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Brief Paper

Pole–Zero Representation of Descriptor Systems*

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Key Words—Descriptor systems; generalized state-space models; singular systems; transfer function; poles and zeros.

Abstract—We present an algorithm for computing the pole–zero representation of descriptor systems whose generalized state-space models are described by the 5-tuple $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d)$, where $\mathbf{E}$ may be a singular matrix but $\det(\mathbf{A} - \lambda\mathbf{E}) \neq 0$. The proposed algorithm uses only orthogonal transformations; hence the computed results are numerically reliable. Numerical examples are included to illustrate the proposed results.

1. *Introduction*

In this paper we consider the problem of pole–zero representation of linear time-invariant *generalized* state space or *descriptor* systems (Dervişoğlu and Desoer, 1975; Luenberger, 1977; Verghese *et al.*, 1981a, b; Bernhard, 1982; Cobb, 1984; Lewis, 1986) described by

$$\begin{aligned} \mathbf{E}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \\ y(t) &= \mathbf{c}\mathbf{x}(t) + du(t), \end{aligned} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $u(t)$, $y(t) \in \mathbb{R}$ and $\det(\lambda\mathbf{E} - \mathbf{A}) \not\equiv 0$, i.e. the pencil $(\lambda\mathbf{E} - \mathbf{A})$ is regular. The transfer function of the system (1) is $G(\lambda) = \mathbf{c}(\lambda\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} + d$. We shall denote the above system by the 5-tuple $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d)$. If the descriptor matrix $\mathbf{E}$ has full rank, the system in (1) is said to be a *nonsingular* system; otherwise, it is called a *singular* system.

If the system is nonsingular, theoretically, we can obtain an equivalent state-space realization of $(\mathbf{E}, \mathbf{A}, \mathbf{c}, d)$ by premultiplying the state equation by $\mathbf{E}^{-1}$ to get an equivalent 4-tuple $(\mathbf{E}^{-1}\mathbf{A}, \mathbf{E}^{-1}\mathbf{b}, \mathbf{c}, d)$. Once we have the 4-tuple $(\mathbf{E}^{-1}\mathbf{A}, \mathbf{E}^{-1}\mathbf{b}, \mathbf{c}, d)$, we can easily obtain its pole–zero representation

$$G(\lambda) = \frac{g\prod_{i=1}^{n_z}(\lambda - \lambda_i^z)}{\prod_{j=1}^{n_p}(\lambda - \lambda_j^p)}. \tag{2}$$

where λ_i^z denotes a zero, λ_j^p denotes a pole, n_z and n_p are respectively the numbers of zeros and poles of $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d)$, and g is the constant gain of the transfer function. Several algorithms for obtaining pole–zero representation of the 4-tuples $(\mathbf{E}^{-1}\mathbf{A}, \mathbf{E}^{-1}\mathbf{b}, \mathbf{c}, d)$, exist (see e.g. Varga and Sima, 1981; Emami-Naeini and Van Dooren, 1982; Misra and Patel, 1987). Varga and Sima (1981) and Emami-Naeini and Van Dooren (1982) compute poles and zeros directly, while Misra and Patel (1987) computed the transfer function, from which the pole–zero representation may be obtained. For descriptor systems, Misra (1989) reported a numerically reliable way of computing the transfer function, from which the pole–zero representation for descriptor systems may be obtained. However, obtaining the form in (2) by first computing the transfer function of the system in (1) can be numerically quite sensitive. A small perturbation in the coefficient of the transfer function can lead to significant loss of accuracy in numerical computation of poles and/or zeros. A pole–zero representation algorithm was proposed by Varga (1989). The present approach is different from Varga's, and, as pointed out below, it has several features that make it more efficient and reliable.

Here we develop a numerical algorithm that uses only orthogonal transformations to obtain the pole–zero representation of a generalized state-space system. The underlying principle is deflation of matrix pencils using orthogonal transformations to obtain subpencils with only finite generalized eigenvalues, followed by their computation using a QZ algorithm. Section 2 contains some relevant results from numerical linear algebra that will be used extensively in developing the proposed algorithm. The algorithm is developed in Section 3. Finally, in Section 4 we discuss the issues associated with efficient implementation of the algorithm developed in Section 3, and illustrate the proposed technique with numerical examples.

2. *Background*

2.1. *Generalized Schur decomposition.* For an arbitrary pencil $(\mathbf{F} - \lambda\mathbf{G})$, there exists *unitary* transformations $\mathbf{Q}$ and $\mathbf{Z}$ yielding the block-triangular decomposition

$$\mathbf{Q}^*(\mathbf{F} - \lambda\mathbf{G})\mathbf{Z} = \left[\begin{array}{c|c|c} \mathbf{F}_r - \lambda\mathbf{G}_r & * & * \\ \hline \mathbf{0} & \mathbf{F}_f - \lambda\mathbf{G}_f & * \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{F}_c - \lambda\mathbf{G}_c \end{array}\right], \tag{3}$$

where $(\mathbf{F}_r - \lambda\mathbf{G}_r)$ has full row rank for all finite λ, $(\mathbf{F}_c - \lambda\mathbf{G}_c)$ has full column rank for all finite λ, and $\mathbf{G}_f$ is invertible. This decomposition is proven in Van Dooren (1979), where an algorithm is also given to find such a decomposition. Moreover, the eigenvalues of $\mathbf{G}_f^{-1}\mathbf{F}_f$ are the only *finite* points where the rank of $\mathbf{F} - \lambda\mathbf{G}$ drops below its normal value, and hence are the finite zeros of the pencil $(\mathbf{F} - \lambda\mathbf{G})$.

