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# Pole-Zero Representation and Transfer Function of Descriptor Systems

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## 1. Introduction

In this paper, we consider the problem of pole-zero representation of linear time-invariant *generalized* state space or *descriptor* systems (Luenberger, 1977, Verghese *et al.*, 1981) described by:

$$E\frac{d}{dt}x(t) = Ax(t) + bu(t) \qquad y(t) = cx(t) + du(t) \tag{1.1}$$

where $x(t) \in \mathbf{R}^n$, $u(t)$, $y(t) \in \mathbf{R}$ and $\det(\lambda E - A) \neq 0$, i.e., the pencil $(\lambda E - A)$ is regular. The transfer function of the system (1.1) is $G(\lambda) = c(\lambda E - A)^{-1}b + d$. We will denote the above system by the 5-tuple $(E, A, b, c, d)$. If the descriptor matrix $(E)$ has full rank, the system in (1.1) is said to be a *nonsingular* system, otherwise it is called a *singular* system.

If the system is nonsingular, theoretically, we can obtain an equivalent state space realization of $(E, A, b, c, d)$ by premultiplying the state equation by $E^{-1}$ to get an equivalent 4-tuple $(E^{-1}A, E^{-1}b, c, d)$. Once we have the 4-tuple $(E^{-1}A, E^{-1}b, c, d)$, we can easily obtain its pole-zero representation

$$G(\lambda) = \frac{g\prod_{i=1}^{n_z}(\lambda - \lambda_i^z)}{\prod_{j=1}^{n_p}(\lambda - \lambda_j^p)}. \tag{1.2}$$

where $\lambda_i^z$ denotes a zero, $\lambda_i^p$ denotes a pole, $n_z$ and $n_p$ are respectively the number of zeros and poles of $(E, A, b, c, d)$ and $g$ is the constant gain of the transfer function. Several algorithms for obtaining pole-zero representation of the 4-tuples $(E^{-1}A, E^{-1}b, c, d)$, exist e.g., see Varga and Sima (1981), Emami-Naeini and Van Dooren (1982), Misra and Patel (1987). In Varga and Sima (1981) and Emami-Naeini and Van Dooren (1982), poles and zeros are computed directly, while in Misra and Patel (1987), the transfer function is computed from where the pole-zero representation may be obtained. For descriptor systems, a numerically reliable way of computing transfer function is reported in (Misra, 1989) from where pole-zero representation for descriptor systems may be obtained. However, obtaining the form in Equation (1.2) by first computing the transfer function of the system in (1.1) can be numerically quite sensitive. A small perturbation in coefficient of the transfer function can lead to significant loss of accuracy in numerical computation of poles and/or zeros. A pole-zero representation algorithm was proposed by Varga (1989). The present approach is different from (Varga, 1989) and as pointed out in the sequel, it has several features that make it more efficient and reliable.

## 2. Main Results

*Observer Hessenberg form of* $(E, A, b, c, d)$

Given a single input single output 5-tuple $(E, A, b, c, d)$, there exist orthogonal transformation matrices $Q$ and $Z$ such that in the transformed 5-tuple,

$$(E, A, b, c, d) := (Q^T EZ, Q^T AZ, Q^T b, cZ, d), \tag{2.1}$$

$E$ is an upper triangular matrix, $A$ is an upper Hessenberg matrix, $b$ is a general dense vector and $c$ has only its last element nonzero. The system has no unobservable finite or infinite modes if and only if $A$ is an unreduced upper Hessenberg matrix, i.e., $a_{i+1,i} \neq 0$. If, however, $a_{i+1,i} = 0$, for some $i$, then the system can be block triangularly partitioned as:

$$E = \left[\begin{array}{c|c} E_{11} & E_{12} \\ \hline & E_{22}\end{array}\right], A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline & A_{22}\end{array}\right], b = \left[\begin{array}{c} b_1 \\ \hline b_2\end{array}\right], c = [\ o \mid c_2\ ] \tag{2.2}$$

where the generalized eigenvalues of the pair $(E_{11}, A_{11})$ are the unobservable modes of the system.

