a time-invariant one, but the transformation may not always exist (Van Dooren and Sreedhar, 1994; Sreedhar and Van Dooren, 1997). Another important case is the Periodic Schur Form (Bojanczyk et al., 1992), where the transformation matrices S_k and T_k are constrained to be unitary, and the resulting matrices $\hat{E}_k$ and $\hat{A}_k$ can all be chosen upper triangular. This second form always exists and can be computed in a numerically stable manner (Bojanczyk et al., 1992).

Both forms are closely linked to the concept of *periodic invariant subspaces*, which play a fundamental role in the solution of important control problem of periodic systems (Sreedhar and Van Dooren, 1994), (Sreedhar and Van Dooren, 1997), (Ferng et al., 1998) :

- solution of periodic Riccati equations and their use in optimal and robust control,
- solution of periodic Lyapunov equations and their use in stability analysis,
- solution of periodic Sylvester equations and their use in decoupling,
- inverse eigenvalue problems and their use in pole placement.

We now describe several characterizations of invariant subspaces of a regular periodic system. A first characterization uses the classical concept of deflating subspace, directly applied to the pencil (2) (we assume $K = 3$ for the sake of simplicity) :

$$\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}(\lambda I - M) (6)$$

where $X \doteq \begin{bmatrix} X_1^T & X_2^T & X_3^T \end{bmatrix}^T, Y \doteq \begin{bmatrix} Y_1^T & Y_2^T & Y_3 \end{bmatrix}^T$ are assumed to have full rank d. With

$$\mathcal{X} \doteq \begin{bmatrix} X_1 & & \\ & X_2 & \\ & & X_3 \end{bmatrix}, \mathcal{Y} \doteq \begin{bmatrix} Y_1 & & \\ & Y_2 & \\ & & Y_3 \end{bmatrix}$$

we can also rewrite this as follows :

$$\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix}\mathcal{X} = \mathcal{Y}\begin{bmatrix} \lambda I & & -M \\ -M & \lambda I & \\ & -M & \lambda I \end{bmatrix} (7)$$

so that every d-dimensional eigenspace of the type (6) induces a $d \cdot K$-dimensional eigenspace of the type (7), *provided* all individual matrices X_k and Y_k have full column rank. The following example shows that one must impose certain conditions on M, since this does not hold in general.

Example 1 Consider $K = 3$ and $A_k = E_k = I_2$. And let $\mathbf{1} = (1,1)^T, M = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$ with $\omega \neq 1$ and $\omega^3 = 1$. Then we have

$$\begin{bmatrix} \lambda I_2 & 0 & -I_2 \\ -I_2 & \lambda I_2 & 0 \\ 0 & -I_2 & \lambda I_2 \end{bmatrix}\begin{bmatrix} \mathbf{1} & \omega^2\mathbf{1} \\ \mathbf{1} & \omega^1\mathbf{1} \\ \mathbf{1} & \omega^0\mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & \omega^2\mathbf{1} \\ \mathbf{1} & \omega^1\mathbf{1} \\ \mathbf{1} & \omega^0\mathbf{1} \end{bmatrix}\left(\lambda I_2 - \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}\right).$$

So (6) and (7) hold with

$$\begin{cases} X_1 = Y_1 = [\mathbf{1}, \omega^2\mathbf{1}] \\ X_2 = Y_2 = [\mathbf{1}, \omega^1\mathbf{1}] \\ X_3 = Y_3 = [\mathbf{1}, \omega^0\mathbf{1}]. \end{cases}$$

Although both the matrices

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

are of full column rank 2, all X_k and Y_k matrices are only of rank 1. □

This example illustrates well that M cannot be arbitrary. The following theorem characterizes which matrix M will yield full column rank matrices X_k and Y_k.

Theorem 1 Let A_k and E_k be nonsingular $n \times n$ matrices, for $k = 1, 2, \cdots, K$. Assume that $X = \left[X_1^T, X_2^T, \cdots, X_K^T\right]^T$ is a full column rank matrix that generates a d-dimensional deflating subspace of $\lambda\mathcal{E} - \mathcal{A}$:

$$\lambda\mathcal{E} - \mathcal{A} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_K \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_K \end{bmatrix} (\lambda I - M). \qquad (8)$$

If M has the property that $\lambda^K \neq \mu^K$ whenever λ and μ are two distinct eigenvalues of M, i.e.,

$$Ker(M - \lambda I) = Ker\left(M^K - \lambda^K I\right), \qquad (9)$$

$\forall \lambda \in \sigma(M)$ (the spectrum of M), then the X_k and Y_k matrices also have full column rank d, for $k = 1, 2, \cdots, K$.