2.2. *QZ algorithm and finite zeros.* For a square matrix pencil $(\mathbf{F}_f - \lambda\mathbf{G}_f)$, with $\det(\mathbf{F}_f - \lambda\mathbf{G}_f) \not\equiv 0$, there exist unitary matrices $\mathbf{Q}_f^*$ and $\mathbf{Z}_f$ of appropriate dimensions such that $\mathbf{Q}_f^*\mathbf{F}_f\mathbf{Z}_f$ and $\mathbf{Q}_f^*\mathbf{G}_f\mathbf{Z}_f$ are both upper-triangular matrices (Moler and Stewart, 1973; Golub and Van Loan, 1989). Let f_{ii} and g_{ii} represent the ith elements along the diagonals of the upper-triangular matrices $\mathbf{Q}_f^*\mathbf{F}_f\mathbf{Z}_f$ and $\mathbf{Q}_f^*\mathbf{G}_f\mathbf{Z}_f$ respectively then the ratios f_{ii}/g_{ii} represent the finite eigenvalues (or *finite zeros*) of $(\mathbf{F}_f - \lambda\mathbf{G}_f)$ and also of $(\mathbf{F} - \lambda\mathbf{G})$. Note that the QZ algorithm that performs this triangularization also works for rank-deficient $\mathbf{G}_f$.

2.3. *Orthogonal transformations and column compression.* Given an arbitrary row vector $\mathbf{x}$ of length n, it is always

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possible to find an orthogonal $n \times n$ matrix $\mathbf{W}$ such that

$$\mathbf{xW} = [0 \quad 0 \quad \ldots \quad 0 \quad x_n]. \tag{4}$$

The matrix $\mathbf{W}$ can be obtained using a Householder transformation or a sequence of Givens rotations between the adjacent elements of the vector $\mathbf{x}$ (Golub and Van Loan, 1989).

2.4. *Compressed representation of matrix pencils.* For a regular matrix pencil $(\lambda\mathbf{E} - \mathbf{A})$, where $\mathbf{E}$ and $\mathbf{A} \in \mathbb{R}^{n\times n}$, we can always find two orthogonal transformation matrices $\mathbf{U}$ and $\mathbf{V} \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{U}^T\mathbf{EV} = \left[\begin{array}{c|c} \mathbf{E}_{11} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right], \quad \mathbf{U}^T\mathbf{AV} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right]. \tag{5}$$

Clearly, if $\mathbf{E}_{11} \in \mathbb{R}^{r\times r}$ has full rank then rank $[\mathbf{A}_{21} \ \mathbf{A}_{22}] = n - r$. Note that singular-value decomposition (Bender and Laub, 1987) or rank-revealing QR factorization (Chan, 1987) can be used to obtain the equivalent pencil representation in (5).

2.5. *Observer Hessenberg form of* $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d)$. Given a single-input single-output 5-tuple $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d)$, there exist orthogonal transformation matrices $\mathbf{Q}$ and $\mathbf{Z}$ such that in the transformed 5-tuple

$$(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d) := (\mathbf{Q}^T\mathbf{EZ}, \mathbf{Q}^T\mathbf{AZ}, \mathbf{Q}^T\mathbf{b}, \mathbf{cZ}, d) \tag{6}$$

$\mathbf{E}$ is an upper-triangular matrix, $\mathbf{A}$ is an upper Hessenberg matrix, $\mathbf{b}$ is a general dense vector and $\mathbf{c}$ has only its last element nonzero. The system has no unobservable finite or infinite modes if and only if $\mathbf{A}$ is an unreduced upper-Hessenberg matrix, i.e. $a_{i+1,i} \neq 0$. If, however, $a_{i+1,i} = 0$, for some i then the system can be block-triangularly partitioned as

$$\mathbf{E} = \left[\begin{array}{c|c} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \hline & \mathbf{E}_{22} \end{array}\right], \quad \mathbf{A} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline & \mathbf{A}_{22} \end{array}\right], \quad \mathbf{b} = \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \end{array}\right], \quad \mathbf{c} = [\mathbf{0} \mid \mathbf{c}_2], \tag{7}$$

where the generalized eigenvalues of the pair $(\mathbf{E}_{11}, \mathbf{A}_{11})$ are the unobservable modes of the system.

A computational algorithm for reducing the given system $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ to the form in (6) (called the observer Hessenberg form) can be easily devised based on the results of Van Dooren and Verhaegen (1985), Misra (1989), Varga (1989) and Miminis (1993).

By duality, a similar statement can be made regarding uncontrollable modes and the reduction of the system to a *controller Hessenberg form*. A singular system that does not have any uncontrollable and/or unobservable finite or infinite modes is said to be *irreducible* (Verghese *et al.*, 1981a, b).

Remark 2.1. The order of an irreducible system $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is equal to the degree of $\det(\lambda\mathbf{E} - \mathbf{A})$ (Verghese *et al.*, 1981a, b). Note that, owing to the presence of poles at infinity, the order of an irreducible generalized state-space system is not necessarily equal to the rank of $\mathbf{E}$.

Remark 2.2. If an nth-order system is controllable and observable at infinity then

$$\text{rank}\,[\mathbf{E} \mid \mathbf{b}] = \text{rank}\left[\begin{array}{c} \mathbf{E} \\ \hline \mathbf{c} \end{array}\right] = n$$

(Verghese *et al.*, 1981a, b; Cobb, 1984). An immediate consequence of this observation is that in the observer Hessenberg form all the diagonal elements $e_{i,i}$ of the matrix $\mathbf{E}$ are nonzero, with the possible exception of $e_{n,n}$.

