Equivalent characterizations of periodical deflating subspaces

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Abstract

In this paper we present several different characterizations of invariant subspaces of periodic eigenvalue problems. We analyze their equivalence and show that in the case of zero and infnite eigenvalues, they are not all equivalent

1 Introduction

Consider the (homogenous) linear time varying system :

$$E_k x_{k+1} = A_k x_k, \qquad k \in \mathbf{N} \tag{1.1}$$

where $\mathbf{N}$ is the set of natural numbers, x_k is an n-dimensional vector of descriptor variables, and the matrices E_k and A_k are $n \times n$. This system is said to be periodic with period K if $E_k = E_{k+K}$ and $A_k = A_{k+K}$, for all $k \in \mathbf{N}$, and K is the smallest positive integer for which this holds. If we allow both the E_k and A_k matrices to be singular, then x_k may still uniquely be defined in the context of a two point boundary value problem. It is shown in [4] that a necessary and sufficient condition for this is that the pencil

$$\lambda \mathcal{E} - \mathcal{A} := \begin{bmatrix} \lambda E_1 & & & -A_1 \\ -A_2 & \lambda E_2 & & \\ & \ddots & \ddots & \\ & & -A_K & \lambda E_K \end{bmatrix} \tag{1.2}$$

is regular (i.e. $\det(\lambda \mathcal{E} - \mathcal{A}) \not\equiv 0$). We will then also call the periodic system *regular*.

In this paper, we describe several characterizations of invariant subspaces of a regular periodic system. Let us assume $K = 3$ for the sake of simplicity. A first characterization uses the classical concept of deflating subspace, directly applied to the pencil (1.2) :

$$\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} (\lambda I - M), \tag{1.3}$$

where $X \doteq \begin{bmatrix} X_1^T & X_2^T & X_3^T \end{bmatrix}^T$ and $Y \doteq \begin{bmatrix} Y_1^T & Y_2^T & Y_3 \end{bmatrix}^T$ are assumed to have full rank d. We can also rewrite this as follows :

$$\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \begin{bmatrix} X_1 & & \\ & X_2 & \\ & & X_3 \end{bmatrix} = \begin{bmatrix} Y_1 & & \\ & Y_2 & \\ & & Y_3 \end{bmatrix} \begin{bmatrix} \lambda I & & -M \\ -M & \lambda I & \\ & -M & \lambda I \end{bmatrix} \tag{1.4}$$

so that every d-dimensional eigenspace of the type (1.3) induces a $d{\cdot}K$-dimensional eigenspace of the type (1.4), *provided* all individual matrices X_k and Y_k have full column rank. Conditions on M for which this holds will be analyzed in the paper.

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A slightly more general form involving again full rank matrices X_k and Y_k is the following :

$$\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \begin{bmatrix} X_1 & & \\ & X_2 & \\ & & X_3 \end{bmatrix} = \begin{bmatrix} Y_1 & & \\ & Y_2 & \\ & & Y_3 \end{bmatrix} \begin{bmatrix} \lambda \bar{E}_1 & & -\bar{A}_1 \\ -\bar{A}_2 & \lambda \bar{E}_2 & \\ & -\bar{A}_3 & \lambda \bar{E}_3 \end{bmatrix} \quad (1.5)$$

or, equivalently :

$$\begin{cases} E_1X_1 = Y_1\bar{E}_1 \\ E_2X_2 = Y_2\bar{E}_2 \\ E_3X_3 = Y_3\bar{E}_3 \end{cases}, \quad \text{and} \quad \begin{cases} A_1X_3 = Y_1\bar{A}_1 \\ A_2X_1 = Y_2\bar{A}_2 \\ A_3X_2 = Y_3\bar{A}_3 \end{cases}. \quad (1.6)$$

This last form (1.5) is very close to the periodic Schur form [1], which says that there always exist unitary matrices Q_k and Z_k, $k = 1, \ldots, K$ such that :

$$\begin{bmatrix} Z_1^* & & \\ & Z_2^* & \\ & & Z_3^* \end{bmatrix} \begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \begin{bmatrix} Q_1 & & \\ & Q_2 & \\ & & Q_3 \end{bmatrix} = \begin{bmatrix} \lambda \bar{E}_1 & & -\bar{A}_1 \\ -\bar{A}_2 & \lambda \bar{E}_2 & \\ & -\bar{A}_3 & \lambda \bar{E}_3 \end{bmatrix} \quad (1.7)$$

where the matrices $\bar{E}_k$ and $\bar{A}_k$ are upper-triangular. Notice that this can be viewed as a *periodic similarity* transformation.

Using the regularity of (1.2) we will show that there exist "swapping" factors $\tilde{A}_k$ and $\tilde{E}_k$ such that

$$\bar{E}_k\tilde{A}_k = \bar{A}_k\tilde{E}_k,$$

which allow us to rewrite (1.5) in a form which does not involve Y_k anymore :

$$\begin{cases} E_1X_1\tilde{A}_1 = A_1X_3\tilde{E}_1 \\ E_2X_2\tilde{A}_2 = A_2X_1\tilde{E}_2 \\ E_3X_3\tilde{A}_3 = A_3X_2\tilde{E}_3 \end{cases}. \quad (1.8)$$

Finally, we point out that (1.6) can be expressed geometrically as the conditions

$$\begin{cases} \dim(E_1\mathcal{X}_1 + A_1\mathcal{X}_3) = d = \dim \mathcal{X}_1 \\ \dim(E_2\mathcal{X}_2 + A_2\mathcal{X}_1) = d = \dim \mathcal{X}_2 \\ \dim(E_3\mathcal{X}_3 + A_3\mathcal{X}_2) = d = \dim \mathcal{X}_3, \end{cases} \quad (1.9)$$

where $\mathcal{X}_k$ is the column space of X_k. We thus obtain equivalences between deflating subspace defined by (1.3), the general form (1.5), the algebraic form (1.8), the geometric form (1.9), and the triangular periodic Schur decomposition (1.7).

The main goal of the paper is to show that (1.8), (1.9), and the block triangular periodic Schur decomposition (1.7) are all equivalent for arbitrary E_k, A_k, as long as (1.2) is a regular pencil. On the other hand, (1.3) and (1.5) only apply to the invariant subspaces with finite nonzero reduced spectrum. For this reason, we propose an algorithm in section 3 that *deflates* the 0 and ∞ eigenvalues, yielding a smaller dimensional periodic pencil without these "troublesome" eigenvalues. We then introduce the concept of regular periodic matrix pairs $((A_k, E_k))_{k=1}^K$ and define their "eigenvalues" and "periodic eigenvectors", in contrast to those of the pencil (1.2). Finally, we give a new "contructive" proof of the periodic Schur theorem.

2 Deflating Subspaces for Finite Nonzero Case

In this section we look at equivalent characterizations of periodic deflating subspaces in the special case that the matrices E_k and A_k are all invertible. That this assumption simplifies the problem is reflected in the following Lemma, proven in [7, 5].

Lemma 2.1 *Let the $n \times n$ matrices E_k and A_k, $k = 1, \ldots, K$ be invertible. Then there always exist a periodic similarity transformation*

$$\begin{bmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_K \end{bmatrix} \begin{bmatrix} \lambda E_1 & & & -A_1 \\ -A_2 & \lambda E_2 & & \\ & \ddots & \ddots & \\ & & -A_K & \lambda E_K \end{bmatrix} \begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda I & & & -M \\ -M & \lambda I & & \\ & \ddots & \ddots & \\ & & -M & \lambda I \end{bmatrix}, \tag{2.10}$$

called the "Floquet transformation", that transforms the system to a time-invariant one.

We first try to show the equivalence between deflating subspaces of the classical type (1.3) and the expanded type of the form (1.4), i.e. we analyze when a d-dimensional eigenspace of the type (1.3) induces a $d{\cdot}K$-dimensional eigenspace of the type (1.4). The following example shows that one must impose certain conditions, since the equivalence does not hold in general.

Example 2.1 *Consider $K = 3$ and $A_k = E_k = I_2$. And let $\mathbf{1} = (1,1)^T, M = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$ with $\omega \neq 1$ and $\omega^3 = 1$. Then we have*

$$\begin{bmatrix} \lambda I_2 & 0 & -I_2 \\ -I_2 & \lambda I_2 & 0 \\ 0 & -I_2 & \lambda I_2 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \omega^2\mathbf{1} \\ \mathbf{1} & \omega^1\mathbf{1} \\ \mathbf{1} & \omega^0\mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \omega^2\mathbf{1} \\ \mathbf{1} & \omega^1\mathbf{1} \\ \mathbf{1} & \omega^0\mathbf{1} \end{bmatrix} \left(\lambda I_2 - \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix} \right).$$

So (1.3) and (1.4) hold with

$$\begin{cases} X_1 = Y_1 = [\mathbf{1}, \omega^2\mathbf{1}] \\ X_2 = Y_2 = [\mathbf{1}, \omega^1\mathbf{1}] \\ X_3 = Y_3 = [\mathbf{1}, \omega^0\mathbf{1}]. \end{cases}$$

Although both the matrices

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

are of full column rank 2, all X_k and Y_k matrices are only of rank 1.

This example illustrates well that M cannot be arbitrary. The following theorem characterizes which matrix M will yield full column rank matrices X_k and Y_k.

Theorem 2.2 *Let A_k and E_k be nonsingular $n \times n$ matrices, for $k = 1, 2, \cdots, K$. Assume that $X = \left[X_1^T, X_2^T, \cdots, X_K^T\right]^T$ is a full column rank matrix that generates a d-dimensional deflating subspace of $\lambda\mathcal{E} - \mathcal{A}$:*

$$\begin{bmatrix} \lambda E_1 & & & -A_1 \\ -A_2 & \lambda E_2 & & \\ & \ddots & \ddots & \\ & & -A_K & \lambda E_K \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_K \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_K \end{bmatrix} (\lambda I - M). \tag{2.1}$$

If M has the property that $\lambda^K \neq \mu^K$ whenever λ and μ are two distinct eigenvalues of M, i.e.,

$$Ker(M - \lambda I) = Ker\left(M^K - \lambda^K I\right), \qquad \text{for all} \quad \lambda \in \sigma(M) \tag{2.2}$$

(here $\sigma(M)$ denotes the spectrum of M), then the X_k and Y_k matrices also have full column rank d, for $k = 1, 2, \cdots, K$.

Proof. Expanding (2.1), yields

$$E_k X_k = Y_k \quad \text{and} \quad A_k X_{k-1} = Y_k M, \tag{2.3}$$

for $k = 1, 2, \cdots, K$. Thus we have

$$A_k X_{k-1} = E_k X_k M, \qquad \text{for all} \quad k. \tag{2.4}$$

Since the E_k matrices are nonsingular, we can define $S_k = E_k^{-1}A_k$. Then (2.4) is equivalent to

$$\begin{bmatrix} 0 & \cdots & 0 & S_1 \\ S_2 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & S_k & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_K \end{bmatrix} M$$

or also

$$\begin{bmatrix} 0 & \cdots & 0 & S_1 \\ S_2 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & S_k & 0 \end{bmatrix} \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0s \\ 0 & \cdots & 0 & X_k \end{bmatrix} = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0s \\ 0 & \cdots & 0 & X_k \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & M \\ M & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & M & 0 \end{bmatrix}. \quad (2.5)$$

Note that M is nonsingular, since the A_k matrices are also nonsingular.

We now show by contradiction that the matrices X_k are full rank if (2.2) holds. Assume e.g. that X_1 does *not* have full column rank and that the columns of the matrix V span $\mathcal{V} := KerX_1$. Then it follows from (2.5) that $X_2MV = S_2X_1V = 0$ and hence $M\mathcal{V} \subset KerX_2$. By similar arguments one shows that $M^k\mathcal{V} \subset KerX_{k+1}$ for $k = 2, \cdots, K-1$ and finally, $M^K\mathcal{V} \subset KerX_1 = \mathcal{V}$. So $\mathcal{V}$ is an invariant subspace of M^K and it must contain an eigenvector v corresponding to some eigenvalue ξ of M^K. Since the eigenvalues of M^K are the K-th powers of those of M, there exists a value $\lambda \in \sigma(M)$ such that $\xi = \lambda^K$. The assumption (2.2) then implies that $(M - \lambda I)v = 0$ and hence v is also an eigenvector of M. Since $\lambda \neq 0$ we found a vector v which is in the kernel of all matrices X_k and hence also of X. So we proved that $rank(X) = d$ implies $rank(X_1) = d$. The same proof also holds for all other matrices X_k since a circulant permutation of column and row blocks in (2.5) can bring any block X_k to the first position.