Proof. Expanding (8), yields

$$E_k X_k = Y_k \quad \text{and} \quad A_k X_{k-1} = Y_k M, \qquad (10)$$

for $k = 1, 2, \cdots, K$. Thus we have

$$A_k X_{k-1} = E_k X_k M, \quad \forall k. \qquad (11)$$

Since the E_k matrices are nonsingular, we can define $S_k = E_k^{-1} A_k$. Then (11) is equivalent to

$$\begin{bmatrix} 0 & \cdots & 0 & S_1 \\ S_2 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & S_k & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_K \end{bmatrix} M$$

or also

$$\begin{bmatrix} 0 & \cdots & 0 & S_1 \\ S_2 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & S_k & 0 \end{bmatrix} \mathcal{X} = \mathcal{X} \begin{bmatrix} 0 & \cdots & 0 & M \\ M & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & M & 0 \end{bmatrix} . \qquad (12)$$

Note that M is nonsingular, since the A_k matrices are also nonsingular.

We now show by contradiction that the matrices X_k are full rank if (9) holds. Assume e.g. that X_1 does *not* have full column rank and that the columns of the matrix V span $\mathcal{V} := Ker X_1$. Then it follows from (12) that $X_2 M V = S_2 X_1 V = 0$ and hence $M\mathcal{V} \subset Ker X_2$. By similar arguments one shows that $M^k \mathcal{V} \subset Ker X_{k+1}$ for $k = 2, \cdots, K-1$ and finally, $M^K \mathcal{V} \subset Ker X_1 = \mathcal{V}$. So $\mathcal{V}$ is an invariant subspace of M^K and it must contain an eigenvector v corresponding to some eigenvalue ξ of M^K. Since the eigenvalues of M^K are the K-th powers of those of M, there exists a value $\lambda \in \sigma(M)$ such that $\xi = \lambda^K$. The assumption (9) then implies that $(M - \lambda I)v = 0$ and hence v is also an eigenvector of M. Since $\lambda \neq 0$ we found a vector v which is in the kernel of all matrices X_k and hence also of X. So we proved that $rank(X) = d$ implies $rank(X_1) = d$. The same proof also holds for all other matrices X_k since a circulant permutation of column and row blocks in (12) can bring any block X_k to the first position.

Observing (10) and using that E_k is nonsingular, it is obvious that Y_k is also of full column rank d. This completes the proof. □

Next, we want to link the system of equations (7) with one which only involves X_k rather than X_k and Y_k. For this, we first rewrite (7) in a slight more general form, which we will retrieve also later on:

$$\begin{cases} E_1 X_1 = Y_1 \bar{E}_1 \\ E_2 X_2 = Y_2 \bar{E}_2 \\ E_3 X_3 = Y_3 \bar{E}_3 \end{cases} ; \qquad \begin{cases} A_1 X_3 = Y_1 \bar{A}_1 \\ A_2 X_1 = Y_2 \bar{A}_2 \\ A_3 X_2 = Y_3 \bar{A}_3 \end{cases} , \quad (13)$$

or

$$\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \mathcal{X} = \mathcal{Y} \begin{bmatrix} \lambda \bar{E}_1 & & -\bar{A}_1 \\ -\bar{A}_2 & \lambda \bar{E}_2 & \\ & -\bar{A}_3 & \lambda \bar{E}_3 \end{bmatrix} (14)$$

where the regularity of $\lambda\mathcal{E} - \mathcal{A}$ implies the regularity of

$$\lambda\bar{\mathcal{E}} - \bar{\mathcal{A}} = \begin{bmatrix} \lambda \bar{E}_1 & & -\bar{A}_1 \\ -\bar{A}_2 & \lambda \bar{E}_2 & \\ & -\bar{A}_3 & \lambda \bar{E}_3 \end{bmatrix} ,$$

provided the X_k and Y_k matrices are of full column rank d. Therefore, we have for all k:

$$\text{rank} \underbrace{\left[\bar{E}_k \ \bar{A}_k\right]}_{2d} = d.$$

And hence there exists a full column rank right null space, which we partition as follows:

$$[\bar{E}_k \ \bar{A}_k] \begin{bmatrix} \tilde{A}_k \\ \\ -\tilde{E}_k \end{bmatrix} = 0. \qquad (15)$$

