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Creating a nilpotent pencil via deadbeat

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We consider the problem of finding a square low rank correction $(\lambda C - B)F$ to a given square pencil $(\lambda E - A)$ such that the new pencil $\lambda(E-CF)-(A-BF)$ has all its generalized eigenvalues at the origin. We give necessary and sufficient conditions for this problem to have a solution and we also provide a constructive algorithm to compute F when such a solution exists. We show that this problem is related to the deadbeat control problem of a discrete-time linear system and that an (almost) equivalent formulation is to find a square embedding that has all its finite generalized eigenvalues at the origin.

Keywords: deadbeat control; generalized state–space systems; numerical methods;

1. Introduction

In this paper we look at the set of implicit difference equations

$$Ex_{i+1} = Ax_i + Bu_i - Cu_{i+1} \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state vector and $u_i \in \mathbb{R}^m$ is the input vector. We will assume $m \ll n$ and consider an affine transformation of the state to the input :

$$u_i = v_i - Fx_i \tag{2}$$

which yields the modified system

$$(E - CF)x_{i+1} = (A - BF)x_i + Bv_i - Cv_{i+1}.$$

We are particularly interested in the homogeneous case where $v_i = 0$, which yields the set of difference equations of the *closed-loop system*

$$(E - CF)x_{i+1} = (A - BF)x_i. \tag{3}$$

We want to choose the *feedback matrix* F such that the state x_i goes to zero *as fast as possible* (i.e. in a minimum number of steps) for any initial condition x_0. In order to have a unique solution to (3) the matrix $(E - CF)$ must be invertible and

$$x_{i+1} = (E - CF)^{-1}(A - BF)x_i. \tag{4}$$

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then converges to zero provided the following spectral radius condition is satisfied

$$\rho_F := \rho[(E - CF)^{-1}(A - BF)] < 1, \tag{5}$$

where $\rho[M] := \max_i |\lambda_i[M]|$. Moreover, if we can make ρ_F equal to zero, then x_k will be zero for some finite value k since the matrix $(E - CF)^{-1}(A - BF)$ will be nilpotent. In control theory, this is known as the *deadbeat control problem* for discrete linear time invariant systems, but it is obviously also relevant to the literature of iterative solvers if we interpret x_i as the approximation error of an iterative solver such as the Jacobi or Gauss-Seidel iteration, see Gander et al. (2012).

A simpler version of this problem was analyzed from a numerical point of view in Van Dooren (1984) for the standard eigenvalue problem (where $E = I_n$ and $C = 0$) and in Beelen et al. (1988) for the generalized eigenvalue problem (where E is invertible and $C = 0$). A description of the solution of these restricted problems and a numerical procedure were given in these papers, but based on the assumption that the system was reachable. In the present paper, we relax this assumption and only require the system to be controllable. We then show that the problem is also linked to the embedding problem of a rectangular pencil $[\lambda E - A | \lambda C - B]$ into the square pencil

$$\begin{bmatrix} \lambda E - A & \lambda C - B \\ F & I \end{bmatrix}$$

with given eigenstructure, which was considered in Boley et al. (1994). The spectral radius ρ_F will be exactly zero if and only if all the generalized eigenvalue of the closed loop pencil

$$\lambda(E - CF) - (A - BF) \tag{6}$$

are all zero. In Section 2 we first describe the solution for $E = I_n$ and $C = 0$ and show this amounts to the deadbeat control problem. In Section 3 we then extend this to the so-called generalized deadbeat problem where E is invertible and $C = 0$. In Section 4 we further extend this to the case where $[E|C]$ is of full row rank, which can be reduced to an embedding problem. Finally, we show in Section 5 that the embedding problem can generically be solved, provided the spectral radius is set equal to an arbitrarily small number $|\alpha|$, rather than equal to 0.

2. The deadbeat problem

The problem of deadbeat control of a standard state-space system was considered from a numerical point of view in Van Dooren (1984). We re-derive some of the results of this papers in a more general setting since we do not require the system to be reachable. For this purpose, it is also appropriate to recall the language of the geometric theory of Wonham (1985).

Definition 1: *The image $\mathcal{B}$ of an arbitrary $n \times m$ matrix B is defined as the linear space*

$$\mathcal{B} := \{y \mid y = Bx,\ x \in \mathbb{R}^m\} \subset \mathbb{R}^n.$$

The pre-image $\mathcal{V}$ of a linear space $\mathcal{S} \subset \mathbb{R}^n$ with respect to an arbitrary $n \times n$ matrix A is defined as the linear space

$$\mathcal{V} = A^{-1}\mathcal{S} := \{v \mid x = Av,\ x \in \mathcal{S}\} \subset \mathbb{R}^n.$$

This is therefore the largest space $\mathcal{V}$ such that $A\mathcal{V} \subset \mathcal{S}$.

