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Model reduction of linear time-varying systems over finite horizons

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ABSTRACT

We consider the problem of approximating a linear time-varying $p \times m$ discrete-time state space model $\mathcal{S}$ of high dimension by another linear time-varying $p \times m$ discrete-time state space model $\widehat{\mathcal{S}}$ of much smaller dimension, using an error criterion defined over a finite time interval. We derive the gradients of the norm of the approximation error and show how this can be solved via a fixed point iteration. We compare this to the classical $\mathcal{H}_2$ norm approximation problem for the infinite horizon time-invariant case and show that our solution extends this to the time-varying and finite horizon case.

1. Introduction

Model reduction is a key analytical and computational component for simulation, design and control of dynamical systems. Theoretical and computational frameworks for the design and analysis of efficient and effective algorithms have been developing steadily over the last few decades. For linear time-invariant dynamical systems there is an extensive and detailed literature and several efficient algorithms for large scale systems. An excellent review of the basic theory and algorithms can be found in the book by Antoulas [1]. Antoulas et al., [2], provide an up-to-date survey of methods for linear time-invariant systems using the interpolatory framework and related projection-based algorithms for computational efficiency.

Recently, there has been an important addition to the interpolatory framework for model reduction. Baur et al., [3], adapted the tangential interpolation framework of [8] to a Hermite tangential interpolation approach and combined it with algorithmic ideas from the efficient $\mathcal{H}_2$ norm model reduction algorithm IRKA, [9], to address the important problem of model reduction of parameterized linear time-invariant systems.

For general nonlinear time-varying and/or parameterized differential equations, proper orthogonal decompositions (PODs) have been used to create reduced order models based on observed trajectories of the solution. Chaturantabut and Sorensen have improved dramatically the efficiency and effectiveness of trajectory-based methods with the recent development of their Discrete Empirical Interpolation Method (DEIM) [6]. Their approach addresses a key problem of PODs: the computational cost of evaluating the nonlinear term in the reduced order model is not necessarily lower than the corresponding term in the full model.

In this paper, we consider the case of model reduction of a linear time-varying dynamical system. These systems have not received quite as much attention as linear time-invariant systems but there have been several attempts to develop an efficient approach to model reduction.

For continuous-time systems, Imae et al., [10], developed the idea of μ-balancing solutions. Rather than computing balancing transformations at each point in time, Riccati equations associated with the balancing problem are solved and under appropriate conditions converge to balancing transformations with accuracy related to the choice of the parameter μ.

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Shokoohi, Silverman and Van Dooren considered continuous-time systems in [16] and [17]. They investigated the development of a uniformly balanced realization of the system and generated a reduced order model by truncation. Of key interest in the work is the investigation of the preservation of stability, controllability and observability in the reduced order model. Shokoohi and Silverman applied the idea of truncated uniformly balanced realizations to discrete-time systems in [15]. They also explored an alternative algorithm based on a Hankel matrix approach and showed the results to be equivalent. In [15] the relationship between the Hankel operator of the system and balanced truncation is explored. Model reduction based on optimization of the approximation in the Hankel norm has been investigated in [7] by Dewilde and van der Veen.

Recently, Lall and Beck, [11] have considered model reduction of time-varying discrete-time systems. They develop balanced realizations using generalized Gramians defined using linear matrix inequalities, specifically Lyapunov inequalities. The realizations are truncated to produce a reduced order model. Error bounds are derived in an induced norm and a procedure for robust synthesis of reduced order models that satisfy desired error bounds is presented.

Sandberg and Rantzer have considered balanced truncation of time-varying systems for both continuous- and discrete-time using Lyapunov inequalities to define controllability and observability Gramians in [14]. They derive generalized error bounds that include those of Lall and Beck and show that input-output stability of the full model is preserved in the reduced order models.

While the approaches above produce a reduced order model for an infinite horizon discrete- or continuous-time system, the associated algorithms have not been shown conclusively to be efficient for large scale systems. Chahlaoui and Van Dooren approached the problem from the point of view of their earlier work on efficiently tracking certain subspaces as they evolve over time. In [5], they propose two methods: the Recursive Low-rank Gramian method (RLRG) and the Recursive Low-rank Hankel method (RLRH). At each instance in time, an efficient update is computed to the dominant subspaces required to form low-rank approximations to the Gramians and to the Hankel map defining the input-output relationship, respectively. Each of these can be used to define a local projection to a reduced order model. Error bounds are derived and the methods are compared to each other and to balanced truncation. It is seen that the RLRH method is preferred over RLRG due to less sensitivity and that RLRH compares favorably with balanced truncation.

We follow an approach similar in spirit to that of RLRH and RLRG in that our objective is to develop a simple algorithm based on a local finite horizon view of the problem and the parameters of the model, i.e., data from a particular solution trajectory is not used. We, however, use a well-defined local optimization approach to model reduction and therefore control directly the accuracy achieved by the reduced order model.

Specifically, we consider the problem of approximating a Linear Time-Varying (LTV) $p \times m$ state space model

$$\mathcal{S} := \begin{cases} x_{k+1} = A_k x_k + B_k u_k, \\ y_k = C_k x_k, \end{cases} \tag{1}$$

with a state x_k of dimension N by another LTV $p \times m$ state-space model

$$\widehat{\mathcal{S}} := \begin{cases} \widehat{x}_{k+1} = \widehat{A}_k \widehat{x}_k + \widehat{B}_k u_k, \\ \widehat{y}_k = \widehat{C}_k \widehat{x}_k \end{cases} \tag{2}$$

with a state $\widehat{x}_k$ of dimension $n \ll N$. If we initialize both systems at time k_i with initial states $x_{k_i} = 0$ and $\widehat{x}_{k_i} = 0$, then over a finite horizon $[k_i, k_f]$, (1) and (2) define linear mappings, respectively from the input sequence u_{k_i,k_f-1} to y_{k_i+1,k_f} and from u_{k_i,k_f-1} to $\widehat{y}_{k_i+1,k_f}$, where

$$u_{k_i,k_f-1} := \begin{bmatrix} u_{k_i} \\ \vdots \\ u_{k_f-1} \end{bmatrix}, \qquad y_{k_i+1,k_f} := \begin{bmatrix} y_{k_i+1} \\ \vdots \\ y_{k_f} \end{bmatrix}, \qquad \widehat{y}_{k_i+1,k_f} := \begin{bmatrix} \widehat{y}_{k_i+1} \\ \vdots \\ \widehat{y}_{k_f} \end{bmatrix}.$$