Observing (2.3) and using that E_k is nonsingular, it is obvious that Y_k is also of full column rank d. This completes the proof. □

Next, we want to link the system of equations (1.4) with one which only involves X_k rather than X_k and Y_k. For this, we first rewrite (1.4) in a slight more general form, which we will retrieve also later on:

$$\begin{cases} E_1X_1 = Y_1\bar{E}_1 \\ E_2X_2 = Y_2\bar{E}_2 \\ E_3X_3 = Y_3\bar{E}_3 \end{cases} ; \qquad \begin{cases} A_1X_3 = Y_1\bar{A}_1 \\ A_2X_1 = Y_2\bar{A}_2 \\ A_3X_2 = Y_3\bar{A}_3 \end{cases}, \quad (2.6)$$

or

$$\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \underbrace{\begin{bmatrix} X_1 & & \\ & X_2 & \\ & & X_3 \end{bmatrix}}_{3d} = \underbrace{\begin{bmatrix} Y_1 & & \\ & Y_2 & \\ & & Y_3 \end{bmatrix}}_{3d} \begin{bmatrix} \lambda \bar{E}_1 & & -\bar{A}_1 \\ -\bar{A}_2 & \lambda \bar{E}_2 & \\ & -\bar{A}_3 & \lambda \bar{E}_3 \end{bmatrix}, \quad (2.7)$$

where the regularity of $\lambda\mathcal{E} - \mathcal{A}$ implies the regularity of

$$\lambda\bar{\mathcal{E}} - \bar{\mathcal{A}} = \begin{bmatrix} \lambda \bar{E}_1 & & -\bar{A}_1 \\ -\bar{A}_2 & \lambda \bar{E}_2 & \\ & -\bar{A}_3 & \lambda \bar{E}_3 \end{bmatrix},$$

provided the X_k and Y_k matrices are of full column rank d. Therefore, we have for all k:

$$\text{rank} \underbrace{\begin{bmatrix} \bar{E}_k & \bar{A}_k \end{bmatrix}}_{2d} = d.$$

And hence there exists a full column rank right null space, which we partition as follows:

$$\begin{bmatrix} \bar{E}_k & \bar{A}_k \end{bmatrix} \begin{bmatrix} \tilde{A}_k \\ -\tilde{E}_k \end{bmatrix} = 0. \quad (2.8)$$

Notice that for invertible matrices $\bar{E}_k, \tilde{E}_k$ this is like " swapping " factors since (2.8) implies

$$\bar{E}_k^{-1}\bar{A}_k = \tilde{A}_k\tilde{E}_k^{-1}.$$

Multiplying the left equations of (2.6) by $\tilde{A}_k$ and the right ones by $\tilde{E}_k$ and equating corresponding terms yields finally:

$$\left\{\begin{array}{l} E_1X_1\tilde{A}_1 = A_1X_3\tilde{E}_1 \\ E_2X_2\tilde{A}_2 = A_2X_1\tilde{E}_2 \\ E_3X_1\tilde{A}_3 = A_3X_2\tilde{E}_3 \end{array}\right. , \qquad (2.9)$$

which does not involve Y_k anymore. To go from (2.9) to (2.6) again, we rewrite (2.9) as

$$[E_kX_k \quad A_kX_{k-1}] \begin{bmatrix} \tilde{A}_k \\ \\ -\tilde{E}_k \end{bmatrix} = 0,$$

which indicates that $[E_kX_k \quad A_kX_{k-1}]$ is in the left null space of $\begin{bmatrix} \tilde{A}_k \\ -\tilde{E}_k \end{bmatrix}$. But a basis for that is given by $\begin{bmatrix}\bar{E}_k & \bar{A}_k\end{bmatrix}$. So there exists a matrix Y_k such that

$$[E_kX_k \quad A_kX_{k-1}] = Y_k \begin{bmatrix}\bar{E}_k & \bar{A}_k\end{bmatrix}, \qquad (2.10)$$

which yields again (2.6). From (2.7) it follows then that the Y_k must be full rank or otherwise we would have

$$\dim(\mathcal{E}\mathcal{X} + \mathcal{A}\mathcal{X}) < \dim \mathcal{X},$$

for

$$\mathcal{X} = Im \begin{bmatrix} X_1 & & \\ & X_2 & \\ & & X_3 \end{bmatrix},$$

which would mean $\lambda\mathcal{E} - \mathcal{A}$ is singular. Finally we point out that (2.10) can be expressed geometrically as the condition

$$\dim(E_k\mathcal{X}_k + A_k\mathcal{X}_{k-1}) = d = \dim \mathcal{X}_k,$$

for $\mathcal{X}_k = Im\,(X_k)$, because $rank \begin{bmatrix}\bar{E}_k & \bar{A}_k\end{bmatrix} = d$.

To go from (2.6) to (1.4), we replace X_k with $\tilde{X}_k$ and Y_k with $\bar{Y}_k$ in (2.6):

$$\left\{\begin{array}{l} E_1\tilde{X}_1 = \bar{Y}_1\bar{E}_1 \\ E_2\tilde{X}_2 = \bar{Y}_2\bar{E}_2 \\ E_3\tilde{X}_3 = \bar{Y}_3\bar{E}_3 \end{array}\right. ; \qquad \left\{\begin{array}{l} A_1\tilde{X}_3 = \bar{Y}_1\bar{A}_1 \\ A_2\tilde{X}_1 = \bar{Y}_2\bar{A}_2 \\ A_3\tilde{X}_2 = \bar{Y}_3\bar{A}_3 \end{array}\right. . \qquad (2.11)$$

Notice that the $\bar{E}_k$ and $\bar{A}_k$ matrices are nonsingular, since the $E_k, A_k, \tilde{X}_k$, and $\bar{Y}_k$ matrices are all full rank. Define

$$\tilde{M}_k = \bar{E}_k^{-1}\bar{A}_k, \quad \text{for} \quad k = 1, 2, 3.$$

And let M be a K-th root (K=3) of nonsingular matrix $\tilde{M}_3\tilde{M}_2\tilde{M}_1$. Then (2.11) induces (1.4) with

$$\left\{\begin{array}{l} X_k = \tilde{X}_k\left(\tilde{M}_k\cdots\tilde{M}_1M^{-k}\right) \\ Y_k = \bar{Y}_k\bar{E}_k\left(\tilde{M}_k\cdots\tilde{M}_1M^{-k}\right), \quad \text{for} \quad k = 1, 2, 3, \end{array}\right.$$

since then

$$\begin{aligned} E_kX_k &= E_k\tilde{X}_k\left(\tilde{M}_k\cdots\tilde{M}_1M^{-k}\right) \\ &= \bar{Y}_k\bar{E}_k\left(\tilde{M}_k\cdots\tilde{M}_1M^{-k}\right) \\ &= Y_k; \end{aligned}$$

and

$$\begin{aligned}
A_k X_{k-1} &= A_k \tilde{X}_{k-1} \left(\tilde{M}_{k-1} \cdots \tilde{M}_1 M^{-(k-1)} \right) \\
&= \bar{Y}_k \bar{A}_k \left(\tilde{M}_{k-1} \cdots \tilde{M}_1 M^{-(k-1)} \right) \\
&= \bar{Y}_k \left(\bar{E}_k \tilde{M}_k \right) \left(\tilde{M}_{k-1} \cdots \tilde{M}_1 M^{-(k-1)} \right) \\
&= \bar{Y}_k \bar{E}_k \left(\tilde{M}_k \cdots \tilde{M}_1 M^{-k} \right) \cdot M \\
&= Y_k M,
\end{aligned}$$

which yield (1.4). Moreover, in the extraction of M such that $M^3 = \tilde{M}_3 \tilde{M}_2 \tilde{M}_1$, we can take any complex number λ that satisfies $\lambda^3 = \xi$ (here ξ is an eigenvalue of $\tilde{M}_3 \tilde{M}_2 \tilde{M}_1$) as an eigenvalue of M (see [3]). Thus, in particular, M can be chosen to satisfy the assumption of Theorem 2.2. That is, for each eigenvalue ξ of $\tilde{M}_3 \tilde{M}_2 \tilde{M}_1$, we extract a fixed number λ from the set of cubic roots of ξ as the only candidate for entering into the set of eigenvalues of M, even if there are multiple Jordan blocks corresponding to ξ.

With M satisfying the assumption of Theorem 2.2, all X_k and Y_k matrices in (1.4) are of full column rank d. Now we construct unitary matrices Q_k and Z_k such that

$$Z_k^* X_k = \hat{X}_k = \begin{bmatrix} R_k \\ 0 \end{bmatrix} \}d$$

and

$$Q_k^* Y_k = \hat{Y}_k = \begin{bmatrix} S_k \\ 0 \end{bmatrix} \}d,$$

where the R_k and S_k matrices are square invertible. If we apply the block transformation to the cyclic pencil

$$\operatorname{diag}(Q_1^*, \cdots, Q_K^*) (\lambda \mathcal{E} - \mathcal{A}) \operatorname{diag}(Z_1, \cdots, Z_K) = \lambda \hat{\mathcal{E}} - \hat{\mathcal{A}},$$

then we find in the new coordinate system that

$$\begin{bmatrix} \lambda \hat{E}_1 & & -\hat{A}_1 \\ -\hat{A}_2 & \lambda \hat{E}_2 & \\ & -\hat{A}_3 & \lambda \hat{E}_3 \end{bmatrix} \begin{bmatrix} \hat{X}_1 & & \\ & \hat{X}_2 & \\ & & \hat{X}_3 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1 & & \\ & \hat{Y}_2 & \\ & & \hat{Y}_3 \end{bmatrix} \begin{bmatrix} \lambda I & & -M \\ -M & \lambda I & \\ & -M & \lambda I \end{bmatrix},$$

which indicates that in this coordinate system the $\hat{E}_k$ and $\hat{A}_k$ matrices are upper block triangular:

$$\hat{E}_k = \left[\begin{array}{c|c} S_k R_k^{-1} & * \\ \hline 0 & * \end{array}\right]; \qquad \hat{A}_k = \left[\begin{array}{c|c} S_k M R_{k-1}^{-1} & * \\ \hline 0 & * \end{array}\right]. \tag{2.12}$$

So (1.3) induces a block triangular periodic Schur decomposition (2.12), provided M satisfies the assumption of Theorem 2.2.

We thus closed the following set of equivalence relations for pencils with invertible matrices E_k, A_k.

(R1) $\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} (\lambda I - M)$

with M satisfying the assumption of Theorem 2.2, X_k and Y_k matrices having full column rank d.

(R2) $\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \begin{bmatrix} \tilde{X}_1 & & \\ & \tilde{X}_2 & \\ & & \tilde{X}_3 \end{bmatrix} = \begin{bmatrix} \bar{Y}_1 & & \\ & \bar{Y}_2 & \\ & & \bar{Y}_3 \end{bmatrix} \begin{bmatrix} \lambda \bar{E}_1 & & -\bar{A}_1 \\ -\bar{A}_2 & \lambda \bar{E}_2 & \\ & -\bar{A}_3 & \lambda \bar{E}_3 \end{bmatrix}$, where

$M^3 = \bar{E}_3^{-1} \bar{A}_3 \bar{E}_2^{-1} \bar{A}_2 \bar{E}_1^{-1} \bar{A}_1$, $\operatorname{Im}\left(\tilde{X}_k\right) = \operatorname{Im}(X_k)$, and $\operatorname{Im}(Y_k) = \operatorname{Im}\left(\bar{Y}_k\right)$, for $k = 1, 2, 3$.

(R3) $\begin{cases} E_1 \tilde{X}_1 \tilde{A}_1 = A_1 \tilde{X}_3 \tilde{E}_1 \\ E_2 \tilde{X}_2 \tilde{A}_2 = A_2 \tilde{X}_1 \tilde{E}_2 \\ E_3 \tilde{X}_3 \tilde{A}_3 = A_3 \tilde{X}_2 \tilde{E}_3 \end{cases}$,

where $\bar{E}_k \tilde{A}_k = \bar{A}_k \tilde{E}_k$, for $k = 1, 2, 3$.

(R4) $\dim(E_k\mathcal{X}_k + A_k\mathcal{X}_{k-1}) = d = \dim \mathcal{X}_k$,
where $\mathcal{X}_k = Im\,(X_k) = Im\left(\tilde{X}_k\right)$, for $k = 1, 2, 3$.

(R5) There exist a block triangular periodic Schur decomposition with leading $d \times d$ blocks.

In the rest of the paper, we show that relations (R3), (R4), and (R5) are still valid for E_k, A_k arbitrary for as long as $\lambda\mathcal{E} - \mathcal{A}$ is regular. In this case (R1) and (R2) only apply to the invariant subspaces with finite nonzero reduced spectrum.

Example 2.2 *The pencil ($n = 1, K = 3$)*

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is clearly regular with triple eigenvalue 0. *It is already in triangular periodic Schur form, but (R1) with $M = 0$ ($d = 1$) has*

$$X_1 = X_2 = 0 \qquad \textit{and} \qquad Y_1 = Y_2 = 0.$$

So (R1) is not equivalent to (R5) in the case, neither is (R2).

The next section describes simple, finite algorithms to deflate out 0 and ∞ eigenvalues. So that the results of this section are applicable to the smaller, deflated system.