3. *Main results*

It is well known (Kailath, 1980) that, using the Schur complement of a matrix,

$$\det\left[\begin{array}{c|c} \mathbf{A} - \lambda\mathbf{E} & \mathbf{b} \\ \hline \mathbf{c} & d \end{array}\right] = \det(\mathbf{A} - \lambda\mathbf{E})\,[d - \mathbf{c}(\mathbf{A} - \lambda\mathbf{E})^{-1}\mathbf{b}], \tag{8}$$

where, by assumption, $\det(\mathbf{A} - \lambda\mathbf{E}) \neq 0$. Equivalently,

$$\mathbf{c}(\lambda\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} + d = \frac{\det\left[\begin{array}{c|c} \mathbf{A} - \lambda\mathbf{E} & \mathbf{b} \\ \hline \mathbf{c} & d \end{array}\right]}{\det(\mathbf{A} - \lambda\mathbf{E})} = G(\lambda). \tag{9}$$

Equation (9) shows how to compute the transfer function of a generalized state-space system. Knowing the transfer function, one can easily compute the pole–zero representation by finding the roots of the numerator and denominator polynomials. However, as discussed earlier, this can potentially lead to considerable numerical inaccuracy in the computed solution. In the rest of this section we discuss how the pole–zero representation can be obtained in a numerically reliable manner without going through the step of computing the transfer function first.

From (9), it is easily seen that the normal rank of the system matrix $\left[\begin{array}{c|c} \mathbf{A} - \lambda\mathbf{E} & \mathbf{b} \\ \hline \mathbf{c} & d \end{array}\right]$ is $n + 1$ if the transfer function $G(\lambda) \neq 0$. Then, from Rosenbrock (1970) and Misra *et al.* (1994), the next result follows.

Theorem 3.1. For an irreducible single-input single-output system $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d)$, the finite transmission zeros are those complex values of λ for which

$$\text{rank}\left[\begin{array}{c|c} \mathbf{A} - \lambda\mathbf{E} & \mathbf{b} \\ \hline \mathbf{c} & d \end{array}\right] < n + 1. \tag{10}$$

Proof. See Rosenbrock (1970) and Misra *et al.* (1994). □

3.1. *Computation of zeros.* The intent here is to obtain a subpencil $(\mathbf{F}_f - \lambda\mathbf{G}_f)$ ($\mathbf{G}_f$ has full rank) as in (3) from the pencil in (10). The zeros of $(\mathbf{F}_f - \lambda\mathbf{G}_f)$ correspond to the finite transmission zeros of the system. To achieve this, we deflate the pencil $\left[\begin{array}{c|c} \mathbf{A} - \lambda\mathbf{E} & \mathbf{b} \\ \hline \mathbf{c} & d \end{array}\right]$ until the required subpencil is obtained. The deflation is performed by column compression of the row vector $[\mathbf{c} \ d]$ recursively using orthogonal transformations.

Since the rank of matrix pencil is unaffected by orthogonal transformations, during the lth recursion

$$\begin{aligned} &\text{rank}\left[\begin{array}{c|c} \mathbf{A}^{(l-1)} - \lambda\mathbf{E}^{(l-1)} & \mathbf{b}^{(l-1)}(\lambda) \\ \hline \mathbf{c}^{(l-1)} & d_n^{(l-1)} \end{array}\right] \\ &= \text{rank}\left[\begin{array}{c|c} \mathbf{A}^{(l-1)} - \lambda\mathbf{E}^{(l-1)} & \mathbf{b}^{(l-1)}(\lambda) \\ \hline \mathbf{c}^{(l-1)} & d_n^{(l-1)} \end{array}\right]\mathcal{G}^{(l)} \\ &= \text{rank}\left[\begin{array}{c|c} \mathbf{A}^{(l)} - \lambda\mathbf{E}^{(l)} & \mathbf{b}^{(l)}(\lambda) \\ \hline \mathbf{0} & d_n^{(l)} \end{array}\right] \end{aligned} \tag{11}$$

where $\mathcal{G}^{(l)}$ is an orthogonal matrix that compresses the columns of the vector $[\mathbf{c}^{(l-1)} \ d_n^{(l-1)}]$. By the assumption on the normal rank of the matrix pencil in (10), $d_n^{(l)}$ is nonzero. Hence the pencils in (10) and (11) have identical finite zeros. Note that owing to the nature of the transformations, during deflation, the constant input vector $\mathbf{b}$ in (10) becomes a pencil. Hence $\mathbf{b}$ is replaced by $\mathbf{b}(\lambda)$.

On compressing the columns of $[\mathbf{c}^{(l-1)} \ d_n^{(l-1)}]$, and transforming the pencil, the following two possibilities may arise.

Case 1. $\mathbf{E}^{(l)}$ *is nonsingular.* In this case the finite zeros of the system are the generalized eigenvalues of the pair $(\mathbf{E}^{(l)}, \mathbf{A}^{(l)})$.