Denoting these mappings by S and $\widehat{S}$, respectively, we are addressing the problem

$$\min \|S - \widehat{S}\|_F^2, \qquad y_{k_i+1,k_f} = S u_{k_i,k_f-1}, \qquad \widehat{y}_{k_i+1,k_f} = \widehat{S} u_{k_i,k_f-1},$$

where we use the Frobenius norm rather than the 2-norm, because it is easier to compute its gradient, and hence to formulate a gradient-based optimization method. Notice also that we can construct an "error" system $\mathcal{S}_e$ with extended state $x_{e,k} = \begin{bmatrix} x_k \\ \widehat{x}_k \end{bmatrix}$, input u_k and output $e_k := y_k - \widehat{y}_k$

$$\mathcal{S}_e := \begin{cases} x_{e,k+1} = A_{e,k} x_{e,k} + B_{e,k} u_k \\ e_k = C_{e,k} x_{e,k} \end{cases} \tag{3}$$

where

$$A_{e,k} := \begin{bmatrix} A_k & \\ & \widehat{A}_k \end{bmatrix}, \qquad B_{e,k} = \begin{bmatrix} B_k \\ \widehat{B}_k \end{bmatrix}, \qquad C_{e,k} = \begin{bmatrix} C_k & -\widehat{C}_k \end{bmatrix}. \tag{4}$$

Its initial state at time k_i is also given by $x_{e,k_i} = 0$. Defining $S_e := S - \widehat{S}$ as the linear mapping of the error model, the problem is then equivalent to

$$\min \|S_e\|_F^2, \qquad e_{k_i+1,k_f} := y_{k_i+1,k_f} - \widehat{y}_{k_i+1,k_f} = S_e u_{k_i,k_f-1}.$$

Notice that we have chosen not to have any direct feed-through terms (D_k and $\widehat{D}_k$) in the systems (1) and (2) since by choosing $D_k = \widehat{D}_k$ this term disappears in the error system $\mathcal{S}_e$ anyway. This also explains why the output vector e_{k_i+1,k_f} is shifted in time with respect to the input vector u_{k_i,k_f-1}.

In order to clarify the link with the $\mathcal{H}_2$-norm, we first recall the equivalent time-invariant case problem, where the dynamical systems are simplified to

$$\mathcal{S} := \begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k \end{cases} \tag{5}$$

and

$$\widehat{\mathcal{S}} := \begin{cases} \widehat{x}_{k+1} = \widehat{A}\widehat{x}_k + \widehat{B}u \\ \widehat{y}_k = \widehat{C}\widehat{x}_k. \end{cases} \tag{6}$$

The time-invariant error system is now given by

$$\mathcal{S}_e := \begin{cases} x_{e,k+1} = A_e x_{e,k} + B_e u_k \\ e_k = C_e x_{e,k} \end{cases}$$

with time-invariant coefficient matrices

$$A_e := \begin{bmatrix} A & \\ & \widehat{A} \end{bmatrix}, \qquad B_e = \begin{bmatrix} B \\ \widehat{B} \end{bmatrix}, \qquad C_e = \begin{bmatrix} C & -\widehat{C} \end{bmatrix}.$$

Since the system is time-invariant one can consider as well the infinite horizon problem, with zero initial state at time $k_i = -\infty$. It is then useful to formulate this problem using the transfer functions of these systems, which are respectively given by

$$H(z) = C(zI - A)^{-1}B, \quad \widehat{H}(z) = \widehat{C}(zI - \widehat{A})^{-1}\widehat{B} \quad \text{and} \quad E(z) := H(z) - \widehat{H}(z).$$

The squared $\mathcal{H}_2$-norm of the error function $E(z)$ is then defined as (see [1])

$$\mathcal{J} := \big\|E(\cdot)\big\|_{\mathcal{H}_2}^2 := \operatorname{tr} \int_0^{2\pi} E\big(e^{j\omega}\big) E\big(e^{j\omega}\big)^H \frac{d\omega}{2\pi} = \operatorname{tr} \sum_{k=0}^{\infty} \big(C_e A_e^k B_e\big)\big(C_e A_e^k B_e\big)^T. \tag{7}$$

This can be rewritten in terms of the solutions of the Stein equations

$$A_e P_e A_e^T + B_e B_e^T = P_e, \qquad A_e^T Q_e A_e + C_e^T C_e = Q_e \tag{8}$$

as

$$\mathcal{J} = \operatorname{tr}\big(C_e P_e C_e^T\big) = \operatorname{tr}\big(B_e^T Q_e B_e\big),$$

which are computable quantities (see [1]). Indeed, one can partition the solutions P_e and Q_e to obtain the Stein equations in the form

$$P_e := \begin{bmatrix} P & X \\ X^T & \widehat{P} \end{bmatrix}, \qquad Q_e := \begin{bmatrix} Q & Y \\ Y^T & \widehat{Q} \end{bmatrix},$$

$$\begin{bmatrix} A & \\ & \widehat{A} \end{bmatrix}\begin{bmatrix} P & X \\ X^T & \widehat{P} \end{bmatrix}\begin{bmatrix} A^T & \\ & \widehat{A}^T \end{bmatrix} + \begin{bmatrix} B \\ \widehat{B} \end{bmatrix}\begin{bmatrix} B^T & \widehat{B}^T \end{bmatrix} = \begin{bmatrix} P & X \\ X^T & \widehat{P} \end{bmatrix},$$

$$\begin{bmatrix} A^T & \\ & \widehat{A}^T \end{bmatrix}\begin{bmatrix} Q & Y \\ Y^T & \widehat{Q} \end{bmatrix}\begin{bmatrix} A & \\ & \widehat{A} \end{bmatrix} + \begin{bmatrix} C^T \\ -\widehat{C}^T \end{bmatrix}\begin{bmatrix} C & -\widehat{C} \end{bmatrix} = \begin{bmatrix} Q & Y \\ Y^T & \widehat{Q} \end{bmatrix}.$$