Lemma 1: *Let A and B be arbitrary $n \times n$ and $n \times m$ matrices and let $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{n\times n}$ be orthogonal matrices such that*

$$U^T AV = \begin{bmatrix} X & Y \\ 0 & C \end{bmatrix}, \quad U^T B = \begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{7}$$

where R has full row rank r and C has full column rank c. If we partition

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}, \quad V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}, \tag{8}$$

where U_1 has r columns and V_2 has c columns then

$$\mathcal{B} = \mathrm{Im}\, U_1 \subset \mathbb{R}^n, \quad A^{-1}\mathcal{B} = \mathrm{Im}\, V_1 \subset \mathbb{R}^n.$$

These linear spaces have dimensions r and $n_1 := n - c$, respectively.

The orthogonal transformations U and V of the above Lemma are easily constructed as follows. A QR factorization of B yields an orthogonal factor U that "compresses" the rows of B to $U^T B = \begin{bmatrix} R \\ 0 \end{bmatrix}$ as in (7). If we then partition U as above, we can construct the matrix $Z := U_2^T A$. The QR factorization of Z^T then yields an orthogonal transformation V^T that compresses the columns of Z to $ZV = \begin{bmatrix} 0 & C \end{bmatrix}$ as in (7). The dimensions of the two spaces trivially follow from this construction. Moreover, if B is not identically zero, it easily follows that $n_1 \geq r > 0$.

Notice that meanwhile we have shown that

$$n_1 = \max_F \dim \ker(A - BF).$$

This follows from the coordinate transformation

$$U^T(A - BF)V = \begin{bmatrix} X & Y \\ 0 & C \end{bmatrix} - \begin{bmatrix} R \\ 0 \end{bmatrix} \begin{bmatrix} F_1 & F_2 \end{bmatrix}$$

where C has full column rank c. Clearly the rank of this transformed matrix is minimized to c by choosing F_1 as any solution of the compatible system $RF_1 = X$. The minimum norm solution for F is therefore given by $\hat{F} := \begin{bmatrix} F_1 & 0 \end{bmatrix} V^T = F_1 V_1^T$, where F_1 is the minimum norm solution of $RF_1 = X$ (we use the Frobenius norm to make the solution unique). Moreover, the first $n - c$ columns of $(A - B\hat{F})V$ are zero and the last c columns form a matrix of rank c. Therefore, the orthogonal similarity transformation V yields a transformed system

$$V^T(A - B\hat{F})V = \begin{bmatrix} 0_{n_1} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad V^T B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where $n_1 := n - c$ and $\mathrm{rank} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = c$. This shows the following result.

Theorem 1: *The space $\mathcal{S}_1 := A^{-1}\mathcal{B}$ is the kernel of largest dimension of $A - BF$ and an orthogonal state space transformation V with first $n_1 := n - c$ columns spanning $\mathcal{S}_1$ yields the transformed system*

$$V^T(A - B\hat{F})V = \begin{bmatrix} 0_{n_1} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \mathrm{rank} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = c.$$

Let us extend this to spaces defined via the recurrence relation

$$\mathcal{S}_0 := \{0\}, \quad \mathcal{S}_{i+1} := A^{-1}(\mathcal{S}_i + \mathcal{B}). \tag{9}$$

The following lemmas – inspired from Wonham (1985) – are crucial for understanding these recurrences and their properties.

Lemma 2: *The spaces defined by* (9) *are nested :*

$$\{0\} = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_k = \mathcal{S}_n$$

and have a supremal element that is reached in $k \leq n$ steps.

Proof. Clearly $\mathcal{S}_0 \subset \mathcal{S}_1$ since $\mathcal{S}_0 = \{0\}$. It then easily follows that

$$\mathcal{S}_i = A^{-1}(\mathcal{S}_{i-1} + \mathcal{B}) \subset A^{-1}(\mathcal{S}_i + \mathcal{B}) = \mathcal{S}_{i+1}$$

which shows by induction that all spaces are nested. The fact that the sequence converges in $k \leq n$ steps, follows from the fact that as soon as $\mathcal{S}_{k+1} = \mathcal{S}_k$, then also $\mathcal{S}_{k+2} = \mathcal{S}_{k+1}$, and therefore the same holds for all subsequent spaces $\mathcal{S}_j, \forall j \geq k$. Hence the number of steps k needed to reach the supremal element must be bounded by the matrix dimension n. □