Case 2. $\mathbf{E}^{(l)}$ *is singular.* In this case it will be possible to deflate the problem by further partitioning the pencil $(\mathbf{A}^{(l)} - \lambda\mathbf{E}^{(l)})$ as

$$\begin{aligned} (\mathbf{A}^{(l)} - \lambda\mathbf{E}^{(l)}) &= \left[\begin{array}{c|c} \mathbf{A}_{11}^{(l)} - \lambda\mathbf{E}_{11}^{(l)} & \mathbf{a}_{12}^{(l)}(\lambda) \\ \hline \mathbf{a}_{21}^{(l)} & a_{22}^{(l)} \end{array}\right] \\ &\triangleq \left[\begin{array}{c|c} \mathbf{A}^{(l)} - \lambda\mathbf{E}^{(l)} & \mathbf{b}^{(l)}(\lambda) \\ \hline \mathbf{c}^{(l)} & d_n^{(l)} \end{array}\right], \end{aligned} \tag{12}$$

where the partitioning conforms to the dimension of the

full-rank matrix $\mathbf{E}_{11}^{(l)}$. The problem can now be further deflated by performing column compression on the bottom row of the pencil in (12). This is illustrated more clearly by means of an example.

Assume that at an intermediate step l, the deflated system is of fourth order. Further assume that the system is in an unreduced observer Hessenberg form. Then the matrix pencil will have the following structure:

$$\left(\left[\begin{array}{c|c}\mathbf{A}^{(l)} & \mathbf{b}^{(l)}\\ \hline \mathbf{c}^{(l)} & d^{(l)}\end{array}\right]-\lambda\left[\begin{array}{c|c}\mathbf{E}^{(l)} & \mathbf{x}\\ \hline \mathbf{0} & 0\end{array}\right]\right)$$

$$=\left[\begin{array}{cccc|c}\times&\times&\times&\times&\times\\ \times&\times&\times&\times&\times\\ 0&\times&\times&\times&\times\\ 0&0&\times&\times&\times\\ \hline 0&0&0&\times_4&\times_5\end{array}\right]-\lambda\left[\begin{array}{cccc|c}\times&\times&\times&\times&\times\\ 0&\times&\times&\times&\times\\ 0&0&\times&\times&\times\\ 0&0&0&\otimes_4&\times\\ \hline 0&0&0&0&0\end{array}\right]. \tag{13}$$

To deflate the pencil, the element $\times_4$ must be reduced to zero by a Givens rotation $\mathcal{G}_{4,5}$ over an appropriate angle θ_4 and between columns 4 and 5. The columns of the matrix pencil in (13) are updated using the same transformation. Note that this transformation does not affect the upper Hessenberg structure of $\mathbf{A}^{(l)}$ or the upper-triangular structure of $\mathbf{E}^{(l)}$.

At this stage, if the element $\otimes_4$ of $\mathbf{E}^{(l)}$ is nonzero, $\mathbf{E}^{(l)}$ has full rank, and further deflation is not possible. Otherwise, the deflation is continued on the following lower-order pencil:

$$\left(\left[\begin{array}{ccc|c}\times&\times&\times&\times\\ \times&\times&\times&\times\\ 0&\times&\times&\times\\ \hline 0&0&\times_3&\times_4\end{array}\right]-\lambda\left[\begin{array}{ccc|c}\times&\times&\times&\times\\ 0&\times&\times&\times\\ 0&0&\otimes_3&\times\\ \hline 0&0&0&0\end{array}\right]\right). \tag{14}$$

Note that the triangular-Hessenberg structure of the pair $(\mathbf{E}, \mathbf{A})$ is preserved. Hence the procedure for deflation described above can be applied to the third order subpencil in (14).

Remark 3.1. Since the system being deflated is assumed irreducible, the matrix $\mathbf{E}^{(l)}$ cannot have nullity greater than 1. Further, at all stages in deflation, the matrices $[\mathbf{E}^{(l)} \mid \mathbf{b}^{(l)}]$ and $\left[\begin{array}{c}\mathbf{E}^{(l)}\\ \hline \mathbf{c}^{(l)}\end{array}\right]$ must have full row and column rank respectively. Therefore there can be only one Jordan block with eigenvalue at infinity. In terms of our algorithm, this implies that $d_n^{(l)}$ will all be scalars.

Once a full-rank matrix $\mathbf{E}^{(\alpha)}$ has been found, the rank condition can be rewritten as

$$\operatorname{rank}\left[\begin{array}{c|c}\mathbf{A}-\lambda\mathbf{E} & \mathbf{b}\\ \hline \mathbf{c} & d\end{array}\right]=\operatorname{rank}\left[\begin{array}{c|ccc}\mathbf{A}^{(\alpha)}-\lambda\mathbf{E}^{(\alpha)} & \multicolumn{3}{c}{*}\\ \hline & d_n^{(\alpha-1)} & * & *\\ \mathbf{0} & & d_n^{(\alpha-2)} & *\\ & & & d_n^{(1)}\end{array}\right]. \tag{15}$$

On the basis of the above discussion, we can state the following.

Theorem 3.2. If in (15) $\mathbf{E}^{(\alpha)}$ has full rank then the finite zeros of the matrix pencil $\left[\begin{array}{c|c}\mathbf{A}-\lambda\mathbf{E} & \mathbf{b}\\ \hline \mathbf{c} & d\end{array}\right]$ are the generalized eigenvalues of the pair $(\mathbf{E}^{(\alpha)}, \mathbf{A}^{(\alpha)})$.