Notice that for invertible matrices $\bar{E}_k, \tilde{E}_k$ this is like " swapping " factors since (15) implies

$$\bar{E}_k^{-1} \bar{A}_k = \tilde{A}_k \tilde{E}_k^{-1}.$$

Multiplying the left equations of (13) by $\tilde{A}_k$ and the right ones by $\tilde{E}_k$ and equating corresponding terms yields finally:

$$\begin{cases} E_1 X_1 \tilde{A}_1 = A_1 X_3 \tilde{E}_1 \\ E_2 X_2 \tilde{A}_2 = A_2 X_1 \tilde{E}_2 \\ E_3 X_1 \tilde{A}_3 = A_3 X_2 \tilde{E}_3 \end{cases} , \qquad (16)$$

which does not involve Y_k anymore. To go from (16) to (13) again, we rewrite (16) as

$$[E_k X_k \quad A_k X_{k-1}] \begin{bmatrix} \tilde{A}_k \\ \\ -\tilde{E}_k \end{bmatrix} = 0,$$

which indicates that $[E_k X_k \quad A_k X_{k-1}]$ is in the left null space of $\begin{bmatrix} \tilde{A}_k \\ -\tilde{E}_k \end{bmatrix}$. But a basis for that is given by $\begin{bmatrix} \bar{E}_k & \bar{A}_k \end{bmatrix}$. So there exists a matrix Y_k such that

$$[E_k X_k \quad A_k X_{k-1}] = Y_k \begin{bmatrix} \bar{E}_k & \bar{A}_k \end{bmatrix}, \qquad (17)$$

which yields again (13). From (14) it follows then that the Y_k must be full rank or otherwise we would have

$$\dim(\mathcal{E}\mathcal{V} + \mathcal{A}\mathcal{V}) < \dim \mathcal{V},$$

for

$$\mathcal{V} = Im \begin{bmatrix} X_1 & & \\ & X_2 & \\ & & X_3 \end{bmatrix},$$

which would mean $\lambda\mathcal{E} - \mathcal{A}$ is singular. Finally we point out that (17) can be expressed geometrically as the condition

$$\dim(E_k \mathcal{V}_k + A_k \mathcal{V}_{k-1}) = d = \dim \mathcal{V}_k,$$

for $\mathcal{V}_k = Im\,(X_k)$, because $rank \begin{bmatrix} \bar{E}_k & \bar{A}_k \end{bmatrix} = d$.

To go from (13) to (7), we replace X_k with $\tilde{X}_k$ and Y_k with $\bar{Y}_k$ in (13):

$$\left\{ \begin{array}{l} E_1 \tilde{X}_1 = \bar{Y}_1 \bar{E}_1 \\ E_2 \tilde{X}_2 = \bar{Y}_2 \bar{E}_2 \\ E_3 \tilde{X}_3 = \bar{Y}_3 \bar{E}_3 \end{array} \right. ; \qquad \left\{ \begin{array}{l} A_1 \tilde{X}_3 = \bar{Y}_1 \bar{A}_1 \\ A_2 \tilde{X}_1 = \bar{Y}_2 \bar{A}_2 \\ A_3 \tilde{X}_2 = \bar{Y}_3 \bar{A}_3 \end{array} \right. . \quad (18)$$

Notice that the $\bar{E}_k$ and $\bar{A}_k$ matrices are nonsingular, since the $E_k, A_k, \tilde{X}_k$, and $\bar{Y}_k$ matrices are all full rank. Define

$$\tilde{M}_k = \bar{E}_k^{-1} \bar{A}_k, \quad \text{for} \quad k = 1, 2, 3.$$

And let M be a K-th root (K=3) of nonsingular matrix $\tilde{M}_3 \tilde{M}_2 \tilde{M}_1$. Then (18) induces (7) with

$$\left\{ \begin{array}{l} X_k = \tilde{X}_k \left(\tilde{M}_k \cdots \tilde{M}_1 M^{-k} \right) \\ Y_k = \bar{Y}_k \bar{E}_k \left(\tilde{M}_k \cdots \tilde{M}_1 M^{-k} \right), \text{ for } k = 1, 2, 3, \end{array} \right.$$

since then

$$\begin{aligned} E_k X_k &= E_k \tilde{X}_k \left(\tilde{M}_k \cdots \tilde{M}_1 M^{-k} \right) \\ &= \bar{Y}_k \bar{E}_k \left(\tilde{M}_k \cdots \tilde{M}_1 M^{-k} \right) \\ &= Y_k; \end{aligned}$$

