On the Stability of the Generalized Schur Algorithm *

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Abstract. The generalized Schur algorithm (GSA) is a fast method to compute the Cholesky factorization of a wide variety of structured matrices. The stability property of the GSA depends on the way it is implemented. In [15] GSA was shown to be as stable as the Schur algorithm, provided one hyperbolic rotation in factored form [3] is performed at each iteration. Fast and efficient algorithms for solving Structured Total Least Squares problems [14, 13] are based on a particular implementation of GSA requiring two hyperbolic transformations at each iteration. In this paper the authors prove the stability property of such implementation provided the hyperbolic transformation are performed in factored form [3].

1 Introduction

The generalized Schur algorithm (GSA) is a fast method to compute the Cholesky decomposition of a wide variety of symmetric positive definite structured matrices, i.e., block–Toeplitz and Toeplitz–block matrices, matrices of the form $T^T T$, where T is a rectangular Toeplitz matrix [9, 7] and to compute the LDL^T factorization of *strongly regular* [1] structured matrices, where L is a triangular matrix and $D = \text{diag}(\pm 1, \ldots, \pm 1)$. The stability property of the GSA depends on the

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way it is implemented [11, 5]. In [15] GSA was shown to be stable, provided one hyperbolic rotation in factored form [3] is performed at each iteration. Similar results were obtained in [5], using the *OD procedure* or the *H procedure* instead of using the hyperbolic rotations in factored form. The computational complexity of GSA is $\mathcal{O}(\alpha N^2)$, where N is the order of the involved matrix and α is its displacement rank (see §2).

The Structured Total Least Squares problem described in [14], can be formulated in this way,

$$\begin{aligned}&\min_{\Delta A, \Delta b, x} \|[\Delta A \ \ \Delta b]\|_F^2 \\ &\text{such that } \ (A+\Delta A)x = b + \Delta b, \ A, \Delta A \in \mathbb{R}^{m\times n}, m \gg n,\end{aligned}$$

with $A, \Delta A$ Toeplitz matrices. The kernel of the algorithm proposed in [14], is the solution of a least squares problem, where the coefficient matrix is a rectangular Toeplitz–block matrix, with dimensions $(2m+n-1)\times(m+2n-1)$. Taking the structure of the generators into account, the complexity of GSA can be reduced to $\mathcal{O}(\alpha mn)$, if two hyperbolic transformations are performed at each iteration. Allowing only one hyperbolic rotation in each iteration, the computational complexity is $\mathcal{O}(\alpha(m+n)^2)$. Hence is worth studying the stability of such an implementation of GSA. In this paper the stability properties of such an implementation are investigated. The paper is organized as follows. In §2 an implementation of GSA requiring two hyperbolic rotations per iteration is described. The stability property of this implementation is analyzed in §3 and in §4 are the conclusions.

2 The Generalized Schur Algorithm

In this section we introduce GSA to compute the $R^T R$ factorization of a symmetric positive definite matrix A, where R is an upper triangular matrix, when two hyperbolic rotations are performed at each iteration.

Given an $n\times n$ symmetric positive definite matrix A, define $D_A = A - ZAZ^T$. We say that the displacement rank of A with respect to Z is α if $\mathrm{rank}(D_A) = \alpha$, where Z is the lower triangular (block) shift matrix of order n (for a more general choice of the matrix Z, see [9, 6]). Clearly D_A will have a decomposition of the form $D_A = G^T J_A G$, where

$$G = \begin{bmatrix} u_{1,1}^{(1)} & {u_{1,2}^{(1)}}^T \\ u_{2,1}^{(1)} & U_{2,2}^{(1)} \\ v_{1,1}^{(1)} & {v_{1,2}^{(1)}}^T \\ z_{1,1}^{(1)} & {z_{1,2}^{(1)}}^T \\ v_{2,1}^{(1)} & V_{2,2}^{(1)} \end{bmatrix}, \quad J_A = I_p \oplus -I_q, \ q = \alpha - p,$$

where $u_{1,1}^{(1)}, v_{1,1}^{(1)}, z_{1,1}^{(1)} \in \mathbb{R}$, $u_{1,2}^{(1)}, v_{1,2}^{(1)}, z_{1,2}^{(1)} \in \mathbb{R}^{n-1}$, $u_{2,1}^{(1)} \in \mathbb{R}^{p-1}$, $v_{2,1}^{(1)} \in \mathbb{R}^{q-2}$, $U_{2,2}^{(1)} \in \mathbb{R}^{(p-1)\times(n-1)}$, $V_{2,2}^{(1)} \in \mathbb{R}^{(q-2)\times(n-1)}$, and I_k is the identity matrix of order

k. The pair $(G, J_A), G \in \mathbb{R}^{\alpha \times n}$ is said to be a generator pair for A [12]. A matrix Θ is said J_A–orthogonal if $\Theta^T J_A \Theta = J_A$.