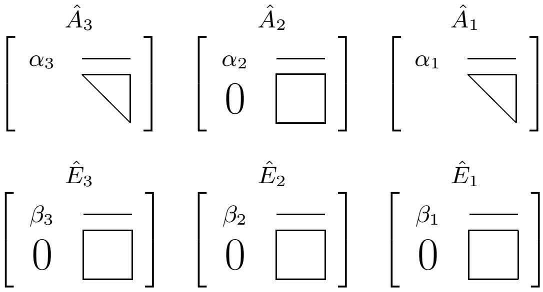
3 Deflation of 0 and ∞ Eigenvalues

In order to develop a new approach to the periodic Schur Theorem and establish its equivalence with " periodically invariant subspaces ", we first propose two simple, finite algorithms to deflate the zero and infinite eigenvalues of (1.2). The first one deflates a zero or infinite eigenvalue. The second is then based on the first and deflates the A_k and E_k matrices into nonsingular ones.

3.1 Deflation of a Zero or Infinite Eigenvalue

We now propose a finite algorithm for the deflation of a zero eigenvalue of the pencil (1.2). We explain the procedure for the case $K = 3$, for simplicity.

First of all, if $\lambda\mathcal{E} - \mathcal{A}$ has a zero eigenvalue, then clearly at least one of the A_k matrices is singular. For illustration we assume A_2 is singular. The idea of the proposed algorithm is to reduce A_1, A_3 to upper triangular form and A_2 as well as E_1, E_2, E_3 to block upper triangular form with leading 1×1 blocks:

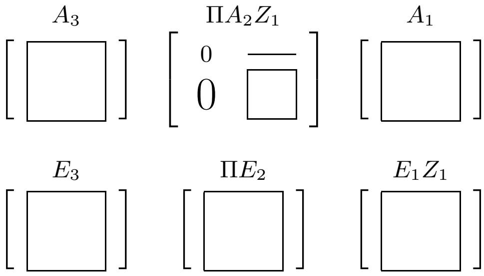


(3.1)

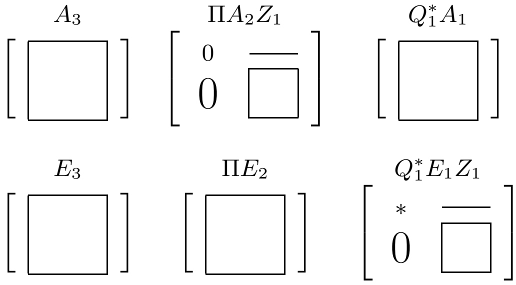
with $\alpha_2 = 0$, via the diagonal block transformation to the cyclic pencil

$$\operatorname{diag}(Q_1^*, \cdots, Q_K^*)\,(\lambda\mathcal{E} - \mathcal{A})\,\operatorname{diag}(Z_1, \cdots, Z_K) = \lambda\hat{\mathcal{E}} - \hat{\mathcal{A}}. \tag{3.2}$$

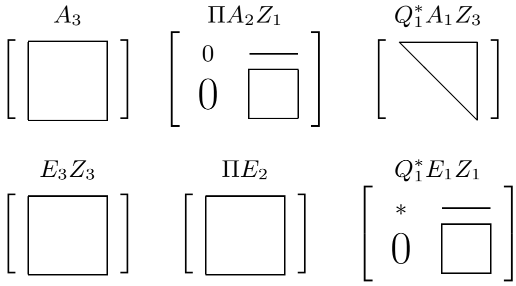
In the first step, we construct a column compression of A_2 to annihilate the first column of A_2 (e.g., from its RQ-decomposition with row pivoting). The transformation Z_1 is also applied to E_1 and the permutation Π is also applied to E_2. We then have:



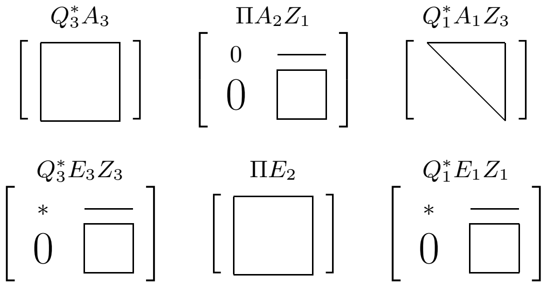
In the second step, we then apply a unitary matrix Q_1^* to the rows of $E_1 Z_1$ to put it in block upper triangular form with leading 1×1 block and apply it to the rows of A_1:



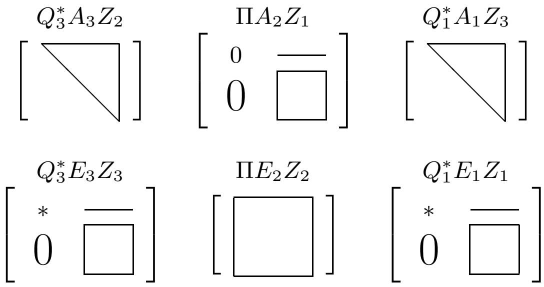
In the third step, we triangularize $Q_1^* A_1$ by a unitary transformation Z_3 and also apply Z_3 to the columns of E_3:



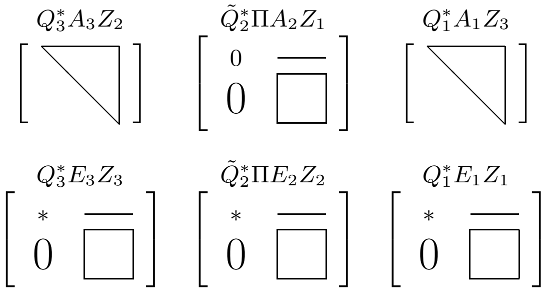
In the fourth step, we apply a unitary matrix Q_3^* to the rows of $E_3 Z_3$ to put it in block upper triangular form with leading 1×1 block and also apply it to the rows of A_3:



In the fifth step, we triangularize $Q_3^* A_3$ by a unitary transformation Z_2 and also apply Z_2 to the columns of ΠE_2:

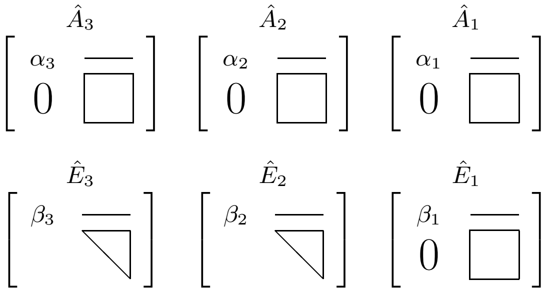


Finally, we apply a unitary matrix $\tilde{Q}_2^*$ to the rows of $\Pi E_2 Z_2$ to put it in block upper triangular form with leading 1×1 block and also apply it to the rows of $\Pi A_2 Z_1$:



Notice the first column of $\tilde{Q}_2^*\Pi A_2 Z_1$ is zero, because the first column of $\Pi A_2 Z_1$ is zero. Thus we obtain (3.1) and (3.2) with $Q_2^* = \tilde{Q}_2^*\Pi$. Since $\alpha_2 = 0$, this completes the deflation of a zero eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$.

If $\lambda\mathcal{E} - \mathcal{A}$ has an infinite eigenvalue, then at least one of the E_k matrices is singular, say E_1 is singular. With a little modification of the above procedure, we then can reduce $\lambda\mathcal{E} - \mathcal{A}$ to the form



(3.3)

with $\beta_1 = 0$. Since $\beta_1 = 0$, this deflates an infinite eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$.

The above mentioned algorithm is finite and simple. It is clear enough to explain the procedure for $K = 3$. So we omit the general description of the algorithm.

3.2 Deflation of A_k and E_k into Nonsingular Ones

This section describes a finite algorithm for the deflation of the A_k and E_k matrices into nonsingular ones.

First of all, assume that the pencil $\lambda\mathcal{E} - \mathcal{A}$ has a finite nonzero eigenvalue. We shall construct a decomposition of the block lower triangular form:

$$\hat{A}_k = Q_k^* A_k Z_{k-1} = \begin{bmatrix} \alpha_k^{(1)} & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \alpha_k^{(m)} & 0 \\ * & \cdots & * & \Sigma_k \end{bmatrix} \tag{3.4}$$

and

$$\hat{E}_k = Q_k^* E_k Z_k = \begin{bmatrix} \beta_k^{(1)} & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \beta_k^{(m)} & 0 \\ * & \cdots & * & T_k \end{bmatrix}, \tag{3.5}$$

for which the Σ_k and T_k matrices are nonsingular and for each fixed $j \in \{1, 2, \cdots, m\}$, at least one of the scalars $\alpha_1^{(j)}, \cdots, \alpha_K^{(j)}, \beta_1^{(j)}, \cdots,$
$\beta_K^{(j)}$ is zero. The proposed algorithm is not an economical one yet. But it is easy to understand and describe. Since the main goal of the paper is not to develop numerical algorithms, we propose here the easy-to-write one only for convenience and the completeness of this work. As before, we omit the general description of the algorithm.

According to (3.1) and (3.3), we can reduce the A_k and E_k matrices to block upper triangular with leading 1×1 blocks, provided $\lambda\mathcal{E} - \mathcal{A}$ has a zero or infinite eigenvalue. Moreover, at least one is 0 among the leading 1×1 blocks (e.g., (3.1) has $\alpha_2 = 0$ and (3.3) has $\beta_1 = 0$). Thus, analogously, we can also reduce A_k and E_k matrices to block lower triangular with leading 1×1 blocks:

$$Q_k^* A_k Z_{k-1} = \begin{bmatrix} \alpha_k & 0 \\ * & \tilde{A}_k \end{bmatrix}; \quad Q_k^* E_k Z_k = \begin{bmatrix} \beta_k & 0 \\ * & \tilde{E}_k \end{bmatrix}, \tag{3.6}$$

for which at least one of the scalars α_k and β_k is zero. This completes one step of the reduction for (3.4) and (3.5). From here on, it is obvious how to proceed further. The reduction then continues inductively. The algorithm preserves all A_k and E_k matrices in block lower triangular form and constructs decompositions of the type (3.6) for the reduced (2,2)-blocks $\tilde{A}_k$'s and $\tilde{E}_k$'s. As soon as the reduced (2,2)-blocks $\tilde{A}_k$'s and $\tilde{E}_k$'s are nonsingular, we stop the reduction.

We remark here the reduction does not require $\lambda\mathcal{E} - \mathcal{A}$ to be a regular pencil, neither does the reduction presented in Section 3.1. However, we merely consider the regular case in the remainder of the paper.

4 Periodic Schur Decomposition

From (3.1) and (3.2), we see there exist unitary Q_k and Z_k matrices that reduce $\lambda\mathcal{E} - \mathcal{A}$ to the block upper triangular form with leading 1×1 block:

$$Q_k^* A_k Z_{k-1} = \begin{bmatrix} \alpha_k & * \\ 0 & * \end{bmatrix}; \qquad Q_k^* E_k Z_k = \begin{bmatrix} \beta_k & * \\ 0 & * \end{bmatrix}, \tag{4.1}$$

provided $\lambda\mathcal{E} - \mathcal{A}$ has a zero eigenvalue. Furthermore, in (4.1) we have $\prod_{j=1}^{K} \alpha_j = 0$. If $\lambda\mathcal{E} - \mathcal{A}$ is regular, this then implies $\prod_{j=1}^{K} \beta_j \neq 0$. So that if we define for complex numbers α and β with $(\alpha, \beta) \neq (0, 0)$,

$$< \alpha, \beta > := \{\tau(\alpha, \beta) |\ \tau \text{ is a nonzero complex number.}\}, \tag{4.2}$$

then

$$< \prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j > = < 0, 1 > . \tag{4.3}$$

Note that (4.2) induces an equivalence relation on $\mathbf{C}^2 \backslash \{(0, 0)\}$, where $\mathbf{C}$ denotes the field of complex numbers. Therefore, when $\beta \neq 0$ we may regard any two elements (representations) in the set $< \alpha, \beta >$ as the *same* finite complex number $\frac{\alpha}{\beta}$. While $\beta = 0$ and $\alpha \neq 0$, we may think of $< \alpha, \beta > = < 1, 0 >$ as ∞. With this convention, the $0, \infty$, and finite nonzero eigenvalue $\frac{\alpha}{\beta}$ of $\lambda\mathcal{E} - \mathcal{A}$ can be rewritten as $< 0, 1 >, < 1, 0 >$, and $< \alpha, \beta >$, respectively.