and

$$\begin{aligned} A_k X_{k-1} &= A_k \tilde{X}_{k-1} \left(\tilde{M}_{k-1} \cdots \tilde{M}_1 M^{-(k-1)} \right) \\ &= \bar{Y}_k \bar{A}_k \left(\tilde{M}_{k-1} \cdots \tilde{M}_1 M^{-(k-1)} \right) \\ &= \bar{Y}_k \left(\bar{E}_k \tilde{M}_k \right) \left(\tilde{M}_{k-1} \cdots \tilde{M}_1 M^{-(k-1)} \right) \\ &= \bar{Y}_k \bar{E}_k \left(\tilde{M}_k \cdots \tilde{M}_1 M^{-k} \right) \cdot M \\ &= Y_k M, \end{aligned}$$

which yield (7). Moreover, in the extraction of M such that $M^3 = \tilde{M}_3 \tilde{M}_2 \tilde{M}_1$, we can take any complex number λ that satisfies $\lambda^3 = \xi$ (here ξ is an eigenvalue of $\tilde{M}_3 \tilde{M}_2 \tilde{M}_1$) as an eigenvalue of M (see (Gantmacher, 1959)). Thus, in particular, M can be chosen to satisfy the assumption of Theorem 1. That is, for each eigenvalue ξ of $\tilde{M}_3 \tilde{M}_2 \tilde{M}_1$, we extract a fixed number λ from the set of cubic roots of ξ as the only candidate for entering into the set of eigenvalues of M, even if there are multiple Jordan blocks corresponding to ξ.

With M satisfying the assumption of Theorem 1, all X_k and Y_k matrices in (7) are of full column rank d. Now we construct unitary matrices Q_k and Z_k such that

$$Z_k^* X_k = \hat{X}_k = \begin{bmatrix} R_k \\ 0 \end{bmatrix} \}d$$

and

$$Q_k^* Y_k = \hat{Y}_k = \begin{bmatrix} S_k \\ 0 \end{bmatrix} \}d,$$

where the R_k and S_k matrices are square invertible. Putting these transformations in block form :

$$\mathcal{Q} \doteq \begin{bmatrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_K \end{bmatrix}, \; \mathcal{Z} \doteq \begin{bmatrix} Z_1 & & & \\ & Z_2 & & \\ & & \ddots & \\ & & & Z_K \end{bmatrix},$$

we apply the block transformation to the cyclic pencil

$$\mathcal{Q}^* (\lambda\mathcal{E} - \mathcal{A}) \mathcal{Z} = \lambda\hat{\mathcal{E}} - \hat{\mathcal{A}},$$

and we find in the new coordinate system that

$$\begin{bmatrix} \lambda\hat{E}_1 & & -\hat{A}_1 \\ -\hat{A}_2 & \lambda\hat{E}_2 & \\ & -\hat{A}_3 & \lambda\hat{E}_3 \end{bmatrix} \hat{\mathcal{X}} = \hat{\mathcal{Y}} \begin{bmatrix} \lambda I & & -M \\ -M & \lambda I & \\ & -M & \lambda I \end{bmatrix},$$

with

$$\hat{\mathcal{X}} \doteq \begin{bmatrix} \hat{X}_1 & & \\ & \hat{X}_2 & \\ & & \hat{X}_3 \end{bmatrix}, \; \hat{\mathcal{Y}} \doteq \begin{bmatrix} \hat{Y}_1 & & \\ & \hat{Y}_2 & \\ & & \hat{Y}_3 \end{bmatrix}.$$

This indicates that in this coordinate system the $\hat{E}_k$ and $\hat{A}_k$ matrices are upper block triangular:

$$\hat{E}_k = \left[\begin{array}{c|c} S_k R_k^{-1} & * \\ \hline 0 & * \end{array} \right], \hat{A}_k = \left[\begin{array}{c|c} S_k M R_{k-1}^{-1} & * \\ \hline 0 & * \end{array} \right].$$

So (6) induces a block triangular periodic Schur decomposition, provided M satisfies the assumption of Theorem 1.

We thus closed the following set of equivalence relations for pencils with invertible matrices E_k, A_k.

R1 $\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} (\lambda I - M)$

with M satisfying the assumption of Theorem 1, X_k and Y_k matrices having full column rank d.