Proof. The result follows immediately on noting that finite zeros of a pencil are unaffected by orthogonal transformations. □

Further, since the deflated pencil on the right-hand side of (15) is obtained using orthogonal transformations on the original (square) pencil,

$$\det\left[\begin{array}{c|c}\mathbf{A}-\lambda\mathbf{E} & \mathbf{b}\\ \hline \mathbf{c} & d\end{array}\right]=\det\left[\begin{array}{c|ccc}\mathbf{A}^{(\alpha)}-\lambda\mathbf{E}^{(\alpha)} & \multicolumn{3}{c}{*}\\ \hline & d_n^{(\alpha-1)} & * & *\\ \mathbf{0} & & d_n^{(\alpha-2)} & *\\ & & & d_n^{(1)}\end{array}\right]$$

$$=\det(\mathbf{A}^{(\alpha)}-\lambda\mathbf{E}^{(\alpha)})\,d_n^{(1)}d_n^{(2)}\cdots d_n^{(\alpha-1)}$$

$$=\det([\mathbf{E}^{(\alpha)}]^{-1}\mathbf{A}^{(\alpha)}-\lambda\mathbf{I})\det(\mathbf{E}^{(\alpha)})\,d_n^{(1)}d_n^{(2)}\cdots d_n^{(\alpha-1)}, \tag{16}$$

where $\det(\mathbf{E}^{(\alpha)})\,d_n^{(1)}d_n^{(2)}\cdots d_n^{(\alpha-1)}$ is a nonzero constant, and the roots of $\det([\mathbf{E}^{(\alpha)}]^{-1}\mathbf{A}^{(\alpha)}-\lambda\mathbf{I})$, or the generalized eigenvalues of the pair $(\mathbf{E}^{(\alpha)}, \mathbf{a}^{(\alpha)})$, are the zeros or the transfer function.

Remark 3.2. The result (16) highlights the major differences between the proposed algorithm and that of Varga (1989).

(1) The present procedure uses only (numerically stable) orthogonal transformations to deflate the pencil; however, in Varga's algorithm the pencil is deflated by applying state feedback to the transformed system. If the element of the input vector used for computing the feedback gain (for deflation) has very small magnitude, the resulting gains will be large, and applying feedback can cause the resulting generalized eigenvalue problem to become numerically ill-conditioned. Hence the proposed approach is numerically more reliable.

(2) In the present case the deflated pencil will contain only finite zeros of the system, while the deflated pencil in Varga (1989) can contain both finite and infinite zeros, requiring that the QZ algorithm be applied to a larger pencil. Therefore the present approach is computationally more efficient.

3.2. *Computation of poles.* Using similar arguments as in Section 3.1, we can transform the pencil $(\mathbf{A}-\lambda\mathbf{E})$ such that

$$\det(\mathbf{A}-\lambda\mathbf{E})=\det\left[\begin{array}{c|ccc}\mathbf{A}^{(\beta)}-\lambda\mathbf{E}^{(\beta)} & \multicolumn{3}{c}{*}\\ \hline & d_d^{(\beta-1)} & * & *\\ \mathbf{0} & & d_d^{(\beta-2)} & *\\ & & & d_d^{(1)}\end{array}\right]$$

$$=\det(\mathbf{A}^{(\beta)}-\lambda\mathbf{E}^{(\beta)})\,d_d^{(1)}\cdots d_d^{(\beta-1)}$$

$$=\det([\mathbf{E}^{(\beta)}]^{-1}\mathbf{A}^{(\beta)}-\lambda\mathbf{I})\det(\mathbf{E}^{(\beta)})\,d_d^{(1)}\cdots d_d^{(\beta-1)} \tag{17}$$

where $\mathbf{E}^{(\beta)}$ is a full-rank matrix. The poles of the system are the roots of the polynomial $\det([\mathbf{E}^{(\beta)}]^{-1}\mathbf{A}^{(\beta)}-\lambda\mathbf{I})$, or equivalently the generalized eigenvalues of the matrix pair $(\mathbf{E}^{(\beta)}, \mathbf{A}^{(\beta)})$, and $\det(\mathbf{E}^{(\beta)})d_d^{(1)}\cdots d_d^{(\beta-1)}$ is a nonzero constant.

From (16) and (17), it is clear that the coefficient g in (2) may be computed as

$$g=\frac{(-1)^{n_z-n_p}\det(\mathbf{E}^{(\alpha)})\,d_n^{(1)}d_n^{(2)}\cdots d_n^{(\alpha-1)}}{\det(\mathbf{E}^{(\beta)})\,d_d^{(1)}\cdots d_d^{(\beta-1)}}$$

$$=(-1)^{n_z-n_p}\frac{\prod_{i=1}^{n_z}e_n^{(\alpha)}\prod_{i=1}^{\alpha-1}d_n^{(i)}}{\prod_{i=1}^{n_p}e_n^{(\beta)}\prod_{i=1}^{\beta-1}d_d^{(i)}}, \tag{18}$$

where n_z and n_p are respectively the orders of the upper-triangular matrices $\mathbf{E}^{(\alpha)}$ and $\mathbf{E}^{(\beta)}$. Knowing the coefficient g and the locations of zeros and poles as the generalized eigenvalues of the pencils $(\mathbf{E}^{(\alpha)}, \mathbf{A}^{(\alpha)})$ and $(\mathbf{E}^{(\beta)}, \mathbf{A}^{(\beta)})$ respectively, the desired pole–zero representation of the single-input single-output system $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is completely determined.

Remark 3.3. Computation of the transfer function gain g as described above requires approximately $2n$ scalar multiplications. Whereas in Varga (1989) one must solve a triangular system of linear algebraic equations, which requires $O(n_{co}^2)$ operations, where n_{co} is the order of the irreducible subsystem. Here again the proposed algorithm is more efficient.

Remark 3.4. For multivariable systems, the pole–zero representation for each element of the transfer function matrix can be obtained by selecting the appropriate input–output pair. As remarked by Misra (1989) and Varga (1989), for maximum computational efficiency, the pole–zero representation of the elements of an entire row (when there are more inputs than outputs) or column (when there are more outputs than inputs) of the transfer-function matrix should be computed successively. By computing the elements in this order, the number of irredicible realizations to be computed is minimized.