Let $x_k(\neq 0)$ and $y_k(\neq 0)$ be, respectively, the first column of Z_k and that of Q_k. Then from (4.1) we get

$$A_k x_{k-1} = \alpha_k y_k \tag{4.4}$$

and

$$E_k x_k = \beta_k y_k. \tag{4.5}$$

Multiplying (4.4) by β_k and (4.5) by α_k and equating, yields

$$\beta_k A_k x_{k-1} = \alpha_k E_k x_k. \tag{4.6}$$

Thus, when the regular pencil $\lambda\mathcal{E} - \mathcal{A}$ has a 0 eigenvalue, there are nonzero x_k vectors and scalars α_k and β_k such that (4.3) and (4.6) hold. Similarly, if the regular pencil $\lambda\mathcal{E} - \mathcal{A}$ has a ∞ eigenvalue, there are nonzero $\tilde{x}_k$ vectors and scalars $\tilde{\alpha}_k$ and $\tilde{\beta}_k$ such that

$$< \prod_{j=1}^{K} \tilde{\alpha}_j, \prod_{j=1}^{K} \tilde{\beta}_j > = < 1, 0 > (= \infty), \tag{4.7}$$

and

$$\tilde{\beta}_k A_k \tilde{x}_{k-1} = \tilde{\alpha}_k E_k \tilde{x}_k. \tag{4.8}$$

Similar results also hold for the finite nonzero eigenvalues of $\lambda\mathcal{E} - \mathcal{A}$:

Theorem 4.1 *Let $\lambda\mathcal{E} - \mathcal{A}$ be a regular pencil. If $< \alpha, \beta >$ is an eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$, then there are complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_K$, $\beta_1, \beta_2, \cdots, \beta_K$ and nonzero vectors $x_1, x_2, \cdots, x_K$ such that*

$$\beta_k A_k x_{k-1} = \alpha_k E_k x_k, \quad \textit{for} \quad k = 1, 2, \cdots, K, \tag{4.9}$$

and

$$< \prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j > = < \alpha^K, \beta^K > . \tag{4.10}$$

Proof. The cases $< \alpha, \beta > = < 0, 1 > = 0$ and $< \alpha, \beta > = < 1, 0 > = \infty$ were shown in (4.3), (4.6) and (4.7), (4.8), respectively. So it remains to show the case $\alpha\beta \neq 0$. Let $< \alpha, \beta > = \frac{\alpha}{\beta}$ be a finite nonzero eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$. According to (3.4) and (3.5), there exist unitary Q_k and Z_k matrices that reduce $\lambda\mathcal{E} - \mathcal{A}$ to the block lower triangular form:

$$Q_k^* A_k Z_{k-1} = \begin{bmatrix} \alpha_k^{(1)} & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \alpha_k^{(m)} & 0 \\ * & \cdots & * & \Sigma_k \end{bmatrix}; \quad Q_k^* E_k Z_k = \begin{bmatrix} \beta_k^{(1)} & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \beta_k^{(m)} & 0 \\ * & \cdots & * & T_k \end{bmatrix}, \tag{4.11}$$

where the Σ_k and T_k matrices are nonsingular and for each fixed $j \in \{1, 2, \cdots, m\}$, at least one of the scalars $\alpha_1^{(j)}, \cdots, \alpha_K^{(j)}, \beta_1^{(j)}, \cdots, \beta_K^{(j)}$ is zero. Since $\alpha\beta \neq 0$, (4.11) implies $\frac{\alpha}{\beta}$ is an eigenvalue of the pencil

$$\begin{bmatrix} \lambda T_1 & & & -\Sigma_1 \\ -\Sigma_2 & \lambda T_2 & & \\ & \ddots & \ddots & \\ & & \Sigma_K & \lambda T_K \end{bmatrix}.$$

That is, there is a nonzero vector $y = \left[y_1^T, y_2^T, \cdots, y_k^T\right]^T \neq 0$ such that

$$\begin{bmatrix} \frac{\alpha}{\beta} T_1 & & & -\Sigma_1 \\ -\Sigma_2 & \frac{\alpha}{\beta} T_2 & & \\ & \ddots & \ddots & \\ & & -\Sigma_k & \frac{\alpha}{\beta} T_K \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = 0. \tag{4.12}$$

However, (4.12) reads

$$\Sigma_k y_{k-1} = \frac{\alpha}{\beta} T_k y_k, \quad \text{for} \quad k = 1, 2, \cdots, K, \tag{4.13}$$

which in turn implies

$$y_k = \frac{\beta}{\alpha} T_k^{-1} \Sigma_k y_{k-1}, \quad \text{for} \quad k = 1, 2, \cdots, K.$$

Thus, if one of the y_k vectors is zero, then the whole vector y will also be a zero vector, which is a contradiction. Therefore we have further that each y_k is a nonzero vector.

From (4.11) and (4.13), it is clear that

$$Q_k^* A_k Z_{k-1} = \begin{bmatrix} 0 \\ y_{k+1} \end{bmatrix} = \frac{\alpha}{\beta} Q_k^* E_k Z_k \begin{bmatrix} 0 \\ y_k \end{bmatrix},$$

or that

$$\beta A_k x_{k-1} = \alpha E_k x_k,$$

where $x_k = Z_k \begin{bmatrix} 0 \\ y_k \end{bmatrix} \neq 0$ (since $y_k \neq 0$). Hence the theorem follows by letting $\alpha_k = \alpha$ and $\beta_k = \beta$ in the case. □

Assume that the matrices E_k are nonsingular so that we can associate with (1.1) a monodromy matrix (starting at step k):

$$\Phi_k = E_{K+k-1}^{-1} A_{K+k-1} \cdots E_{k+1}^{-1} A_{k+1} E_k^{-1} A_k.$$

¿From Theorem 4.1 we know that if $< \alpha, \beta >= \frac{\alpha}{\beta}$ is an eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$ ($\beta \neq 0$, since the matrices E_k are invertible), then (4.9) and (4.10) hold with some nonzero x_k vectors. And hence

$$E_k^{-1} A_k x_{k-1} = \frac{\alpha_k}{\beta_k} x_k, \quad \text{for} \quad k = 1, 2, \cdots, K. \tag{4.14}$$

Successively using (4.14) and taking into account the periodicity of the matrices, we get

$$\Phi_k x_{k-1} = \frac{\alpha_1 \alpha_2 \cdots \alpha_K}{\beta_1 \beta_2 \cdots \beta_K} x_{k-1} = \frac{\alpha^K}{\beta^K} x_{k-1}, \quad \text{for} \quad k = 1, 2, \cdots, K. \tag{4.15}$$

That is, x_{k-1} is an eigenvector of Φ_k with corresponding eigenvalue $\frac{\alpha^K}{\beta^K}$. Notice that all the Φ_k matrices have the same set of eigenvalues, which is completely determined by the eigenvalues of $\lambda\mathcal{E} - \mathcal{A}$. Specifically, each eigenvalue of Φ_k is just the K-th power of an eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$.

From now on, we use the notation $((A_k, E_k))_{k=1}^K$ to indicate that the A_k and E_k matrices are of period K. And similarly, we use $(S_k)_{k=1}^K$ to denote a periodical sequence $\{S_k\}_{k=1}^\infty$ of period K. We sometimes identify S_0 with S_K for convenience. And we say periodic matrix pairs $((A_k, E_k))_{k=1}^K$ are regular if $\lambda\mathcal{E} - \mathcal{A}$ is a regular pencil.

According to (4.9), it is clear that

$$\begin{bmatrix} \alpha_1 E_1 & & & -\beta_1 A_1 \\ -\beta_2 A_2 & \alpha_2 E_2 & & \\ & \ddots & \ddots & \\ & & -\beta_K A_K & \alpha_K E_K \end{bmatrix} \tag{4.16}$$

is a singular matrix. It is then well-known from the theory of K-cyclic pencil that $< \prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j >$ is the K-th power of an eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$, which consists with (4.10). On the other hand, (4.15) given an eigenvalue-eigenvector interpretation of Theorem 4.1 in terms of monodromy matrix Φ_k. So, in order to keep these relationships, we impose on $((A_k, E_k))_{k=1}^K$ the following definitions about eigenvalues and eigenvectors.

Definition 4.1 *Let $((A_k, E_k))_{k=1}^K$ be regular periodic $n \times n$ matrix pairs. If there exist complex numbers $\alpha_1, \cdots, \alpha_K$ and $\beta_1, \cdots, \beta_K$ such that $\left(\prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j\right) \neq (0,0)$ and (4.16) is singular, then we say $<\prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j>$ is an eigenvalue of $((A_k, E_k))_{k=1}^K$. The set of eigenvalues of $((A_k, E_k))_{k=1}^K$ is denoted as $\sigma\left((A_k, E_k)\right)_{k=1}^K$.*

Definition 4.2 *Let $((A_k, E_k))_{k=1}^K$ be regular periodic $n \times n$ matrix pairs. If there exist complex numbers $\alpha_1, \cdots, \alpha_K, \beta_1, \cdots, \beta_K$, and nonzero vectors $x_1, \cdots, x_K$ such that*

$$\beta_k A_k x_{k-1} = \alpha_k E_k x_k, \quad \text{for} \quad k = 1, 2, \cdots, K \tag{4.17}$$

with $(\prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j) \neq (0,0)$, then we say that $(x_k)_{k=1}^K$ is an eigenvector sequence of $((A_k, E_k))_{k=1}^K$ with corresponding eigenvalue $<\prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j>$.

The determinant of (4.16) is a homogeneous polynomial in $<\prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j>$ of degree n:

$$\sum_{k=0}^{n} C_k \left(\alpha_1 \alpha_2 \cdots \alpha_K\right)^k \left(\beta_1 \beta_2 \cdots \beta_K\right)^{n-k},$$

where $C_0, C_1, \cdots, C_n$ are complex numbers uniquely determined by $((A_k, E_k))_{k=1}^K$. For regular $((A_k, E_k))_{k=1}^K$, this then implies at least one of the C_k's is not zero and hence we see from Definition 4.1 that there are exactly n eigenvalues (counting multiplicity) for $((A_k, E_k))_{k=1}^K$. Furthermore, Theorem 4.1 shows every eigenvalue of regular $((A_k, E_k))_{k=1}^K$ has a corresponding eigenvector sequence. So the above two definitions are well-defined. Recall that the form (4.16) is a mere slightly more general form of (1.2). One reason that we define the eigenvalues of $((A_k, E_k))_{k=1}^K$ in terms of (4.16) rather than (1.2) is to reveal its relationship with the eigenvector sequence defined by Definition 4.2. Actually, if we define the K-th power of an eigenvalue λ of (1.2) as an eigenvalue of $((A_k, E_k))_{k=1}^K$, then in some situations there no longer exist *nonzero* x_k vectors such that

$$\begin{bmatrix} \lambda E_1 & & & -A_1 \\ -A_2 & \lambda E_2 & & \\ & \ddots & \ddots & \\ & & -A_K & \lambda E_K \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_K \end{bmatrix} = 0. \tag{4.18}$$

Example 4.1 *Consider $n = 1$, $K = 3$, and*

$$\lambda \mathcal{E} - \mathcal{A} = \begin{bmatrix} \lambda & 0 & 0 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{bmatrix}.$$

Then $((A_k, E_k))_{k=1}^3$ is regular and 0 is its unique eigenvalue. But then the solution of (4.18) with $\lambda = 0$ has $x_1 = x_2 = 0$. So it is impossible to find nonzero *x_1, x_2, and x_3 such that (4.18) holds. However, $(x_k)_{k=1}^3$ with $x_1 = x_2 = x_3 = 1$ is an eigenvector sequence of $((A_k, E_k))_{k=1}^3$ corresponding to the 0 eigenvalue, since then (4.17) holds with $\alpha_1 = 0$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 1$.*

Hence in Definition 4.1 we use (4.16) instead of (1.2) to make connection with Definition 4.2.

On the other hand, according to Definition 4.2 the eigenvector sequence $(x_k)_{k=1}^K$ can be chosen to satisfy $\|x_k\| = 1$ for each k, where $\|\cdot\|$ is any vector norm in $\mathbf{C}^n$. As a matter of fact, if we let $\tilde{\alpha}_k = \alpha_k \|x_k\|$, $\tilde{\beta}_k = \beta_k \|x_{k-1}\|$, and $\tilde{x}_k = x_k / \|x_k\|$, then $(\tilde{x}_k)_{k=1}^K$ will be a "normalized" eigenvector sequence corresponding to the eigenvalue $<\prod_{j=1}^{K} \tilde{\alpha}_j, \prod_{j=1}^{K} \tilde{\beta}_j>$ $(=<\prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j>)$, provided (4.17) holds. This is why we require each x_k is nonzero.

In addition to the basic requirement $x_k \neq 0$ for each k, the eigenvector sequence $(x_k)_{k=1}^K$ has the following particular property that will help us construct the periodic Schur decomposition for regular periodic matrix pairs $((A_k, E_k))_{k=1}^K$.

Theorem 4.2 *Let $((A_k, E_k))_{k=1}^K$ be regular periodic $n \times n$ matrix pairs. Assume that $(x_k)_{k=1}^K$ is an eigenvector sequence of $((A_k, E_k))_{k=1}^K$. Then either $(A_k x_{k-1})_{k=1}^K$ or $(E_k x_k)_{k=1}^K$ is a sequence of nonzero vectors.*

Proof. In view of Definition 4.2, there exist complex numbers $\alpha_1, \cdots, \alpha_K$ and $\beta_1, \cdots, \beta_K$ such that

$$\beta_k A_k x_{k-1} = \alpha_k E_k x_k, \quad \text{for} \quad k = 1, 2, \cdots, K.$$

Moreover, we may assume x_k is of unit modulus and that either $\prod_{j=1}^{K} \alpha_j \neq 0$ or $\prod_{j=1}^{K} \beta_j \neq 0$. Thus for each k, the two vectors $A_k x_{k-1}$ and $E_k x_k$ are parallel. So there exist unitary Q_k matrices such that

$$Q_k^* A_k x_{k-1} = [\tilde{\alpha}_k, 0, \cdots, 0]^T \tag{4.19}$$

and

$$Q_k^* E_k x_k = \left[\tilde{\beta}_k, 0, \cdots, 0\right]^T, \tag{4.20}$$

for $k = 1, 2, \cdots, K$, where $\tilde{\alpha}_k$ and $\tilde{\beta}_k$ are complex numbers.