The GSA requires n iterations to compute the factor R. Let $G_{0,Z} = G$. At the ith iteration, $i = 1, \ldots, n$, a J_A–orthogonal matrix Θ_i is chosen such that the ith column of $G_i = \Theta_i G_{i-1,Z}$ has all the elements equal to zero with the exception of a single pivot element in the first row (the first $i-1$ columns of G_i are zero). The generator matrix G_i is said to be in a proper form. Then the first row of G_i becomes the ith row of R. The generator matrix $G_{i,Z}$ at the next iteration is given by

$$G_{i,Z}(1,:) = G_i(1,:)Z^T, \quad G_{i,Z}([2:\alpha],:) = G_i([2:\alpha],:).$$

Without loss of generality, the matrices $\Theta_i,\ i = 1, \ldots, n$, can be factored as the product of two hyperbolic rotations and an orthogonal one, i.e.,

$$\Theta_i = H_{i,1} H_{i,2} Q_i, \quad \text{where} \quad Q_i = \left[\begin{array}{c|c} Q_{i,1} & \\ \hline & Q_{i,2} \end{array}\right],$$

with $Q_{i,1}$ and $Q_{i,2}$ orthogonal matrices of order p and q, such that

$$Q_i G_{i-1,Z} = \begin{bmatrix} 0_{i-1}^T & u_{1,1}^{(i,1)} & {u_{1,2}^{(i,1)}}^T \\ 0_{p-1,i-1} & 0_{p-1} & U_{2,2}^{(i,1)} \\ {0_{i-1}}^T & v_{1,1}^{(i,1)} & {v_{1,2}^{(i,1)}}^T \\ 0_{i-1}^T & z_{1,1}^{(i,1)} & {z_{1,2}^{(i,1)}}^T \\ 0_{q-2,i-1} & 0_{q-2} & V_{2,2}^{(i,1)} \end{bmatrix} \tag{1}$$

respectively, and

$$H_{i,1} = \frac{1}{\sqrt{1-\rho_{i,1}^2}} \begin{bmatrix} 1 & 0_{p-1}^T & \rho_{i,1} & 0_{q-1}^T \\ 0_{p-1} & I_{p-1} & 0 & 0_{p-1,q-1} \\ \rho_{i,1} & 0_{p-1}^T & 1 & 0_{q-1}^T \\ 0_{q-1} & 0_{q-1,p-1} & 0_{q-1} & I_{q-1} \end{bmatrix},$$

$$H_{i,2} = \frac{1}{\sqrt{1-\rho_{i,2}^2}} \begin{bmatrix} 1 & 0_p^T & \rho_{i,2} & 0_{q-2}^T \\ 0_p & I_p & 0 & 0_{p,q-2} \\ \rho_{i,2} & 0_p^T & 1 & 0_{q-2}^T \\ 0_{q-2} & 0_{q-2,p} & 0_{q-2} & I_{q-2} \end{bmatrix},$$

$|\rho_{i,1}|, |\rho_{i,2}| < 1$, such that

$$G_i = H_{i,1} H_{i,2} Q_i G_{i-1,Z} = \begin{bmatrix} 0_{i-1}^T & u_{1,1}^{(i,2)} & {u_{1,2}^{(i,2)}}^T \\ 0_{p-1,i-1} & 0_{p-1} & U_{2,2}^{(i,2)} \\ {0_{i-1}}^T & 0 & {v_{1,2}^{(i,2)}}^T \\ 0_{i-1}^T & 0 & {z_{1,2}^{(i,2)}}^T \\ 0_{q-2,i-1} & 0_{q-2} & V_{2,2}^{(i,2)} \end{bmatrix},$$

where $0_{r,s}$ denotes the rectangular null matrix wit r rows and s columns. As mentioned in §1, the computation of the hyperbolic rotation in a stable way is crucial for the stability of the algorithm. For implementation details of hyperbolic rotations in factored form see [3, 15]. In the next section we will show that GSA is stable, provided in each iteration the J_A–orthogonal matrix is computed as previously described, and the hyperbolic rotations are implemented in factored form. Similar stability results hold considering either the H–procedure or the OD–procedure to implement the hyperbolic rotations [5, 12].