Definition 2: *The discrete-time system $x_{k+1} = Ax_k + Bu_k$ is said to be controllable, if for any initial state x_0 there exists a sequence of inputs $u_i, i = 0, \ldots, k$ that can drive x_0 to the zero state in finite time k. This is the case if and only if $\mathcal{S}_k = \mathbb{R}^n$, where the spaces $\mathcal{S}_i$ are defined by $\mathcal{S}_0 := \{0\}$, and the recurrence $\mathcal{S}_i := A^{-1}(\mathcal{S}_{i-1} + \mathcal{B}), i = 1, \ldots, k$.*

Lemma 3: *Let $\hat{\mathcal{V}}$ be defined as*

$$\hat{\mathcal{V}} := A^{-1}(\mathcal{S} + \mathcal{B}).$$

Then $\hat{\mathcal{V}}$ is the space of largest dimension that satisfies

$$(A - BF)\mathcal{V} \subset \mathcal{S}, \quad \text{for any matrix } F.$$

It is also the sum of all spaces $\mathcal{V}$ satisfying the above inclusion.

Proof. Let V and S be orthonormal bases for the spaces $\mathcal{V}$ and $\mathcal{S}$, respectively, satisfying $(A - BF)\mathcal{V} \subset \mathcal{S}$. Then there exists a matrix $X \in \mathbb{R}^{s \times v}$ such that $(A - BF)V = SX$ which implies

$$AV = \left[\begin{array}{cc} S & B \end{array}\right] \left[\begin{array}{c} X \\ Y \end{array}\right], \quad \text{for } FV = Y \tag{10}$$

and therefore also

$$\mathcal{V} \subset A^{-1}(\mathcal{S} + \mathcal{B}).$$

Moreover, the matrix $F := YV^{\dagger}$ solves the compatible system $FV = Y$ in a least Frobenius norm sense. If there are two such spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ then it follows that $\mathcal{V} := \mathcal{V}_1 + \mathcal{V}_2 \subset A^{-1}(\mathcal{S} + \mathcal{B})$ and

$$A\left[\begin{array}{cc} V_1 & V_2 \end{array}\right] = \left[\begin{array}{cc} S & B \end{array}\right] \left[\begin{array}{cc} X_1 & X_2 \\ Y_1 & Y_2 \end{array}\right], \quad \text{for } FV_i = Y_i, \ i = 1, 2.$$

An orthogonal basis for the sum $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$ is moreover obtained from

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

implying also

$$AV = \begin{bmatrix} S & B \end{bmatrix} \begin{bmatrix} X_1 Z_1 + X_2 Z_2 \\ Y_1 Z_1 + Y_2 Z_2 \end{bmatrix}, \quad \text{for } FV = (Y_1 Z_1 + Y_2 Z_2).$$

This is an alternate proof that the set of such spaces have a supremal element $\hat{\mathcal{V}}$ under space addition, which also must be given by

$$\hat{\mathcal{V}} = A^{-1}(\mathcal{S} + \mathcal{B}) = \arg \sup_F \{\mathcal{V} : (A - BF)\mathcal{V} \subset \mathcal{S}\}.$$

□

Using these two Lemmas, we finally obtain the following Theorem, for which a proof can also be found in Wonham (1985).

Theorem 2: *The largest null space of the matrix $(A - BF)^i$ for any feedback matrix F is given by $\mathcal{S}_i$ of the recurrence* (9) *and there exists a feedback F such that $(A - BF)^k = 0$ if and only if $\mathcal{S}_k = \mathbb{R}^n$.*

Proof. It follows from the above Lemmas that $(A - BF)^i \mathcal{S}_i = \{0\}$ if and only if there exists a decreasing sequence of spaces $\mathcal{S}_j, j = i, \ldots, 0$ satisfying

$$(A - BF)\mathcal{S}_j = \mathcal{S}_{j-1}, \ j = i, \ldots, 1 \quad \mathcal{S}_0 = \{0\}.$$

This corresponds exactly to the sequence (9). Moreover, the kernel is the whole space $\mathbb{R}^n$ if and only if $(A - BF)^k = 0$ and the smallest index k for which this happens is the smallest index for which $\mathcal{S}_k = \mathbb{R}^n$. □

The following equivalent statements for the existence of a deadbeat control of a given pair (A, B) are recalled from Wonham (1985) and Boley et al. (1994).

Corollary 1: *There exists a deadbeat feedback F for (A, B) such that $(A - BF)^k = 0$ if and only if one the following equivalent conditions hold :*
i) the (A, B) pair is controllable, i.e. $\exists k : \mathcal{S}_k = \mathbb{R}^n$
ii) the pencil $[\lambda I_n - A|B]$ has full rank n for all $\lambda \neq 0$.