Now extend x_k to a unitary matrix Z_k with x_k at its first column. Then (4.19) and (4.20) imply $((A_k, E_k))_{k=1}^k$ is reduced to the block upper triangular form:

$$Q_k^* A_k Z_{k-1} = \begin{bmatrix} \tilde{\alpha}_k & * \\ 0 & * \end{bmatrix}; \qquad Q_k^* E_k Z_k = \begin{bmatrix} \tilde{\beta}_k & * \\ 0 & * \end{bmatrix}.$$

Thus the regularity of $((A_k, E_k))_{k=1}^K$ shows either $\prod_{j=1}^{K} \tilde{\alpha}_j \neq 0$ or $\prod_{j=1}^{K} \tilde{\beta}_j \neq 0$, which in turn implies from (4.19) and (4.20) that either $(A_k x_{k-1})_{k=1}^K$ or $(E_k x_k)_{k=1}^K$ is a sequence of nonzero vectors. □

Now we use the eigenvalue-eigenvector approach to construct the periodic Schur decomposition for regular periodic matrix pairs. And we also point out that the eigenvalues may be arranged in any desired order in the periodic Schur form.

Theorem 4.3 (Periodic Schur Theorem [1]) *Let $((A_k, E_k))_{k=1}^K$ be regular periodic $n \times n$ matrix pairs. Then there exist unitary matrices Q_k and Z_k such that*

$$\hat{A}_k := Q_k^* A_k Z_{k-1} \qquad \textit{and} \qquad \hat{E}_k := Q_k^* E_k Z_k$$

are all upper triangular. Moreover, the diagonal parts

$$\left(\left(\begin{bmatrix} \alpha_1^{(k)} & & \\ & \ddots & \\ & & \alpha_n^{(k)} \end{bmatrix}, \begin{bmatrix} \beta_1^{(k)} & & \\ & \ddots & \\ & & \beta_n^{(k)} \end{bmatrix} \right) \right)_{k=1}^K$$

of $((\hat{A}_k, \hat{E}_k))_{k=1}^K$ determine all the eigenvalues of $((A_k, E_k))_{k=1}^K$ and the eigenvalues $< \prod_{j=1}^{K} \alpha_i^{(j)}, \prod_{j=1}^{K} \beta_i^{(j)} >$, for $i = 1, 2, \cdots, n$, may be arranged to appear in any desired order.

Proof. We prove the theorem by construction. Let $< \alpha, \beta >$ be the first eigenvalue in some prescribed order of the eigenvalues of $((A_k, E_k))_{k=1}^K$, and let $(x_k)_{k=1}^K$ be a "normalized" eigenvector sequence corresponding to the eigenvalue $< \alpha, \beta >$. That is, there are complex numbers $\alpha_1, \cdots, \alpha_K, \beta_1, \cdots, \beta_K$ such that

$$\beta_k A_k x_{k-1} = \alpha_k E_k x_k, \qquad \text{for} \quad k = 1, 2, \cdots, K, \tag{4.21}$$

and

$$< \prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j > = < \alpha, \beta >, \tag{4.22}$$

where $\|x_k\| = 1$. Then in view of Theorem 4.2, either $(A_k x_{k-1})_{k=1}^K$ or $(E_k x_k)_{k=1}^K$ is a sequence of nonzero vectors. Without loss of generality, assume $A_k x_{k-1} \neq 0$ for all k.

If $\alpha_{k_0} = 0$ for some $k_0 \in \{1, 2, \cdots, K\}$, then (4.21) implies $\beta_{k_0} A_{k_0} x_{k_0 - 1} = \alpha_{k_0} E_{k_0} x_{k_0} = 0$, which in turn implies $\beta_{k_0} = 0$ since $A_{k_0} x_{k_0-1} \neq 0$. But this is impossible because $\left(\prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j \right) \neq (0, 0)$. Thus we see $\alpha_k \neq 0$ for all k. Consequently, (4.21) can be rewritten as

$$E_k x_k = \frac{\beta_k}{\alpha_k} A_k x_{k-1}, \qquad \text{for} \quad k = 1, 2, \cdots, K. \tag{4.23}$$

For each k, let Q_k and Z_k be two unitary matrices with x_k at the first column of Z_k and $\frac{A_k x_{k-1}}{\|A_k x_{k-1}\|}$ at the first column of Q_k. Using (4.23) as well as the unitariness of Z_k, we see that the first columns of $Q_k^* A_k Z_{k-1}$ and $Q_k^* E_k Z_k$ are then

$$\begin{bmatrix} \alpha_1^{(k)} \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \beta_1^{(k)} \\ 0 \end{bmatrix},$$

respectively, where

$$\alpha_1^{(k)} = \|A_k x_{k-1}\| \qquad \text{and} \qquad \beta_1^{(k)} = \frac{\beta_k}{\alpha_k} \|A_k x_{k-1}\|. \tag{4.24}$$

Additionally, using (4.22) and (4.24) we obtain

$$< \prod_{j=1}^{K} \alpha_1^{(j)}, \prod_{j=1}^{K} \beta_1^{(j)} > = < \prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j > = < \alpha, \beta > .$$

This completes the first step of the construction. The theorem then follows by induction. □

5 Periodical Eigenspaces

We now assume $((A_k, E_k))_{k=1}^K$ are regular periodic $n \times n$ matrix pairs and come back to the problem of showing the relations (R3), (R4), and (R5) given in section 2.

For $d = 1$, the problem reduces to the one of showing the equivalence among (R3), (R4), (R5) for eigenvector sequence. Let $(x_k)_{k=1}^K$ be a *normalized* eigenvector sequence of $((A_k, E_k))_{k=1}^K$. Then there exist complex numbers $\alpha_1, \cdots, \alpha_K$ and $\beta_1, \cdots, \beta_K$ such that

$$\beta_k A_k x_{k-1} = \alpha_k E_k x_k, \quad \text{for} \quad k = 1, 2, \cdots, K, \tag{5.1}$$

and

$$\left(\Pi_{j=1}^K \alpha_j, \Pi_{j=1}^K \beta_j \right) \neq (0, 0).$$

Equation (5.1) is just the (R3). But then (4.19) and (4.20) shows $A_k x_{k-1}$ and $E_k x_k$ are contained in a one dimensional subspace of $\mathbf{C}^n$ simultaneously. Therefore,

$$dim\,(E_k \mathcal{X} + A_k \mathcal{X}) \leq 1 = dim \mathcal{X}_k, \tag{5.2}$$

where $\mathcal{X}_k$ is the space spanned by x_k. However, using Theorem 4.2 we see the inequality (5.2) itself is in fact an equality

$$dim\,(E_k \mathcal{X}_k + A_k \mathcal{X}_{k-1}) = 1 = dim \mathcal{X}_k. \tag{5.3}$$

This yields (R4). To go from (R4) to (R5), we see from (5.3) that the two vectors $A_k x_{k-1}$ and $E_k x_k$ are parallel so that (4.19) and (4.20) hold. Thus, after extending x_k to a unitary matrix with x_k at its first column, one obtains (R5) from (4.19) and (4.20). Finally, it is obvious to go from (R5) to (5.1). Thus (R3), (R4), and (R5) are equivalent in case $d = 1$.

For the general case, we attach the additional inequality like (5.2),

$$dim\,(E_k\mathcal{X}_k + A_k\mathcal{X}_{k-1}) \le d = dim\mathcal{X}_k, \tag{5.4}$$

and show that (R3), (R4), (R5), and (5.4) are all equivalent. Before proving this, we recall the F/B decomposition of regular periodic matrix pairs:

Theorem 5.1 *([5]) Let $((A_k, E_k))_{k=1}^{k}$ be regular periodic $n\times n$ matrix pairs. Then there are K-periodic nonsingular matrices V_k and W_k such that*

$$V_kA_kW_{k-1} = \begin{bmatrix} A_k^f & 0 \\ 0 & I \end{bmatrix}; \quad V_kE_kW_k = \begin{bmatrix} I & 0 \\ 0 & E_k^b \end{bmatrix},$$

where A_k^f and E_k^b are upper triangular, the eigenvalues of $((A_k^f, I))_{k=1}^K$ lie on or inside the unit circle, and those of $((I, E_k^b))_{k=1}^K$ lie outside the unit circle.

Following [6, pp. 304-305], we then have the following result.

Theorem 5.2 *Let $((A_k, E_k))_{k=1}^K$ be regular periodic $n\times n$ matrix pairs. And let $\mathcal{X}_k$ be K-periodic d-dimensional subspaces of $\mathbf{C}^n$. Then the following five statements are equivalent:*

(i) *$dim(E_k\mathcal{X}_k + A_k\mathcal{X}_{k-1}) \le d = dim\mathcal{X}_k$, for $k = 1, 2, \cdots, K$.*

(ii) *(Block triangular periodic Schur decomposition) For each k, there exist unitary (or nonsingular) matrices*

$$Q_k^{-*} = [\underbrace{Y_k}_{d} \mid *] \quad and \quad Z_k = [\underbrace{X_k}_{d} \mid *]$$

such that

$$Q_k^*A_kZ_{k-1} = \begin{bmatrix} \underbrace{\begin{matrix}\Phi_k \\ 0\end{matrix}}_{d} & \begin{matrix}* \\ *\end{matrix} \end{bmatrix}\}d; \quad Q_k^*E_kZ_k = \begin{bmatrix} \underbrace{\begin{matrix}\Psi_k \\ 0\end{matrix}}_{d} & \begin{matrix}* \\ *\end{matrix} \end{bmatrix}\}d, \tag{5.5}$$

where the columns of X_k span $\mathcal{X}_k$.

(iii) *(Swapping factors) If $\tilde{X}_k \in \mathbf{C}^{n\times d}$ forms a basis of $\mathcal{X}_k$, then there are $d\times d$ matrix pairs $((A_k^{(0)}, E_k^{(0)}))_{k=1}^K$, K-periodic nonsingular $d\times d$ matrices T_k, and K-periodic $n\times d$ matrices $\tilde{Y}_k$ of full column rank such that*

$$A_k\tilde{X}_{k-1}T_{k-1} = \tilde{Y}_kA_k^{(0)} \tag{5.6}$$

and

$$E_k\tilde{X}_kT_k = \tilde{Y}_kE_k^{(0)}, \tag{5.7}$$

where

$$A_k^{(0)} = \begin{bmatrix} \Phi_k^f & 0 \\ 0 & I \end{bmatrix}; \quad E_k^{(0)} = \begin{bmatrix} I & 0 \\ 0 & \Psi_k^b \end{bmatrix}, \tag{5.8}$$

and Φ_k^f, Ψ_k^b are upper triangular.

(iv) *(Algebraic form) If $\tilde{X}_k \in \mathbf{C}^{n\times d}$ forms a basis of $\mathcal{X}_k$, then there are regular periodic $d\times d$ matrix pairs $((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$ such that*

$$A_k\tilde{X}_{k-1}\tilde{E}_k = E_k\tilde{X}_k\tilde{A}_k, \quad for \quad k = 1, 2, \cdots, K. \tag{5.9}$$

(v) *(Geometric form)*

$$dim\,(E_k\mathcal{X}_k + A_k\mathcal{X}_{k-1}) = d = dim\mathcal{X}_k, \quad for \quad k = 1, 2, \cdots, K.$$

Proof.

(i)⇒(ii): Suppose (i) holds. For each k, let $\mathcal{Y}_k$ a d-dimensional subspace of $\mathbf{C}^n$ that contains both $A_k\mathcal{X}_{k-1}$ and $E_k\mathcal{X}_k$, and let

$$Q_k = [\underbrace{Y_k}_{d} \mid *] \quad \text{and} \quad Z_k = [\underbrace{X_k}_{d} \mid *]$$

be two unitary matrices with $Im(Y_k) = \mathcal{Y}_k$ and $Im(X_k) = \mathcal{X}_k$. Then we have

$$A_k X_{k-1} = Y_k \Phi_k \qquad \text{and} \qquad E_k X_k = Y_k \Psi_k, \tag{5.10}$$

for some $d \times d$ matrices Φ_k, Ψ_k. So by the unitariness of Q_k and Z_k, we obtain (5.5).