3 Stability Analysis

A stability analysis of the GSA with a single hyperbolic rotation in factored form per iteration is presented in [15]. The stability analysis for the algorithm described in the previous section can be done in a similar way. It is split up into two parts: one which shows how local error propagates through the algorithm and one which bounds the local error. We consider the same notation as introduced in §2 but denote by the superscript the corresponding quantities as stored in the computer. Hence $\tilde{G}_i = \begin{bmatrix} \tilde{u}_i & \tilde{U}_i^T & \tilde{v}_i & \tilde{z}_i^T & \tilde{V}_i \end{bmatrix}^T$.

The local errors, generated by computing $\tilde{G}_{i+1}$ by means of orthogonal and hyperbolic transformations, are given by

$$\epsilon F_i = \tilde{G}_{i+1}^T J_A \tilde{G}_{i+1} - \tilde{G}_{i,Z}^T J_A \tilde{G}_{i,Z} + O(\epsilon^2), \quad i = 1, \ldots, n, \tag{2}$$

where ϵ is the machine precision. In [15] is proved that

$$A - \tilde{R}^T \tilde{R} = \sum_{j=0}^{n-1} Z_j (G^T J_A G - \tilde{G}^T J_A \tilde{G}) Z_j^T - \epsilon \sum_{j=0}^{n-1} \sum_{k=1}^{n-j-1} Z_j F_k Z_j^T + O(\epsilon^2), \tag{3}$$

where $Z_j = Z^j$ and $\tilde{R}$ is the computed Cholesky factor. This means that if the error in the computation of the initial generator matrix and the local errors are bounded, the algorithm is stable. The error in the initial generator matrix is not a problem, since often it is explicitly known or can be computed in a backward stable way [8]. In the following, we assume that the initial generator matrix is computed exactly and restrict ourselves to the effects of local errors due to the orthogonal and hyperbolic transformations.

Because any bounds on the errors produced by the transformations will depend on the norm of the generators, it is essential to bound the generators.

Theorem 1. *When the generators are computed by applying a block diagonal orthogonal matrix and two hyperbolic transformations, they satisfy*

$$\|G_i\|_F \leq 2\sqrt{i-1}\|A\|_F + \|G\|_F \tag{4}$$

Proof. Let $\hat{u}_i, \hat{v}_i$ and $\hat{z}_i$ be the generator vectors in (1) that will be modified by the two hyperbolic rotations $H_{i,2}$ and $H_{i,1}$,

$$\begin{aligned}
\begin{bmatrix} u_i^T \\ v_i^T \\ z_i^T \end{bmatrix} &= \tfrac{1}{\delta_{i,1}\delta_{i,2}} \begin{bmatrix} 1 & 0 & \rho_{i,2} \\ 0 & \delta_{i,2} & 0 \\ \rho_{i,2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \rho_{i,1} & 0 \\ \rho_{i,1} & 1 & 0 \\ 0 & 0 & \delta_{i,1} \end{bmatrix} \begin{bmatrix} \hat{u}_i^T \\ \hat{v}_i^T \\ \hat{z}_i^T \end{bmatrix} \\
&= \tfrac{1}{\delta_{i,1}\delta_{i,2}} \begin{bmatrix} 1 & \rho_{i,1} & \rho_{i,2}\delta_{i,1} \\ \rho_{i,1}\delta_{i,2} & \delta_{i,2} & 0 \\ \rho_{i,2} & \rho_{i,1}\rho_{i,2} & \delta_{i,1} \end{bmatrix} \begin{bmatrix} \hat{u}_i^T \\ \hat{v}_i^T \\ \hat{z}_i^T \end{bmatrix},
\end{aligned}$$