Let us now assume that we can construct an orthogonal coordinate transformation V such that its first j subblocks span the space $\mathcal{S}_j$:

$$V := \begin{bmatrix} V_1 & V_2 & \ldots & V_k \end{bmatrix}, \quad \mathcal{S}_j = \text{span} \begin{bmatrix} V_1 & \ldots & V_j \end{bmatrix}, \ j = 1, ..., k.$$

Then the matrix $V^T(A-BF)V$ must have the following form :

$$V^T(A-BF)V := \begin{bmatrix} 0_{n_1} & A_{1,2} & A_{1,3} & \dots & A_{1,k} \\ & 0_{n_2} & A_{2,3} & \dots & A_{2,k} \\ & & 0_{n_3} & \ddots & \vdots \\ & & & \ddots & A_{k-1,k} \\ & & & & 0_{n_k} \end{bmatrix}, \tag{11}$$

with $\dim \mathcal{S}_i = \sum_{j=1}^{i} n_j,\ i=1,...,k$ and $\mathrm{rank} A_{i-1,i} = n_i, i=2,...,k$. Notice that this rank condition also implies that the null space dimensions n_i are non-increasing :

$$n_1 \geq n_2 \geq \ldots \geq n_k.$$

Let us also partition the matrix $F_v := FV$ conformably :

$$F_v := \begin{bmatrix} F_1 & F_2 & \ldots & F_k \end{bmatrix},$$

then the following algorithm constructs the matrix F_v recursively such that $V^T(A-BF)V$ has the above form (11). The construction of the transformation matrix V uses the basic ideas explained in Theorem 1 in a recursive fashion. Let us assume that the first transformation V_1 implementing the basic step for computing $\mathcal{S}_1 := A^{-1}(\mathcal{S}_0 + \mathcal{B})$ has been performed as described in Theorem 1. We then have

$$\left[\begin{array}{c|c} V_1^T(A-BF_1)V_1 & V_1^T B \\ \hline F_1 V_1 & \end{array}\right] = \left[\begin{array}{cc|c} 0_{n_1} & A_{12} & B_{11} \\ 0 & A_{22} & B_{12} \\ \hline F_{11} & 0 & \end{array}\right].$$

We then proceed in a similar fashion for the subsystem (A_{22}, B_{12}). Let V_2 be constructed such that

$$\left[\begin{array}{c|c} V_2^T(A_{22}-B_{12}F_2)V_2 & V_2^T B_{12} \\ \hline F_2 V_2 & \end{array}\right] = \left[\begin{array}{cc|c} 0_{n_2} & A_{23} & B_{22} \\ 0 & A_{33} & B_{23} \\ \hline F_{22} & 0 & \end{array}\right],$$

then embedding this decomposition into the previous step and defining

$$\hat{V}_2 := V_1 \cdot \begin{bmatrix} I_{n_1} & \\ & V_2 \end{bmatrix}$$

yields

$$\left[\begin{array}{c|c} \hat{V}_2^T(A-BF)\hat{V}_2 & \hat{V}_2^T B \\ \hline F\hat{V}_2 & \end{array}\right] = \left[\begin{array}{ccc|c} 0_{n_1} & A_{12} & A_{13} & B_{11} \\ & 0_{n_2} & A_{23} & B_{22} \\ & & A_{33} & B_{23} \\ \hline F_{11} & F_{22} & 0 & \end{array}\right].$$

and this continues in this fashion until all diagonal blocks are 0 (i.e. at step k).

3. The generalized deadbeat problem

Since we want all the generalized eigenvalues of the closed loop pencil

$$\lambda E - (A - BF) \tag{12}$$

to lie at 0, we have to assume here that E is invertible since otherwise there is no solution. Let us then define the matrices

$$\tilde{A} := E^{-1}A, \quad \tilde{B} := E^{-1}B,$$

and look at the equivalent pencil

$$\lambda E - (A - BF) = E[\lambda I_n - (\tilde{A} - \tilde{B}F)]$$

which is now in the standard form considered in the previous section. This reduction was already proposed in Beelen et al. (1988) but again it was assumed that the generalized state-space model was reachable. We consider the problem in a more general geometric setting and show that this problem has a solution if and only if the matrix pair $(\tilde{A}, \tilde{B})$ is controllable and that it does not require to be reachable.

It follows from the invertibility of the matrix E that the recurrence relation defined earlier for a standard pair $(\tilde{A}, \tilde{B})$:

$$\mathcal{S}_0 := \{0\}, \quad \mathcal{S}_{i+1} := \tilde{A}^{-1}(\mathcal{S}_i + \mathrm{Im}\tilde{B}) \tag{13}$$

is equivalent to the definition

$$\mathcal{S}_0 := \{0\}, \quad \mathcal{S}_{i+1} := A^{-1}(E\mathcal{S}_i + \mathcal{B}) \tag{14}$$

Using these connections, we trivially extend the results of the previous section to a triple (E, A, B).