R2 $\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \tilde{\mathcal{X}} = \tilde{\mathcal{Y}} \begin{bmatrix} \lambda \bar{E}_1 & & -\bar{A}_1 \\ -\bar{A}_2 & \lambda \bar{E}_2 & \\ & -\bar{A}_3 & \lambda \bar{E}_3 \end{bmatrix}$,
where $M^3 = \bar{E}_3^{-1}\bar{A}_3\bar{E}_2^{-1}\bar{A}_2\bar{E}_1^{-1}\bar{A}_1$,
$\mathrm{Im}\left(\tilde{X}_k\right)$=$\mathrm{Im}(X_k)$, and $\mathrm{Im}(Y_k)$=$\mathrm{Im}\left(\bar{Y}_k\right)$,
for $k = 1, 2, 3$.

R3 $\begin{cases} E_1\tilde{X}_1\tilde{A}_1 = A_1\tilde{X}_3\tilde{E}_1 \\ E_2\tilde{X}_2\tilde{A}_2 = A_2\tilde{X}_1\tilde{E}_2 \\ E_3\tilde{X}_3\tilde{A}_3 = A_3\tilde{X}_2\tilde{E}_3 \end{cases}$,
where $\bar{E}_k\tilde{A}_k = \bar{A}_k\tilde{E}_k$, for $k = 1, 2, 3$.

R4 $\dim\left(E_k\mathcal{V}_k + A_k\mathcal{V}_{k-1}\right) = d = \dim \mathcal{V}_k$,
where $\mathcal{V}_k = Im\left(X_k\right) = Im\left(\tilde{X}_k\right)$,
for $k = 1, 2, 3$.

R5 There exist a block triangular periodic Schur decomposition with leading $d \times d$ blocks.

We can also show that relations (R3), (R4), and (R5) are still valid for E_k, A_k arbitrary for as long as $\lambda\mathcal{E} - \mathcal{A}$ is regular. In this case (R1) and (R2) only apply to the invariant subspaces with finite nonzero reduced spectrum.

Example 2 The pencil $(n = 1, K = 3)$

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is clearly regular with triple eigenvalue 0. It is already in triangular periodic Schur form, but (R1) with $M = 0$ $(d = 1)$ has

$$X_1 = X_2 = 0 \qquad \text{and} \qquad Y_1 = Y_2 = 0.$$

So (R1) is not equivalent to (R5) in the case, neither is (R2). $\square$

Finally, this also allows to give a new definition of eigenvalue/eigenvector pairs for periodic pencils. Proofs of the validity of these definition are given in the full paper.

Definition 1 Let $((A_k, E_k))_{k=1}^K$ be regular periodic $n \times n$ matrix pairs. If there exist complex numbers $\alpha_1, \cdots, \alpha_K$ and $\beta_1, \cdots, \beta_K$ such that $\left(\prod_{j=1}^K \alpha_j, \prod_{j=1}^K \beta_j\right) \neq (0, 0)$ and

$$\begin{bmatrix} \alpha_1 E_1 & & & -\beta_1 A_1 \\ -\beta_2 A_2 & \alpha_2 E_2 & & \\ & \ddots & \ddots & \\ & & -\beta_K A_K & \alpha_K E_K \end{bmatrix} \tag{19}$$

is singular, then we say $< \prod_{j=1}^K \alpha_j, \prod_{j=1}^K \beta_j >$ is an eigenvalue of $(A_k, E_k)_{k=1}^K$. The set of eigenvalues of $(A_k, E_k)_{k=1}^K$ is denoted as $\sigma\left(A_k, E_k\right)_{k=1}^K$. $\square$

Definition 2 Let $(A_k, E_k)_{k=1}^K$ be regular periodic $n \times n$ matrix pairs. If there exist complex numbers $\alpha_1, \cdots, \alpha_K, \beta_1, \cdots, \beta_K$, and *nonzero* vectors $x_1, \cdots, x_K$ such that

$$\beta_k A_k x_{k-1} = \alpha_k E_k x_k, \text{ for } k = 1, 2, \cdots, K \tag{20}$$

with $(\prod_{j=1}^K \alpha_j, \prod_{j=1}^K \beta_j) \neq (0, 0)$, we say that $(x_k)_{k=1}^K$ is an eigenvector sequence of $((A_k, E_k))_{k=1}^K$ with eigenvalue $< \prod_{j=1}^K \alpha_j, \prod_{j=1}^K \beta_j >$. $\square$

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