Remark 3.5. The proposed algorithm is equally applicable to standard state-space systems, where the descriptor matrix $\mathbf{E}$ is an identity matrix. In addition, for systems described by the matrix-valued linear differential equation

$$\left(\sum_{i=1}^{r} \mathbf{E}_i \frac{\mathrm{d}^{(i)}}{\mathrm{d}t^{(i)}}\right)\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \qquad y(t) = \mathbf{c}\mathbf{x}(t) + du(t), \tag{19}$$

the preceding algorithm can be used by rewriting the system (19) as

$$\begin{bmatrix} \mathbf{I} & & & \\ & \mathbf{I} & & \\ & & \ddots & \\ & & & \mathbf{E}_r \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_{r-1}(t) \\ \mathbf{x}_r(t) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{O} & \mathbf{I} & & & \\ & \mathbf{O} & \mathbf{I} & & \\ & & \ddots & \ddots & \\ & & & \mathbf{O} & \mathbf{I} \\ \mathbf{A} & -\mathbf{E}_1 & \cdots & -\mathbf{E}_{r-2} & -\mathbf{E}_{r-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_{r-1}(t) \\ \mathbf{x}_r(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{b} \end{bmatrix} u(t),$$

$$y(t) = [\mathbf{c} \quad \mathbf{0} \quad \cdots \quad \mathbf{0} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_{r-1}(t) \\ \mathbf{x}_r(t) \end{bmatrix} + du(t).$$

where $\mathbf{x}_1(t) = \mathbf{x}(t)$, $\mathbf{x}_2(t) = \mathrm{d}\mathbf{x}(t)/\mathrm{d}t$ etc

4. *Implementation and examples*

4.1. *Implementation of algorithm and operations count.* Assuming that we have an irreducible 5-tuple $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d)$, then, from the discussion in Section 3, it is clear that the main step in computation of zeros and poles (and the constant coefficient) is the recursive deflation of the pencils $\left[\begin{array}{c|c} \mathbf{A}-\lambda\mathbf{E} & \mathbf{b} \\ \hline \mathbf{c} & d \end{array}\right]$ and $(\mathbf{A} - \lambda\mathbf{E})$ respectively. Obtaining irreducible realization is already well documented in the literature (Van Dooren and Verhaegen 1981; Misra, 1989; Varga, 1989; Miminis, 1993). On the basis of the discussion in Section 3.1, we next present an efficient implementation of the deflation step. Note that the deflation procedure presented next is computationally the most efficient one for the problem at hand.

By first eliminating the finite and infinite uncontrollable modes and then the unobservable modes, we can ensure that the irreducible realization of the system is in an *unreduced observer Hessenberg form* (6). Once the zeros at infinity have been eliminated, the finite zeros can be computed by applying the QZ algorithm to the finite subpencil. Note that the first step of the QZ algorithm is the reduction of the pair $(\mathbf{E}, \mathbf{A})$ to a triangular-Hessenberg form. Since the irreducible system in (6) is already in a triangular-Hessenberg form, it will be computationally efficient if the condensed structure of the pair $(\mathbf{E}, \mathbf{A})$ is preserved.

The algorithm DEFLATE given below performs deflation on a matrix pencil $(\mathbf{A} - \lambda\mathbf{E})$, while preserving the condensed structure of the pair $(\mathbf{E}, \mathbf{A})$. By our assumption, $\mathbf{E}^{(0)}$ is an upper-triangular matrix and $\mathbf{A}^{(0)}$ is an unreduced upper-Hessenberg matrix.

Algorithm DEFLATE.
input $(\mathbf{E}^{(0)}, \mathbf{A}^{(0)}, n^{(0)})$, **output** $(\mathbf{E}^{(\theta)}, \mathbf{A}^{(\theta)}, n^{(\theta)}, \Delta)$
step-initialize
set $\Delta = 1$; $l = 0$
if rank $(\mathbf{E}^{(0)}) = n^{(0)}$ **then**
go to **exit**
else
step-l
$l := l + 1$
comment: *partition the singular pencil* $[\mathbf{A}^{(l-1)} - \lambda\mathbf{E}^{(l-1)}]$

$$\textbf{set} \left[\begin{array}{c|c} \mathbf{A}^{(l-1)} - \lambda\mathbf{E}^{(l-1)} & \mathbf{b}^{(l-1)}(\lambda) \\ \hline \mathbf{c}^{(l-1)} & d^{(l-1)} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{11}^{(l-1)} - \lambda\mathbf{E}_{11}^{(l-1)} & \mathbf{a}_{12}^{(l-1)}(\lambda) \\ \hline \mathbf{a}_{21}^{(l-1)} & a_{22}^{(l-1)} \end{array}\right]$$

comment: *compress columns of* $[0 \ \ldots \ 0 \ c_{n^{(l-1)}}^{(l-1)} \ d^{(l-1)}]$ *by postmultiplication with* $\mathcal{G}_{n^{(l-1)}, n^{(l-1)}+1}^{(l)}$

$$[\mathbf{0} \quad d^{(l)}] = [\mathbf{c}^{(l-1)} \quad d^{(l-1)}]\mathcal{G}_{n^{(l-1)}, n^{(l-1)}+1}^{(l)}$$

comment: *update*

$$\left[\begin{array}{c|c} \mathbf{A}_{11}^{(l-1)} - \lambda\mathbf{E}_{11}^{(l-1)} & \mathbf{b}^{(l-1)}(\lambda) \\ \hline \mathbf{c}^{(l-1)} & d^{(l-1)} \end{array}\right] \mathcal{G}_{n^{(l-1)}, n^{(l-1)}+1}^{(l)} \stackrel{\text{def}}{=} \left[\begin{array}{c|c} \mathbf{A}^{(l)} - \lambda\mathbf{E}^{(l)} & \mathbf{b}^{(l)}(\lambda) \\ \hline \mathbf{0} & d^{(l)} \end{array}\right]$$