Lemma 4: *Let E be invertible and $\hat{\mathcal{V}}$ be defined as*

$$\hat{\mathcal{V}} := A^{-1}(E\mathcal{S} + \mathcal{B}).$$

Then $\hat{\mathcal{V}}$ is the space of largest dimension that satisfies

$$(A - BF)\mathcal{V} \subset E\mathcal{S}, \quad \text{for any matrix } F.$$

It is also the sum of all spaces $\mathcal{V}$ satisfying the above inclusion.

Corollary 2: *The spaces defined by (14) are nested :*

$$\{0\} = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_k = \mathcal{S}_n$$

and have a supremal element that is reached in $k \leq n$ steps.

Definition 3: *The discrete-time system $Ex_{k+1} = Ax_k + Bu_k$ is said to be controllable, if for any initial state x_0 there exists a sequence of inputs $u_i, i = 0, \ldots, k$ that can drive x_0 to the zero state in finite time k. This is the case if and only if $\mathcal{S}_k = \mathbb{R}^n$, where the spaces $\mathcal{S}_i$ are defined by $\mathcal{S}_0 := \{0\}$, and the recurrence $\mathcal{S}_i := A^{-1}(E\mathcal{S}_{i-1} + \mathcal{B}), i = 1, \ldots, k$.*

Theorem 3: *Let E be an invertible matrix, then the space $\mathcal{S}_1 := A^{-1}\mathcal{B}$ is the kernel of largest dimension of $A-BF$ and the orthogonal transformations V and W whose first $n_1 := n-c$ columns span respectively $\mathcal{S}_1$ and $E\mathcal{S}_1$ yield the transformed system*

$$W^T(\lambda E - (A - B\hat{F}))V = \begin{bmatrix} \lambda E_{11} & \lambda E_{12} - A_{12} \\ 0 & \lambda E_{22} - A_{22} \end{bmatrix}, \quad \text{rank} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = c.$$

Corollary 3: *Let E be invertible, then there exists a feedback F such that $\lambda E - (A - BF) = 0$ has all its eigenvalues at 0 if and only if one the following equivalent conditions hold :*
i) the (E, A, B) triple is controllable, i.e. $\exists k : \mathcal{S}_k = \mathbb{R}^n$
ii) the pencil $[\lambda E - A|B]$ has full rank n for all $\lambda \neq 0$.
The smallest index k for which this is the case is the smallest index for which $\mathcal{S}_k = \mathbb{R}^n$.

Assume again that we can construct an orthogonal coordinate transformation V such that its first j subblocks span the space $\mathcal{S}_j$:

$$V := \begin{bmatrix} V_1 & V_2 & \dots & V_k \end{bmatrix}, \quad \mathcal{S}_j = \text{span} \begin{bmatrix} V_1 & \dots & V_j \end{bmatrix}, \; j = 1, ..., k.$$

Then the pencil $W^T [\lambda E - (A - BF)] V$ must have the following form :

$$\begin{bmatrix} \lambda E_{1,1} & \lambda E_{1,2} - A_{1,2} & \lambda E_{1,3} - A_{1,3} & \dots & \lambda E_{1,k} - A_{1,k} \\ & \lambda E_{2,2} & \lambda E_{2,3} - A_{2,3} & \dots & \lambda E_{2,k} - A_{2,k} \\ & & \lambda E_{3,3} & \ddots & \vdots \\ & & & \ddots & \lambda E_{k-1,k} - A_{k-1,k} \\ & & & & \lambda E_{k,k} \end{bmatrix}, \tag{15}$$

with $E_{i,i} \in \mathbb{R}^{n_i \times n_i}$ and of full rank n_i, $\dim \mathcal{S}_i = \sum_{j=1}^{i} n_j$, $i = 1, ..., k$ and $\text{rank} A_{i-1,i} = n_i, i = 2, ..., k$. Notice that the rank conditions on A again imply that the null space dimensions n_i are non-increasing :

$$n_1 \geq n_2 \geq \ldots \geq n_k.$$

The recursive construction to obtain this form is completely analogous to the problem described in Section 2.

4. The general embedding problem

The generalized deadbeat problem consists of finding a feedback matrix F such that the pencil

$$(\lambda E - A) - (\lambda C - B)F \tag{16}$$

has all its n generalized eigenvalues at 0. An equivalent way to state this is to say that we look for a matrix F such that the pencil

$$\begin{bmatrix} \lambda E - A & \lambda C - B \\ F & I_m \end{bmatrix} \tag{17}$$

has n generalized eigenvalues at 0 since by the Schur complement lemma we have that the above two pencils have the same determinant and hence finite generalized eigenvalues. Clearly the pencil

(17) can only have n finite eigenvalues if rank $\left[\begin{array}{cc} E & C \end{array} \right] = n$ since the pencil obviously has m infinite eigenvalues.