The expansions above can be used to express first order optimality conditions for the squared $\mathcal{H}_2$-norm in terms of the gradients of $\mathcal{J}$ versus $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$. This theorem is originally due to Wilson [19] (see also [18,9,4]):

Theorem 1.1. *The gradients $\nabla_{\widehat{A}}\mathcal{J}$, $\nabla_{\widehat{B}}\mathcal{J}$ and $\nabla_{\widehat{C}}\mathcal{J}$ of $\mathcal{J} := \|E(\cdot)\|_{\mathcal{H}_2}^2$ are given by*

$$\nabla_{\widehat{A}}\mathcal{J} = 2\big(\widehat{Q}\widehat{A}\widehat{P} + Y^T A X\big), \qquad \nabla_{\widehat{B}}\mathcal{J} = 2\big(\widehat{Q}\widehat{B} + Y^T B\big), \qquad \nabla_{\widehat{C}}\mathcal{J} = 2(\widehat{C}\widehat{P} - CX), \tag{9}$$

where

$$A^T Y \widehat{A} - C^T \widehat{C} = Y, \qquad \widehat{A}^T \widehat{Q} \widehat{A} + \widehat{C}^T \widehat{C} = \widehat{Q}, \tag{10}$$

$$\widehat{A} X^T A^T + \widehat{B} B^T = X^T, \qquad \widehat{A} \widehat{P} \widehat{A}^T + \widehat{B} \widehat{B}^T = \widehat{P}. \tag{11}$$

It follows from the gradient expression that the following equations hold at the stationary points (see [18] for a proof).

Theorem 1.2. *At every stationary point of $\mathcal{J}$ where $\widehat{P}$ and $\widehat{Q}$ are invertible, we have the following identities*

$$\widehat{A} = W^T A V, \qquad \widehat{B} = W^T B, \qquad \widehat{C} = C V, \qquad W^T V = I_n \tag{12}$$

with $W := -Y \widehat{Q}^{-1}$, $V := X \widehat{P}^{-1}$ and where X, Y, $\widehat{P}$ and $\widehat{Q}$ satisfy the Stein equations (10)–(11).

We can rewrite this as a projection since we are constructing a projector $\Pi := V W^T$ (implying $W^T V = I_n$) where $V := X \widehat{P}^{-1}$ and $W := -Y \widehat{Q}^{-1}$ are derived from the Stein equations. Those ideas also lead to algorithms. One can view (10)–(12) as two coupled systems of equations

$$(X, Y, \widehat{P}, \widehat{Q}) = F(\widehat{A}, \widehat{B}, \widehat{C}) \quad \text{and} \quad (\widehat{A}, \widehat{B}, \widehat{C}) = G(X, Y, \widehat{P}, \widehat{Q})$$

for which we have a fixed point $(\widehat{A}, \widehat{B}, \widehat{C}) = G(F(\widehat{A}, \widehat{B}, \widehat{C}))$ at every stationary point of $\mathcal{J}(\widehat{A}, \widehat{B}, \widehat{C})$. This automatically suggests an iterative procedure

$$(X, Y, \widehat{P}, \widehat{Q})_{i+1} = F\big((\widehat{A}, \widehat{B}, \widehat{C})_{i+1}\big), \qquad (\widehat{A}, \widehat{B}, \widehat{C})_{i+1} = G\big((X, Y, \widehat{P}, \widehat{Q})_i\big),$$

which is expected to converge to a nearby fixed point. This is essentially the idea behind existing algorithms using Stein equations in their iterations (see [1]) and behind the IRKA algorithm of [9].

In the following section, we extend this to the time-varying case. Since the matrices A_k, B_k and C_k are varying with time we must consider the finite horizon case in order to be tractable.

2. The time-varying approximation problem

We consider the discrete-time linear systems $\mathcal{S}$ and $\widehat{\mathcal{S}}$ given in (1) and (2) and associate with it the "error" system $\mathcal{S}_e$ given in (3) and (4). Its state for initial condition $x^e_{k_i} = 0$ is given by

$$x^e_k = \sum_{i=k_i}^{k-1} \Phi_{k,i+1} B_{e,i} u_i$$

where

$$\Phi_{k+1,i} = A_{e,k} \Phi_{k,i} \quad \text{for } k \geqslant i, \quad \text{and} \quad \Phi_{k,k} = I. \tag{13}$$

We now simplify the notation by using $\tilde{u}$ and $\tilde{e}$ for the inputs and outputs of $\mathcal{S}_e$ over a time interval $[k_i, k_f]$, respectively. This yields

$$\tilde{e} = S_e \tilde{u} \quad \text{where} \quad \tilde{e} := \begin{bmatrix} e_{k_i+1} \\ \vdots \\ \vdots \\ e_{k_f} \end{bmatrix}, \quad \tilde{u} := \begin{bmatrix} u_{k_i} \\ \vdots \\ \vdots \\ u_{k_f-1} \end{bmatrix},$$

and

$$S_e = D_C \Phi_e D_B, \qquad D_C = \begin{bmatrix} C_{e,k_i+1} & & 0 \\ & \ddots & \\ 0 & & C_{e,k_f} \end{bmatrix},$$

$$\Phi_e = \begin{bmatrix} \Phi_{k_i+1,k_i+1} & & 0 \\ \vdots & \ddots & \\ \Phi_{k_f,k_i+1} & \cdots & \Phi_{k_f,k_f} \end{bmatrix}, \qquad D_B = \begin{bmatrix} B_{e,k_i} & & 0 \\ & \ddots & \\ 0 & & B_{e,k_f-1} \end{bmatrix}.$$

We will define the squared $\mathcal{H}_2(k_i, k_f)$ norm, simply denoted by $\mathcal{H}_2$, of the finite horizon time-varying system $\mathcal{S}_e$ as the trace of $S_e^T S_e$ (or the Frobenius norm squared of S_e)

$$\|\mathcal{S}_e\|^2_{\mathcal{H}_2} := \mathcal{J}(k_i, k_f) := \operatorname{tr}\big(S_e^T S_e\big) = \|S_e\|_F^2.$$

Notice that

$$\|\tilde{e}\|_2^2 = \|S_e\tilde{u}\|_2^2 \leqslant \|S_e\|_2^2\|\tilde{u}\|_2^2 \leqslant \|S_e\|_F^2\|\tilde{u}\|_2^2. \tag{14}$$

For LTI systems, one can show that $\|S_e\|_2$ is related to the $\mathcal{H}_\infty$-norm, while $\|S_e\|_F$ is shown here to be related to the $\mathcal{H}_2$-norm. Since the 2-norm is not so simple to differentiate versus its matrix elements, we choose here to minimize $\|S_e\|_F$ in order to guarantee nearby responses for a given input energy.