(ii)⇒(iii): Suppose (ii) holds. Then clearly $((\Phi_k, \Psi_k))_{k=1}^K$ is regular. So, by Theorem 5.1, there are K-periodic nonsingular matrices V_k and W_k such that

$$A_k^{(0)} := V_k \Phi_k W_{k-1} = \begin{bmatrix} \Phi_k^f & 0 \\ 0 & I \end{bmatrix}; \quad E_k^{(0)} := V_k \Psi_k W_k = \begin{bmatrix} I & 0 \\ 0 & \Psi_k^b \end{bmatrix}, \tag{5.11}$$

where Φ_k^f and Ψ_k^b are upper triangular. Since $Im(X_k) = Im(\tilde{X}_k) = \mathcal{X}_k$, there are K-periodic nonsingular matrices S_k such that

$$\tilde{X}_k = X_k S_k, \qquad \text{for} \quad k = 1, 2, \cdots, K.$$

In addition, we see from (5.5) that (5.10) holds. Now rewrite (5.10) as

$$A_k \left(X_{k-1} S_{k-1}\right) \left(S_{k-1}^{-1} W_{k-1}\right) = \left(Y_k V_k^{-1}\right) \left(V_k \Phi_k W_{k-1}\right)$$

and

$$E_k \left(X_k S_k\right) \left(S_k^{-1} W_k\right) = \left(Y_k V_k^{-1}\right) \left(V_k \Psi_k W_k\right).$$

Claim (iii) then follows by letting

$$T_k = S_k^{-1} W_k \qquad \text{and} \quad \tilde{Y} = Y_k V_k^{-1}.$$

(iii)⇒(iv): Suppose (iii) holds. Then we see from (5.8) that $A_k^{(0)} E_k^{(0)} = E_k^{(0)} A_k^{(0)}$. Post-multiplying both sides of equations (5.6) and (5.7) by $E_k^{(0)}$ and $A_k^{(0)}$, respectively, and equating, yields

$$A_k \tilde{X}_{k-1} \left(T_{k-1} E_k^{(0)}\right) = E_k \tilde{X}_k \left(T_k A_k^{(0)}\right). \tag{5.12}$$

So (5.9) holds by letting

$$\tilde{E}_k = T_{k-1} E_k^{(0)} \qquad \text{and} \quad \tilde{A}_k = T_k A_k^{(0)}. \tag{5.13}$$

It is obvious from the form of (5.8) that $((A_{k-1}^{(0)}, E_k^{(0)}))_{k=1}^K$ is regular. So, in view of (5.13), $((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$ is regular, too.

(iv)⇒(i): Suppose (iv) holds. Applying Theorem 5.1 to $(\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$, we have for some K-periodic nonsingular matrices V_k and W_k,

$$V_k \tilde{A}_{k-1} W_{k-1} = \begin{bmatrix} \tilde{A}_{k-1}^f & 0 \\ 0 & I \end{bmatrix} \qquad \text{and} \quad V_k \tilde{E}_k W_k = \begin{bmatrix} I & 0 \\ 0 & \tilde{E}_k^b \end{bmatrix}. \tag{5.14}$$

Equation (5.9) can be rewritten as

$$\left(A_k \tilde{X}_{k-1} V_k^{-1}\right) \left(V_k \tilde{E}_k W_k\right) = \left(E_k \tilde{X}_k V_{k+1}^{-1}\right) \left(V_{k+1} \tilde{A}_k W_k\right).$$

Now let $P_k = A_k \tilde{X}_{k-1} V_k^{-1}$ and $Q_k = E_k \tilde{X}_k V_{k+1}^{-1}$. And partition $P_k = \left[P_k^{(1)} | P_k^{(2)}\right]$ and $Q_k = \left[Q_k^{(1)} | Q_k^{(2)}\right]$ conformably with (5.14). We then obtain

$$\left[\overbrace{P_k^{(1)}}^{d} | P_k^{(2)}\right] \begin{bmatrix} I & 0 \\ 0 & \tilde{E}_k^b \end{bmatrix} = \left[\overbrace{Q_k^{(1)}}^{d} | Q_k^{(2)}\right] \begin{bmatrix} \tilde{A}_k^f & 0 \\ 0 & I \end{bmatrix}.$$

It immediately follows that $Im\left(Q_k^{(1)}\right) \supseteq Im\left(P_k^{(1)}\right)$ and $Im\left(Q_k^{(2)}\right) \subseteq Im\left(P_k^{(2)}\right)$. Hence

$$\begin{aligned} & \dim\left(A_k\mathcal{X}_{k-1} + E_k\mathcal{X}_k\right) \\ = \; & \dim\left(Im\left(P_k\right) + Im\left(Q_k\right)\right) \\ = \; & \dim\left(Im\left(P_k^{(1)}\right) + Im\left(P_k^{(2)}\right) + Im\left(Q_k^{(1)}\right) + Im\left(Q_k^{(2)}\right)\right) \\ = \; & \dim\left(Im\left(Q_k^{(1)}\right) + Im\left(P_k^{(2)}\right)\right) \\ \leq \; & d. \end{aligned}$$

(iii)⇒(v): Suppose (iii) holds. Then by (5.6), (5.7), and (5.8) we have

$$Im\left(A_k\tilde{X}_{k-1}T_{k-1}\right) + Im\left(E_k\tilde{X}_kT_k\right) = Im\left(\tilde{Y}_k\right).$$

Thus

$$A_k\mathcal{X}_{k-1} + E_k\mathcal{X}_k = Im\left(\tilde{Y}_k\right),$$

which in turn implies $\dim(A_k\mathcal{X}_{k-1} + E_k\mathcal{X}_k) = d$.

(v)⇒(i): Trivial. □

The algebraic form (5.9) is very close to Definition 4.2. For this reason, we can use Theorem 5.2 to define periodical eigenspaces for $((A_k, E_k))_{k=1}^K$ as follows.

Definition 5.1 *Let $((A_k, E_k))_{k=1}^K$ be regular periodic $n \times n$ matrix pairs. And let $(\mathcal{X}_k)_{k=1}^K$ be a K-periodic sequence of d-dimensional subspaces of $\mathbf{C}^n$. If there exist regular periodic $d \times d$ matrix pairs $((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$ such that*

$$A_kX_{k-1}\tilde{E}_k = E_kX_k\tilde{A}_k, \qquad \textit{for} \quad k = 1, 2, \cdots, K,$$

where $X_k \in \mathbf{C}^{n\times d}$ and $Im(X_k) = \mathcal{X}_k$, then we say that $(\mathcal{X}_k)_{k=1}^K$ is an eigenspace sequence of $((A_k, E_k))_{k=1}^K$ or that the $\mathcal{X}_k$ spaces are periodical eigenspaces of $((A_k, E_k))_{k=1}^K$.

In order to determine the corresponding eigenvalues inherited from a given eigenspace sequence, one can use Theorem 5.2 to obtain a block triangular periodic Schur decomposition (5.5) and says that $\sigma((\Phi_k, \Psi_k))_{k=1}^K$ ($\subseteq \sigma((A_k, E_k))_{k=1}^K$) gives the desired answer. But this has the disadvantage of expressing the answer in terms of the transformed matrices. So we want to answer the question directly in terms of $((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$. As a matter of fact, $\sigma((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$ itself is just what we want. To show this and to get deeper insight into Theorem 5.2, we first show three preliminary lemmata.

Lemma 5.3 *Let $((A_k, E_k))_{k=1}^K$ be regular. And assume the algebraic form (iv) of Theorem 5.2 holds. Then there are K-periodic nonsingular matrices $\tilde{K}_k$ and $\tilde{V}_k$ such that*

$$\tilde{K}_k^*\left(A_k\tilde{X}_{k-1}\right)\tilde{V}_k^{-1} = \begin{bmatrix} \tilde{A}_k^{(0)} \\ 0 \end{bmatrix} \quad \textit{and} \quad \tilde{K}_k^*\left(E_k\tilde{X}_k\right)\tilde{V}_{k+1}^{-1} = \begin{bmatrix} \tilde{E}^{(0)} \\ 0 \end{bmatrix}, \tag{5.15}$$

where

$$\tilde{A}_k^{(0)} = \begin{bmatrix} \tilde{A}_k^f & 0 \\ 0 & I \end{bmatrix} \quad \textit{and} \quad \tilde{E}_k^{(0)} = \begin{bmatrix} I & 0 \\ 0 & \tilde{E}_k^b \end{bmatrix},$$

and $\tilde{A}_k^f, \tilde{E}_k^b$ are upper triangular with $\sigma((\tilde{A}_k^f, I))_{k=1}^K$ contained in the unit disk and $\sigma((I, \tilde{E}_k^b))_{k=1}^K$ outside the unit disk.

Proof. Applying Theorem 5.1 to $((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$, we see there exist K-periodic nonsingular matrices $\tilde{V}_k$ and $\tilde{W}_k$ such that

$$\tilde{A}_{k-1}^{(0)} := \tilde{V}_k\tilde{A}_{k-1}\tilde{W}_{k-1} = \begin{bmatrix} \tilde{A}_{k-1}^f & 0 \\ 0 & I \end{bmatrix}; \quad \tilde{E}_k^{(0)} := \tilde{V}_k\tilde{E}_k\tilde{W}_k = \begin{bmatrix} I & 0 \\ 0 & \tilde{E}_k^b \end{bmatrix}, \tag{5.16}$$

where $\tilde{A}_k^f$ and $\tilde{E}_k^b$ are upper triangular with $\sigma((\tilde{A}_k^f, I))_{k=1}^K$ contained in the unit disk and $\sigma((I, \tilde{E}_k^b))_{k=1}^K$ outside the unit disk. From (5.9) we obtain

$$\left(A_k\tilde{X}_{k-1}\tilde{V}_k^{-1}\right)\left(\tilde{V}_k\tilde{E}_k\tilde{W}_k\right) = \left(E_k\tilde{X}_k\tilde{V}_{k+1}^{-1}\right)\left(\tilde{V}_{k+1}\tilde{A}_k\tilde{W}_k\right).$$

It then follows that

$$\underbrace{\left[\tilde{P}_k^{(1)}|\tilde{P}_k^{(2)}\right]}_{d}\begin{bmatrix} I & 0 \\ 0 & \tilde{E}_k^b \end{bmatrix} = \underbrace{\left[\tilde{Q}_k^{(1)}|\tilde{Q}_k^{(2)}\right]}_{d}\begin{bmatrix} \tilde{A}_k^f & 0 \\ 0 & I \end{bmatrix}, \tag{5.17}$$

where

$$\left[\tilde{P}_k^{(1)}|\tilde{P}_k^{(2)}\right] = \tilde{P}_k = A_k\tilde{X}_{k-1}\tilde{V}_k^{-1}; \quad \left[\tilde{Q}_k^{(1)}|\tilde{Q}_k^{(2)}\right] = \tilde{Q}_k = E_k\tilde{X}_k\tilde{V}_{k+1}^{-1}.$$

Similar to the proof of " (iv) $\Longrightarrow$ (i) " in Theorem 5.2, it can be shown from (5.17) that

$$\begin{aligned} & \dim\left(A_k\mathcal{X}_{k-1} + E_k\mathcal{X}_k\right) \\ = \;& \dim\left(Im\left(\tilde{P}_k\right) + Im\left(\tilde{Q}_k\right)\right) \\ = \;& \dim\left(Im\left(\tilde{P}_k^{(2)}\right) + Im\left(\tilde{Q}_k^{(1)}\right)\right) \\ \leq \;& d. \end{aligned}$$

However, Theorem 5.2 implies the equality

$$\dim\left(A_k\mathcal{X}_{k-1} + E_k\mathcal{X}_A\right) = d.$$

So the $n \times d$ matrices $[\tilde{Q}_k^{(1)}|\tilde{P}_k^{(2)}]$ are of full column rank d and hence these exist K-periodic nonsingular matrices $\tilde{K}_k$ that reduce them to the form

$$\tilde{K}_k^*\left[\tilde{Q}_k^{(1)}|\tilde{P}_k^{(2)}\right] = \left[\begin{array}{c|c} I & 0 \\ 0 & I \\ 0 & 0 \end{array}\right].$$

Thus

$$\begin{aligned} \tilde{K}_k^*\left(A_k\tilde{X}_{k-1}\right)\tilde{V}_k^{-1} &= \tilde{K}_k^*\tilde{P}_k = \tilde{K}_k^*\left[\tilde{P}_k^{(1)}|\tilde{P}_k^{(2)}\right] \\ &= \tilde{K}_k^*\left[Q_k^{(1)}\tilde{A}_k^f|\tilde{P}_k^{(2)}\right] \\ &= \begin{bmatrix} \tilde{A}_k^f & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{A}^{(0)} \\ 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \tilde{K}_k^*\left(E_k\tilde{X}_k\right)\tilde{V}_{k+1}^{-1} &= \tilde{K}_k^*\tilde{Q}_k = \tilde{K}_k^*\left[\tilde{Q}_k^{(1)}\middle|\tilde{Q}_k^{(2)}\right] \\ &= \tilde{K}_k^*\left[\tilde{Q}_k^{(1)}\middle|P_k^{(2)}\tilde{E}_k^b\right] \\ &= \begin{bmatrix} I & 0 \\ 0 & \tilde{E}_k^b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{E}_k^{(0)} \\ 0 \end{bmatrix}. \end{aligned}$$

This completes the proof. □

Lemma 5.4 *Let $(A_k, E_k))_{k=1}^K$ be regular. And assume the block triangular periodic Schur decomposition (ii) of Theorem 5.2 holds. Then $((\Phi_k, \Psi_k))_{k=1}^K$ is uniquely determined up to periodic Schur transformations.*

Proof. It suffices to show that if

$$\begin{cases} A_kX_{k-1} = Y_k\Phi_k \\ E_kX_k = Y_k\Psi_k \end{cases} \quad \text{and} \quad \begin{cases} A_k\hat{X}_{k-1} = \hat{Y}_k\hat{\Phi}_k \\ E_k\hat{X}_k = \hat{Y}_k\hat{\Psi}_k \end{cases} \tag{5.18}$$

hold with $Im(X_k) = Im(\hat{X}_k) = \mathcal{X}_k$ and full column rank d matrices $Y_k, \hat{Y}_k$, then $((\Phi_k, \Psi_k))_{k=1}^K \sim ((\hat{\Phi}_k, \hat{\Psi}_k))_{k=1}^K$ (here $\sim$ denotes $((\Phi_k, \Psi_k))_{k=1}^K$ and $((\hat{\Phi}_k, \hat{\Psi}_k))_{k=1}^K$ can be transformed to each other via periodic Schur transformations).