If we relax the problem now to finding a rank m matrix $\left[\begin{array}{cc} F & G \end{array} \right]$ such that the pencil

$$\left[\begin{array}{cc} \lambda E - A & \lambda C - B \\ F & G \end{array} \right]$$

has n infinite eigenvalues, we can reduce this to the previous problem with $C = 0$. Indeed, let us find a transformation Q (which we can choose orthogonal) such that

$$\left[\begin{array}{cc} \lambda E - A & \lambda C - B \\ F & G \end{array} \right] Q = \left[\begin{array}{cc} \lambda \hat{E} - \hat{A} & -\hat{B} \\ \hat{F} & \hat{G} \end{array} \right]. \tag{18}$$

Since $\hat{E}$ will be invertible, we can now choose the construction of the previous Section to find a feedback $\hat{F}$ and choose $\hat{G} = I_m$, to make sure that this pencil has n generalized eigenvalues at 0, and m ones at ∞. Notice that if we perform the back-transformation to the original pencil, then G may no longer be invertible and hence the solution may not be acceptable.

Corollary 4: *There exists a bordering (F, G) for a quadruple (E, A, B, C) such that the embedded pencil*

$$\left[\begin{array}{cc} \lambda E - A & \lambda C - B \\ F & G \end{array} \right]$$

is regular and has all its finite generalized eigenvalues at $\lambda = 0$, if and only if the rectangular pencil

$$\left[\begin{array}{cc} \lambda E - A & \lambda C - B \end{array} \right]$$

has full rank n for all finite $\lambda \neq 0$.

Proof. This follows easily from the transformation (18) since this reduces the problem to a generalized deadbeat problem, and the rank condition is clearly not affected by the constant invertible column transformation. □

The following example shows that the embedding problem is more general and may have a solution, while the pencil $\lambda(E - CF) - (A - BF)$ can not be made nilpotent. Let

$$\lambda E - A = \left[\begin{array}{cc} -1 & \lambda \\ 0 & -1 \end{array} \right], \quad \lambda C - B = \left[\begin{array}{c} 0 \\ \lambda \end{array} \right].$$

Then for any feedback matrix F we have that $A - BF = I_2$ and hence $\lambda(E - CF) - (A - BF)$ can not be nilpotent. Nevertheless, the embedding

$$\left[\begin{array}{cc} \lambda E - A & \lambda C - B \\ F & G \end{array} \right] = \left[\begin{array}{cc|c} -1 & \lambda & 0 \\ 0 & -1 & \lambda \\ \hline 1 & 0 & 0 \end{array} \right],$$

has clearly 2 generalized eigenvalues at 0 and one at ∞.

5. Placing other poles

In the previous section it was shown that the row rank of the compound matrix $\left[\begin{array}{cc} E & C \end{array}\right]$ has to be full (i.e. n) in order to be able to assign n finite generalized eigenvalues for the embedding pencil and therefore also for the generalized deadbeat control problem. But the smallest index k at which the controllable space $\mathcal{S}_k$ reaches the full state dimension n, can still be quite high : it is $n-1$ at worst if the dimensions of the spaces $\mathcal{S}_i, i=1\ldots,n-1$ increase by one at each step.

An alternative solution is to make a *random* shift to a point α of small magnitude $|\alpha|$ and to consider the deadbeat control problem for the modified embedding problem

$$\left[\begin{array}{cc} \tilde{\lambda}E-\tilde{A} & \tilde{\lambda}C-\tilde{B} \\ \tilde{F} & \tilde{G} \end{array}\right], \quad \tilde{A}:=A-\alpha E, \quad \tilde{B}:=B-\alpha C. \tag{19}$$

If α is a random shift, the rank increases of the modified spaces will be generically equal to m and k will then be equal to n/m, or the smallest integer larger or equal to this. Moreover, the matrix $\tilde{G}$ is then likely to be invertible and the embedding problem is then equivalent to a generalized deadbeat control problem. On the other hand, the state x_n is not really *beaten to death* in n steps, but is is reduced in norm to about a factor $|\alpha|^k$ after k steps. For the convergence of iterative schemes, this is certainly an acceptable alternative.