We now consider how $\mathrm{tr}(S_e^T S_e)$ can be rewritten and possibly minimized versus changes in $\widehat{A}_i, \widehat{B}_i$ and $\widehat{C}_i$. Therefore we define the Gramians $P_{e,k}$ and $Q_{e,k}$ as follows:

$$P_{e,k} := \sum_{i=k_i}^{k-1} (\Phi_{k,i+1}B_{e,i})(\Phi_{k,i+1}B_{e,i})^T, \quad k = k_i+1,\ldots,k_f, \quad P_{e,k_i} = 0,$$

$$Q_{e,k} := \sum_{i=k+1}^{k_f} (C_{e,i}\Phi_{i,k+1})^T(C_{e,i}\Phi_{i,k+1}), \quad k = k_i,\ldots,k_f-1, \quad Q_{e,k_f} = 0.$$

These Gramians satisfy also the difference equations (see [5])

$$\begin{cases} P_{e,k+1} = \begin{bmatrix} A_k & \\ & \widehat{A}_k \end{bmatrix} P_{e,k} \begin{bmatrix} A_k^T & \\ & \widehat{A}_k^T \end{bmatrix} + \begin{bmatrix} B_k \\ \widehat{B}_k \end{bmatrix} \begin{bmatrix} B_k^T & \widehat{B}_k^T \end{bmatrix}, \\ P_{e,k_i} = 0 \end{cases} \tag{15}$$

$$\begin{cases} Q_{e,k-1} = \begin{bmatrix} A_k^T & \\ & \widehat{A}_k^T \end{bmatrix} Q_{e,k} \begin{bmatrix} A_k & \\ & \widehat{A}_k \end{bmatrix} + \begin{bmatrix} C_k^T \\ \widehat{C}_k^T \end{bmatrix} \begin{bmatrix} C_k & \widehat{C}_k \end{bmatrix}. \\ Q_{e,k_f} = 0 \end{cases} \tag{16}$$

They can be seen to correspond to the diagonal blocks of $\Phi_e D_B D_B^T \Phi_e^T$ and $\Phi_e^T D_C^T D_C \Phi_e$:

$$\mathrm{diag}\{\Phi_e D_B D_B^T \Phi_e^T\} = \{P_{e,k_i+1},\ldots,P_{e,k_f}\},$$
$$\mathrm{diag}\{\Phi_e^T D_C^T D_C \Phi_e\} = \{Q_{e,k_i},\ldots,Q_{e,k_f-1}\}.$$

From this it follows that $\mathcal{J}(k_i,k_f)$ can also be written as

$$\mathcal{J}(k_i,k_f) := \sum_{k=k_i+1}^{k_f} \mathrm{tr}\big(C_{e,k}P_{e,k}C_{e,k}^T\big) = \sum_{k=k_i}^{k_f-1} \mathrm{tr}\big(B_{e,k}^T Q_{e,k} B_{e,k}\big) \tag{17}$$

which looks very much like the time-invariant case (in fact, one can relate both quantities in the case of time-invariant systems by letting the window length go to infinity). Let us again partition $P_{e,k}$ and $Q_{e,k}$ as follows

$$P_{e,k} = \begin{bmatrix} P_k & X_k \\ X_k^T & \widehat{P}_k \end{bmatrix} \quad \text{and} \quad Q_{e,k} = \begin{bmatrix} Q_k & Y_k \\ Y_k^T & \widehat{Q}_k \end{bmatrix}.$$

The following theorem gives then the gradients of $\mathcal{J}(k_i,k_f)$ versus $\widehat{A}_k, \widehat{B}_k$ and $\widehat{C}_k$.

Theorem 2.1. *The gradients $\nabla_{\widehat{A}_k}\mathcal{J}$, $\nabla_{\widehat{B}_k}\mathcal{J}$ and $\nabla_{\widehat{C}_k}\mathcal{J}$ of $\mathcal{J}(k_i,k_f) = \|\mathcal{S}_e\|_{\mathcal{H}_2}^2$ are given by*

$$\nabla_{\widehat{A}_k}\mathcal{J} = 2\big(\widehat{Q}_k\widehat{A}_k\widehat{P}_k + Y_k^T A_k X_k\big), \tag{18}$$
$$\nabla_{\widehat{B}_k}\mathcal{J} = 2\big(\widehat{Q}_k\widehat{B}_k + Y_k^T B_k\big), \tag{19}$$
$$\nabla_{\widehat{C}_k}\mathcal{J} = 2(\widehat{C}_k\widehat{P}_k - C_k X_k) \tag{20}$$

where X_k, Y_k, $\widehat{Q}_k$, $\widehat{P}_k$ satisfy the recurrence relations

$$\begin{aligned} X_{k+1} &= A_k X_k \widehat{A}_k^T + B_k\widehat{B}_k^T, \quad k = k_i,\ldots,k_f-1, \quad X_{k_i} = 0, \\ \widehat{P}_{k+1} &= \widehat{A}_k\widehat{P}_k\widehat{A}_k^T + \widehat{B}_k\widehat{B}_k^T, \quad k = k_i,\ldots,k_f-1, \quad \widehat{P}_{k_i} = 0, \\ Y_{k-1} &= A_k^T Y_k \widehat{A}_k - C_k^T\widehat{C}_k, \quad k = k_f,\ldots,k_i+1, \quad Y_{k_f} = 0, \end{aligned} \tag{21}$$

$$\widehat{Q}_{k-1} = \widehat{A}_k^T Q_k \widehat{A}_k + \widehat{C}_k^T\widehat{C}_k, \quad k = k_f,\ldots,k_i+1, \quad \widehat{Q}_{k_f} = 0. \tag{22}$$

Proof. The proofs for $\nabla_{\widehat{B}_k}\mathcal{J}$ and $\nabla_{\widehat{C}_k}\mathcal{J}$ are similar. We therefore only derive it for $\nabla_{\widehat{B}_k}\mathcal{J}$. We use the fact that $\mathcal{J}(k_i,k_f) := \sum_{k=k_i}^{k_f-1}\operatorname{tr}(B_{e,k}^T Q_{e,k} B_{e,k})$ has only one term that depends on variations in $\widehat{B}_k$. The k-th term in this equation can be worked out as

$$\operatorname{tr}\big(B_k^T Q_k B_k + 2B_k^T Y_k \widehat{B}_k + \widehat{B}_k^T \widehat{Q}_k \widehat{B}_k\big),$$

and the gradient of this term is clearly given by $2(\widehat{Q}_k\widehat{B}_k + Y_k^T B_k)$.