A computational algorithm for reducing the given system $(E, A, b, c, d)$ to the form in (2.1) (called the observer Hessenberg form) can easily devised based on the results in (Van Dooren and Verhaegen 1985, Misra 1989, Varga 1989).

By duality, a similar statement can be made regarding uncontrollable modes and the reduction of the system to a *controller Hessenberg form*. A singular system that does not have any uncontrollable and/or unobservable finite or infinite modes is said to be *irreducible* (Verghese *et al.*, 1981).

**Remark 2.1.** If an $n$-th order system is controllable/observable at infinity, then $\text{rank}[E\ |b] = \text{rank}\left[\frac{E}{c}\right] = n$ (Verghese *et al.*, 1981). An immediate consequence of this observation is that in the observer Hessenberg form, all the diagonal elements $e_{i,i}$ of the matrix $E$ are nonzero; with the possible exception of $e_{n,n}$.

Next, we develop the computational procedure. It is well known that if $\det(A - \lambda E) \neq 0$, then

$$c(\lambda E - A)^{-1}b + d = \frac{\det\left[\begin{array}{c|c} A - \lambda E & b \\ \hline c & d\end{array}\right]}{\det[A - \lambda E]} = G(\lambda). \tag{2.3}$$

Equation (2.3) shows how to compute the transfer function of a generalized state space system. In the rest of this section, we discuss how the pole-zero representation can be obtained in a numerically reliable manner without going through the step of computing the transfer function first.

From (2.3) it is easily seen that the normal rank of the system matrix $\left[\begin{array}{c|c} A - \lambda E & b \\ \hline c & d\end{array}\right] = n + 1$ if the transfer function $G(\lambda) \neq 0$. Then, from Rosenbrock (1970) and Misra *et al.* (1994), the next result follows:

**Theorem 2.1.** *For an irreducible single input single output system* $(E, A, b, c, d)$, *the finite transmission zeros are those complex values of* $\lambda$ *for which*

$$\text{rank}\left[\begin{array}{c|c} A - \lambda E & b \\ \hline c & d\end{array}\right] < n + 1. \tag{2.4}$$

*Proof.* See Rosenbrock (1970), Misra *et al.* (1994). □

### 2.1. Computation of Zeros

The intent here is to obtain a subpencil $(F_f - \lambda G_f)$, ($G_f$ has full rank), from the pencil in Equation (2.4). The zeros of $(F_f - \lambda G_f)$ correspond to the finite transmission zeros of the system. To achieve this, we deflate the pencil $\left[\begin{array}{c|c} A - \lambda E & b \\ \hline c & d\end{array}\right]$ until the required subpencil is obtained. The deflation is performed by column compression of the row vector $[c\ \ d]$ recursively using orthogonal transformations.

Since, the rank of matrix pencil is unaffected by orthogonal transformations, during $(\ell)$-th recursion

$$\rho\left[\begin{array}{c|c} A^{\ell-1} - \lambda E^{\ell-1} & b^{\ell+1}(\lambda) \\ \hline c^{\ell-1} & d_n^{\ell-1}\end{array}\right] = \rho\left[\begin{array}{c|c} A^{\ell-1} - \lambda E^{\ell-1} & b^{\ell-1}(\lambda) \\ \hline c^{\ell-1} & d_n^{\ell-1}\end{array}\right]\mathcal{G}^{\ell} = \rho\left[\begin{array}{c|c} A^{\ell} - \lambda E^{\ell} & b^{\ell}(\lambda) \\ \hline o & d_n^{\ell}\end{array}\right]$$

where $\mathcal{G}^{(\ell)}$ is an orthogonal matrix that compresses the columns of the vector $[c^{(\ell-1)}\ \ d_n^{(\ell-1)}]$. By the assumption on the normal rank of the matrix pencil in Equation (2.4), $d_n^{(\ell)}$ is nonzero. Hence, the pencil above and the one in Equation (2.4) have identical finite zeros. Note that due to the nature of the transformations, during deflation, the constant input vector $b$ in (2.4) becomes a pencil. Hence $b$ is replaced by $b(\lambda)$.