If in the example given in the previous section we replace λ by $\tilde{\lambda}-\alpha$, then we obtain the shifted pencil

$$\tilde{\lambda}E-\tilde{A}=\left[\begin{array}{cc} -1 & \tilde{\lambda}-\alpha \\ 0 & -1 \end{array}\right], \quad \tilde{\lambda}C-\tilde{B}=\left[\begin{array}{c} 0 \\ \tilde{\lambda}-\alpha \end{array}\right]$$

for which there is an embedding

$$\left[\begin{array}{cc} \tilde{\lambda}E-\tilde{A} & \tilde{\lambda}C-\tilde{B} \\ \tilde{F} & \tilde{G} \end{array}\right]=\left[\begin{array}{cc|c} -1 & \tilde{\lambda}-\alpha & 0 \\ 0 & -1 & \tilde{\lambda}-\alpha \\ \hline 1 & 2\alpha & \alpha^2 \end{array}\right],$$

which clearly has 2 generalized eigenvalues at α and one at ∞. Moreover, $\tilde{G}$ is now invertible.

A related problem is obtained when swapping the role of $\left[\begin{array}{cc} E & C \end{array}\right]$ and $\left[\begin{array}{cc} A & B \end{array}\right]$ and hence considering the so-called *reversed* pencil

$$\left[\begin{array}{cc} E-\mu A & C-\mu B \\ F & G \end{array}\right]=\left[\begin{array}{cc} \mu I_n & \\ & I_m \end{array}\right]\left[\begin{array}{cc} 1/\mu E-A & 1/\mu C-B \\ F & G \end{array}\right]. \tag{20}$$

The following result is then a consequence of our earlier discussions.

Corollary 5: *There exists a bordering* (F,G) *for a quadruple* (E,A,B,C) *such that the embedded pencil* (20) *is unimodular, if and only if the embedding of the pencil*

$$\left[\begin{array}{cc} \lambda E-A & \lambda C-B \\ F & G \end{array}\right]$$

is regular and has all its finite generalized eigenvalues at $\lambda=0$.

Proof. This follows easily from the transformation $\mu=1/\lambda$ and the identity (20). The finite roots of the pencil $\left[\begin{array}{cc} E-\mu A & C-\mu B \\ F & G \end{array}\right]$ are indeed the nonzero roots of the pencil $\left[\begin{array}{cc} \lambda E-A & \lambda C-B \\ F & G \end{array}\right]$.

But since $\lambda = 0$ is the only root of the second pencil, then $\mu = \infty$ is the only root of the first pencil, and hence it is unimodular. $\square$

6. Numerical experiments

We used the DeadBeat algorithm described in the appendix on the following random matrices (only 4 digits are given)

$$A = \begin{bmatrix} 0.538 & 0.343 & 0.715 & -1.208 & 0.294 & 1.438 & 0.319 \\ 1.834 & 3.578 & -0.205 & 0.717 & -0.787 & 0.325 & 0.313 \\ -2.259 & 2.769 & -0.124 & 1.630 & 0.888 & -0.755 & -0.865 \\ 0.862 & -1.350 & 1.490 & 0.489 & -1.147 & 1.370 & -0.030 \\ 0.319 & 3.035 & 1.409 & 1.035 & -1.069 & -1.711 & -0.165 \\ -1.308 & 0.725 & 1.417 & 0.727 & -0.809 & -0.102 & 0.628 \\ -0.434 & -0.063 & 0.671 & -0.303 & -2.944 & -0.241 & 1.093 \end{bmatrix}, B = \begin{bmatrix} 1.109 & -0.770 \\ -0.864 & 0.371 \\ 0.077 & -0.226 \\ -1.214 & 1.117 \\ -1.113 & -1.089 \\ -0.007 & 0.033 \\ 1.533 & 0.552 \end{bmatrix},$$

of dimension 7×7 and 7×2. The Matlab call

```
[Av,Bv,Fv,V,nk]=DeadBeat(A,B,tol)
```

returned the $k = 4$ indices

$$n_1 = 2, n_2 = 2, n_3 = 2, n_4 = 1$$

and the closed loop matrix

$$A_v - B_v F_v = \begin{bmatrix} 0.000 & 0.000 & -1.615 & -1.035 & -1.656 & -0.605 & 7.146 \\ 0.000 & 0.000 & -3.298 & -0.270 & -1.512 & -0.863 & 3.878 \\ 0.000 & 0.000 & 0.000 & 0.000 & -0.4820 & -3.849 & 0.064 \\ 0.000 & 0.000 & 0.000 & 0.000 & 3.5724 & -1.728 & 0.188 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & -6.181 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.040 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix},$$

and the norms of the successive powers $(A_v - B_v F_v), i = 1, \ldots, k$ gave

$$11.7737, 36.0680, 85.5020, 7.8802e - 013,$$

indicating that the matrix $A_v - B_v F_v$ is *essentially* nilpotent. The Matlab code implementing this method is described in the Appendix. Notice that this code is not optimized for speed, since it uses the singular value decomposition for the rank tests while an efficient implementation should use rank revealing QR factorizations as well as low rank updating techniques for such decompositions. We also did not give codes for the generalized deadbeat problem or the pencil embedding problem, since these are just variations on the same theme.