Since X_k and $\hat{X}_k$ span the same space $\mathcal{X}_k$, there are K-periodic nonsingular matrices S_k such that $\hat{X}_k = X_k S_k$. Hence (5.18) implies

$$\hat{Y}_k \hat{\Phi}_k = Y_k \Phi_k S_{k-1} \quad \text{and} \quad \hat{Y}_k \hat{\Psi}_k = Y_k \Psi_k S_k. \tag{5.19}$$

Because $\hat{Y}_k$ has linearly independent columns, $\hat{Y}_k^T \hat{Y}_k$ is a nonsingular matrix. Therefore we obtain from (5.19) the following relationships between $((\hat{\Phi}_k, \hat{\Psi}))_{k=1}^K$ and $((\Phi_k, \Psi_k))_{k=1}^K$:

$$\hat{\Phi}_k = \left(\hat{Y}_k^T \hat{Y}_k\right)^{-1} \hat{Y}_k^T Y_k \Phi_k S_{k-1} \quad \text{and} \quad \hat{\Psi}_k = \left(\hat{Y}_k^T \hat{Y}_k\right)^{-1} \hat{Y}_k^T Y_k \Psi_k S_k,$$

which then imply $((\Phi_k, \Psi_k))_{k=1}^K \sim ((\hat{\Phi}_k, \hat{\Psi}_k))_{k=1}^K$. □

Following the notation used in Lemma 5.4, from now on the relation $((A_k, E_k))_{k=1}^K \sim ((\tilde{A}_k, \tilde{E}_k))_{k=1}^K$ means $((A_k, E_k))_{k=1}^K$ and $((\tilde{A}_k, \tilde{E}_k))_{k=1}^K$ can be related via periodic Schur transformations.

Lemma 5.5 *Consider the two sequences of K-periodic matrix pairs*

$$((A_k, E_k))_{k=1}^K := \left(\left(\begin{bmatrix} A_k^f & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & E_k^b \end{bmatrix}\right)\right)_{k=1}^K$$

and

$$\left((\tilde{A}_k, \tilde{E}_k)\right)_{k=1}^K := \left(\left(\begin{bmatrix} \tilde{A}_k^f & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & \tilde{E}_k^b \end{bmatrix}\right)\right)_{k=1}^K$$

where $A_k^f, E_k^b, \tilde{A}_k^f, \tilde{E}_k^b$ have consistent dimensions and satisfy

$$\sigma\left(\left(A_k^f, I\right)\right)_{k=1}^K \cap \sigma\left(\left(I, \tilde{E}_k^b\right)\right)_{k=1}^K = \phi \tag{5.20}$$

and

$$\sigma\left(\left(\tilde{A}_k^f, I\right)\right)_{k=1}^K \cap \sigma\left(\left(I, E_k^b\right)\right)_{k=1}^K = \phi. \tag{5.21}$$

If $((A_k, E_k))_{k=1}^K \sim ((\tilde{A}_k, \tilde{E}_k))_{k=1}^K$, then we have further $((A_{k-1}, E_k))_{k=1}^K \sim ((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$.

Proof. Assume that $((A_k, E_k))_{k=1}^K \sim ((\tilde{A}_k, \tilde{E}_k))_{k=1}^K$. Then they can be related via periodic Schur transformations like

$$\begin{bmatrix} P_{11}^{(k)} & P_{12}^{(k)} \\ P_{21}^{(k)} & P_{22}^{(k)} \end{bmatrix} \begin{bmatrix} A_k^f & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{A}_k^f & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_{11}^{(k-1)} & Q_{12}^{(k-1)} \\ Q_{21}^{(k-1)} & Q_{22}^{(k-1)} \end{bmatrix} \tag{5.22}$$

and

$$\begin{bmatrix} P_{11}^{(k)} & P_{12}^{(k)} \\ P_{21}^{(k)} & P_{22}^{(k)} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & E_k^b \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \tilde{E}_k^b \end{bmatrix} \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} \\ Q_{21}^{(k)} & Q_{22}^{(k)} \end{bmatrix}. \tag{5.23}$$

Equations (5.22) and (5.23) then give rise to the following two systems of periodic Sylvester equations:

$$\begin{cases} \tilde{A}_k^f Q_{12}^{(k-1)} - P_{12}^{(k)} = 0 \\ Q_{12}^{(k)} - P_{12}^{(k)} E_k^b = 0, \qquad \text{for } k = 1, 2, \cdots, K, \end{cases}$$

and

$$\begin{cases} Q_{21}^{(k-1)} - P_{21}^{(k)} A_k^f = 0 \\ \tilde{E}_k^b Q_{21}^{(k)} - P_{21}^{(k)} = 0, \qquad \text{for } k = 1, 2, \cdots, K. \end{cases}$$

Since we have the two conditions (5.20) and (5.21), the two systems have only the trivial solutions, namely,

$$Q_{12}^{(k)} = P_{12}^{(k)} = 0, \quad Q_{21}^{(k)} = P_{21}^{(k)} = 0, \quad \text{for } k = 1, 2, \cdots, K.$$

Thus (5.22) and (5.23) can be rewritten as

$$P_{11}^{(k)} A_k^f = \tilde{A}_k^f Q_{11}^{(k-1)}; \quad P_{22}^{(k)} = Q_{22}^{(k-1)}$$

and

$$P_{11}^{(k)} = Q_{11}^{(k)}; \quad P_{22}^{(k)} E_k^b = \tilde{E}_k^b Q_{22}^{(k)},$$

respectively. We then have

$$P_{11}^{(k)} A_k^f = \tilde{A}_k^f P_{11}^{(k-1)} \quad \text{and} \quad P_{22}^{(k)} E_k^b = \tilde{E}_k^b P_{22}^{(k+1)}.$$

So that the following two identities hold:

$$\begin{bmatrix} P_{11}^{(k-1)} & 0 \\ 0 & P_{22}^{(k)} \end{bmatrix} \begin{bmatrix} A_{k-1}^f & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{A}_{k-1}^f & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{11}^{(k-2)} & 0 \\ 0 & P_{22}^{(k)} \end{bmatrix}$$

and

$$\begin{bmatrix} P_{11}^{(k-1)} & 0 \\ 0 & P_{22}^{(k)} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & E_k^b \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \tilde{E}_k^b \end{bmatrix} \begin{bmatrix} P_{11}^{(k-1)} & 0 \\ 0 & P_{22}^{(k+1)} \end{bmatrix},$$

which tell us $((A_{k-1}, E_k))_{k=1}^K \sim ((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$. □

We now come back to the determination of the corresponding eigenvalues inherited from a given eigenspace sequence. Let $((A_k, E_k))_{k=1}^K$ be regular periodic $n \times n$ matrix pair. And let $(\mathcal{X}_k)_{k=1}^K$ be an eigenspace sequence of $((A_k, E_k))_{k=1}^K$ with $dim\mathcal{X}_k = d$. Then there exist regular periodic $d \times d$ matrix pairs $((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$ and full column rank matrices $\tilde{X}_k$ such that

$$A_k \tilde{X}_{k-1} \tilde{E}_k = E_k \tilde{X}_k \tilde{A}_k, \quad \text{for} \quad k = 1, 2, \cdots, K, \tag{5.24}$$

and $Im(\tilde{X}_k) = \mathcal{X}_k$. We are going to show that $\sigma((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K \subseteq \sigma((A_k, E_k))_{k=1}^K$ using the direct approach of eigenvalue-eigenvector interpretation.

Let $< \alpha, \beta > \in \sigma((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$. Then Theorem 4.1 shows there associates the periodical eigenvectors y_k and complex numbers $\alpha_1, \cdots, \alpha_K, \beta_1, \cdots, \beta_K$ such that

$$\beta_k \tilde{A}_{k-1} y_{k-1} = \alpha_{k-1} \tilde{E}_k y_k, \quad \text{for} \quad k = 1, 2, \cdots, K, \tag{5.25}$$

and

$$< \prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j > = < \alpha, \beta > .$$

According to Theorem 4.2, either $(\tilde{E}_k y_k)_{k=1}^K$ or $(\tilde{A}_{k-1} y_{k-1})_{k=1}^K$ is a sequence of nonzero vectors. Assume that $\tilde{E}_k y_k \neq 0$ for all k. Then from (5.24) and (5.25) we see

$$\beta_{k+1} A_k \tilde{X}_{k-1} \tilde{E}_k y_k = \beta_{k+1} E_k \tilde{X}_k \tilde{A}_k y_k = \alpha_k E_k \tilde{X}_k \tilde{E}_{k+1} y_{k+1}.$$

Therefore we have

$$\beta_{k+1} A_k x_{k-1} = \alpha_k E_k x_k, \quad \text{for} \quad k = 1, 2, \cdots, K,$$

where $x_k = \tilde{X}_k \tilde{E}_{k+1} y_{k+1}$. Notice that $x_k \neq 0$ because $\tilde{E}_{k+1} y_{k+1} \neq 0$ and $\tilde{X}_k$ is of full column rank. So $< \alpha, \beta > = < \prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j >$ is an eigenvalue of $((A_k, E_k))_{k=1}^K$. Similarly, if $\tilde{A}_{k-1} y_{k-1} \neq 0$ for all k, then

$$\beta_k A_k \left(\tilde{X}_{k-1} \tilde{A}_{k-1} y_{k-1} \right) = \alpha_{k-1} E_k \left(\tilde{X}_k \tilde{A}_k y_k \right), \quad \text{for} \quad k = 1, 2, \cdots, K,$$

and $\tilde{X}_k \tilde{A}_k y_k \neq 0$. So, again, $< \alpha, \beta > = < \prod_{j=1}^{K} \alpha_j, \prod_{j=1}^{K} \beta_j > \in \sigma((A_k, E_k))_{k=1}^K$. Thus we established $\sigma((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K \subseteq ((A_k, E_k))_{k=1}^K$.

Although we can identify the corresponding eigenvalues of $(\mathcal{X}_k)_{k=1}^K$ directly from the algebraic form (5.24), the $\tilde{A}_k$ and $\tilde{E}_k$ matrices are not unique! So we have to show that the set $\sigma((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$ is independent of the $\tilde{A}_k$ and $\tilde{E}_k$ matrices. On the other hand, we have shown in Lemma 5.4 that $\sigma((\Phi_k, \Psi_k))_{k=1}^K$ is independent of the Φ_k and Ψ_k matrices. Because $\sigma((\Phi_k, \Psi_k))_{k=1}^K$ also gives the eigenvalues inherited from $(\mathcal{X}_k)_{k=1}^K$, the set $\sigma((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K$ has to coincide with $\sigma((\Phi_k, \Psi_k))_{k=1}^K$. So we have the following theorem.

Theorem 5.6 *Let $((A_k, E_k))_{k=1}^K$ be regular periodic $n\times n$ matrix pairs. And let $\mathcal{X}_k$ be K-periodic d-dimensional subspaces of $\mathbf{C}^n$. Assume that the algebraic form (iv) of Theorem 5.2 holds. Then we have the following properties:*

(i) $\sigma((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K \subseteq \sigma((A_k, E_k))_{k=1}^K$.

(ii) *The matrices $\tilde{A}_{k-1}$ and $\tilde{E}_k$ in (5.9) are uniquely determined up to periodic Schur transformations.*

(iii) $\sigma((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K = \sigma((\Phi_k, \Psi_k))_{k=1}^K$, *where the Φ_k and Ψ_k matrices are given by the block triangular periodic Schur decomposition (5.5).*

Proof. We have proven (i) in the previous discussion. So it remains to show (ii) and (iii).