On compressing the columns of $[c^{(\ell-1)}\ \ d_n^{(\ell-1)}]$, and transforming the pencil, following two possibilities may occur:

**Case 1:** $E^{(\ell)}$ *is non-singular.* In this case finite zeros of the system are the generalized eigenvalues of the pair $(E^{(\ell)}, A^{(\ell)})$.

**Case 2:** $E^{(\ell)}$ *is singular.* In this case it will be possible to deflate the problem by further partitioning the pencil $[A^{(\ell)} - \lambda E^{(\ell)}]$ as

$$[A^{\ell} - \lambda E^{\ell}] = \left[\begin{array}{c|c} A_{11}^{\ell} - \lambda E_{11}^{\ell} & a_{12}^{\ell}(\lambda) \\ \hline a_{21}^{\ell} & a_{22}^{\ell}\end{array}\right] \stackrel{\text{def}}{=} \left[\begin{array}{c|c} A^{\ell} - \lambda E^{\ell} & b^{\ell}(\lambda) \\ \hline c^{\ell} & d_n^{\ell}\end{array}\right] \tag{2.5}$$

where the partitioning conforms to the dimension of the full rank matrix $E_{11}^{(\ell)}$. The problem can now be further deflated by performing column compression on the bottom row of the pencil in Equation (2.5). This is illustrated more clearly by means of an example.

**Remark 2.2.** Since the system being deflated is assumed irreducible, the matrix $E^{(\ell)}$ can not have nullity greater than 1. Further, at all stages in deflation, the matrices $[\ E^{(\ell)} \mid b^{(\ell)}\ ]$ and $\left[\begin{array}{c} E^{(\ell)} \\ \hline c^{(\ell)}\end{array}\right]$ must have full row and column rank respectively. Therefore, there can be only one Jordan block with eigenvalue at infinity. In terms of our algorithm, this implies that $d_n^{(i)}$ will all be scalars.

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Once a full rank matrix $E^{(\alpha)}$ is found, rank condition can be rewritten as

$$\rho\left[\begin{array}{c|c} A-\lambda E & b \\ \hline c & d \end{array}\right] = \rho \left[\begin{array}{c|cccc} A^{\alpha}-\lambda E^{\alpha} & & * & & \\ \hline & d_n^{\alpha-1} & * & \cdots & * \\ O & & d_n^{\alpha-2} & \cdots & * \\ & & & \ddots & \vdots \\ & & & & d_n^{1} \end{array}\right].$$

Based on the above discussion can we can state the following:

**Theorem 2.2.** *If in above equation, $E^{(\alpha)}$ has full rank, then finite zeros of the matrix pencil $\left[\begin{array}{c|c} A-\lambda E & b \\ \hline c & d \end{array}\right]$ are the generalized eigenvalues of the pair $(E^{(\alpha)}, A^{(\alpha)})$.*

*Proof.* On noting that finite zeros of a pencil are unaffected by orthogonal transformations, the result follows immediately. □

Further, since the deflated pencil on the right hand side in above equation is obtained using orthogonal transformations on the original (square) pencil,

$$\det\left[\begin{array}{c|c} A-\lambda E & b \\ \hline c & d \end{array}\right] = \det \left[\begin{array}{c|cccc} A^{\alpha}-\lambda E^{\alpha} & & * & & \\ \hline & d_n^{\alpha-1} & * & \cdots & * \\ O & & d_n^{\alpha-2} & \cdots & * \\ & & & \ddots & \vdots \\ & & & & d_n^{1} \end{array}\right]$$

$$= \det(A^{\alpha}-\lambda E^{\alpha}) d_n^1 \cdots d_n^{\alpha-1}$$

$$= \det([E^{\alpha}]^{-1} A^{\alpha} - \lambda I) \det(E^{\alpha}) d_n^1 \cdots d_n^{\alpha-1} \tag{2.6}$$

where, $\det(E^{(\alpha)}) d_n^{(1)} d_n^{(2)} \cdots d_n^{(\alpha-1)}$ is a nonzero constant and generalized eigenvalues of the pair $(E^{(\alpha)}, A^{(\alpha)})$, are the zeros of the transfer function.