$n^{(l)} = n^{(l-1)} - 1$; $\Delta := \Delta \times d^{(l)}$
if $e_{n^{(l)}, n^{(l)}}^{(l)} \neq 0$ **then**
go to **exit**
else
go to *step-l*
exit
$\mathbf{E}^{(\theta)} = \mathbf{E}^{(l)}$; $\mathbf{A}^{(\theta)} = \mathbf{A}^{(l)}$; $n^{(\theta)} = n^{(l)}$ and $\Delta = \Delta \times \det(\mathbf{E}^{(l)})$

Using the above algorithm, an algorithm for obtaining pole–zero representation of a descriptor system may be stated formally as follows. It is again assumed that the system $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is irreducible.

Algorithm POLE-ZERO.
input $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}, d, n)$, **output** $(\mathbf{z}, \mathbf{p}, g)$
comment: *compute zeros and numerator constant*

$$\textbf{set } \mathbf{E}^{(0)} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \mathbf{A}^{(0)} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix} \quad \text{and} \quad n^{(0)} = n$$

$[\mathbf{E}^{(\alpha)}, \mathbf{A}^{(\alpha)}, n^{(\alpha)}, \Delta] = \mathrm{DEFLATE}\ (\mathbf{E}^{(0)}, \mathbf{A}^{(0)}, n^{(0)})$
comment: *generalized eigenvalue decomposition*
$[\mathbf{E}_f, \mathbf{A}_f] = \mathrm{QZ}\ (\mathbf{E}^{(\alpha)}, \mathbf{A}^{(\alpha)})$
set $g_n = \Delta$; $n_z = n^{(\alpha)}$; $z_i = a_{ii}/e_{ii}$, $i = 1 : n_z$
comment: *compute poles and denominator constant*
set $\mathbf{E}^{(0)} = \mathbf{E}$, $\mathbf{A}^{(0)} = \mathbf{A}$ and $n^{(0)} = n$
$[\mathbf{E}^{(\beta)}, \mathbf{A}^{(\beta)}, n^{(\beta)}, \Delta] = \mathrm{DEFLATE}\ (\mathbf{E}^{(0)}, \mathbf{A}^{(0)}, n^{(0)})$
comment: *generalized eigenvalue decomposition*
$[\mathbf{E}_f, \mathbf{A}_f] = \mathrm{QZ}\ (\mathbf{E}^{(\beta)}, \mathbf{A}^{(\beta)})$
set $g_d = \Delta$; $n_p = n^{(\beta)}$; $p_i = a_{ii}/e_{ii}$, $i = 1 : n_p$
set $g = (-1)^{(n_z - n_p)} g_n / g_d$

Remark 4.1. Note that, since coalescent poles and zeros cancel, the above algorithm will only need to deflate either the poles or the zeros at infinity.

The total computational cost can be broken down as follows:

(a) *Obtaining observer Hessenberg form.* The initial reduction of **E** to an upper-triangular matrix requires $\frac{2}{3}n^3$ operations using QR factorization and an additional n^3 operations to update the system matrix. Reduction of **A** to an upper-Hessenberg matrix and condensation of the columns of **c** requires $\frac{16}{3}n^3$, leading to a total of $7n^3$ operations. The above count assumes that the transformations are not stored.

(b) *Deflation.* Owing to the triangular-Hessenberg structure of the matrix pair (**E**, **A**), each deflation of a zero at ∞ for the pencil $(\mathbf{A}-\lambda\mathbf{E})$ requires approximately $8n^{(l)}$ operations, where $n^{(l)}$ is the dimension of the pencil during the lth iteration. Note that the infinite zeros can also be deflated using the QZ algorithm; however, the latter would require $O(n^2)$ operations for each deflation. Hence the proposed implementation is considerably more efficient.

(c) *QZ decompositions.* Starting with triangular-Hessenberg structure of the pair (**E**, **A**), the computation of zeros and poles requires $10(n^{(\alpha)})^3$ and $10(n^{(\beta)})^3$ operations respectively.

In the worst case, where $n^{(\alpha)} = n^{(\beta)} = n$, the total operations count for the algorithm is approximately $27n^3$.

It should be pointed out that the entire algorithm essentially involves deflation of matrix pencil and two separate QZ decompositions using only unitary operations. In view of the analysis of numerical properties of the former in Van Dooren (1979) and of the latter in Moler and Stewart (1973), it is clear that computation of poles and zeros as separate entities is numerically backward stable. Unfortunately, it does not seem possible to attain the same backward error for both zeros and poles.

4.2. *Illustrative examples.* We next provide some examples to illustrate the performance of the proposed technique. All calculations were performed on a DEC Alpha using Matlab 4.1. For the sake of presentation, all numerical values have been rounded off to four significant decimal places.