where $\delta_{i,k} = \sqrt{1-\rho_{i,k}^2},\ k = 1, 2$. Then we have

$$\begin{aligned}
\begin{bmatrix} u_i^T \\ v_i^T \\ z_i^T \end{bmatrix} &= \frac{1}{\delta_{i,1}\delta_{i,2}} \begin{bmatrix} 1 & \rho_{i,1} & \rho_{i,2}\delta_{i,1} \\ \rho_{i,1}\delta_{i,2} & \rho_{i,1}^2\delta_{i,2} & \rho_{i,1}\rho_{i,2}\delta_{i,1}\delta_{i,2} \\ \rho_{i,2} & \rho_{i,1}\rho_{i,2} & \rho_{i,2}^2\delta_{i,1} \end{bmatrix} \begin{bmatrix} \hat{u}_i^T \\ \hat{v}_i^T \\ \hat{z}_i^T \end{bmatrix} \\
&\quad + \frac{1}{\delta_{i,1}\delta_{i,2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta_{i,1}^2\delta_{i,2} & -\rho_{i,1}\rho_{i,2}\delta_{i,1}\delta_{i,2} \\ 0 & 0 & \delta_{i,1}\delta_{i,2}^2 \end{bmatrix} \begin{bmatrix} \hat{u}_i^T \\ \hat{v}_i^T \\ \hat{z}_i^T \end{bmatrix} \\
&= \begin{bmatrix} u_i^T \\ \rho_{i,1}\delta_{i,2}u_i^T \\ \rho_{i,2}u_i^T \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta_{i,1} & -\rho_{i,1}\rho_{i,2} \\ 0 & 0 & \delta_{i,2} \end{bmatrix} \begin{bmatrix} \hat{u}_i^T \\ \hat{v}_i^T \\ \hat{z}_i^T \end{bmatrix}.
\end{aligned}$$

Consider the Givens rotations

$$\begin{aligned}
U &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_l & -s_l \\ 0 & s_l & c_l \end{bmatrix}, \quad \begin{cases} c_l = \rho_{i,2}/\sqrt{1-\delta_{i,1}^2\delta_{i,2}^2} \\ s_l = \rho_{i,1}\delta_{i,2}/\sqrt{1-\delta_{i,1}^2\delta_{i,2}^2} \end{cases} \\
V &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_r & -s_r \\ 0 & s_r & c_r \end{bmatrix}, \quad \begin{cases} c_r = \rho_{i,2}\delta_{i,1}/\sqrt{1-\delta_{i,1}^2\delta_{i,2}^2} \\ s_r = \rho_{i,1}/\sqrt{1-\delta_{i,1}^2\delta_{i,2}^2} \end{cases}.
\end{aligned}$$

Then

$$\begin{aligned}
U \begin{bmatrix} u_i^T \\ v_i^T \\ z_i^T \end{bmatrix} &= U \begin{bmatrix} u_i^T \\ \rho_{i,1}\delta_{i,2}u_i^T \\ \rho_{i,2}u_i^T \end{bmatrix} + U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta_{i,1} & -\rho_{i,1}\rho_{i,2} \\ 0 & 0 & \delta_{i,2} \end{bmatrix} V^T V \begin{bmatrix} \hat{u}_i^T \\ \hat{v}_i^T \\ \hat{z}_i^T \end{bmatrix} \\
&= \begin{bmatrix} u_i^T \\ 0 \\ \sqrt{1-\delta_{i,1}^2\delta_{i,2}^2}\,u_i^T \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta_{i,1}\delta_{i,2} \end{bmatrix} \begin{bmatrix} \tilde{u}_i^T \\ \tilde{v}_i^T \\ \tilde{z}_i^T \end{bmatrix} \\
&= \begin{bmatrix} u_i^T \\ \tilde{v}^T \\ \sqrt{1-\delta_{i,1}^2\delta_{i,2}^2}\,u_i^T + \delta_{i,1}\delta_{i,2}\tilde{z}_i^T \end{bmatrix},
\end{aligned}$$

where

$$\begin{bmatrix} \tilde{u}_i^T \\ \tilde{v}_i^T \\ \tilde{z}_i^T \end{bmatrix} = V \begin{bmatrix} \hat{u}_i^T \\ \hat{v}_i^T \\ \hat{z}_i^T \end{bmatrix}.$$

Then

$$\left\| U \begin{bmatrix} u_i^T \\ v_i^T \\ z_i^T \end{bmatrix} \right\|_F^2 = \left\| \begin{bmatrix} u_i^T \\ v_i^T \\ z_i^T \end{bmatrix} \right\|_F^2 = \left\| \begin{bmatrix} u_i^T \\ \tilde{v}_i^T \\ \sqrt{1-\delta_{i,1}^2\delta_{i,2}^2} u_i^T + \delta_{i,1}\delta_{i,2}\tilde{z}_i^T \end{bmatrix} \right\|_F^2 ,$$