The proof for $\nabla_{\widehat{A}_k}\mathcal{J}$ is more involved since $\widehat{A}_k$ appears in many terms of $\operatorname{tr}(S_e^T S_e)$. The matrix $\widehat{A}_k$ enters only in $A_{e,k}$, and hence only in the factor Φ_e of S_e. It follows from the repetitive Hankel structure of Φ_e that its first order perturbation Δ_Φ is given by

$$\Delta_\Phi = \Phi_{:,k+1}\begin{bmatrix}0 & 0\\ 0 & \Delta_{\widehat{A}_k}\end{bmatrix}\Phi_{k,:}$$

where $k = k_i+1,\dots,k_f-1$. The corresponding perturbation Δ_S of S_e then equals $D_C\Delta_\Phi D_B$ which yields

$$\Delta_{\operatorname{tr}(S_e^T S_e)} = 2\operatorname{tr}\big(S_e^T\Delta_S\big) = 2\operatorname{tr}\left(\Phi_{k,:}D_B D_B^T\Phi_e^T D_C^T D_C \Phi_{:,k+1}\begin{bmatrix}0 & 0\\ 0 & \Delta_{\widehat{A}_k}\end{bmatrix}\right).$$

But the expression $\Phi_{k,:}D_B D_B^T\Phi_e^T D_C^T D_C\Phi_{:,k+1}$ is the same as

$$\Phi_{k,:}D_B D_B^T\begin{bmatrix}0 & 0\\ \Phi_{k+1:k_f,k_i+1:k} & 0\end{bmatrix}^T D_C^T D_C\Phi_{:,k+1}$$

since that is the only part of Φ_e that gets multiplied by nonzero elements. From the Hankel structure of Φ_e, it again follows that

$$\begin{bmatrix}0 & 0\\ \Phi_{k+1:k_f,k_i+1:k} & 0\end{bmatrix} = \Phi_{:,k+1}A_{e,k}\Phi_{k,:},$$

which finally yields

$$\Phi_{k,:}D_B D_B^T\Phi_{k,:}^T A_{e,k}^T\Phi_{:,k+1}^T D_C^T D_C\Phi_{:,k+1} = P_{e,k}A_{e,k}^T Q_{e,k}.$$

From this, we obtain

$$\Delta_{\operatorname{tr}(S_e^T S_e)} = 2\operatorname{tr}\left(P_{e,k}A_{e,k}^T Q_{e,k}\begin{bmatrix}0 & 0\\ 0 & \Delta_{\widehat{A}_k}\end{bmatrix}\right) = \operatorname{tr}\big(2\big(X_k^T A_k^T Y_k + \widehat{P}_k\widehat{A}_k^T\widehat{Q}_k\big)\Delta_{\widehat{A}}\big)$$

which expresses that $\nabla_{\widehat{A}_k}\mathcal{J} = 2(Y_k^T A_k X_k + \widehat{Q}_k\widehat{A}_k\widehat{P}_k)$. □

The initialization of the Gramians implies that the gradient with respect to $\widehat{A}_{k_i}, \widehat{C}_{k_i}, \widehat{A}_{k_f}$ and $\widehat{B}_{k_f}$, are automatically 0. Indeed, these matrices are not used in the definition of $\mathcal{J}(k_i,k_f)$ since the mapping over the finite horizon $[k_i,k_f]$ does not depend on them.

The recurrences (21)–(22) and the associated matrices Y_k, $\widehat{Q}_k$, X_k and $\widehat{P}_k$ for $k = k_i,\dots,k_f$ provide a method to compute projections

$$W_k := -Y_k\widehat{Q}_k^{-1}, \qquad V_k = X_k\widehat{P}_k^{-1}$$

that satisfy $W_k^T V_{k+1} = I$. Therefore, these can be used to construct a reduced order model

$$(\widehat{A}_k,\widehat{B}_k,\widehat{C}_k) := \big(W_k^T A_k V_k, W_k^T B_k, C_k V_k\big).$$

This follows from filling in the state approximation $x_k \approx V_k\widehat{x}_k$ for $k = k_i+1,\dots,k_f$ in (1) to obtain:

$$\begin{cases} V_{k+1}\widehat{x}_{k+1} = A_k V_k\widehat{x}_k + B_k u_k,\\ \widehat{y}_k = C_k V_k\widehat{x}_k.\end{cases}$$

Notice that V_{k_i} can be chosen arbitrarily since x_{k_i} is known ($x_{e,k_i}=0$). Multiplying the top equation by W_k^T then yields

$$\begin{cases} W_k^T V_{k+1}\widehat{x}_{k+1} = W_k^T A_k V_k\widehat{x}_k + W_k^T B_k u_k,\\ \widehat{y}_k = C_k V_k\widehat{x}_k.\end{cases}$$

This is essentially identical to the interpolation result that was obtained for the time-invariant case.

The production of these projectors, in turn, suggests a relaxation iteration, similar to that used for the time-invariant case, to produce a series of reduced order models

$$\widehat{\mathcal{S}}_{(r)} = (\widehat{A}_{k,(r)}, \widehat{B}_{k,(r)}, \widehat{C}_{k,(r)}), \quad r = 0, 1, 2, \ldots$$

that converge to a fixed point defining the final reduced order model. The simplest relaxation schedule keeps the matrices defining $\widehat{\mathcal{S}}_{(r)}$ fixed while computing the next set of X_k and Y_k for $k = k_i, \ldots, k_f$ and then applying the associated projectors to compute

$$(\widehat{A}_{k,(r+1)}, \widehat{B}_{k,(r+1)}, \widehat{C}_{k,(r+1)}), \quad \text{for } k = k_i, \ldots, k_f$$

to define the next reduced order model $\widehat{\mathcal{S}}_{(r+1)}$.