Example 4.1. The first example illustrates the results for a *singular nonproper system.* Since **E** is singular, clearly the system cannot be transformed to the standard state-space description by premultiplying the state equation by the inverse of descriptor matrix **E**. The data for this example is given by the following matrices (Misra and Patel, 1989):

$$\mathbf{E} = \begin{bmatrix} 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 2 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 \\ 2 & 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 & 1 \end{bmatrix},$$

$\mathbf{b} = [1 \quad 0 \quad 1 \quad 1 \quad 1]^{\mathrm{T}}$, $\mathbf{c} = [1 \quad 5 \quad 3 \quad 1 \quad 2]$, $d = 0$.

On applying the algorithm developed in Section 4.1 the following pole–zero representation is obtained:

$$G(\lambda) = \frac{(\lambda - 0.1226 + 0.7449\mathrm{i})(\lambda - 0.1226 - 0.7449\mathrm{i})(\lambda - 1.7549)}{(\lambda - 1)(\lambda - 1)}.$$

If necessary, from the above pole–zero representation, the transfer function can be easily computed as

$$G(\lambda) = \frac{\lambda^3 - 2\lambda^2 + \lambda - 1}{\lambda^2 - 2\lambda + 1}.$$

For this example, the coefficients of the transfer function can be obtained analytically. The coefficients computed from the pole–zero representation were accurate to 15–16 decimal places.

Example 4.2. This example illustrates a case when inverting the descriptor matrix to obtain a 4-tuple of the standard state-space system can lead to extremely poor results. The parameters of the system (**E**, **A**, **b**, **c**, d) are as follows:

$$\mathbf{E} = \begin{bmatrix} 1.0 & 8.7 & 6.3 & 9.1 & 3.2 \\ 0.0 & 10^{-1} & 7.3 & 8.7 & 3.2 \\ 0.0 & 0.0 & 10^{-2} & 7.9 & 5.9 \\ 0.0 & 0.0 & 0.0 & 10^{-3} & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.0 & 10^{-4} \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} 7.8 & 9.2 & 7.1 & 1.2 & 6.3 \\ 2.9 & 2.7 & 2.2 & 5.1 & 6.7 \\ 5.5 & 5.9 & 7.7 & 6.8 & 3.3 \\ 8.6 & 8.2 & 7.2 & 0.5 & 7.3 \\ 7.1 & 4.8 & 1.7 & 5.5 & 3.7 \end{bmatrix},$$

$\mathbf{b} = [0.0 \quad 6.8 \quad 1.8 \quad 4.4 \quad 9.7]^{\mathrm{T}}$,

$\mathbf{c} = [7.7 \quad 3.3 \quad 1.2 \quad 6.0 \quad 7.3]$, $d = 1.0$

For this example, since $d = 1$, the zeros of the system can be computed from the eigenvalues of $\mathbf{E}^{-1}(\mathbf{A} - \mathbf{b}d^{-1}\mathbf{c})$ using the QR algorithm (Golub and Van Loan, 1989). The zeros computed using the QR algorithm and the proposed method are shown in Table 1. To compare he numerical accuracy of the zeros obtained by the two methods, we computed the singular values of the matrix $\left[\begin{array}{c|c} \mathbf{A}-\lambda\mathbf{E} & \mathbf{b} \\ \hline \mathbf{c} & d \end{array}\right]$, by replacing λ by the zeros computed using the eigenvalues of $\mathbf{E}^{-1}(\mathbf{A} - \mathbf{b}d^{-1}\mathbf{c})$ and the proposed method. The smallest singular values for each case are presented in Table 2.

It is evident from the above results that, for these data, the transformation of the 5-tuple (**E**, **A**, **b**, **c**, d) to standard state-space form leads to incorrect results.

In a similar manner, the finite poles computed using the two approaches are shown in Table 3, and their numerical accuracy is compared in Table 4. Of course, in this case the singular-value decomposition was performed on the pencil $(\mathbf{A} - \lambda\mathbf{E})$ by replacing λ by the computed finite poles.

The computed values of poles were considerably closer to the actual locations, unlike the zeros. However, the results tabulated above clearly indicate a several orders of

Table 1. Finite transmission zeros in Example 4.2

Using QR algorithm	Using proposed method
0.27468	0.41387
0.53993	±i0.11672
−1.5335	−1.1721
20.949	20.578

Table 2. Smallest singular values (for zeros) in Example 4.2

Using QR algorithm	Using proposed method
4.0231×10^{-1}	1.9946×10^{-14}
3.7486×10^{-1}	2.1146×10^{-14}
1.0489	1.5360×10^{-14}
9.7264×10^{-2}	1.4045×10^{-12}

Table 3. Finite poles in Example 4.2

Using QR algorithm	Using proposed method
0.34236	0.34143
±i0.088047	±i0.051349
−0.87699	−0.89473
25.140	25.160

Table 4. Smallest singular values (for poles) in Example 4.2

Using QR algorithm	Using proposed method
8.1745×10^{-2}	1.9741×10^{-14}
8.1745×10^{-2}	1.8228×10^{-14}
6.7607×10^{-2}	7.6535×10^{-14}
4.0443×10^{-2}	1.1160×10^{-12}

magnitude improvement in the accuracy of the computed poles using the proposed approach. The results presented in Tables 1–4 are for finite zeros and poles only. For the sake of completeness, the constant coefficient for the pole–zero representation was found to be $b = -9.848$.

5. *Concluding remarks*

In this paper we have presented an algorithm for transforming a single-input, single-output generalized state-space model (possibly singular) to its pole–zero representation using orthogonal transformations only. The results are easily modified for state-space systems represented by matrix-valued differential equations or standard-state space systems where **E** is an identity matrix of appropriate order. The proposed algorithm can also be used for multivariable systems by selecting appropriate input–output pairs.

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