(ii) By Lemma 5.3, there exist nonsingular matrices $\tilde{K}_k$ and $\tilde{V}_k$ such that (5.15) holds:

$$\tilde{K}_k^* \left(A_k \tilde{X}_{k-1}\right) \tilde{V}_k^{-1} = \begin{bmatrix} \tilde{A}_k^{(0)} \\ 0 \end{bmatrix}; \quad \tilde{K}_k^* \left(E_k \tilde{X}_k\right) \tilde{V}_{k+1}^{-1} = \begin{bmatrix} \tilde{E}_k^{(0)} \\ 0 \end{bmatrix}, \tag{5.26}$$

where

$$\tilde{A}_k^{(0)} = \begin{bmatrix} \tilde{A}_k^f & 0 \\ 0 & I \end{bmatrix}, \quad \tilde{E}_k^{(0)} = \begin{bmatrix} I & 0 \\ 0 & \tilde{E}_k^b \end{bmatrix},$$

and the eigenvalues of $((\tilde{A}_k^f, I))_{k=1}^K$ lie on or inside the unit circle and those of $((I, \tilde{E}_k^b))_{k=1}^K$ lie outside the unit circle. Furthermore we see from (5.16) that

$$((\tilde{A}_{k-1}, \tilde{E}_k))_{k=1}^K \sim ((\tilde{A}_{k-1}^{(0)}, \tilde{E}_k^{(0)}))_{k=1}^K. \tag{5.27}$$

On the other hand, (5.26) can be rewritten as

$$\tilde{K}_k^* A_k \tilde{X}_{k-1} = \begin{bmatrix} \tilde{\Phi}_k \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{K}_k^* E_k \tilde{X}_k = \begin{bmatrix} \tilde{\Psi}_k \\ 0 \end{bmatrix},$$

where $\tilde{\Phi}_k = \tilde{A}_k^{(0)} \tilde{V}_k$ and $\tilde{\Psi}_k = \tilde{E}_k^{(0)} \tilde{V}_{k+1}$. Thus we can construct a block triangular periodic Schur decomposition of $((A_k, E_k))_{k=1}^K$ with leading $d\times d$ blocks $\tilde{\Phi}_k$ and $\tilde{\Psi}_k$. In particular, we have

$$\left(\left(\tilde{\Phi}_k, \tilde{\Psi}_k\right)\right)_{k=1}^K \sim \left(\left(\tilde{A}_k^{(0)}, \tilde{E}_k^{(0)}\right)\right)_{k=1}^K, \tag{5.28}$$

since $\tilde{\Phi}_k = \tilde{A}_k^{(0)} \tilde{V}_k$ and $\tilde{\Psi}_k = \tilde{E}_k^{(0)} \tilde{V}_{k+1}$.

Similarly, if $\hat{X}_k$ is another basis of $\mathcal{X}_k$ and that there are regular periodic $d\times d$ matrix pairs $((\hat{A}_{k-1}, \hat{E}_k))_{k=1}^K$ such that

$$A_k \hat{X}_{k-1} \hat{E}_k = E_k \hat{X}_k \hat{A}_k, \quad \text{for} \quad k = 1, 2, \cdots, K.$$

Then above results still hold with the " tilde " matrices replaced by the " hat " ones. Thus we have

$$\left(\left(\hat{A}_{k-1}, \hat{E}_k\right)\right)_{k=1}^K \sim \left(\left(\hat{A}_{k-1}^{(0)}, \hat{E}_k^{(0)}\right)\right)_{k=1}^K, \tag{5.29}$$

and

$$\left(\left(\hat{\Phi}_k, \hat{\Psi}_k\right)\right)_{k=1}^K \sim \left(\left(\hat{A}_k^{(0)}, \hat{E}_k^{(0)}\right)\right)_{k=1}^K. \tag{5.30}$$

Moreover, we can also construct a block triangular periodic Schur decomposition of $((A_k, E_k))_{k=1}^K$ with leading $d\times d$ blocks $\hat{\Phi}_k$ and $\hat{\Psi}_k$. But then Lemma 5.4 shows that

$$\left(\left(\hat{\Phi}_k, \hat{\Psi}_k\right)\right)_{k=1}^K \sim \left(\left(\tilde{\Phi}_k, \tilde{\Psi}_k\right)\right)_{k=1}^K. \tag{5.31}$$

Thus we obtain from (5.28), (5.30), and (5.31) that

$$\left(\left(\hat{A}_k^{(0)}, \hat{E}_k^{(0)}\right)\right)_{k=1}^K \sim \left(\left(\tilde{A}_k^{(0)}, \tilde{E}_k^{(0)}\right)\right)_{k=1}^K.$$

However, Lemma 5.5 gives further that

$$\left(\left(\hat{A}_{k-1}^{(0)}, \hat{E}_k^{(0)}\right)\right)_{k=1}^{K} \sim \left(\left(\tilde{A}_{k-1}^{(0)}, \tilde{E}_k^{(0)}\right)\right)_{k=1}^{K}. \tag{5.32}$$

So that by (5.27), (5.29), and (5.32) we get

$$\left(\left(\hat{A}_{k-1}, \hat{E}_k\right)\right)_{k=1}^{K} \sim \left(\left(\tilde{A}_{k-1}, \tilde{E}_k\right)\right)_{k=1}^{K}.$$

This shows the desired result.

(iii) Observing the proofs of " (ii) $\Longrightarrow$ (iii) " and " (iii) $\Longrightarrow$ (iv) " in Theorem 5.2, we have

$$\sigma\left((\Phi_k, \Psi_k)\right)_{k=1}^{K} = \sigma\left(\left(A_k^{(0)}, E_k^{(0)}\right)\right)_{k=1}^{K}, \tag{5.33}$$

and

$$A_k \tilde{X}_{k-1}\left(T_{k-1}E_k^{(0)}\right) = E_k \tilde{X}_k \left(T_k A_k^{(0)}\right), \tag{5.34}$$

where the matrices $A_k^{(0)}$ and $E_k^{(0)}$ are defined by (5.11) and (5.34) is just a copy of (5.12). Because $A_k^{(0)}$ and $E_k^{(0)}$ are upper triangular matrices of the form (5.8), it is easy to see that $((A_{k-1}^{(0)}, E_k^{(0)}))_{k=1}^{K}$ is regular and that

$$\sigma\left(\left(A_{k-1}^{(0)}, E_k^{(0)}\right)\right)_{k=1}^{K} = \sigma\left(\left(A_k^{(0)}, E_k^{(0)}\right)\right)_{k=1}^{K}. \tag{5.35}$$

Since

$$\left(\left(T_{k-1}A_{k-1}^{(0)}, T_{k-1}E_k^{(0)}\right)\right)_{k=1}^{K} \sim \left(\left(A_{k-1}^{(0)}, E_k^{(0)}\right)\right)_{k=1}^{K}, \tag{5.36}$$

it follows that $((T_{k-1}A_{k-1}^{(0)}, T_{k-1}E_k^{(0)}))_{k=1}^{K}$ is regular. Thus, in view of (5.34), the result of (ii) implies

$$\sigma\left(\left(T_{k-1}A_{k-1}^{(0)}, T_{k-1}E_k^{(0)}\right)\right)_{k=1}^{K} = \sigma\left(\left(\tilde{A}_{k-1}, \tilde{E}_k\right)\right)_{k=1}^{K}. \tag{5.37}$$

Therefore, by (5.33), (5.35), (5.36), and (5.37), we have

$$\sigma\left((\Phi_k, \Psi_k)\right)_{k=1}^{K} = \sigma\left(\left(\tilde{A}_{k-1}, \tilde{E}_k\right)\right)_{k=1}^{K}.$$

This completes of proof. □

The uniqueness properties in Theorem 5.6 justify the terminology " eigenspace ". The most straightforward application is to provide ways of describing the inherent eigenvalues. We now use this to define simple periodic eigenspaces and close the section with their uniqueness property.

Definition 5.2 *Let $((A_k, E_k))_{k=1}^{K}$ be regular periodic matrix pairs. And let $(\mathcal{X})_{k=1}^{K}$ be an eigenspace sequence of $((A_k, E_k))_{k=1}^{K}$. Denote by $\Lambda(\mathcal{X}_k)_{k=1}^{K}$ the inherent eigenvalues of $(\mathcal{X}_k)_{k=1}^{K}$. Then $(\mathcal{X}_k)_{k=1}^{K}$ is said to be simple if*

$$\Lambda\left(\mathcal{X}_k\right)_{k=1}^{K} \cap \left(\sigma\left((A_k, E_k)\right)_{k=1}^{K} \backslash \Lambda\left(\mathcal{X}_k\right)_{k=1}^{K}\right) = \phi.$$

Theorem 5.7 *Let $(\mathcal{X}_k)_{k=1}^{K}$ and $(\tilde{\mathcal{X}}_k)_{k=1}^{K}$ be two simple eigenspace sequences of the regular periodic matrix pairs $((A_k, E_k))_{k=1}^{K}$. If $\Lambda(\mathcal{X}_k)_{k=1}^{K} = \Lambda(\tilde{\mathcal{X}}_k)_{k=1}^{K}$, then it holds $\mathcal{X}_k = \tilde{\mathcal{X}}_k$ for each k.*

Proof. Since $(\mathcal{X}_k)_{k=1}^{K}$ is simple, there exist K-periodic nonsingular matrices P_k and Q_k such that

$$P_k A_k Q_{k-1} = \begin{bmatrix} \Phi_k & 0 \\ 0 & \Gamma_k \end{bmatrix} \quad \text{and} \quad P_k E_k Q_k = \begin{bmatrix} \Psi_k & 0 \\ 0 & \Delta_k \end{bmatrix}, \tag{5.38}$$

where $\sigma((\Phi_k, \Psi_k))_{k=1}^K = \Lambda(\mathcal{X}_k)_{k=1}^K$ (see, e.g., [5]). Analogously, there exist K-periodic nonsingular matrices $\tilde{P}_k$ and $\tilde{Q}_k$ such that

$$\tilde{P}_k A_k \tilde{Q}_{k-1} = \begin{bmatrix} \tilde{\Phi}_k & 0 \\ 0 & \tilde{\Gamma}_k \end{bmatrix} \quad \text{and} \quad \tilde{P}_k E_k \tilde{Q}_k = \begin{bmatrix} \tilde{\Psi}_k & 0 \\ 0 & \tilde{\Delta}_k \end{bmatrix} \tag{5.39}$$

with $\sigma((\tilde{\Phi}_k, \tilde{\Psi}_k))_{k=1}^K = \Lambda(\tilde{\mathcal{X}}_k)_{k=1}^K$.

Using (5.38) and (5.39), we get

$$\begin{cases} P_k^{-1} \begin{bmatrix} \Phi_k & 0 \\ 0 & \Gamma_k \end{bmatrix} Q_{k-1}^{-1} = \tilde{P}_k^{-1} \begin{bmatrix} \tilde{\Phi}_k & 0 \\ 0 & \tilde{\Gamma}_k \end{bmatrix} \tilde{Q}_{k-1}^{-1} \\ P_k^{-1} \begin{bmatrix} \Psi_k & 0 \\ 0 & \Delta_k \end{bmatrix} Q_k^{-1} = \tilde{P}_k^{-1} \begin{bmatrix} \tilde{\Psi}_k & 0 \\ 0 & \tilde{\Delta}_k \end{bmatrix} \tilde{Q}_k^{-1}, \end{cases}$$

or also

$$\begin{cases} \tilde{P}_k P_k^{-1} \begin{bmatrix} \Phi_k & 0 \\ 0 & \Gamma_k \end{bmatrix} = \begin{bmatrix} \tilde{\Phi}_k & 0 \\ 0 & \tilde{\Gamma}_k \end{bmatrix} \tilde{Q}_{k-1}^{-1} Q_{k-1} \\ \tilde{P}_k P_k^{-1} \begin{bmatrix} \Psi_k & 0 \\ 0 & \Delta_k \end{bmatrix} = \begin{bmatrix} \tilde{\Psi}_k & 0 \\ 0 & \tilde{\Delta}_k \end{bmatrix} \tilde{Q}_k^{-1} Q_k. \end{cases}$$

Now partition $\tilde{P}_k P_k^{-1}$ and $\tilde{Q}_k^{-1} Q_k$ as

$$\tilde{P}_k P_k^{-1} = \begin{bmatrix} P_{11}^{(k)} & P_{12}^{(k)} \\ P_{21}^{(k)} & P_{[22}^{(k)} \end{bmatrix} \quad \text{and} \quad \tilde{Q}_k^{-1} Q_k = \begin{bmatrix} Q_{11}^{(k)} & Q^{(k)} \\ Q_{21}^{(k)} & Q_{22}^{(k)} \end{bmatrix}.$$

Similar to Lemma 5.5, one can show that

$$Q_{12}^{(k)} = P_{12}^{(k)} = 0 \quad \text{and} \quad Q_{21}^{(k)} = P_{21}^{(k)} = 0, \quad \text{for} \quad k = 1, 2, \cdots, K.$$

So we have in particular,

$$Q_k = \tilde{Q}_k \begin{bmatrix} Q_{11}^{(k)} & 0 \\ 0 & Q_{22}^{(k)} \end{bmatrix}, \quad \text{for} \quad k = 1, 2, \cdots, K.$$

This shows that $X_k = \tilde{X}_k Q_{11}^{(k)}$ for each k, where X_k and $\tilde{X}_k$ are the first d (d =dim $\mathcal{X}_k$) columns of Q_k and $\tilde{Q}_k$, respectively. Thus $\mathcal{X}_k = \tilde{\mathcal{X}}_k$ for each k. □

6 Concluding Remarks

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