Applying the inequality

$$\left\| [\sqrt{1-\alpha^2}, \alpha] \begin{bmatrix} u_i^T \\ \tilde{z}_i^T \end{bmatrix} \right\|_2^2 \le \left\| \begin{bmatrix} u_i^T \\ \tilde{z}_i^T \end{bmatrix} \right\|_2^2$$

with $\alpha = \delta_{i,1}\delta_{i,2}$, $|\alpha| \le 1$, we finally obtain

$$\left\| \begin{bmatrix} u_i^T \\ v_i^T \\ z_i^T \end{bmatrix} \right\|_F^2 \le 2\|u_i\|_2^2 + \left\| \begin{bmatrix} \tilde{v}_i^T \\ \tilde{z}_i^T \end{bmatrix} \right\|_F^2 = 2\|u_i\|_2^2 + \left\| \begin{bmatrix} \hat{v}_i^T \\ \hat{z}_i^T \end{bmatrix} \right\|_F^2 \le 2\|u_i\|_2^2 + \left\| \begin{bmatrix} \hat{u}_i^T \\ \hat{v}_i^T \\ \hat{z}_i^T \end{bmatrix} \right\|_F^2 .$$

Since the orthogonal transformations don't affect the norm of the generators and $\|Z\|_2 = 1$, then $\|G_i\|_F^2 \le 2\|u_i\|^2 + \|G_{i-1}\|_F^2$, and recursively we have

$$\|G_i\|_F^2 \le 2\sum_{j=1}^{i} \|u_j\|_2^2 + \|G\|_F^2 = 2\|R(1:k,:)\|_F^2 + \|G\|_F^2.$$

Then (4) follows since, for an arbitrary positive semi–definite, rank $i-1$ matrix with a factorization $A = R^T R$, (see [15]), $\|R\|_F^2 \le \sqrt{i}\|A\|_F^2$ $\square$

To complete the stability analysis we need to show that the orthogonal and hyperbolic transformations, applied in factored form, produce a local error, ϵF_i, which is proportional to the norm of the generator matrix. An error analysis of hyperbolic transformations applied in factored form is given in [3]. Denoted by

$$H_{i,j} = \frac{1}{\sqrt{1-\rho_{i,j}^2}} \begin{bmatrix} 1 & 0 \\ \rho_{i,j} & \sqrt{1-\rho_{i,j}^2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1-\rho_{i,j}^2}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \rho_{i,j} \\ 0 & 1 \end{bmatrix}, \quad j = 1, 2,$$

the hyperbolic transformations applied in factored form, then

$$\begin{bmatrix} \tilde{u}_{1,i+1}^T + \widehat{\Delta u}^T \\ \tilde{v}_{1,i+1}^T \\ \tilde{z}_{1,i+1}^T \end{bmatrix} = H_{i,2}H_{i,1} \begin{bmatrix} \hat{u}_{1,i}^T \\ \hat{v}_{1,i}^T + \widehat{\Delta v}^T \\ \hat{z}_{1,i}^T + \widehat{\Delta z}^T \end{bmatrix}. \tag{5}$$

The mixed error vectors $\widehat{\Delta u}$, $\widehat{\Delta v}$ and $\widehat{\Delta z}$ satisfy

$$\left\| \begin{bmatrix} \widehat{\Delta u}^T \\ \widehat{\Delta v}^T \\ \widehat{\Delta z}^T \end{bmatrix} \right\|_F \leq 12.5\epsilon \left\| \begin{bmatrix} \tilde{u}_{1,i+1}^T \\ \hat{v}_{1,i}^T \\ \hat{z}_{1,i}^T \end{bmatrix} \right\|_F , \tag{6}$$

where ϵ is the roundoff unit. Furthermore, concerning the application of the orthogonal transformations, it can be proved [15, 16] that there exist orthogonal matrices $\hat{Q}_{i,1}$ and$\hat{Q}_{i,2}$ such that

$$\left[\begin{array}{c|c} \hat{Q}_{i,1} & \\ \hline & \hat{Q}_{i,2} \end{array}\right] \left[\begin{array}{c} \tilde{u}_{1,i}^T + \Delta u_1^T \\ \tilde{U}_{2,i} + \Delta U_2 \\ \hline \tilde{v}_{1,i}^T + \Delta v_1 \\ \tilde{z}_{1,i}^T + \Delta z_1 \\ \tilde{V}_{2,i} + \Delta V_2 \end{array}\right] = \left[\begin{array}{c} \hat{u}_{1,i}^T \\ \hat{U}_{2,i} \\ \hline \hat{v}_{1,i}^T \\ \hat{z}_{1,i}^T \\ \hat{V}_{2,i} \end{array}\right] = \hat{G}_i, \tag{7}$$