**Remark 2.3.** The result in Equation (2.6) highlights the major differences between the proposed algorithm and the one by Varga (1989). (1) The present procedure uses only (numerically stable) orthogonal transformations to deflate the pencil; however, in the latter, the pencil is deflated by applying state feedback to the transformed system. If the element of the input vector used for computing the feedback gain (for deflation) has very small magnitude, the resulting gains will be large and applying feedback *can* cause the resulting generalized eigenvalue problem to become numerically ill-conditioned. Hence the proposed approach is numerically more reliable. (2) In the present case, the deflated pencil will contain only finite zeros of the system, while the deflated pencil in Varga (1989) can contain both finite as well as infinite zeros, requiring that $QZ$ algorithm be applied to a larger pencil. Therefore the present approach is computationally more efficient.

### 2.2. Computation of Poles

Using similar arguments as in Section 2.1, we can transform the pencil $(A - \lambda E)$ such that

$$\det[A-\lambda E] = \det \left[\begin{array}{c|cccc} A^{(\beta)}-\lambda E^{(\beta)} & & * & & \\ \hline & d_d^{(\beta-1)} & * & \cdots & * \\ O & & d_d^{(\beta-2)} & \cdots & * \\ & & & \ddots & \vdots \\ & & & & d_d^{(1)} \end{array}\right]$$

$$= \det(A^{(\beta)}-\lambda E^{(\beta)}) d_d^{(1)} \cdots d_d^{(\beta-1)}$$

$$= \det([E^{(\beta)}]^{-1} A^{(\beta)} - \lambda I) \det(E^{(\beta)}) d_d^{(1)} \cdots d_d^{(\beta-1)} \tag{2.7}$$

where $E^{(\beta)}$ is a full rank matrix. Poles of the system are the roots of the polynomial $\det([E^{(\beta)}]^{-1} A^{(\beta)} - \lambda I)$ or equivalently, the generalized eigenvalues of the matrix pair $(E^{(\beta)}, A^{(\beta)})$ and $\det(E^{(\beta)}) d_d^{(1)} \cdots d_d^{(\beta-1)}$ is a non-zero constant.

From Equations (2.6) and (2.7), it is clear that the coefficient $g$ in (1.2) may be computed as

$$g = \frac{(-1)^{n_z-n_p} \det(E^{(\alpha)}) d_n^{(1)} \cdots d_n^{(\alpha-1)}}{\det(E^{(\beta)}) d_d^{(1)} \cdots d_d^{(\beta-1)}} = (-1)^{n_z-n_p} \frac{\prod_{i=1}^{n_z} e_{ii}^{(\alpha)} \prod_{i=1}^{\alpha-1} d_n^{(i)}}{\prod_{i=1}^{n_p} e_{ii}^{(\beta)} \prod_{i=1}^{\beta-1} d_d^{(i)}}$$

where $n_z$ and $n_p$ are respectively the orders of the *upper triangular* matrices $E^{(\alpha)}$ and $E^{(\beta)}$. Knowing the coefficient $g$ and the locations of zeros and poles as the generalized eigenvalues of the pencils $(E^{(\alpha)}, A^{(\alpha)})$ and $(E^{(\beta)}, A^{(\beta)})$ respectively, the desired pole-zero representation of the single input single output system $(E, A, b, c, d)$ is completely determined.