The resulting algorithm is given by

Initialize $\widehat{\mathcal{S}}_{(0)} = (\widehat{A}_{k,(0)}, \widehat{B}_{k,(0)}, \widehat{C}_{k,(0)})$ and $r = 0$
repeat until convergence
  Set $X_{k_i} = 0$, $\widehat{P}_{k_i} = 0$ and compute for $k = k_i, \ldots, k_f - 1$
    $X_{k+1} = A_k X_k \widehat{A}^T_{k,(r)} + B_k \widehat{B}^T_{k,(r)}$ and $\widehat{P}_{k+1} = \widehat{A}_{k,(r)} \widehat{P}_k \widehat{A}^T_{k,(r)} + \widehat{B}_{k,(r)} \widehat{B}^T_{k,(r)}$
  Set $Y_{k_f} = 0$, $\widehat{Q}_{k_f} = 0$ and compute for $k = k_f, \ldots, k_i + 1$
    $Y_{k-1} = A^T_{k,(r)} Y_k \widehat{A}_{k,(r)} - C_k^T \widehat{C}_{k,(r)}$ and $\widehat{Q}_{k-1} = \widehat{A}^T_{k,(r)} Q_k \widehat{A}_{k,(r)} + \widehat{C}^T_{k,(r)} \widehat{C}_{k,(r)}$
  Compute projectors W_k and V_k and $\widehat{\mathcal{S}}_{(r+1)} = (\widehat{A}_{k,(r+1)}, \widehat{B}_{k,(r+1)}, \widehat{C}_{k,(r+1)})$
  Place $\widehat{\mathcal{S}}_{(r+1)}$ in an appropriate coordinate system (see Remarks below)
  and compare to $\widehat{\mathcal{S}}_{(r)}$ to check convergence
  $r \leftarrow r + 1$
end repeat

Remarks.

1. It appears from the definition (17) that the cost function is independent of the coordinate systems used for the state models for (A_k, B_k, C_k) and $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k)$, but clearly the gradients do depend on this coordinate system. In order to check convergence, one therefore has to use a unique coherent coordinate system for all iterates $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k)$. In our experiments, we used a form of balanced realization for the reduced order system. This choice removes the degrees of freedom related to the choice of coordinate system in which to describe the reduced order system.

2. Notice that, just as for the LTI case [9], there is no proof of convergence for this iteration and that there may be several local minima for the cost function we try to minimize. Nevertheless, in practice we observe linear convergence to a local minimum, which is presumed to yield a reasonable approximation of the global minimum.

3. The rank of the matrices $P_{e,k}$ and $Q_{e,k}$ increase linearly at each step of their recurrences (15)–(16) until saturation since their initial condition is zero. If the number of state variables of the reduced order model is kept constant, the stationary conditions on the linear time varying $\mathcal{H}_2$ norm remains compatible. For instance, the projectors can be computed using the generalized inverse of the low rank Gramians of the reduced order model. But in that case, the inner product $W_k^T V_{k+1}$ is rank deficient instead of equal to the identity matrix. Alternatively, it is possible to choose the number of state variables equal to the rank of the Gramians of $\mathcal{S}_e$. Hence, it increases linearly at both extremities of the time interval $[k_i, k_f]$.

Also note that one can run the recurrences (21)–(22) forward and backward and compute at the same time the gradients for corrections to the reduced order model. These can be used to define various gradient-related optimization algorithms that would probably converge faster than the simple relaxation-based approach above.

3. Time-invariant versus time-varying

In this section we compare the time-invariant $\mathcal{H}_2$ approximation problem with the finite horizon approximation problem described in this paper. It is clear from the similarities between all the formulas that there should be a relation between the two cases. We make this more precise in this section. For this we consider a time-invariant system $\{A, B, C\}$ over a finite horizon (of length $\ell = k_f - k_i$) and apply our time-varying method to it. The first result relates the norms used in both approaches.

Theorem 3.1. *The Frobenius norm squared of the matrix with ℓ block rows*

$$S = \begin{bmatrix} CB & & 0 \\ \vdots & \ddots & \\ CA^{\ell-1}B & \ldots & CB \end{bmatrix},$$

associated with a stable LTI system $\{A, B, C\}$ for a finite horizon tends to the $\mathcal{H}_2$ norm squared of the infinite horizon system multiplied by ℓ, i.e.,

$$\lim_{\ell\to\infty} \frac{1}{\sqrt{\ell}}\|S\|_F = \big\|H(\cdot)\big\|_{\mathcal{H}_2}.$$

Proof. The following semi-infinite matrix with ℓ block rows

$$S_\infty = \begin{bmatrix} \cdots & CA^2B & CAB & CB & & 0 \\ & \cdots & \ddots & \ddots & \ddots & \\ & & \cdots & CA^2B & CAB & CB \end{bmatrix},$$

has bounded Frobenius norm squared which is exactly equal to the $\mathcal{H}_2$ norm squared multiplied by ℓ of the infinite horizon system:

$$\frac{1}{\ell}\|S_\infty\|_F^2 = \big\|H(\cdot)\big\|_{\mathcal{H}_2}^2,$$

since the nonzero part of each block-row can be written as

$$C\mathcal{C} := C\begin{bmatrix}\cdots & A^2B & AB & B\end{bmatrix},$$

which has Frobenius norm squared equal to $\operatorname{tr} C\mathcal{C}\mathcal{C}^TC^T = \operatorname{tr} CPC^T$, with P the solution of the Stein equation $P = APA^T + BB^T$. We thus need to show that $\frac{1}{\ell}(\|S_\infty\|_F^2 - \|S\|_F^2)$ tends to zero as ℓ goes to infinity. But the portions of the block rows of S_∞ not in S can be written as $CA^i\mathcal{C}$ for $i = 1, \ldots, \ell$ and hence we have that

$$\big\|H(\cdot)\big\|_{\mathcal{H}_2}^2 - \frac{1}{\ell}\|S\|_F^2 = \frac{1}{\ell}\big(\|S_\infty\|_F^2 - \|S\|_F^2\big) = \frac{1}{\ell}\operatorname{tr} C\left(\sum_{i=1}^{\ell} A^iPA^{iT}\right)C^T$$

which for $\ell\to\infty$ tends to zero as $1/\ell$, since the summation remains bounded for a stable matrix A. □

If we know that the norms are becoming the same as the window size increases, one can expect that the time-varying finite horizon approximation $\{\widehat{A}_k, \widehat{B}_k, \widehat{C}_k\}$ of an LTI system $\{A, B, C\}$ will also converge to a time-invariant system, but because of the boundary effects at the borders of the interval, this can only be expected near the center of the interval. This can be expected also from the equations for the time-varying Gramians P_k and Q_k of the large scale system. Clearly P_k tends to P for increasing k and Q_k tends to Q for decreasing k. Near the middle of the approximation interval, we are thus working with essentially a time-invariant linear system and one can expect the optimal approximation to be nearly time-invariant as well. Moreover, the difference between the approximation errors of both approaches, will tend to disappear. We illustrate this phenomenon in our test results.