where, for $m = \max\{p, q-1\}$,

$$\left\| \begin{bmatrix} \Delta u_1^T \\ \Delta U_2 \end{bmatrix} \right\|_F \leq 6m\epsilon \left\| \begin{bmatrix} \tilde{u}_{1,i}^T Z^T \\ \tilde{U}_{2,i} \end{bmatrix} \right\|_F \text{ and } \left\| \begin{bmatrix} \Delta v_1^T \\ \Delta z_1^T \\ \Delta V_2 \end{bmatrix} \right\|_F \leq 6m\epsilon \left\| \begin{bmatrix} \tilde{v}_{1,i}^T Z^T \\ \tilde{z}_{1,i}^T Z^T \\ \tilde{V}_{2,i} \end{bmatrix} \right\|_F .$$

Letting $\Delta G_i = [\Delta u_1 \ \Delta U_2^T \ \Delta v_1 \ \Delta z_1 \ \Delta V_2^T]^T$, then $\|\Delta G_i\|_F \leq 6m\epsilon \|G_{i,Z}\|_F \leq 6m\epsilon\|G_i\|_F$. Analogously, letting $\widehat{\Delta G_i} = [\widehat{\Delta u} \ \widehat{\Delta v} \ \widehat{\Delta z}]^T$, then the error bounds (5) and (7) can be used to show that

$$\begin{aligned} \hat{G}_i^T J_A \hat{G}_i &= (\tilde{G}_{i,Z} + \Delta G_i)^T J_A (\tilde{G}_{i,Z} + \Delta G_i), \\ (\tilde{G}_{i+1} + e_1 \widehat{\Delta u}^T)^T J_A (\tilde{G}_{i+1} + e_1 \widehat{\Delta u}^T) &= (\hat{G}_i^T + e_{p+1}\widehat{\Delta v}^T + e_{p+2}\widehat{\Delta z}^T)^T J_A \\ &\quad \times (\hat{G}_i^T + e_{p+1}\widehat{\Delta v}^T + e_{p+2}\widehat{\Delta z}^T), \end{aligned}$$

where e_1, e_{p+1} and e_{p+2} are standard basis vectors. Then

$$\epsilon F_i = \tilde{G}_{i,Z}^T J_A \Delta G_i + \Delta G_i^T J_A \tilde{G}_{i,Z} - [\tilde{u}_{1,i+1} \ \hat{v}_{1,i} \ \hat{z}_{1,i}] \widehat{\Delta G}_i - \widehat{\Delta G}_i^T \begin{bmatrix} \tilde{u}_{1,i+1}^T \\ \hat{v}_{1,i}^T \\ \hat{z}_{1,i}^T \end{bmatrix},$$

corresponding to the bound

$$\begin{aligned} \|\epsilon F_i\|_F &\leq 2\|\tilde{G}_{i,Z}\|_F \|\Delta G_i\|_F + 2\left(\|\hat{G}_i\|_F + \|\tilde{G}_{i+1}\|_F\right) \|\widehat{\Delta G}_i\|_F \\ &\leq 12m\epsilon \|\tilde{G}_{i,Z}\|_F^2 + 25\epsilon \left(\|\hat{G}_i\|_F + \|\tilde{G}_{i+1}\|_F\right)^2 \\ &\leq 12m\epsilon \|G_i\|_F^2 + 25\epsilon(\|G_i\|_F + \|G_{i+1}\|_F)^2 + O(\epsilon^2). \end{aligned}$$

By Theorem 1, $\|G_i\|_F^2, \|G_{i+1}\|_F^2 \leq 2\sqrt{i}\|A\|_F + \|G\|_F^2$, the following bound holds

$$\|\epsilon F_i\|_F \leq (12m + 100)\epsilon \left(2\sqrt{i}\|A\|_F + \|G\|_F^2\right).$$

From (3), we have $\|A - \tilde{R}^T \tilde{R}\|_F \leq (6m + 50)(n-1)n\epsilon \left(2\sqrt{n}\|A\|_F + \|G\|_F^2\right)$.

4 Conclusion

Fast and efficient algorithms for solving Structured Total Least Squares problems [14, 13] are based on a particular implementation of GSA requiring two hyperbolic transformations at each iteration.

In this paper the stability of such implementation is discussed. It is proved that if the hyperbolic transformations are performed in factored form, the considered implementation is as stable as the implementation studied in [15] that requires only one hyperbolic transformation at each iteration.

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