7. Concluding remarks

We considered the problem of finding a square low rank correction $(\lambda C - B)F$ to a given square pencil $(\lambda E - A)$ such that the new pencil $\lambda(E - CF) - (A - BF)$ has all its generalized eigenvalues at the origin, or more generally such that the bordered pencil

$$\begin{bmatrix} \lambda E - A & \lambda C - B \\ F & G \end{bmatrix}$$

has a determinant equal to λ^n (i.e. has n generalized eigenvalues at the origin). This problem was shown to be closely related to the (generalized) deadbeat control problem and to the eigenvalue assignment problem for embedded pencils. For both these problems we have derived necessary and sufficient conditions for the existence of a solution and we have presented numerical algorithms to solve them. These results in fact extend earlier results of the first author (see Van Dooren (1984), Boley et al. (1994), Beelen et al. (1988)) to the more general setting of controllable – but not necessarily reachable – systems.

The motivation for revisiting this problem, came from a paper of Gander et al. (see Gander et al. (2012)), showing that the convergence speed of block-iterative methods for PDEs can be significantly accelerated by using preconditioners based on this nilpotent pencil correction problem. These techniques should be useful as well for these kind of problems. A second application is the unimodular embedding problem described at the end of Section 5. If the rows of a pencil $\begin{bmatrix} E - \mu A & C - \mu B \end{bmatrix}$ have full row rank for all finite μ then there exists a unimodular embedding of this pencil, as shown in that section. Finding the embedding with Jordan chains at $\mu = \infty$ of minimal degree k, will yield a unimodular embedding whose unimodular inverse has an as small as possible degree k. Both these problems clearly are of interest, both from a theoretical and numerical point of view.

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8. Appendix

```
function [Av,Bv,Fv,V,nk] = DeadBeat(A,B,tol)
%
% Function [Av,Bv,Fv,V,nk]=DeadBeat(A,B,tol)
% computes the largest kernel of (A-B*F)^k for a pair (A,B).
% That kernel is returned as the first sum(nk) columns of
% the orthogonal matrix V.
% The transformed matrices Av=V'*A*V, Bv=V'*B and
% Fv=F*V are returned as well as the Weyr characteristic
% of the closed loop system (A+BF) via the integer vector nk.
% The rank decisions are using the given tolerance tol.
%
[n1,n2]=size(A);[n,m]=size(B);
if n~=n1, disp('Incorrect dimensions'),return; end
if n~=n2, disp('Incorrect dimensions'),return; end
% Initialize Av, Bv, Fv, V and nk
Av=A;Bv=B;Fv=zeros(m,n);V=eye(n,n);nk=[];one=1;
while n >= one,
    ncur=n-one+1;
    % Compress rows of B
    [U,S,W]=svd(B);r=rank(S,tol);
    % Treat the extreme cases for r (0 and n) separately
    if r==ncur,   % This is a controllable case
        Fv(:,one:n)=pinv(B)*A; nk=[nk,ncur]; break;
    end
    if r==0,
        [U,S,Vup]=svd(Av);c=rank(S,tol);
        if c==n, disp('Uncontrollable'); break; end
        if c==0,   % This is also a controllable case
            nk=[nk,ncur]; break;
        end
        % Here we found a nontrivial Weyr index
        Vup=Vup(:,[c+1:ncur,1:c]);
        V(:,one:n)=V(:,one:n)*Vup;
        Av(:,one:n)=Av(:,one:n)*Vup;
        Av(one:n,:)=Vup'*Av(one:n,:);
        Bv(one:n,:)=Vup'*Bv(one:n,:);
        one=one+ncur-c; nk=[nk,ncur-c]; ncur=c;
        break;
    end
    % Now treat the general case 0<r<n
    U1=U(:,1:r);B1=U1'*B;
    UA=U'*A;UA1=UA(1:r,:);UA2=UA(r+1:ncur,:);
    [U,S,Vup]=svd(UA2);c=rank(S,tol);
    Fv(:,one:one-1+ncur-c)=pinv(B1)*UA1*Vup(:,c+1:ncur);
    Vup=Vup(:,[c+1:ncur,1:c]);
    V(:,one:n)=V(:,one:n)*Vup;
    Av(:,one:n)=Av(:,one:n)*Vup;
    Av(one:n,:)=Vup'*Av(one:n,:);
    Bv(one:n,:)=Vup'*Bv(one:n,:);
    % Now define the subsystem (A,B) to continue on
    one=one+ncur-c; nk=[nk,ncur-c]; ncur=c;
    A=Av(one:n,one:n)
    B=Bv(one:n,:)
end
return
```