4. Results

In this section, we give a simple example to show the effectiveness of the algorithm. We take a linear time-invariant system (rather than a time-varying one) where A has spectral radius equal to 0.98 and then applied our algorithm for a time interval of 50 samples (the example is a 3x2 MIMO system of order 9 taken from [13]). The reduced order model is nevertheless time-varying since we are solving a finite horizon problem. But since the system is also relatively damped, the finite horizon reduced order model will tend to be close to the time-invariant infinite horizon problem. In Fig. 1 we show the convergence of the error and its gradients with respect to $\widehat{A}_k$, $\widehat{B}_k$ and $\widehat{C}_k$. One can see that the error function converges to a steady state value in a few steps for a reduced order model of degree 3. The gradient, on the other hand, continues to decrease in a linear fashion, which is to be expected since it is a fixed point iteration. Moreover this convergence was similar to that of the infinite horizon time-invariant problem (not shown in this paper). In dashed line, we also indicated the finite horizon error level obtained for a linear time-invariant reduced order model and one can see that the finite horizon time-invariant model behaves indeed better. This is of course to be expected since the time-varying reduced order model minimizes the correct criterion. The fixed point iteration is initialized using the balanced truncation of $\mathcal{S}_{inv}$ for the reduced order model at each time k.

Using this method, the reduced model of order $n = 1, \ldots, 7$ can be computed. The corresponding values of $\mathcal{J}$ are shown in Fig. 2. Since the curve is monotonically decreasing and reaches machine precision for $n = 7$, it tends to indicate that each fixed point is the global minimum of this criterion.

Theorem 3.1 suggests that, in the middle of the time interval, the time-varying reduced order model of a time-invariant system converges to the time-invariant reduced order model when the length of the time window increases. In Fig. 3, we show that the difference between their eigenvalues after a quarter of the time interval and at the middle decreases accordingly.

The advantage of this model reduction scheme for linear time-varying systems, compared to computing the optimal $\mathcal{H}_2$ model corresponding to a time-invariant approximation of the initial one, is shown in [12].

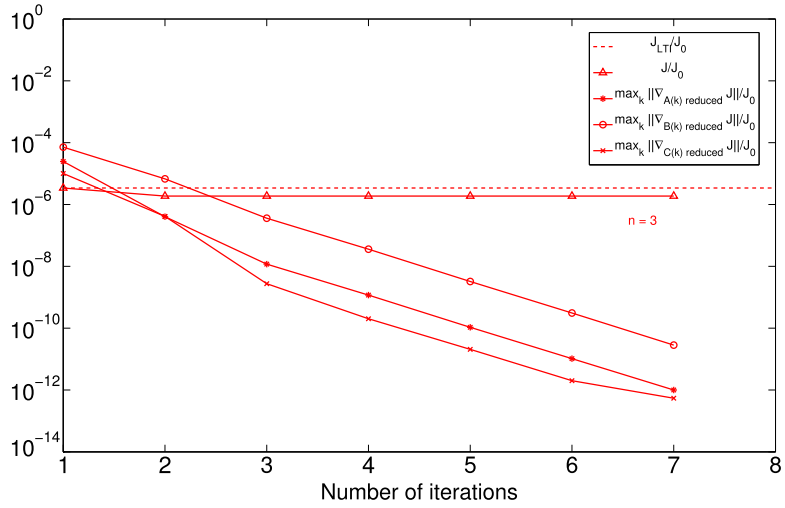

Fig. 1. Convergence of $\mathcal{J}$ and its gradients for third order approximation, normalized by $\mathcal{J}_0$ the $\mathcal{H}_2$ norm of the initial model.

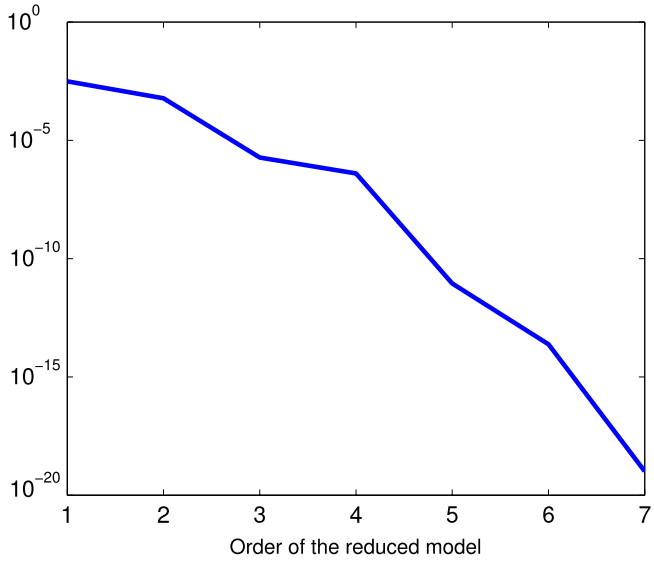

Fig. 2. Convergence of $\mathcal{J}$ for reduced model of order $n = 1$ to 7, normalized by $\mathcal{J}_0$ the $\mathcal{H}_2$ norm of the initial model.