## 3. Example

**Example 3.1.** This example illustrates a case when inverting the descriptor matrix to obtain 4-tuple of standard state space system can lead to extremely poor results. The parameters of the system $(E, A, b, c, d)$ are given below:

$$E = \begin{bmatrix} 1.0 & 8.7 & 6.3 & 9.1 & 3.2 \\ 0.0 & 1.0e^{-1} & 7.3 & 8.7 & 3.2 \\ 0.0 & 0.0 & 1.0e^{-2} & 7.9 & 5.9 \\ 0.0 & 0.0 & 0.0 & 1.0e^{-3} & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0e^{-4} \end{bmatrix},$$

$$\left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right] = \left[\begin{array}{ccccc|c} 7.8 & 9.2 & 7.1 & 1.2 & 6.3 & 0.0 \\ 2.9 & 2.7 & 2.2 & 5.1 & 6.7 & 6.8 \\ 5.5 & 5.9 & 7.7 & 6.8 & 3.3 & 1.8 \\ 8.6 & 8.2 & 7.2 & 0.5 & 7.3 & 4.4 \\ 7.1 & 4.8 & 1.7 & 5.5 & 3.7 & 9.7 \\ \hline 7.7 & 3.3 & 1.2 & 6.0 & 7.3 & 1.0 \end{array}\right]$$

For this example, since $d = 1$, the zeros of the system can be computed by eigenvalues of $E^{-1}(A - bd^{-1}c)$ using $QR$ algorithm (Golub, Van Loan, 1989). The zeros computed using $QR$ algorithm and the proposed method are tabulated in Table 1. To compare the numerical accuracy cf zeros obtained by the two methods we computed the singular values of the matrix $\left[\begin{array}{c|c} A-\lambda E & b \\ \hline c & d \end{array}\right]$, by replacing $\lambda$ by the zeros computed using eigenvalues of $E^{-1}(A - bd^{-1}c)$ and the proposed method. The smallest singular values for each case are presented in Table 2.

TABLE 1

*Finite transmission zeros*

| QR | Proposed |
|---|---|
| $2.7468e^{-01}$ | $4.1387e^{-01}$ |
| $5.3993e^{-01}$ | $\pm i1.1672e^{-01}$ |
| $-1.5335e^{+00}$ | $-1.1721e^{+00}$ |
| $2.0949e^{+01}$ | $2.0578e^{+01}$ |

TABLE 2

*Smallest sing. values (zeros)*

| QR | Proposed |
|---|---|
| $4.0231e^{-01}$ | $1.9946e^{-14}$ |
| $3.7486e^{-01}$ | $2.1146e^{-14}$ |
| $1.0489e^{+00}$ | $1.5360e^{-14}$ |
| $9.7264e^{-02}$ | $1.4045e^{-12}$ |

It is evident from these results that for this data, the transformation of the 5-tuple $(E, A, b, c, d)$ to standard state space form leads to incorrect results.

In a similar manner, the finite poles computed using the two approaches are tabulated in Table 3 and their numerical accuracy is compared in Table 4. Of course, in this case, singular value decomposition was performed on the pencil $(A - \lambda E)$ by replacing $\lambda$ by the computed finite poles.

TABLE 3

*Finite poles*

| QR | Proposed |
|---|---|
| $3.4236e^{-01}$ | $3.4143e^{-01}$ |
| $\pm i8.8047e^{-02}$ | $\pm i5.1349e^{-02}$ |
| $-8.7699e^{-01}$ | $-8.9473e^{-01}$ |
| $2.5140e^{+01}$ | $2.5160e^{+01}$ |

TABLE 4

*Smallest sing. values (poles)*

| QR | Proposed |
|---|---|
| $8.1745e^{-02}$ | $1.9741e^{-14}$ |
| $8.1745e^{-02}$ | $1.8228e^{-14}$ |
| $6.7607e^{-02}$ | $7.6535e^{-14}$ |
| $4.0443e^{-03}$ | $1.1160e^{-12}$ |

The computed values of poles were considerably closer to the actual locations, unlike zeros. Yet, the results tabulated above clearly indicate several orders of magnitude improvement in accuracy of computed poles using proposed approach. The results presented in Tables 1 – 4 are for finite zeros and poles only. For the sake of completeness, the constant coefficient for the pole-zero representation was found to be $g = -9.848$.

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