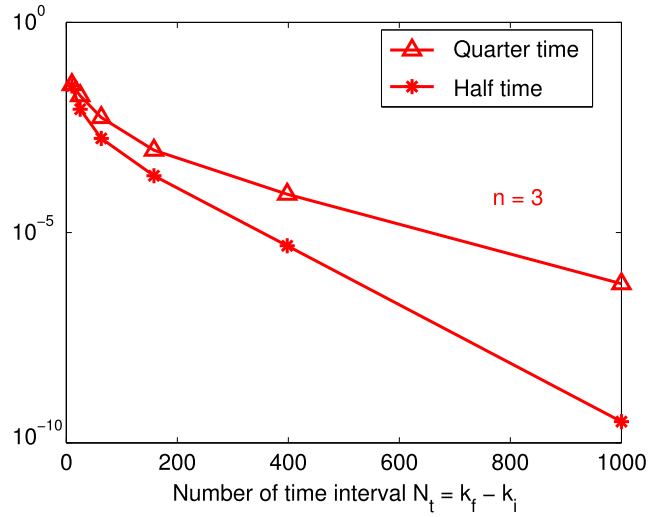

Fig. 3. Convergence of the difference between the eigenvalues of the third order time-invariant reduced order model and the time-varying reduced order model after a quarter of the time interval and at the middle.

5. Conclusion

The contribution of this paper is the extension of the $\mathcal{H}_2$ problem to time-varying discrete-time systems, including the characterization of the stationary points of $\mathcal{J}$ based on Stein-like recurrences. Unlike the methods in [5,11] and [14], we base our reduced order model's construction on the minimization of a well-defined error criterion.

Like the work in [5], we predicate the development of our approach on the requirement of a computationally efficient algorithm based on the parameters of the model. However, in our case, the characterization of a stationary point of the error criterion provides the fixed point iteration algorithm, evaluated here, and the basis for more sophisticated gradient-based algorithms that allow us to reduce the error criterion to required values in an efficient manner.

There are several areas of interest for future work on this topic. These include assessing the effectiveness of more rapidly converging minimization algorithms and interval selection. Perhaps the most important, however, is investigating scaling the algorithms to systems with much larger dimension, N, and longer time intervals, ultimately addressing infinite horizon problems. Systems with matrices of moderate dimension such as the numerical tests included in this paper and the related references are state-of-the-art for model reduction algorithms that produce a time-varying reduced order model via explicit construction of its sequence of defining matrices. Unlike methods based on PODs or the more sophisticated DEIM that compress information about the behavior of large systems over large time intervals into one or two projectors, the methods of interest here must determine a set of parameters with cardinality proportional to N and the size of the time interval defining the finite horizon. As a result, exploitation of structure in the problem, e.g., sparsity and properties of the manner in which the coefficients defining the full system evolve, is crucial and is under investigation. The effectiveness of the scaled up algorithms will be assessed on more substantial application problems.

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References

[1] A. Antoulas, Approximation of Large-Scale Dynamical Systems, SIAM Publications, Philadelphia, 2005.
[2] A. Antoulas, C. Beattie, S. Gugercin, Interpolatory model reduction of large-scale dynamical systems, in: J. Mohammadour, K.A. Grigoriadis (Eds.), Efficient Modeling and Control of Large-Scale Systems, Springer, Berlin, 2010, pp. 3–58.
[3] U. Baur, C. Beattie, P. Benner, S. Gugercin, Interpolatory projection methods for parameterized model reduction, SIAM J. Sci. Comput. 33 (2011) 2489–2518.
[4] A. Bunse-Gerstner, D. Kubalinska, G. Vossen, D. Wilczek, $\langle_2$-norm optimal model reduction for large-scale discrete dynamical MIMO systems, J. Comput. Appl. Math. 233 (2010) 1202–1216.
[5] Y. Chahlaoui, P. Van Dooren, Model reduction of time-varying systems, in: P. Benner, V. Mehrmann, S.D. C (Eds.), Dimension Reduction of Large-Scale Systems, in: Lecture Notes in Computational Science and Engineering, vol. 45, Springer, Berlin, 2005, pp. 131–148.
[6] S. Chaturantabut, D.C. Sorensen, Nonlinear model reduction via discrete empirical interpolation, SIAM J. Sci. Comput. 32 (2010) 2737–2764.
[7] P. Dewilde, A. Van der Veen, Time-Varying Systems and Computations, Kluwer Academic Publishers, Boston, 1998.
[8] K.A. Gallivan, A. Vandendorpe, P. Van Dooren, Model reduction of MIMO systems via tangential interpolation, SIAM J. Matrix Anal. Appl. 26 (2004) 328–349.
[9] S. Gugercin, A. Antoulas, C. Beattie, h_2 model reduction for large-scale linear dynamical systems, SIAM J. Matrix Anal. Appl. 30 (2008) 609–638.
[10] J. Imae, J. Perkins, J. Moore, Toward time-varying balanced realization via Riccati equations, Math. Control Signals Syst. 5 (1992) 313–326.
[11] S. Lall, C. Beck, Error-bounds for balanced model-reduction of linear time-varying systems, IEEE Trans. Autom. Control 48 (2003) 946–956.
[12] S.A. Melchior, P. Van Dooren, K.A. Gallivan, Finite horizon approximation of linear time-varying systems, in: Proceedings of 16th IFAC Symposium on System Identification, Elsevier Limited, 2012, pp. 734–738.
[13] L. Patnaik, N. Viswanadham, I. Sarma, Computer control algorithms for a tubular ammonia reactor, IEEE Trans. Autom. Control 25 (1980) 642–651.
[14] H. Sandberg, A. Rantzer, Balanced truncation of linear time-varying systems, IEEE Trans. Autom. Control 49 (2004) 217–229.
[15] S. Shokoohi, L. Silverman, Identification and model reduction of time-varying discrete-time systems, Automatica 23 (1987) 509–521.
[16] S. Shokoohi, L. Silverman, P. Van Dooren, Linear time-variable systems: Balancing and model reduction, IEEE Trans. Autom. Control 28 (1983) 810–822.
[17] S. Shokoohi, L.M. Silverman, P. Van Dooren, Linear time-variable systems: Stability of reduced models, Automatica 20 (1984) 59–67.
[18] P. Van Dooren, K.A. Gallivan, P.A. Absil, $\mathcal{H}_2$-optimal model reduction of MIMO systems, Appl. Math. Lett. 21 (2008) 1267–1273.
[19] D.A. Wilson, Optimum solution of model-reduction problem, Proc. IEEE 117 (1970) 1161–1165.